05 – p2 (Appendix A) – Area Moments of Inertia

STATICS, AGE-1330

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Watch these videos first:

- Basic concept of MOI
- Application 1
- Application 2

Rectangular and Polar Moments of Inertia

Consider the area A in the x-y plane, Fig. A/2. The moments of inertia of the element dA about the x- and y-axes are, by definition, $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively. The moments of inertia of A about the same axes are therefore

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \qquad \textbf{(A/1)}$$

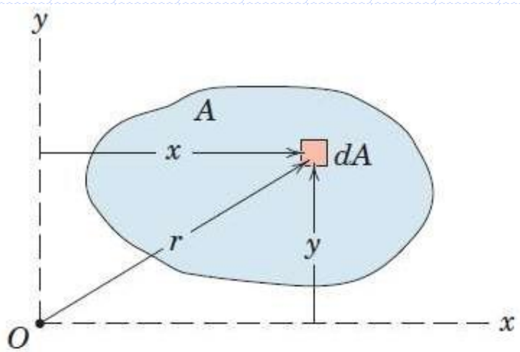


Figure A/2

The moment of inertia of dA about the pole O (z-axis) is, by similar definition, $dI_z = r^2\,dA$. The moment of inertia of the entire area about O is

$$I_z = \int r^2\,dA \qquad \textbf{(A/2)}$$

The expressions defined by Eqs. A/1 are called *rectangular* moments of inertia, whereas the expression of Eq. A/2 is called the *polar* moment of inertia.* Because $x^2 + y^2 = r^2$, it is clear that

$$I_z = I_x + I_y \qquad \textbf{(A/3)}$$

For an area whose boundaries are more simply described in rectangular coordinates than in polar coordinates, its polar moment of inertia is easily calculated with the aid of Eq. A/3.

Radius of Gyration

Consider an area A, Fig. A/3a, which has rectangular moments of inertia I_x and I_y and a polar moment of inertia I_z about O. We now visualize this area as concentrated into a long narrow strip of area A a distance k_x from the x-axis, Fig. A/3b. By definition the moment of inertia of the strip about the x-axis will be the same as that of the original area if $k_x^2 A = I_x$. The distance k_x is called the *radius of gyration* of the area

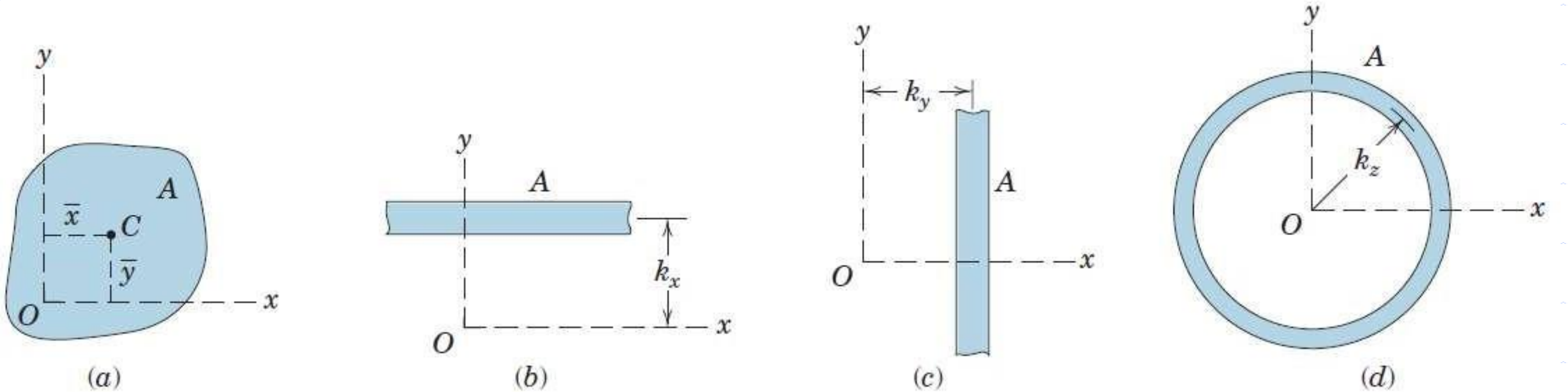


Figure A/3

about the x-axis. A similar relation for the y-axis is written by considering the area as concentrated into a narrow strip parallel to the y-axis as shown in Fig. A/3c. Also, if we visualize the area as concentrated into a narrow ring of radius k_z as shown in Fig. A/3d, we may express the polar moment of inertia as $k_z^2A = I_z$. In summary we write

$$\boxed{\begin{aligned} I_x &= k_x^2A \\ I_y &= k_y^2A \\ I_z &= k_z^2A \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} k_x &= \sqrt{I_x/A} \\ k_y &= \sqrt{I_y/A} \\ k_z &= \sqrt{I_z/A} \end{aligned}} \qquad \textbf{(A/4)}$$

The radius of gyration, then, is a measure of the distribution of the area from the axis in question. A rectangular or polar moment of inertia may be expressed by specifying the radius of gyration and the area.

When we substitute Eqs. A/4 into Eq. A/3, we have

$$\boxed{k_z^2 = k_x^2 + k_y^2} \qquad \textbf{(A/5)}$$

Transfer of Axes

The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Fig. A/4 the x_0-y_0 axes pass through the centroid C of the area. Let us now determine the moments of inertia of the area about the

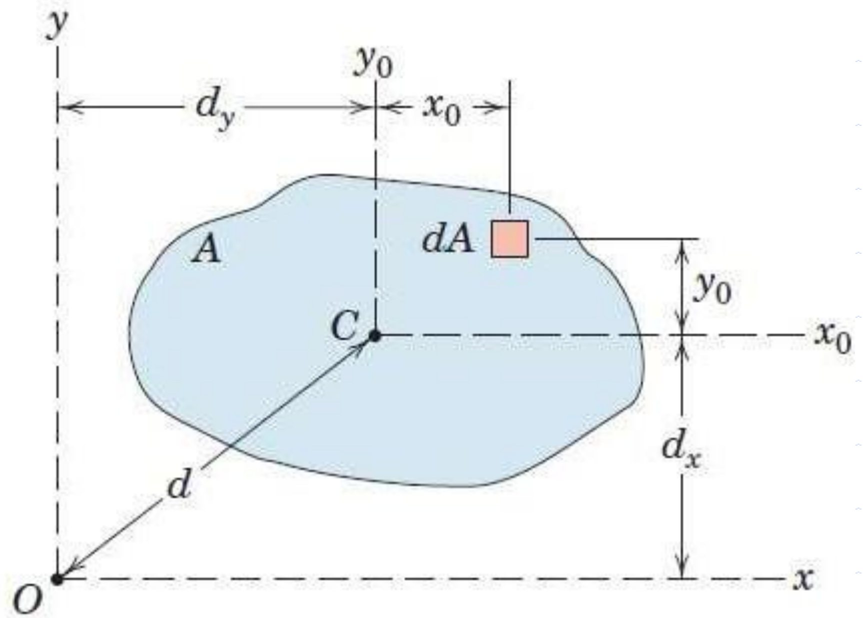


Figure A/4

parallel x-y axes. By definition, the moment of inertia of the element dA about the x-axis is

$$dI_x = (y_0 + d_x)^2\, dA$$

Expanding and integrating give us

$$I_x = \int y_0^2\, dA + 2d_x \int y_0\, dA + d_x^2 \int dA$$

We see that the first integral is by definition the moment of inertia $\bar{I}_x$ about the centroidal x_0-axis. The second integral is zero, since $\int y_0\, dA = A\bar{y}_0$ and $\bar{y}_0$ is automatically zero with the centroid on the x_0-axis. The third term is simply Ad_x^2. Thus, the expression for I_x and the similar expression for I_y become

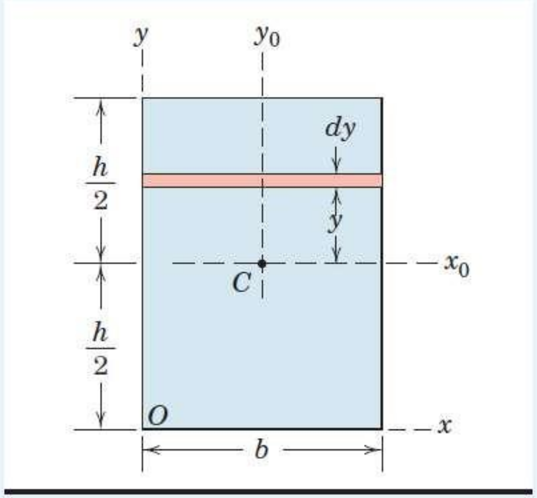
$$I_x = \bar{I}_x + Ad_x^2$$
$$I_y = \bar{I}_y + Ad_y^2 \qquad \textbf{(A/6)}$$

By Eq. A/3 the sum of these two equations gives

$$I_z = \bar{I}_z + Ad^2 \qquad \textbf{(A/6a)}$$

SAMPLE PROBLEM A/1

Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Solution. For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$\left[I_x = \int y^2\,dA\right] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2 \qquad \textit{Ans.}$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

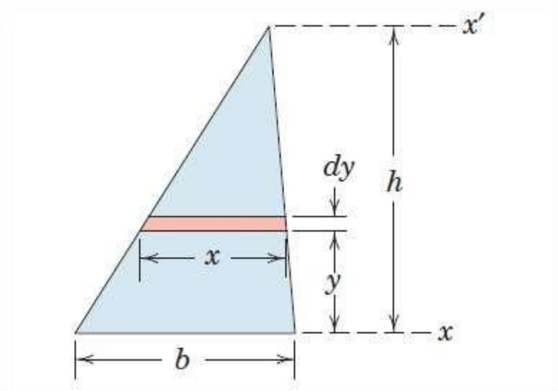
$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

Helpful Hint

1 If we had started with the second-order element $dA = dx\,dy$, integration with respect to x holding y constant amounts simply to multiplication by b and gives us the expression $y^2b\,dy$, which we chose at the outset.

SAMPLE PROBLEM A/2

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



1 **Solution.** A strip of area parallel to the base is selected as shown in the figure,
2 and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

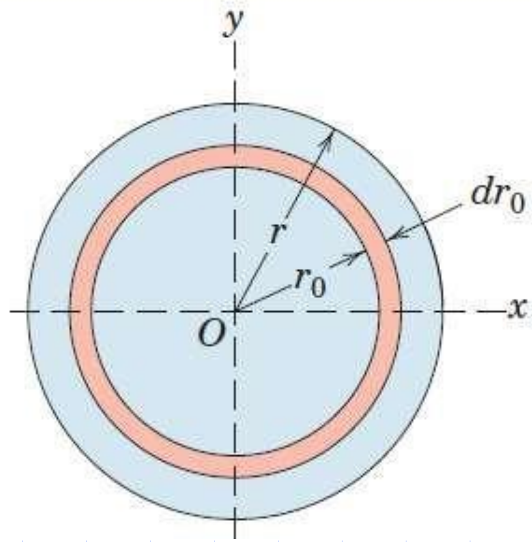
$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$

Helpful Hints

1. Here again we choose the simplest possible element. If we had chosen $dA = dx\,dy$, we would have to integrate $y^2\,dx\,dy$ with respect to x first. This gives us $y^2x\,dy$, which is the expression we chose at the outset.
2. Expressing x in terms of y should cause no difficulty if we observe the proportional relationship between the similar triangles.

SAMPLE PROBLEM A/3

Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.



Solution. A differential element of area in the form of a circular ring may be
1 used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\, dr_0$, and thus,

$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \tfrac{1}{2}Ar^2 \qquad \textit{Ans.}$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \tfrac{1}{2}I_z = \frac{\pi r^4}{4} = \tfrac{1}{4}Ar^2 \qquad \textit{Ans.}$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2} \qquad \textit{Ans.}$$

A/3 Composite Areas

It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape. Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity. The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

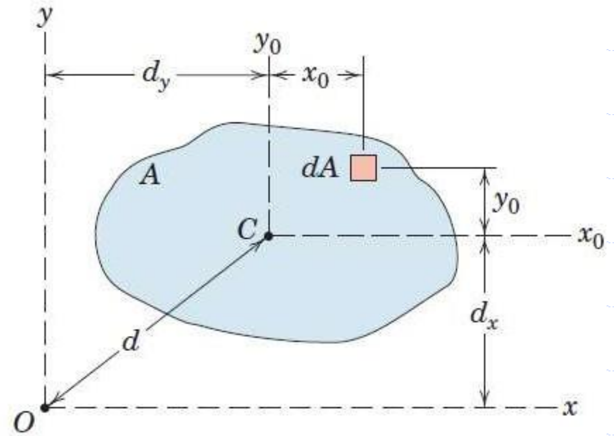


Figure A/4

For such an area in the x-y plane, for example, and with the notation of Fig. A/4, where $\bar{I}_x$ is the same as I_{x_0} and $\bar{I}_y$ is the same as I_{y_0} the tabulation would include

Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma\bar{I}_x$	$\Sigma\bar{I}_y$

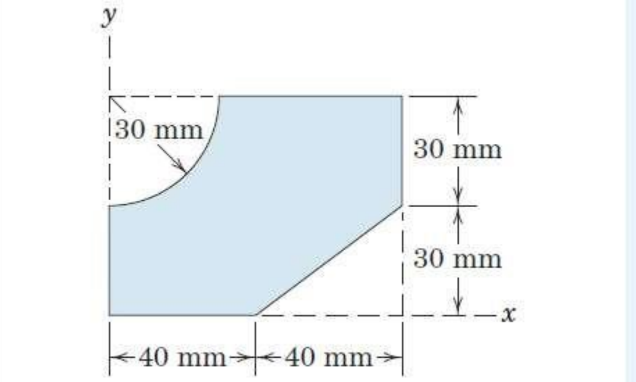
From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

$$I_x = \Sigma\bar{I}_x + \Sigma Ad_x^2$$

$$I_y = \Sigma\bar{I}_y + \Sigma Ad_y^2$$

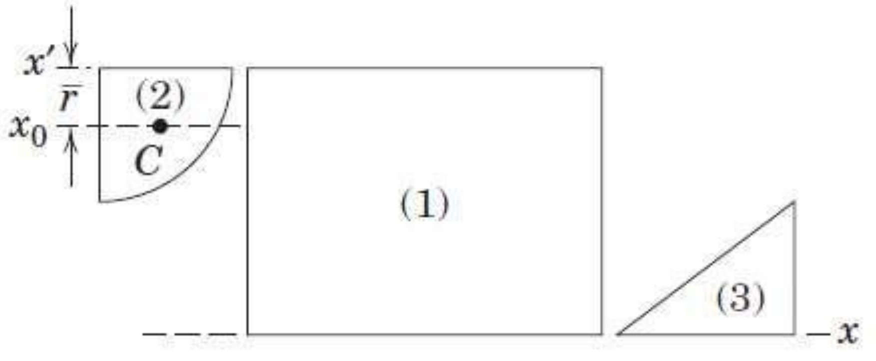
SAMPLE PROBLEM A/7

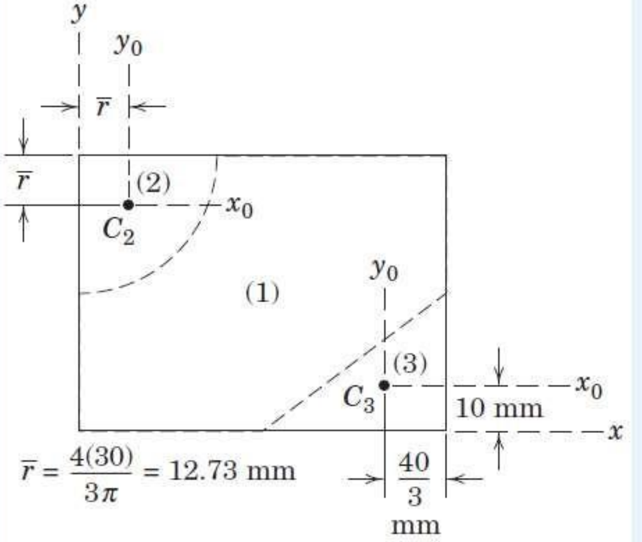
Determine the moments of inertia about the x- and y-axes for the shaded area. Make direct use of the expressions given in Table D/3 for the centroidal moments of inertia of the constituent parts.



Solution. The given area is subdivided into the three subareas shown—a rectangular (1), a quarter-circular (2), and a triangular (3) area. Two of the subareas are "holes" with negative areas. Centroidal x_0–y_0 axes are shown for areas (2) and (3), and the locations of centroids C_2 and C_3 are from Table D/3.

The following table will facilitate the calculations.





PART	A mm^2	d_x mm	d_y mm	Ad_x^2 mm^3	Ad_y^2 mm^3	$\bar{I}_x$ mm^4	$\bar{I}_y$ mm^4
1	80(60)	30	40	$4.32(10^6)$	$7.68(10^6)$	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$-\frac{1}{4}\pi(30)^2$	(60 – 12.73)	12.73	$-1.579(10^6)$	$-0.1146(10^6)$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80-\frac{40}{3}\right)$	$-0.06(10^6)$	$-2.67(10^6)$	$-\frac{1}{36}40(30)^3$	$-\frac{1}{36}(30)(40)^3$
TOTALS	3490			$2.68(10^6)$	$4.90(10^6)$	$1.366(10^6)$	$2.46(10^6)$

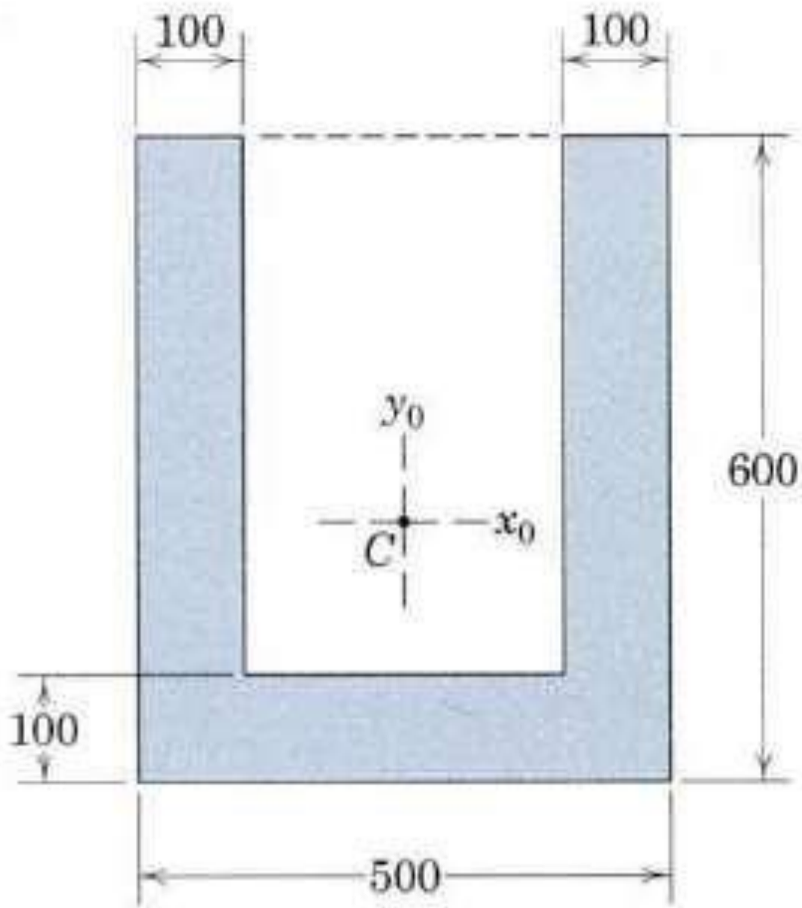
$[I_x = \Sigma \bar{I}_x + \Sigma A d_x^2]$ $\quad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6)\ \text{mm}^4$ *Ans.*

$[I_y = \Sigma \bar{I}_y + \Sigma A d_y^2]$ $\quad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6)\ \text{mm}^4$ *Ans.*

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \text{mm}^2$ so that the radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm}$$ *Ans.*

A/50 Calculate the polar radius of gyration of the shaded area about its centroid C.



Dimensions in millimeters

A/50

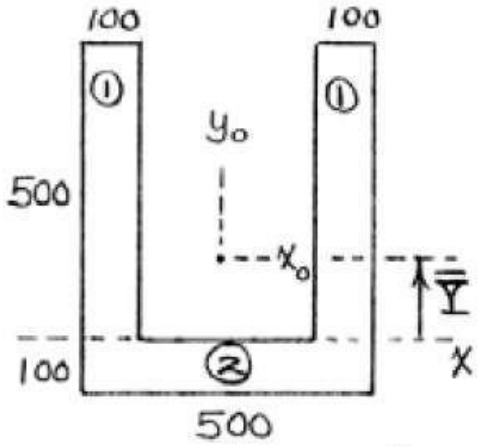
$$\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}$$

$$= \frac{2[(100)(500)(250)] + 500(100)(-50)}{2(100)(500) + 100(500)}$$

$$= 150 \text{ mm}$$

$$A = 2(100)(500) + 100(500)$$

$$= 15(10^4) \text{ mm}^2$$



(Dim. in mm)

① + ① $I_{x_o} = 2[\frac{1}{12}(100)(500)^3 + 100(500)(250-150)^2]$
$= 30.8(10^8) \text{ mm}^4$

$I_{y_o} = 2[\frac{1}{12}(500)(100)^3 + 100(500)(150+50)^2] = 40.8(10^8) \text{ mm}^4$

② $I_{x_o} = \frac{1}{12}(500)(100)^3 + 100(500)(50+150)^2 = 20.4(10^8) \text{ mm}^4$

$I_{y_o} = \frac{1}{12}(100)(500)^3 = 10.42(10^8) \text{ mm}^4$

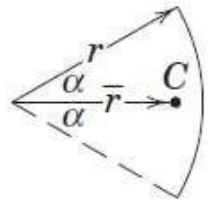
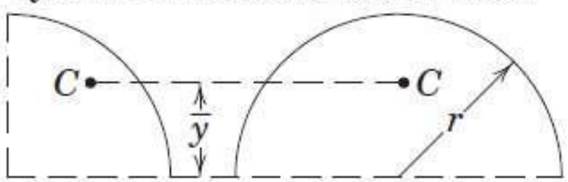
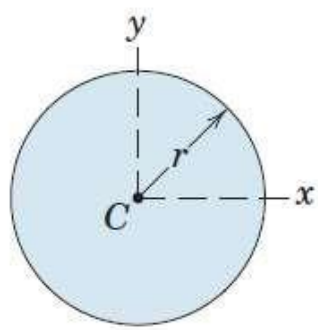
Totals ① + ① + ② : $I_{x_o} = 51.2(10^8) \text{ mm}^4$

$I_{y_o} = 51.2(10^8) \text{ mm}^4$

$$I_c = I_{x_o} + I_{y_o} = 102.5(10^8) \text{ mm}^4$$

$$k_c = \sqrt{I_c/A} = \sqrt{\frac{102.5(10^8)}{15(10^4)}} = \underline{261 \text{ mm}}$$

TABLE D/3 PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin\alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$

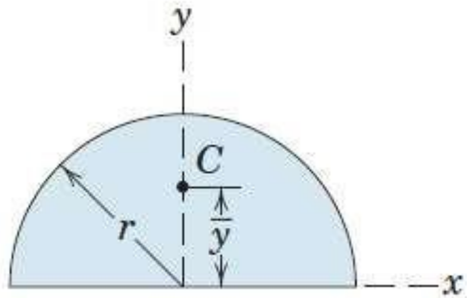
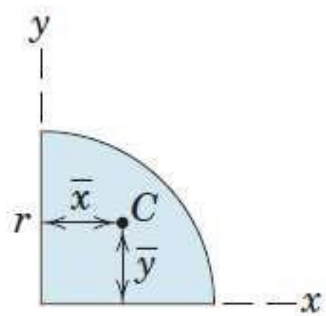
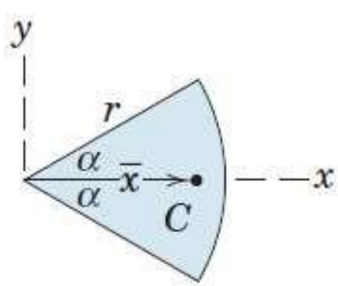
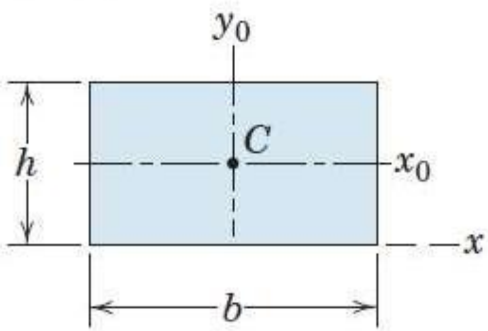
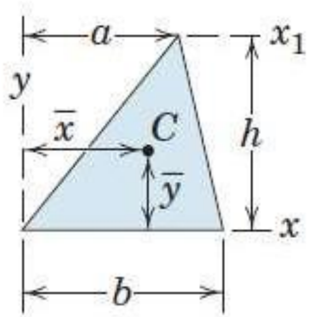
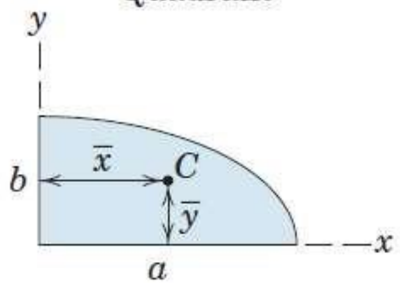
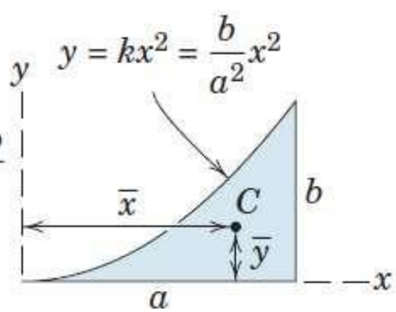
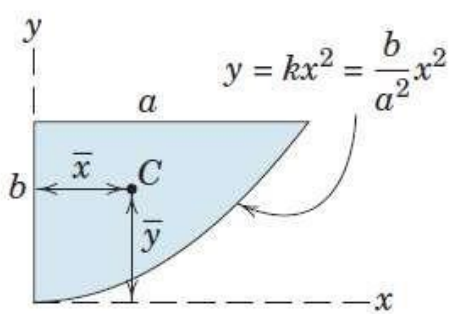
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3}\,\dfrac{r \sin\alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}\left(\alpha - \dfrac{1}{2}\sin 2\alpha\right)$ $I_y = \dfrac{r^4}{4}\left(\alpha + \dfrac{1}{2}\sin 2\alpha\right)$ $I_z = \dfrac{1}{2} r^4 \alpha$

TABLE D/3 PROPERTIES OF PLANE FIGURES *Continued*

<table>
<tr><th>FIGURE</th><th>CENTROID</th><th>AREA MOMENTS OF INERTIA</th></tr>
<tr><td>Rectangular Area

</td><td>—</td><td>$I_x = \frac{bh^3}{3}$
$\bar{I}_x = \frac{bh^3}{12}$
$\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$</td></tr>
<tr><td>Triangular Area

</td><td>$\bar{x} = \frac{a+b}{3}$
$\bar{y} = \frac{h}{3}$</td><td>$I_x = \frac{bh^3}{12}$
$\bar{I}_x = \frac{bh^3}{36}$
$I_{x_1} = \frac{bh^3}{4}$</td></tr>
</table>

Area of Elliptical Quadrant 	$\bar{x} = \dfrac{4a}{3\pi}$ $\bar{y} = \dfrac{4b}{3\pi}$	$I_x = \dfrac{\pi ab^3}{16},\quad \bar{I}_x = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)ab^3$ $I_y = \dfrac{\pi a^3 b}{16},\quad \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)a^3 b$ $I_z = \dfrac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \dfrac{ab}{3}$ 	$\bar{x} = \dfrac{3a}{4}$ $\bar{y} = \dfrac{3b}{10}$	$I_x = \dfrac{ab^3}{21}$ $I_y = \dfrac{a^3 b}{5}$ $I_z = ab\left(\dfrac{a^3}{5} + \dfrac{b^2}{21}\right)$
Parabolic Area Area $A = \dfrac{2ab}{3}$ 	$\bar{x} = \dfrac{3a}{8}$ $\bar{y} = \dfrac{3b}{5}$	$I_x = \dfrac{2ab^3}{7}$ $I_y = \dfrac{2a^3 b}{15}$ $I_z = 2ab\left(\dfrac{a^2}{15} + \dfrac{b^2}{7}\right)$