MATH 343 · GROUP THEORY · CHAPTER 1

Introduction to Groups

Symmetries of a Square · The Dihedral Groups

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Part 1 — Symmetries of a Square

GOAL. Describe *every* way a square region can be removed from the plane and put back into the space it originally occupied.

- We record only the **net effect** of a motion, not the motion itself
- We will find that there are exactly **eight** such motions
- Composing two of them always returns one of the eight
- Out of this one example the whole definition of a group will emerge

The Question

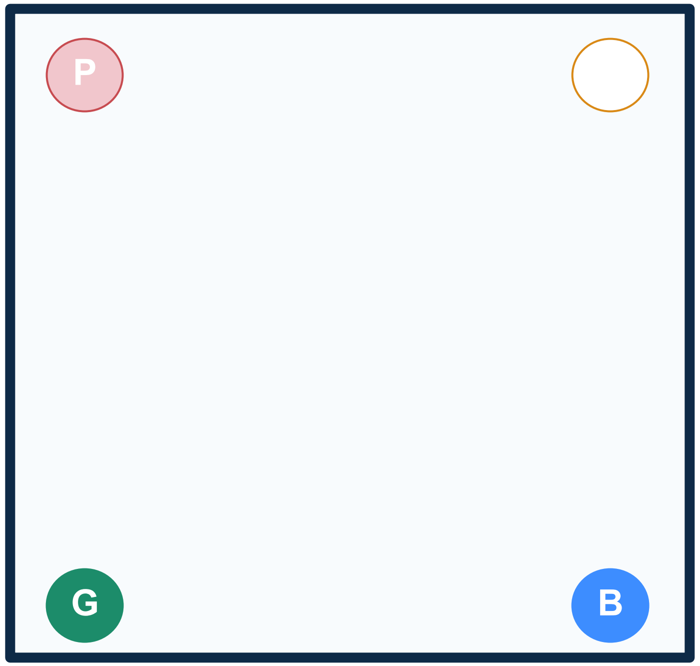
Remove a square region from the plane, move it in some way, and put it back into the space it originally occupied.

QUESTION. What are all the possible relationships between the starting position and the final position?

- The answer is to be given **in terms of motions**
- Two motions are regarded as **equal** when they have the same net effect on every point
- So a rotation of 90° and a rotation of 450° are the *same* motion

Making the Motions Visible

Think of the square as transparent glass with the four corners colored **P**, **W**, **G**, **B**:



- The colors let us distinguish motions with different effects
- A motion is completely recorded by where the four colors end up

Why Exactly Eight

CLAIM. Every motion of the square — however complicated — has the same net effect as one of eight motions.

PROOF. The final position of the square is completely determined by the location and the orientation (face up or face down) of any one chosen corner.

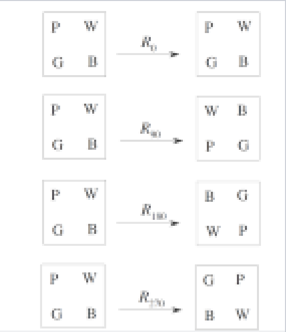
There are 4 possible locations for that corner and 2 possible orientations, hence at most

$$4 \cdot 2 = 8$$

distinct final positions. Exhibiting eight motions with distinct effects shows that all eight occur.

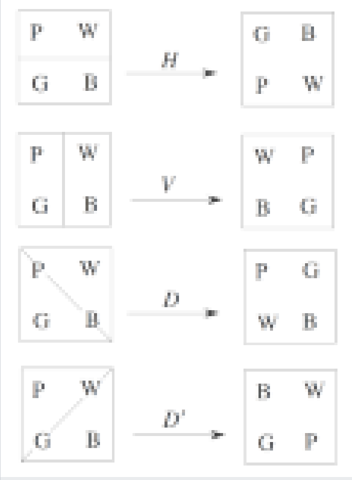
The Four Rotations

All rotations are **counterclockwise** about the centre of the square.

Motion	Description	Result
R_0	rotation of 0° (no change)	
R_{90}	rotation of 90°	
R_{180}	rotation of 180°	
R_{270}	rotation of 270°	

The Four Reflections

Each is a flip in three dimensions about an axis of symmetry — equivalently, a reflection across a line in the plane.

Motion	Axis	Result
H	horizontal axis	
V	vertical axis	
D	main diagonal (top-left to bottom-right)	
D'	other diagonal (top-right to bottom-left)	

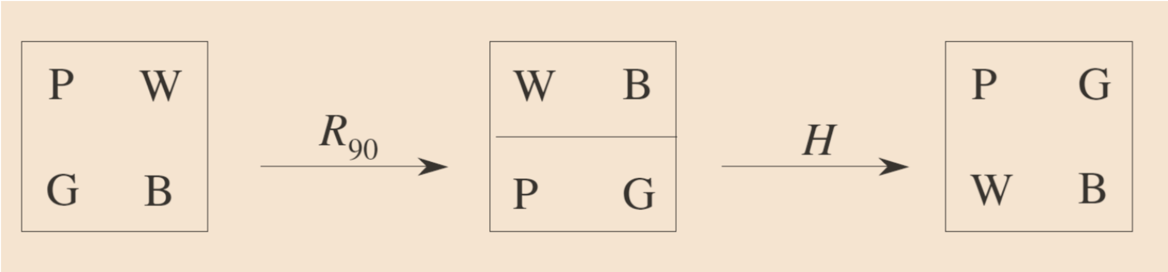
Composing Two Motions

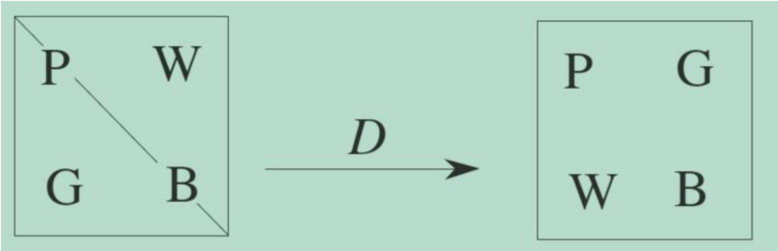
The eight motions are **functions** from the square region to itself, so they can be combined by **function composition**.

CONVENTION. As in earlier courses, $f \circ g$ means "g followed by f". We suppress the $\circ$ and write fg.

$$HR_{90} \quad \text{means} \quad \text{first } R_{90}, \text{ then } H$$

- Read products **right to left**





The Cayley Table of D_4

Entry in row a, column b is the product ab, meaning b first and then a.

$HR_{90}=D$

↗	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

D_4 Is Not Abelian

Compare the two products of H and D, read off the table:

$$HD = R_{90}, \qquad DH = R_{270}.$$

Since $R_{90} \neq R_{270}$ we have $HD \neq DH$.

TERMINOLOGY. If $ab = ba$ for *all* a, b, the group is **commutative**, or **Abelian**, after Niels Abel. Otherwise it is **non-Abelian**.

- One failing pair is enough to make a group non-Abelian
- Some pairs still commute: $R_{90}R_{180} = R_{180}R_{90} = R_{270}$

Summary — the Group D_4

The eight motions $R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'$ under composition form the **dihedral group of order 8**, written D_4.

- **closed** under composition
- **associative**, because composition of functions is
- has an **identity** R_0
- every element has an **inverse**
- **non-Abelian**, since $HD \neq DH$

The *order* of a group is the number of elements it contains, so $|D_4| = 8$.

Part 2 — The Dihedral Groups

GOAL. Carry out the same analysis for a regular n-gon and describe the resulting group algebraically.

- The symmetries of a regular n-gon form the **dihedral group** D_n, of order $2n$
- For large n a Cayley table is useless, so we need an algebraic description
- Two relations do all the work: $F^2 = R_0$ and $FR = R^{-1}F$

The Dihedral Group D_n

The analysis carried out for the square works for an equilateral triangle, a regular pentagon, and indeed any regular n-gon with $n \geq 3$.

DEFINITION. For $n \geq 3$, the group of symmetries of a regular n-gon under composition is the **dihedral group of order** $2n$, denoted D_n.

- n rotations: $R_0, R_{360/n}, R_{2(360)/n}, \ldots, R_{(n-1)360/n}$
- n reflections, one in each axis of symmetry
- hence $|D_n| = 2n$

The Axes of a Regular n-gon

Where the n reflection axes lie depends on the parity of n.

n **odd.** Every axis joins one vertex to the midpoint of the opposite side. There are n such axes, and they are all of the same type.

n **even.** There are $n/2$ axes through pairs of opposite vertices and $n/2$ axes through midpoints of pairs of opposite sides — n axes in all, of two distinct types.

Rotation or Reflection?

Assign $+1$ to each rotation and -1 to each reflection. Then the type of a product is the **product of the signs**.

- rotation $\cdot$ rotation $=$ rotation
- reflection $\cdot$ reflection $=$ rotation
- rotation $\cdot$ reflection $=$ reflection $=$ reflection $\cdot$ rotation

CONSEQUENCE. A product of m rotations and n reflections is a rotation if and only if n is **even**.

The Algebraic Form of D_n

In D_n write $R = R_{360/n}$ and let F be **any one** reflection. Then

$$D_n = \{\, R_0,\, R,\, R^2,\, \dots,\, R^{n-1},\; F,\, RF,\, R^2F,\, \dots,\, R^{n-1}F \,\}.$$

Two relations govern all computation:

$$R^n = R_0, \qquad F^2 = R_0, \qquad FR^k = R^{-k}F.$$

Using the Relations

The rule $FR^k = R^{-k}F$ lets us move every F to the right, changing the sign of the exponent each time it passes an R.

EXAMPLE. In D_{10}, where $R^{10} = R_0$:

$$R^3FR^7F \;=\; R^3R^{-7}FF \;=\; R^{-4} \;=\; R^6,$$

$$FR^4 = R^{-4}F = R^6F, \qquad FR^{-3}F = R^3FF = R^3.$$

What to Take Away

- Every motion of a square is one of **eight**; they form D_4, of order 8
- Products are read **right to left**: HR_{90} means R_{90} first
- The Cayley table exhibits **closure**, an **identity**, **inverses**, and the **Latin square** property; associativity comes free from function composition
- D_4 is **non-Abelian**: $HD = R_{90}$ but $DH = R_{270}$
- D_n has order $2n$ and is generated by R and F subject to $R^n = F^2 = R_0$ and $FR = R^{-1}F$

Next: these four properties become the **DEFINITION OF A GROUP**.