

05 – p2 (Appendix A) – Area Moments of Inertia

STATICS, AGE-1330

Ahmed M El-Sherbeeny, PhD

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Watch these videos first:

- **Basic concept of MOI**



- **Application 1**



- **Application 2**



Rectangular and Polar Moments of Inertia

Consider the area A in the x - y plane, Fig. A/2. The moments of inertia of the element dA about the x - and y -axes are, by definition, $dI_x = y^2 dA$ and $dI_y = x^2 dA$, respectively. The moments of inertia of A about the same axes are therefore

$$I_x = \int y^2 dA$$

$$I_y = \int x^2 dA$$

(A/1)

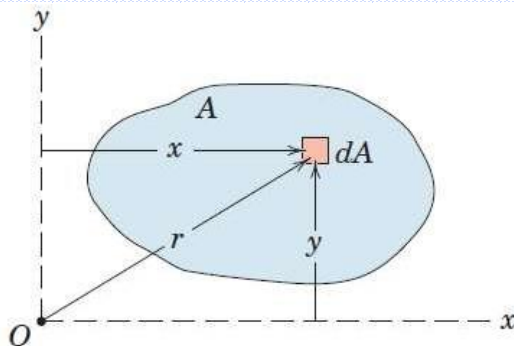


Figure A/2

The moment of inertia of dA about the pole O (z -axis) is, by similar definition, $dI_z = r^2 dA$. The moment of inertia of the entire area about O is

$$I_z = \int r^2 dA \quad (\text{A/2})$$

The expressions defined by Eqs. A/1 are called *rectangular* moments of inertia, whereas the expression of Eq. A/2 is called the *polar* moment of inertia.* Because $x^2 + y^2 = r^2$, it is clear that

$$I_z = I_x + I_y \quad (\text{A/3})$$

For an area whose boundaries are more simply described in rectangular coordinates than in polar coordinates, its polar moment of inertia is easily calculated with the aid of Eq. A/3.

Radius of Gyration

Consider an area A , Fig. A/3a, which has rectangular moments of inertia I_x and I_y and a polar moment of inertia I_z about O . We now visualize this area as concentrated into a long narrow strip of area A a distance k_x from the x -axis, Fig. A/3b. By definition the moment of inertia of the strip about the x -axis will be the same as that of the original area if $k_x^2 A = I_x$. The distance k_x is called the *radius of gyration* of the area

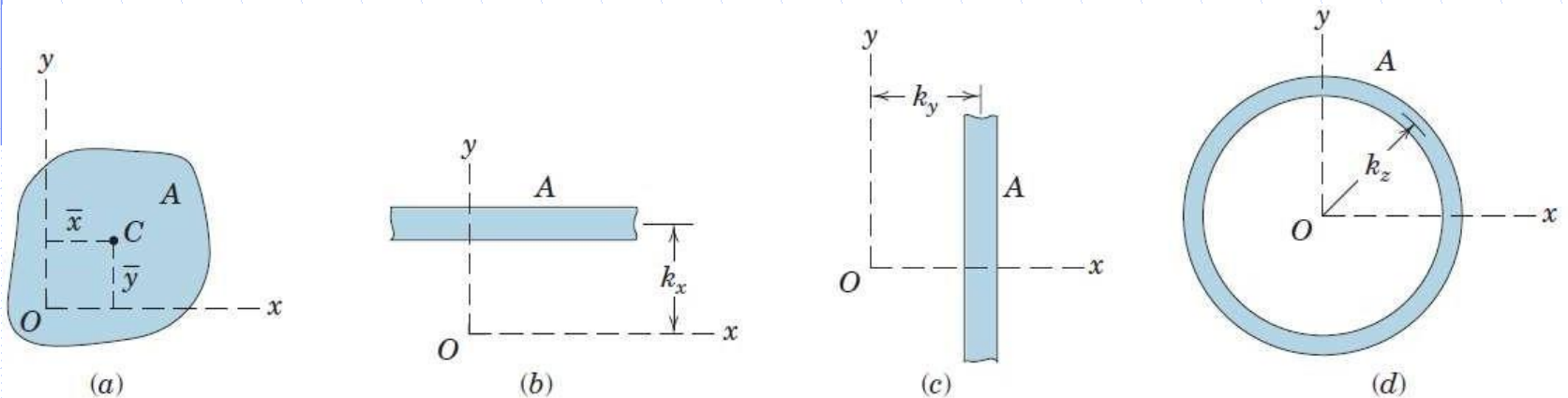


Figure A/3

about the x -axis. A similar relation for the y -axis is written by considering the area as concentrated into a narrow strip parallel to the y -axis as shown in Fig. A/3c. Also, if we visualize the area as concentrated into a narrow ring of radius k_z as shown in Fig. A/3d, we may express the polar moment of inertia as $k_z^2 A = I_z$. In summary we write

$$I_x = k_x^2 A$$

$$I_y = k_y^2 A$$

$$I_z = k_z^2 A$$

or

$$k_x = \sqrt{I_x/A}$$

$$k_y = \sqrt{I_y/A}$$

$$k_z = \sqrt{I_z/A}$$

(A/4)

The radius of gyration, then, is a measure of the distribution of the area from the axis in question. A rectangular or polar moment of inertia may be expressed by specifying the radius of gyration and the area.

When we substitute Eqs. A/4 into Eq. A/3, we have

$$k_z^2 = k_x^2 + k_y^2$$

(A/5)

Transfer of Axes

The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Fig. A/4 the x_0 - y_0 axes pass through the centroid C of the area. Let us now determine the moments of inertia of the area about the

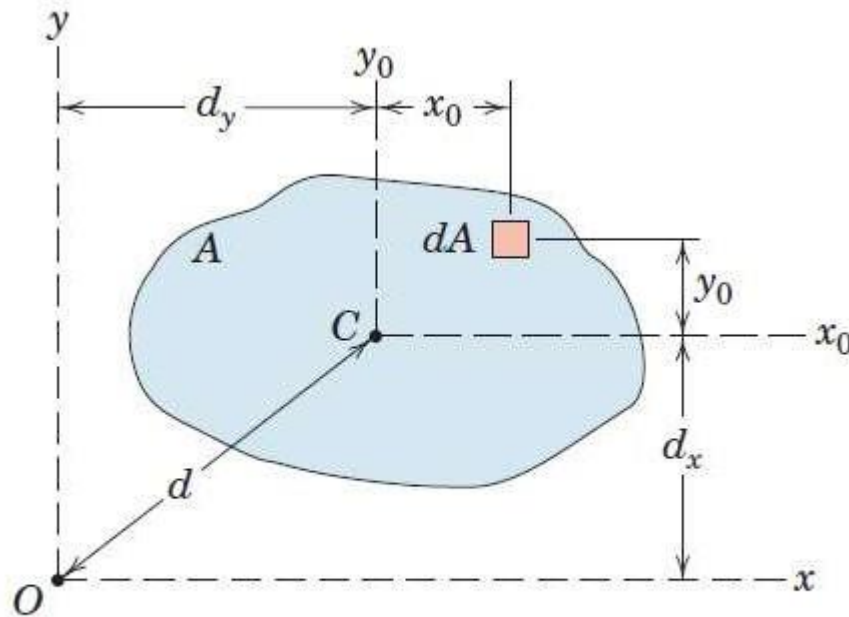


Figure A/4

parallel x - y axes. By definition, the moment of inertia of the element dA about the x -axis is

$$dI_x = (y_0 + d_x)^2 dA$$

Expanding and integrating give us

$$I_x = \int y_0^2 dA + 2d_x \int y_0 dA + d_x^2 \int dA$$

We see that the first integral is by definition the moment of inertia \bar{I}_x about the centroidal x_0 -axis. The second integral is zero, since $\int y_0 dA = A\bar{y}_0$ and \bar{y}_0 is automatically zero with the centroid on the x_0 -axis. The third term is simply Ad_x^2 . Thus, the expression for I_x and the similar expression for I_y become

$$\begin{aligned} I_x &= \bar{I}_x + Ad_x^2 \\ I_y &= \bar{I}_y + Ad_y^2 \end{aligned} \tag{A/6}$$

By Eq. A/3 the sum of these two equations gives

$$I_z = \bar{I}_z + Ad^2 \tag{A/6a}$$

SAMPLE PROBLEM A/1

Determine the moments of inertia of the rectangular area about the centroidal x_0 - and y_0 -axes, the centroidal polar axis z_0 through C , the x -axis, and the polar axis z through O .

Solution. For the calculation of the moment of inertia \bar{I}_x about the x_0 -axis, a horizontal strip of area $b \, dy$ is chosen so that all elements of the strip have the same y -coordinate. Thus,

$$[I_x = \int y^2 \, dA] \quad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b \, dy = \frac{1}{12} b h^3 \quad \text{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0 -axis is

$$\bar{I}_y = \frac{1}{12} h b^3 \quad \text{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \quad \bar{I}_z = \frac{1}{12} (b h^3 + h b^3) = \frac{1}{12} A (b^2 + h^2) \quad \text{Ans.}$$

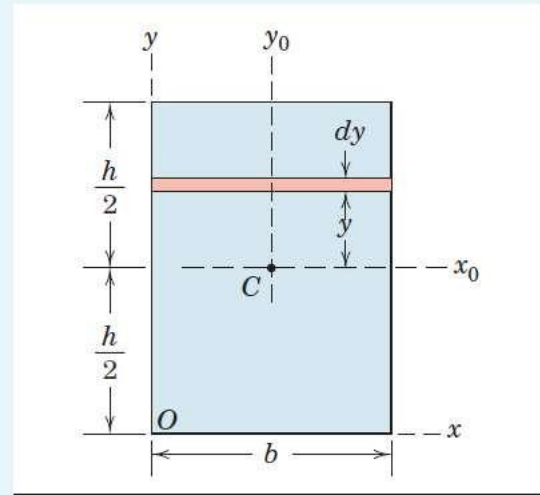
By the parallel-axis theorem the moment of inertia about the x -axis is

$$[I_x = \bar{I}_x + A d_x^2] \quad I_x = \frac{1}{12} b h^3 + b h \left(\frac{h}{2} \right)^2 = \frac{1}{3} b h^3 = \frac{1}{3} A h^2 \quad \text{Ans.}$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + A d^2] \quad I_z = \frac{1}{12} A (b^2 + h^2) + A \left[\left(\frac{b}{2} \right)^2 + \left(\frac{h}{2} \right)^2 \right]$$

$$I_z = \frac{1}{3} A (b^2 + h^2) \quad \text{Ans.}$$



Helpful Hint

- 1 If we had started with the second-order element $dA = dx \, dy$, integration with respect to x holding y constant amounts simply to multiplication by b and gives us the expression $y^2 b \, dy$, which we chose at the outset.

SAMPLE PROBLEM A/2

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.

- 1 Solution.** A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x dy = [(h - y)b/h] dy$. By definition

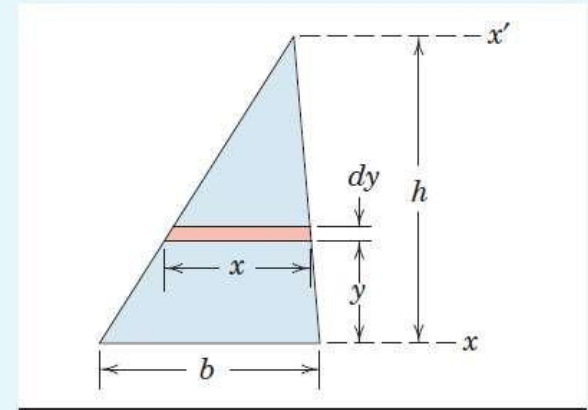
$$[I_x = \int y^2 dA] \quad I_x = \int_0^h y^2 \frac{h-y}{h} b dy = b \left[\frac{y^3}{3} - \frac{y^4}{4h} \right]_0^h = \frac{bh^3}{12} \quad \text{Ans.}$$

By the parallel-axis theorem the moment of inertia \bar{I} about an axis through the centroid, a distance $h/3$ above the x -axis, is

$$[\bar{I} = I - Ad^2] \quad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2} \right) \left(\frac{h}{3} \right)^2 = \frac{bh^3}{36} \quad \text{Ans.}$$

A transfer from the centroidal axis to the x' -axis through the vertex gives

$$[I = \bar{I} + Ad^2] \quad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2} \right) \left(\frac{2h}{3} \right)^2 = \frac{bh^3}{4} \quad \text{Ans.}$$



Helpful Hints

- Here again we choose the simplest possible element. If we had chosen $dA = dx dy$, we would have to integrate $y^2 dx dy$ with respect to x first. This gives us $y^2 x dy$, which is the expression we chose at the outset.
- Expressing x in terms of y should cause no difficulty if we observe the proportional relationship between the similar triangles.

SAMPLE PROBLEM A/3

Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.

- 1 **Solution.** A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z -axis through O since all elements of the ring are equidistant from O . The elemental area is $dA = 2\pi r_0 dr_0$, and thus,

$$[I_z = \int r^2 dA] \quad I_z = \int_0^r r_0^2 (2\pi r_0 dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2 \quad \text{Ans.}$$

The polar radius of gyration is

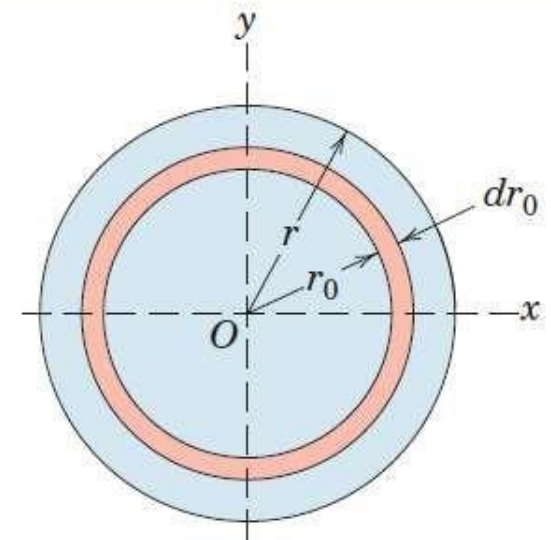
$$\left[k = \sqrt{\frac{I}{A}} \right] \quad k_z = \frac{r}{\sqrt{2}} \quad \text{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \quad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 \quad \text{Ans.}$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}} \right] \quad k_x = \frac{r}{2} \quad \text{Ans.}$$



A/3 Composite Areas

It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape. Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity. The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

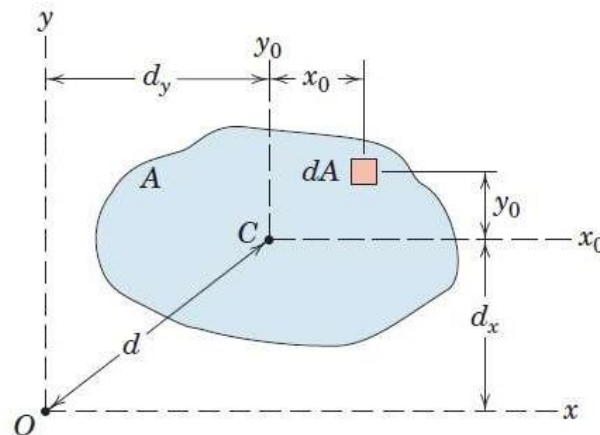


Figure A/4

For such an area in the x - y plane, for example, and with the notation of Fig. A/4, where \bar{I}_x is the same as I_{x_0} and \bar{I}_y is the same as I_{y_0} the tabulation would include

Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	\bar{I}_x	\bar{I}_y
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma \bar{I}_x$	$\Sigma \bar{I}_y$

From the sums of the four columns, then, the moments of inertia for the composite area about the x - and y -axes become

$$I_x = \Sigma \bar{I}_x + \Sigma Ad_x^2$$

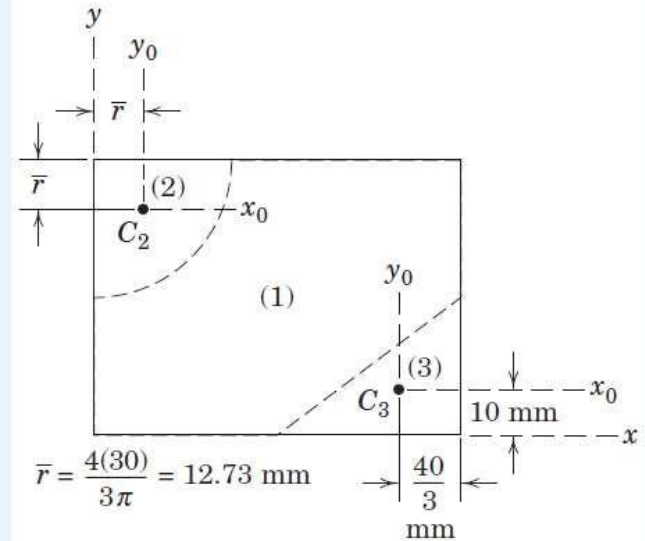
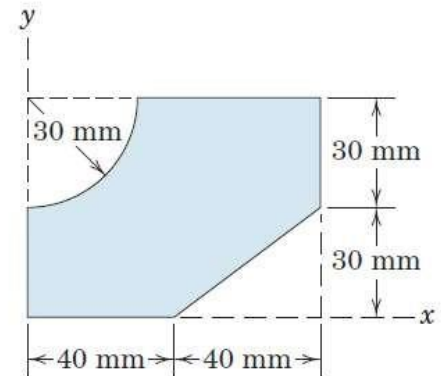
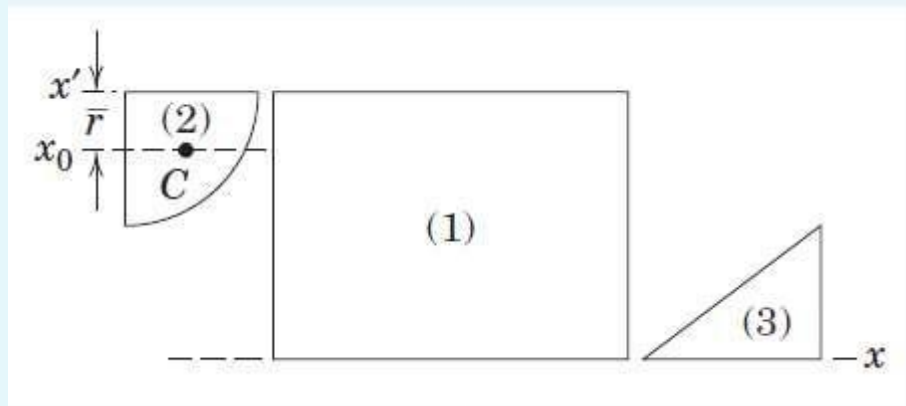
$$I_y = \Sigma \bar{I}_y + \Sigma Ad_y^2$$

SAMPLE PROBLEM A/7

Determine the moments of inertia about the x - and y -axes for the shaded area. Make direct use of the expressions given in Table D/3 for the centroidal moments of inertia of the constituent parts.

Solution. The given area is subdivided into the three subareas shown—a rectangular (1), a quarter-circular (2), and a triangular (3) area. Two of the subareas are “holes” with negative areas. Centroidal x_0 - y_0 axes are shown for areas (2) and (3), and the locations of centroids C_2 and C_3 are from Table D/3.

The following table will facilitate the calculations.



PART	A mm ²	d_x mm	d_y mm	Ad_x^2 mm ³	Ad_y^2 mm ³	\bar{I}_x mm ⁴	\bar{I}_y mm ⁴
1	80(60)	30	40	4.32(10 ⁶)	7.68(10 ⁶)	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$-\frac{1}{4}\pi(30)^2$	(60 - 12.73)	12.73	-1.579(10 ⁶)	-0.1146(10 ⁶)	$-\left(\frac{\pi}{16} - \frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16} - \frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80 - \frac{40}{3}\right)$	-0.06(10 ⁶)	-2.67(10 ⁶)	$-\frac{1}{36}40(30)^3$	$-\frac{1}{36}(30)(40)^3$
TOTALS	3490			2.68(10 ⁶)	4.90(10 ⁶)	1.366(10 ⁶)	2.46(10 ⁶)

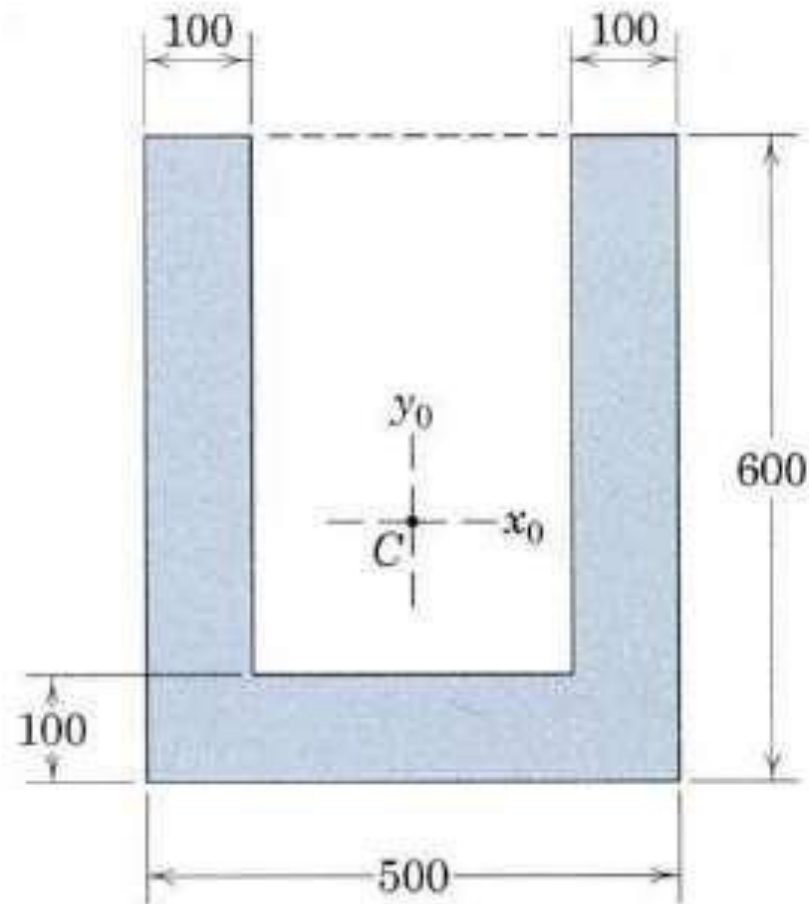
$$[I_x = \Sigma \bar{I}_x + \Sigma Ad_x^2] \quad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6) \text{ mm}^4 \quad \text{Ans.}$$

$$[I_y = \Sigma \bar{I}_y + \Sigma Ad_y^2] \quad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6) \text{ mm}^4 \quad \text{Ans.}$$

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490 \text{ mm}^2$ so that the radius of gyration about the x -axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0 \text{ mm} \quad \text{Ans.}$$

A/50 Calculate the polar radius of gyration of the shaded area about its centroid C .



Dimensions in millimeters

$$\frac{A}{50}$$

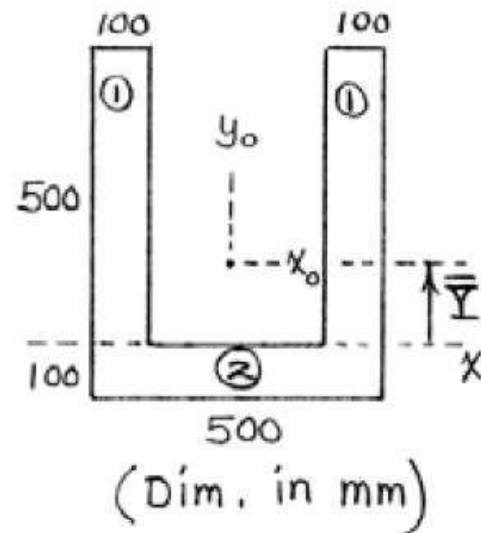
$$\bar{Y} = \frac{\sum A\bar{y}}{\sum A}$$

$$= \frac{2[(100)(500)(250) + 500(100)(-50)]}{2(100)(500) + 100(500)}$$

$$= 150 \text{ mm}$$

$$A = 2(100)(500) + 100(500)$$

$$= 15(10^4) \text{ mm}^2$$



$$\textcircled{1} + \textcircled{1} \quad I_{x_0} = 2 \left[\frac{1}{12} (100)(500)^3 + 100(500)(250 - 150)^2 \right]$$

$$= 30.8(10^8) \text{ mm}^4$$

$$I_{y_0} = 2 \left[\frac{1}{12} (500)(100)^3 + 100(500)(150 + 50)^2 \right] = 40.8(10^8) \text{ mm}^4$$

$$\textcircled{2} \quad I_{x_0} = \frac{1}{12} (500)(100)^3 + 100(500)(50 + 150)^2 = 20.4(10^8) \text{ mm}^4$$

$$I_{y_0} = \frac{1}{12} (100)(500)^3 = 10.42(10^8) \text{ mm}^4$$

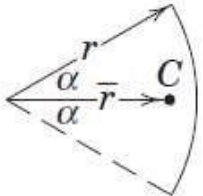
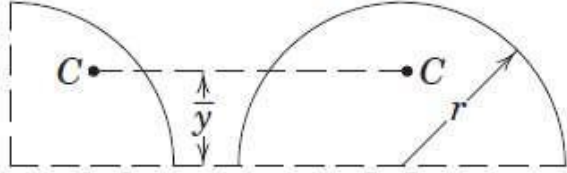
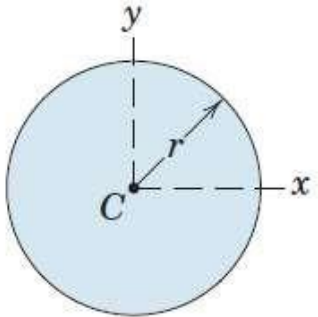
$$\text{Totals } \textcircled{1} + \textcircled{1} + \textcircled{2} : \quad I_{x_0} = 51.2(10^8) \text{ mm}^4$$

$$I_{y_0} = 51.2(10^8) \text{ mm}^4$$

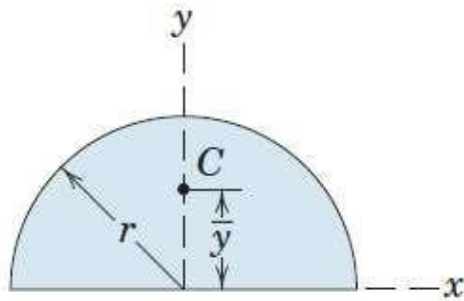
$$I_c = I_{x_0} + I_{y_0} = 102.5(10^8) \text{ mm}^4$$

$$k_c = \sqrt{I_c/A} = \sqrt{\frac{102.5(10^8)}{15(10^4)}} = \underline{261 \text{ mm}}$$

TABLE D/3 PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
<p>Arc Segment</p> 	$\bar{r} = \frac{r \sin \alpha}{\alpha}$	<p>—</p>
<p>Quarter and Semicircular Arcs</p> 	$\bar{y} = \frac{2r}{\pi}$	<p>—</p>
<p>Circular Area</p> 	<p>—</p>	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$

Semicircular Area



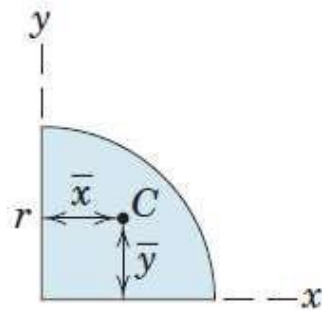
$$\bar{y} = \frac{4r}{3\pi}$$

$$I_x = I_y = \frac{\pi r^4}{8}$$

$$\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi} \right) r^4$$

$$I_z = \frac{\pi r^4}{4}$$

Quarter-Circular Area



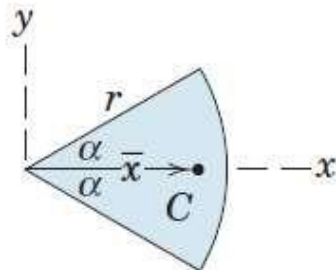
$$\bar{x} = \bar{y} = \frac{4r}{3\pi}$$

$$I_x = I_y = \frac{\pi r^4}{16}$$

$$\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi} \right) r^4$$

$$I_z = \frac{\pi r^4}{8}$$

Area of Circular Sector



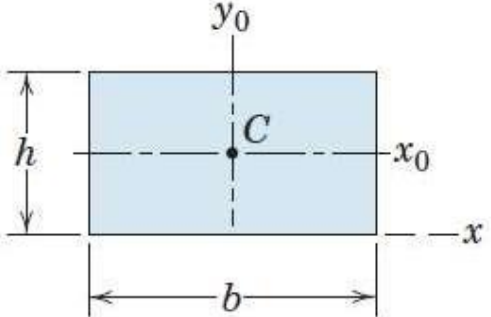
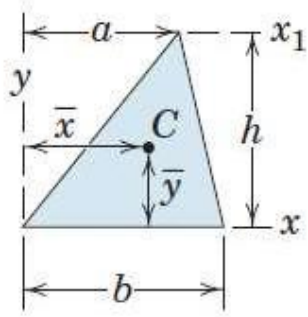
$$\bar{x} = \frac{2}{3} \frac{r \sin \alpha}{\alpha}$$

$$I_x = \frac{r^4}{4} \left(\alpha - \frac{1}{2} \sin 2\alpha \right)$$

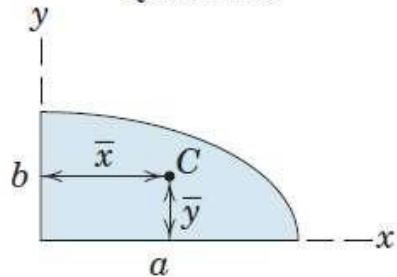
$$I_y = \frac{r^4}{4} \left(\alpha + \frac{1}{2} \sin 2\alpha \right)$$

$$I_z = \frac{1}{2} r^4 \alpha$$

TABLE D/3 PROPERTIES OF PLANE FIGURES *Continued*

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
<p>Rectangular Area</p> 	<p>—</p>	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
<p>Triangular Area</p> 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$

Area of Elliptical Quadrant



$$\bar{x} = \frac{4a}{3\pi}$$

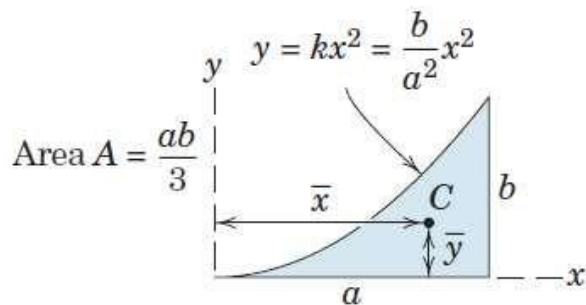
$$\bar{y} = \frac{4b}{3\pi}$$

$$I_x = \frac{\pi ab^3}{16}, \quad \bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) ab^3$$

$$I_y = \frac{\pi a^3 b}{16}, \quad \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) a^3 b$$

$$I_z = \frac{\pi ab}{16} (a^2 + b^2)$$

Subparabolic Area



$$\bar{x} = \frac{3a}{4}$$

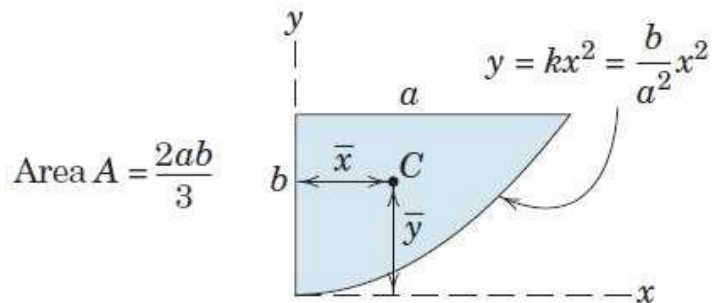
$$\bar{y} = \frac{3b}{10}$$

$$I_x = \frac{ab^3}{21}$$

$$I_y = \frac{a^3 b}{5}$$

$$I_z = ab \left(\frac{a^3}{5} + \frac{b^2}{21} \right)$$

Parabolic Area



$$\bar{x} = \frac{3a}{8}$$

$$\bar{y} = \frac{3b}{5}$$

$$I_x = \frac{2ab^3}{7}$$

$$I_y = \frac{2a^3 b}{15}$$

$$I_z = 2ab \left(\frac{a^2}{15} + \frac{b^2}{7} \right)$$