Chapter 5 – Distributed Forces Part 1 – Centers of Mass – Centroids

STATICS, AGE-1330
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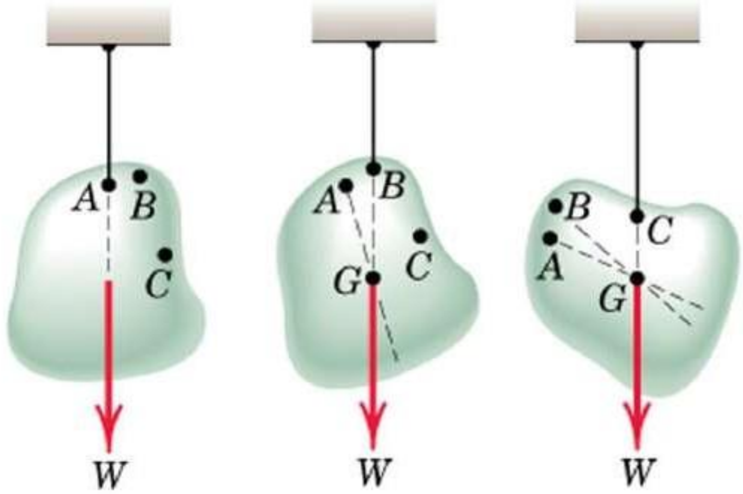
Center of Mass and Centroids

Center of Mass

A body of mass *m* in equilibrium under the action of tension in the cord, and resultant *W* of the gravitational forces acting on all particles of the body.
-The resultant is collinear with the cord



Suspend the body at different points

-Dotted lines show lines of action of the resultant force in each case.
-These lines of action will be concurrent at a single point G

As long as dimensions of the body are smaller compared with those of the earth.

- we assume uniform and parallel force field due to the gravitational attraction of the earth.

The unique **Point G** is called the Center of Gravity of the body (CG)

Watch this video:
https://youtube.com/shorts/vLhYIEPz4lk?si=lUPE3dn1A4CCdy5x

Center of Mass and Centroids

Determination of CG

- Apply Principle of Moments

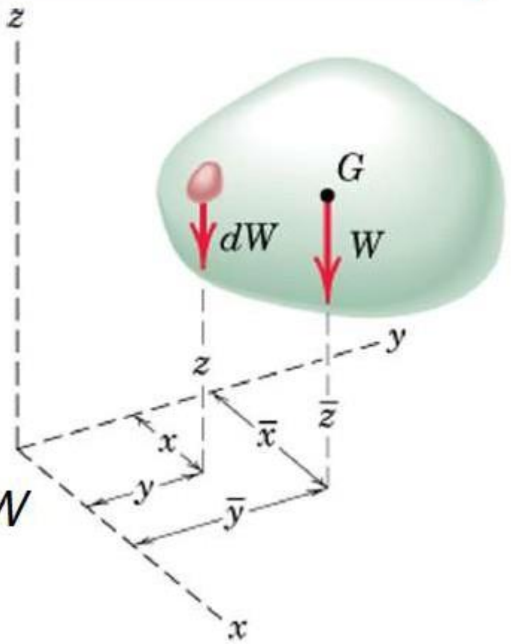
Moment of resultant gravitational force W about any axis equals sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements.

Weight of the body $W = \int dW$

Moment of weight of an element (dW) @ x-axis = ydW

Sum of moments for all elements of body = $\int ydW$

From Principle of Moments: $\int ydW = \bar{y}\, W$



$$\bar{x} = \frac{\int x dW}{W} \qquad \bar{y} = \frac{\int y dW}{W} \qquad \bar{z} = \frac{\int z dW}{W}$$

Moment of dW @ z axis???

= 0 or, ≠ 0

→ Numerator of these expressions represents the sum of the moments; Product of W and corresponding coordinate of G represents the moment of the sum → Moment Principle.

Center of Mass and Centroids

Determination of CG

$$\bar{x} = \frac{\int x dW}{W} \quad \bar{y} = \frac{\int y dW}{W} \quad \bar{z} = \frac{\int z dW}{W}$$

Substituting $W = mg$ and $dW = gdm$

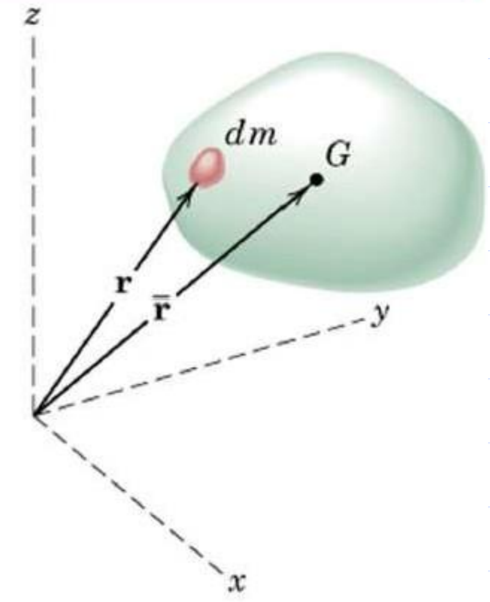
→ $$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

In vector notations:

Position vector for elemental mass: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Position vector for mass center G: $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m}$$



Density ρ of a body = **mass per unit volume**

→Mass of a differential element of volume dV → $dm = \rho dV$

→ ρ may not be constant throughout the body

$$\bar{x} = \frac{\int x\rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y\rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z\rho dV}{\int \rho dV}$$

Center of Mass and Centroids

Center of Mass: Following equations independent of *g*

$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m}$$ (Vector representation)

$$\bar{x} = \frac{\int x \rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y \rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z \rho dV}{\int \rho dV}$$

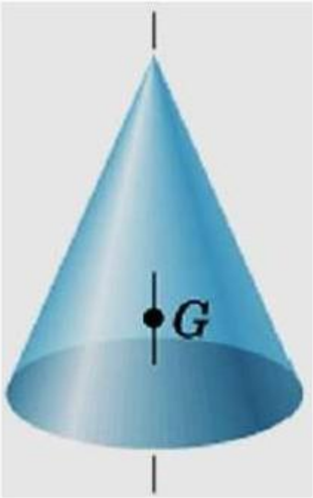
→**Unique point** [$= f(\rho)$] :: Centre of Mass (CM)

→CM coincides with CG as long as gravity field is treated as uniform and parallel

→CG or CM may lie outside the body

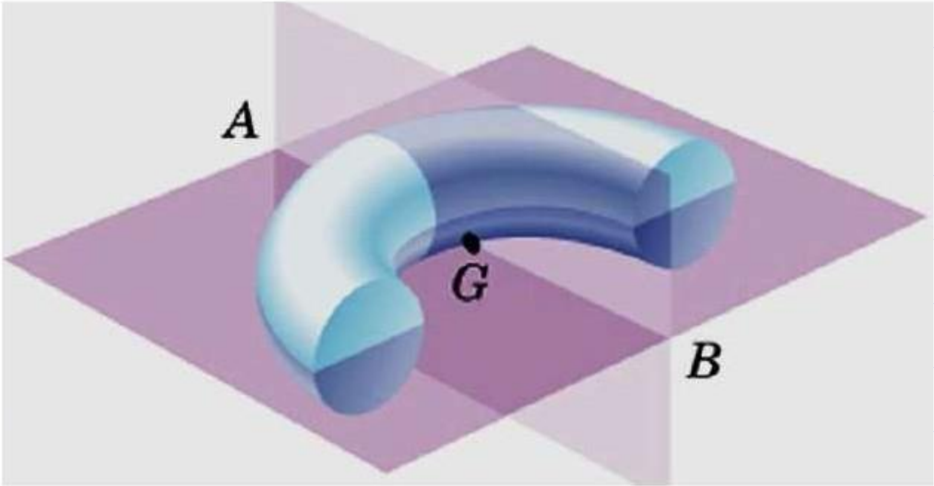
Center of Mass and Centroids

- **Symmetry**
  - *CM always lie on a* ***line*** *or a* ***plane of symmetry*** *in a homogeneous body*



Right Circular Cone
CM on central axis

Half Right Circular Cone
CM on vertical plane of symmetry



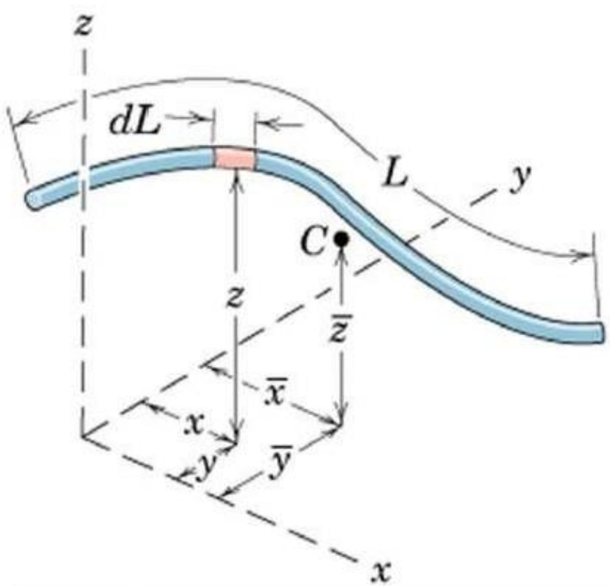
Half Ring
CM on intersection of two planes of symmetry (line AB)

Center of Mass and Centroids

Centroid

- Geometrical property of a body

- Body of uniform density :: Centroid and CM coincide



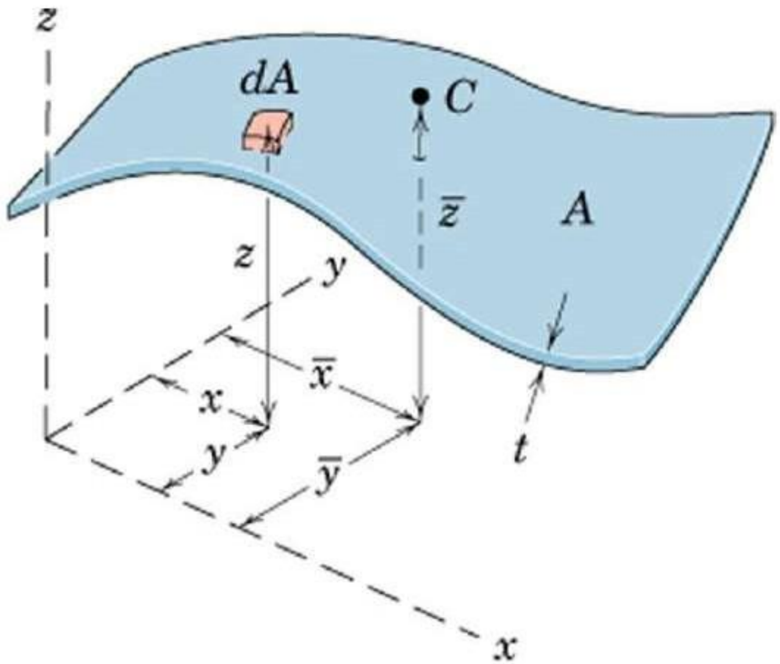
$$\bar{x} = \frac{\int x dm}{m} \qquad \bar{y} = \frac{\int y dm}{m} \qquad \bar{z} = \frac{\int z dm}{m}$$

Lines: Slender rod, Wire
Cross-sectional area = *A*
ρ and *A* are constant over *L*
dm = *ρAdL*
Centroid and CM are the same points

$$\bar{x} = \frac{\int x dL}{L} \qquad \bar{y} = \frac{\int y dL}{L} \qquad \bar{z} = \frac{\int z dL}{L}$$

Center of Mass and Centroids

- Centroid



Areas: Body with small but constant thickness *t*

Cross-sectional area = *A*

ρ and *A* are constant over *A*

$dm = \rho t dA$

Centroid and CM are the same points

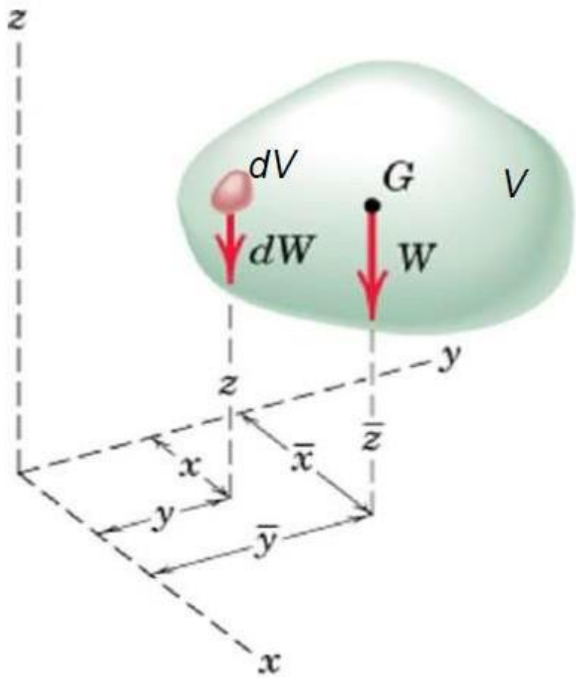
$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

$$\bar{x} = \frac{\int x dA}{A} \quad \bar{y} = \frac{\int y dA}{A} \quad \bar{z} = \frac{\int z dA}{A}$$

Numerator = First moments of Area

Center of Mass and Centroids

- Centroid



Volumes: Body with volume *V*

ρ constant over *V*

dm = *ρdV*

Centroid and CM are the same point

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

$$\bar{x} = \frac{\int x\,dV}{V} \qquad \bar{y} = \frac{\int y\,dV}{V} \qquad \bar{z} = \frac{\int z\,dV}{V}$$

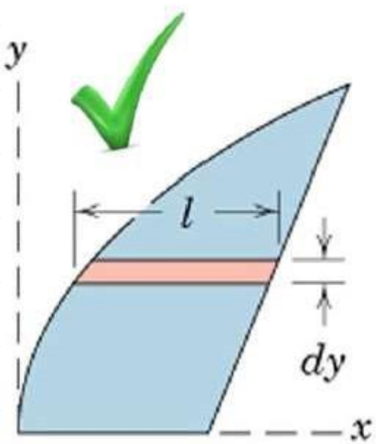
Numerator = First moments of Volume

Center of Mass and Centroid :: Guidelines

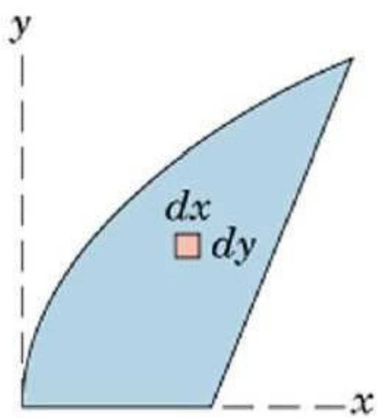
(a) Element Selection for Integration

- Order of Element

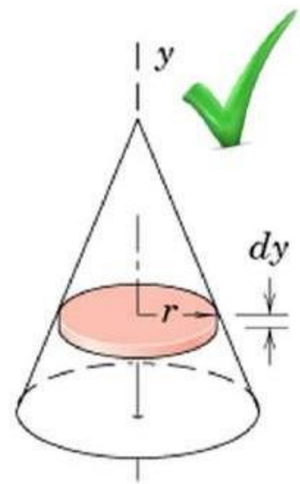
- First order differential element preferred over higher order element
- only one integration should cover the entire figure



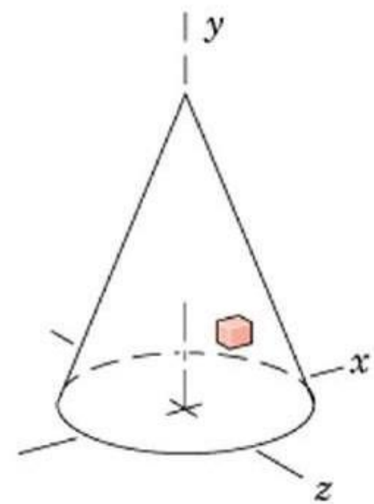
$A = \int dA = \int l\,dy$



$A = \int\int dx\,dy$



$V = \int dV = \int \pi r^2\,dy$



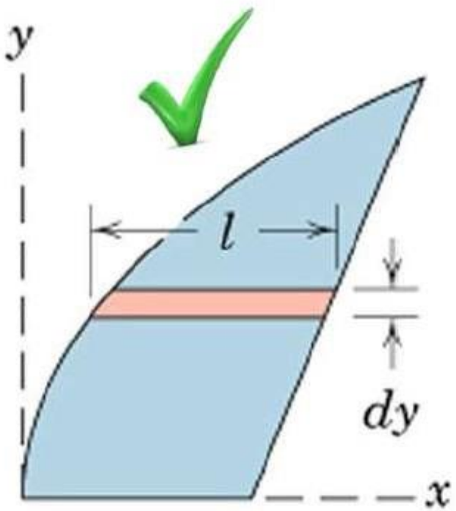
$V = \int\int\int dxdydz$

Center of Mass and Centroids :: Guidelines

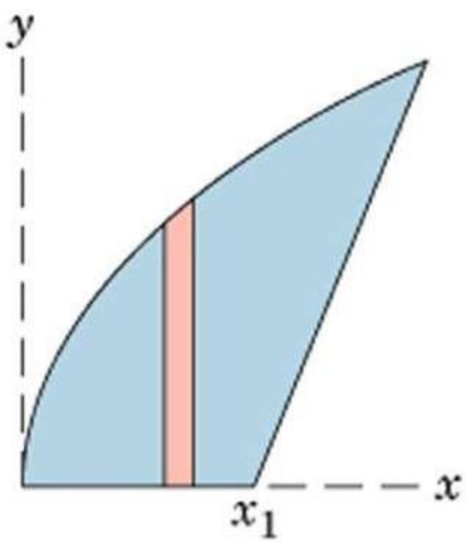
(b) Element Selection for Integration

- Continuity

- Integration of a single element over the entire area
- Continuous function over the entire area



Continuity in the expression for the width of the strip



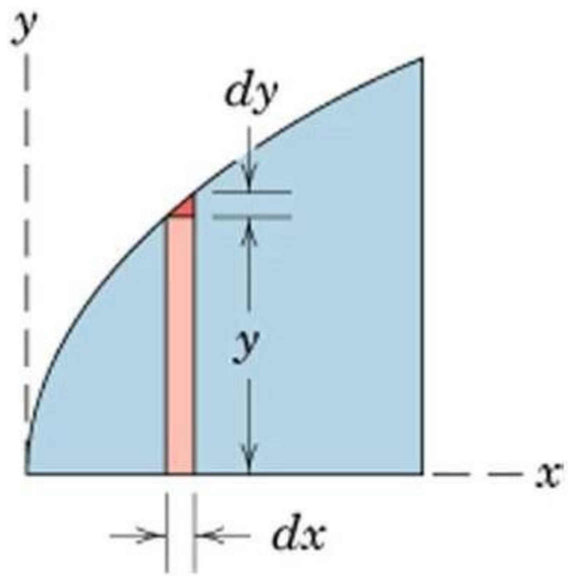
Discontinuity in the expression for the height of the strip at $x = x_1$

Center of Mass and Centroids :: Guidelines

(c) Element Selection for Integration

- Discarding higher order terms

- No error in limits



:: Vertical strip of area under the curve → $dA = ydx$

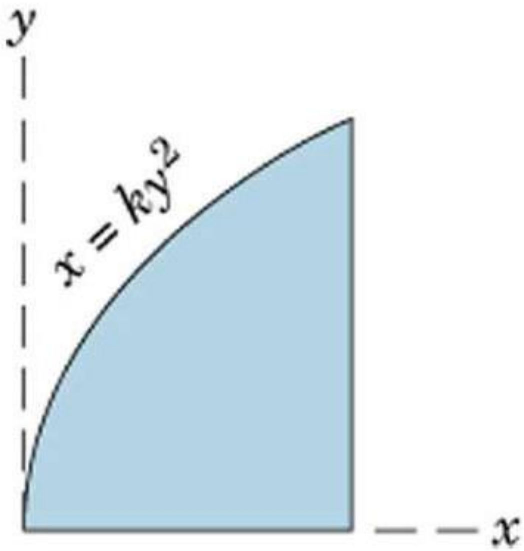
:: Ignore 2nd order triangular area $0.5dxdy$

Center of Mass and Centroids :: Guidelines

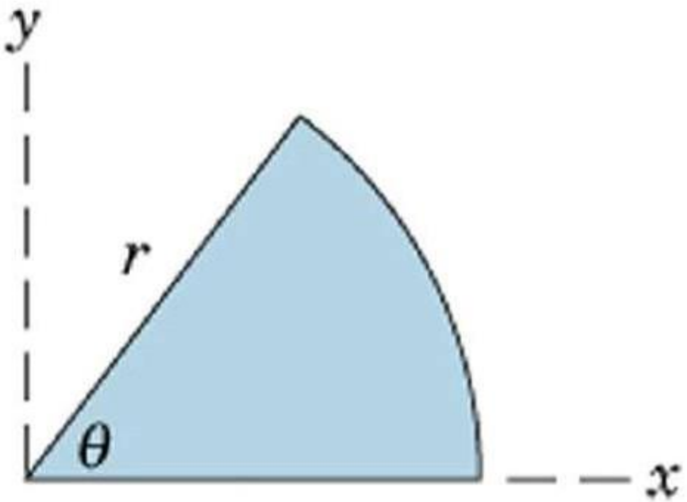
(d) Element Selection for Integration

- Coordinate system

- Convenient to match it with the boundaries of the shape



Curvilinear boundary
(Rectangular Coordinates)



Circular boundary
(Polar coordinates)

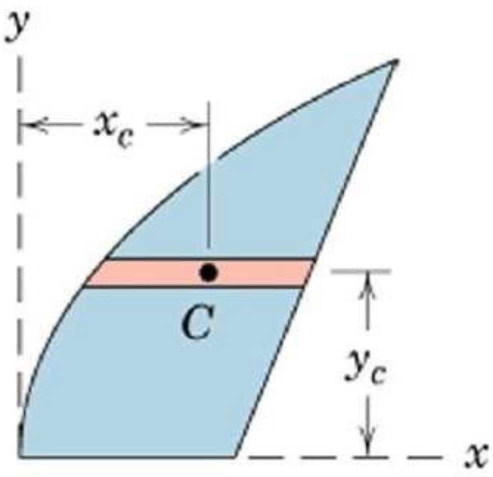
Center of Mass and Centroids :: Guidelines

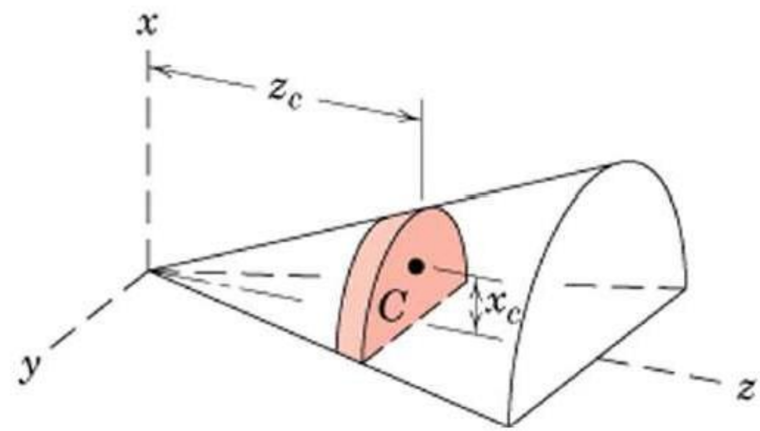
(e) Element Selection for Integration

- Centroidal coordinate (x_c, y_c, z_c) of element

- x_c, y_c, z_c to be considered for lever arm

  :: not the coordinates of the area boundary





$$\bar{x} = \frac{\int x_c dV}{V} \qquad \bar{y} = \frac{\int y_c dV}{V} \qquad \bar{z} = \frac{\int z_c dV}{V}$$

Modified Equations

$$\bar{x} = \frac{\int x_c dA}{A} \qquad \bar{y} = \frac{\int y_c dA}{A} \qquad \bar{z} = \frac{\int z_c dA}{A}$$

Center of Mass and Centroids :: Guidelines

Centroids of Lines, Areas, and Volumes

1. Order of Element Selected for Integration
2. Continuity
3. Discarding Higher Order Terms
4. Choice of Coordinates
5. Centroidal Coordinate of Differential Elements

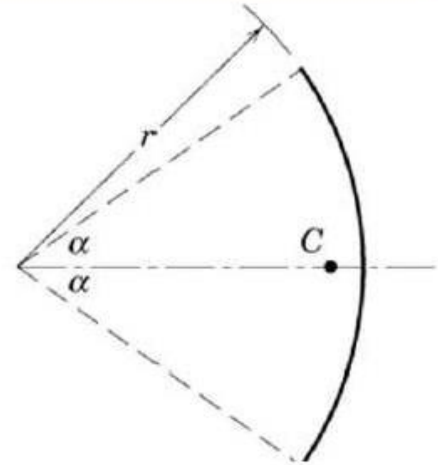
$$\bar{x} = \frac{\int x dL}{L} \qquad \bar{y} = \frac{\int y dL}{L} \qquad \bar{z} = \frac{\int z dL}{L}$$

$$\bar{x} = \frac{\int x_c dA}{A} \qquad \bar{y} = \frac{\int y_c dA}{A} \qquad \bar{z} = \frac{\int z_c dA}{A}$$

$$\bar{x} = \frac{\int x_c dV}{V} \qquad \bar{y} = \frac{\int y_c dV}{V} \qquad \bar{z} = \frac{\int z_c dV}{V}$$

Example on Centroid :: Circular Arc

Locate the centroid of the circular arc



Solution: Polar coordinate system is better

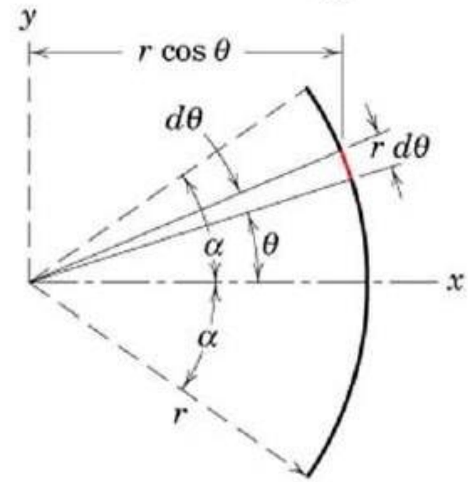
Since the figure is symmetric: centroid lies on the x axis

Differential element of arc has length $dL = rd\Theta$

Total length of arc: $L = 2\alpha r$

x-coordinate of the centroid of differential element: $x = r\cos\Theta$

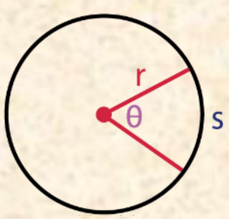
$$\bar{x} = \frac{\int x dL}{L} \qquad \bar{y} = \frac{\int y dL}{L} \qquad \bar{z} = \frac{\int z dL}{L}$$



$$[L\bar{x} = \int x\, dL] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\, r\, d\theta$$

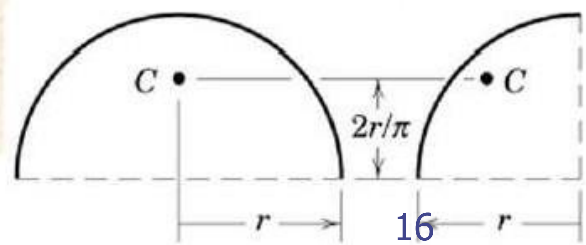
$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$



Arc Length : $s = r\theta$

For a semi-circular arc: $2\alpha = \pi \rightarrow$ centroid lies at $2r/\pi$

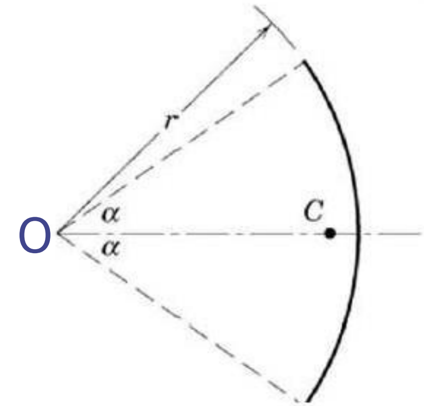


More detailed solution:

Problem

Locate the centroid C of a *circular arc* of radius r that subtends total angle 2α (radians).

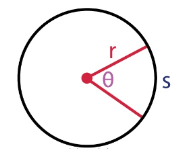
The arc is symmetric about the horizontal bisector.

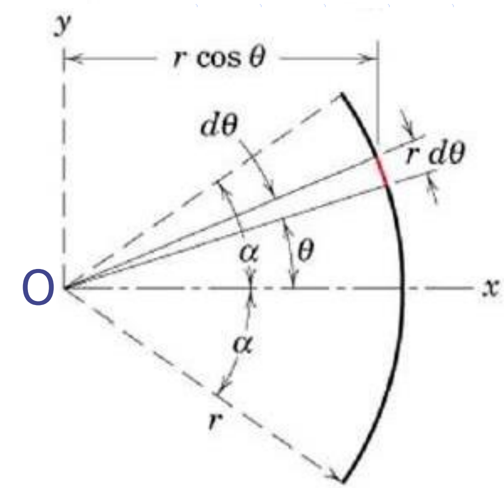


Setup

- Place the origin at the circle center O.
- Let the x-axis lie along the bisector of the arc, pointing toward the arc.
- Parameterize the arc by angle $\theta \in [-\alpha, \alpha]$.
- Differential length: $ds = r\,d\theta$.
- Coordinates of a point on the arc: $(x, y) = (r\cos\theta,\ r\sin\theta)$.

Arc Length : s = rθ





By symmetry, $\bar{y} = 0$. We only need $\bar{x}$.

Centroid of a line (arc)

$$\bar{x} = \frac{1}{L}\int x\,ds, \qquad L = \int ds = r\int_{-\alpha}^{\alpha} d\theta = 2r\alpha.$$

Compute the first moment:

$$\int x\,ds = \int_{-\alpha}^{\alpha} (r\cos\theta)\,(r\,d\theta) = r^2\int_{-\alpha}^{\alpha} \cos\theta\,d\theta = r^2\big[\,\sin\theta\,\big]_{-\alpha}^{\alpha} = 2r^2\sin\alpha.$$

More detailed solution (cont.):

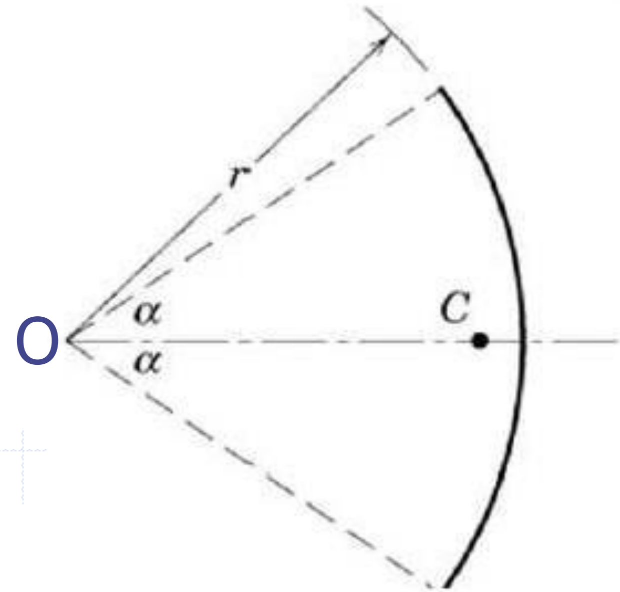
Therefore

$$\boxed{\bar{x} = \frac{2r^2 \sin\alpha}{2r\alpha} = r\,\frac{\sin\alpha}{\alpha}}, \qquad \boxed{\bar{y} = 0}.$$

So the centroid C lies on the bisector at a distance

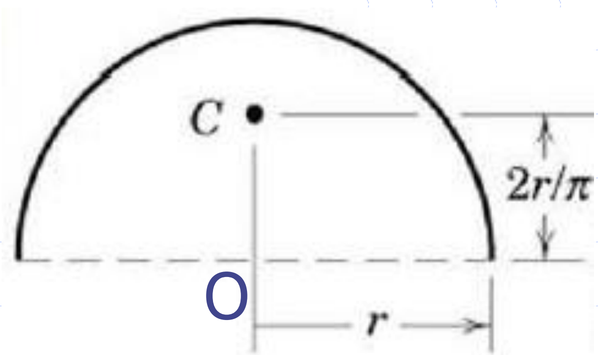
$$\boxed{OC = r\,\frac{\sin\alpha}{\alpha}}$$

from the center, toward the arc.



Quick checks and special cases

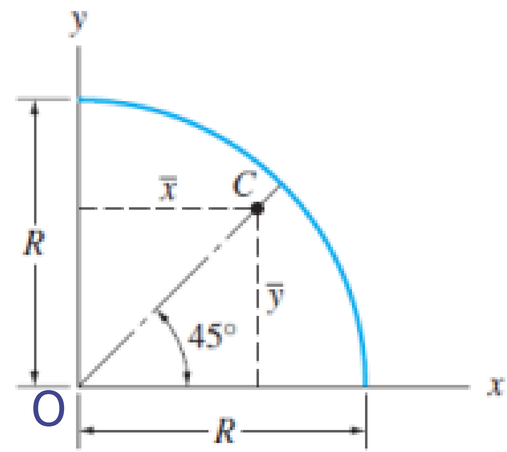
- Very small arc: $\alpha \to 0 \Rightarrow \sin\alpha \sim \alpha$, so $OC \to r$. The centroid approaches the circle, as expected.
- Semicircular arc: $\alpha = \pi/2 \Rightarrow OC = \dfrac{2r}{\pi}$.
- Full circle: $\alpha = \pi \Rightarrow OC = 0$. The centroid is at the center.



More detailed solution (cont.):

Problem

Find the centroid of a **quarter-circle arc** of radius r lying in the first quadrant, with center at the origin. The arc runs from $\theta = 0$ to $\theta = \pi/2$.



Setup

Parameterize by the polar angle θ.

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad ds = r\,d\theta, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

Total arc length:
$$\bar{x} = \frac{1}{L}\int x\,ds$$

$$L = \int_0^{\pi/2} ds = r\int_0^{\pi/2} d\theta = \frac{\pi r}{2}.$$

By symmetry about the line $y = x$, $\bar{x} = \bar{y}$. Compute one and copy to the other.

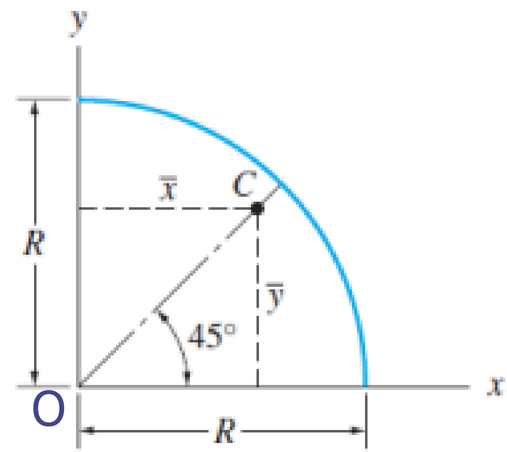
Centroid coordinates

$$\bar{x} = \frac{1}{L}\int_0^{\pi/2} x\,ds = \frac{1}{L}\int_0^{\pi/2} (r\cos\theta)(r\,d\theta) = \frac{r^2}{L}\int_0^{\pi/2} \cos\theta\,d\theta = \frac{r^2}{L}\left[\sin\theta\right]_0^{\pi/2} = \frac{r^2}{L}.$$

More detailed solution (cont.):

Insert $L = \pi r/2$:

$$\boxed{\bar{x} = \frac{2r}{\pi}}.$$



By symmetry,

$$\boxed{\bar{y} = \frac{2r}{\pi}}.$$

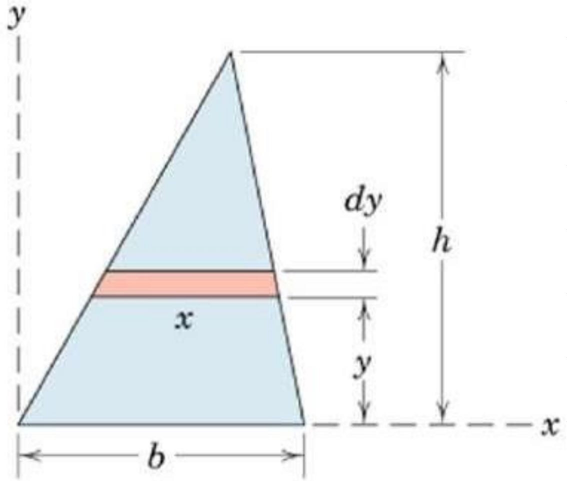
Result

The centroid C of a quadrant arc is at

$$\boxed{C\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)} \approx \boxed{(0.637r,\ 0.637r)}.$$

Example on Centroid :: Triangle

Locate the centroid of the triangle along h from the base



Solution:

$dA = xdy$; $x/(h-y) = b/h$

Total Area, A = ½(bh)

$$\bar{x} = \frac{\int x_c dA}{A} \qquad \bar{y} = \frac{\int y_c dA}{A} \qquad \bar{z} = \frac{\int z_c dA}{A}$$

$y_c = y$

$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{h} dy = \frac{bh^2}{6}$$

and $$\bar{y} = \frac{h}{3}$$

More detailed solution:

Problem

A triangle of base b and height h. Locate the centroid coordinate $\bar{y}$ measured upward from the base.

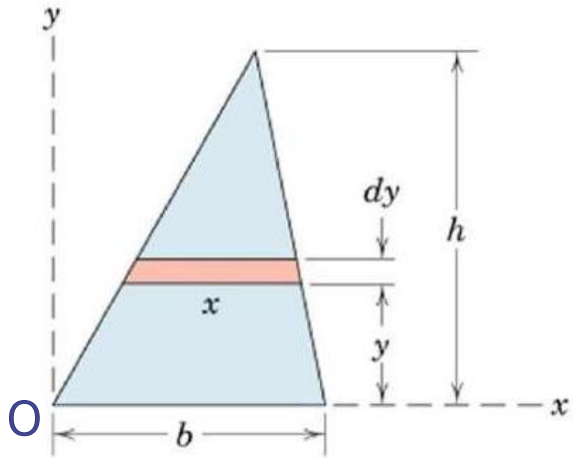
Setup (strip method)

$$\bar{y} = \frac{\int y_c dA}{A}$$

- Put the origin on the base at the left, y pointing upward.
- Take a thin **horizontal** strip at height y with thickness dy.
  Its length is $x(y)$, so $dA = x(y)\,dy$.

Because the triangle tapers linearly to zero at the top, **similar triangles** give the strip length:

$$\frac{x(y)}{b} = \frac{h-y}{h} \quad \Rightarrow \quad x(y) = b\left(1 - \frac{y}{h}\right).$$

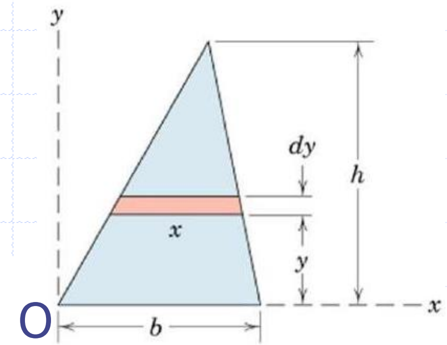


Area (check)

$$A = \int_0^h dA = \int_0^h b\left(1 - \frac{y}{h}\right) dy = b\left[h - \frac{h}{2}\right] = \frac{bh}{2},$$

which matches the known area of a triangle.

More detailed solution (cont.):



First moment about the base

$$\int y\,dA = \int_0^h y\,b\left(1-\frac{y}{h}\right)dy = b\left[\int_0^h y\,dy - \frac{1}{h}\int_0^h y^2\,dy\right] = b\left[\frac{h^2}{2}-\frac{h^3}{3h}\right] = \frac{bh^2}{6}.$$

Centroid coordinate

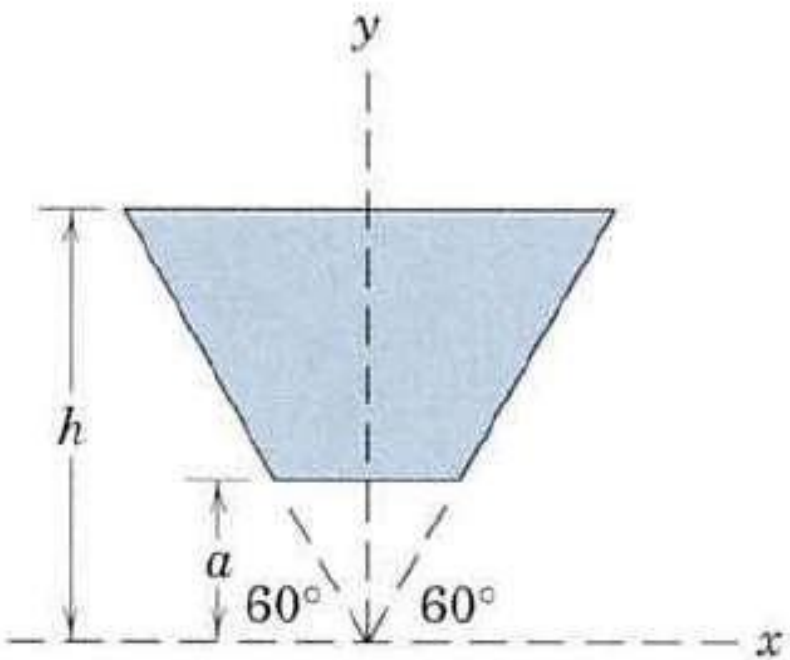
$$\bar{y} = \frac{1}{A}\int y\,dA = \frac{\frac{bh^2}{6}}{\frac{bh}{2}} = \boxed{\frac{h}{3}}.$$

Interpretation and checks

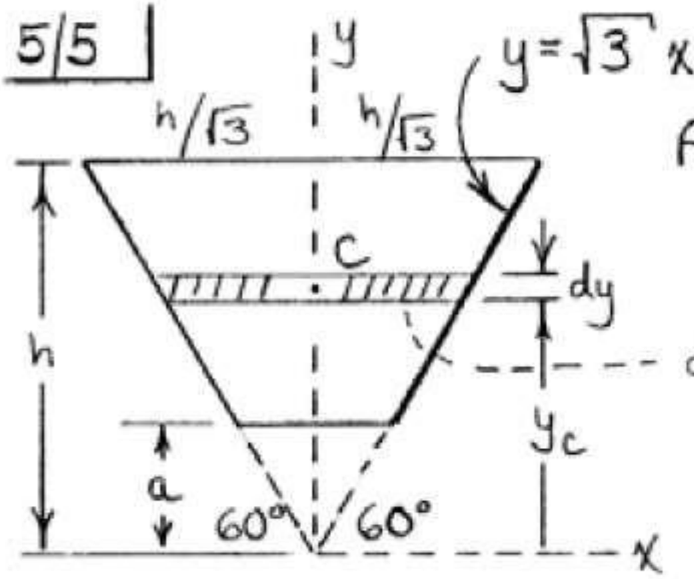
- The centroid of a triangular **area** is one third of the height **up from the base**, equivalently two thirds of the way **down from the apex.**
- This matches the general theorem: the centroid lies at the intersection of medians, at $1/3$ of each median measured from the base.

5/5 Determine the y-coordinate of the centroid of the shaded area. Check your result for the special case $a = 0$.

$$Ans.\ \bar{y} = \frac{2(h^3 - a^3)}{3(h^2 - a^2)}$$



5/5



$$A = \frac{1}{2}\frac{2}{\sqrt{3}}hh - \frac{1}{2}\frac{2}{\sqrt{3}}aa$$

$$= \frac{1}{\sqrt{3}}(h^2 - a^2)$$

$$dA = 2x\,dy$$

$$= 2\left(\frac{y}{\sqrt{3}}\right)dy$$

$$\int y_c\,dA = \int_a^h y\frac{2}{\sqrt{3}}y\,dy = \frac{2}{\sqrt{3}}\left.\frac{y^3}{3}\right|_a^h$$

$$= \frac{2}{3\sqrt{3}}(h^3 - a^3)$$

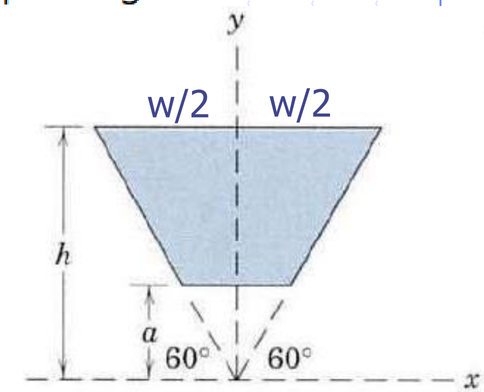
$$\bar{y} = \frac{\int y_c\,dA}{A} = \frac{\frac{2}{3\sqrt{3}}(h^3 - a^3)}{\frac{1}{\sqrt{3}}(h^2 - a^2)} = \underline{\frac{2(h^3 - a^3)}{3(h^2 - a^2)}}$$

(For $a = 0$, $\bar{y} = \frac{2}{3}h$, the correct value.)

More detailed solution:

Goal

Find $\bar{y}$ of the shaded frustum measured from the x-axis. The region is bounded between $y = a$ and $y = h$. Each side makes a 60° angle with the x-axis, so the half-apex angle is 30°.



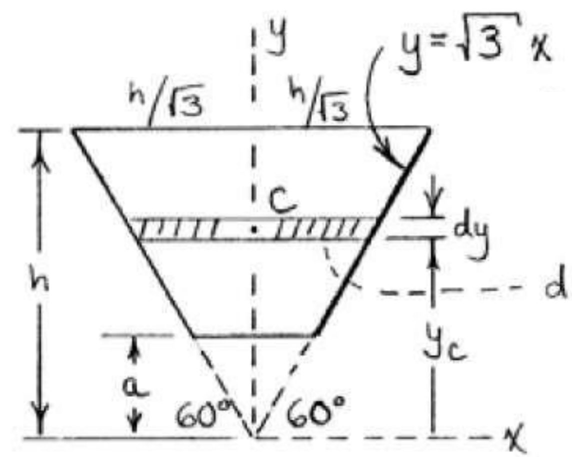
Geometry

At any height y above the apex, the similar right triangle gives

$$\text{half-width} = y\,\tan 30^\circ = \frac{y}{\sqrt{3}}.$$

Hence the full width at height y is

$$w(y) = 2\left(\frac{y}{\sqrt{3}}\right) = \frac{2y}{\sqrt{3}}.$$

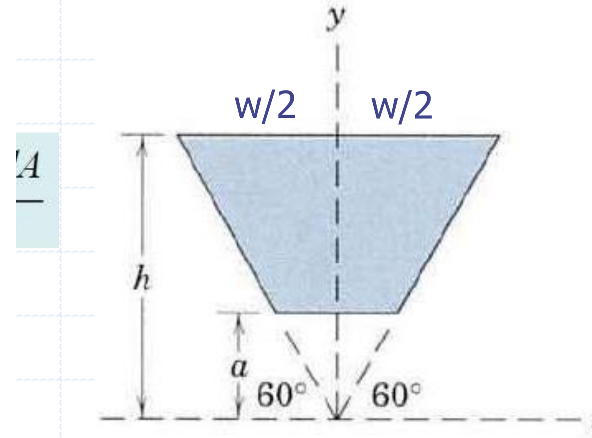


A horizontal strip of thickness dy at height y has area

$$dA = w(y)\,dy = \frac{2y}{\sqrt{3}}\,dy, \qquad a \le y \le h.$$

More detailed solution (cont.):

$$dA = w(y)\,dy = \frac{2y}{\sqrt{3}}\,dy, \qquad a \le y \le h.$$

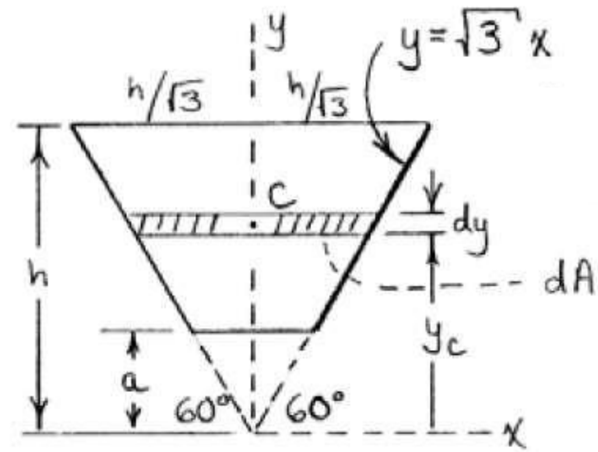


Area

$$A = \int_a^h dA = \int_a^h \frac{2y}{\sqrt{3}}\,dy = \frac{2}{\sqrt{3}}\left[\frac{y^2}{2}\right]_a^h = \frac{1}{\sqrt{3}}\,(h^2 - a^2).$$

First moment about the x-axis

$$\int y\,dA = \int_a^h y \cdot \frac{2y}{\sqrt{3}}\,dy = \frac{2}{\sqrt{3}}\int_a^h y^2\,dy = \frac{2}{\sqrt{3}}\left[\frac{y^3}{3}\right]_a^h = \frac{2}{3\sqrt{3}}\,(h^3 - a^3).$$



Centroid coordinate

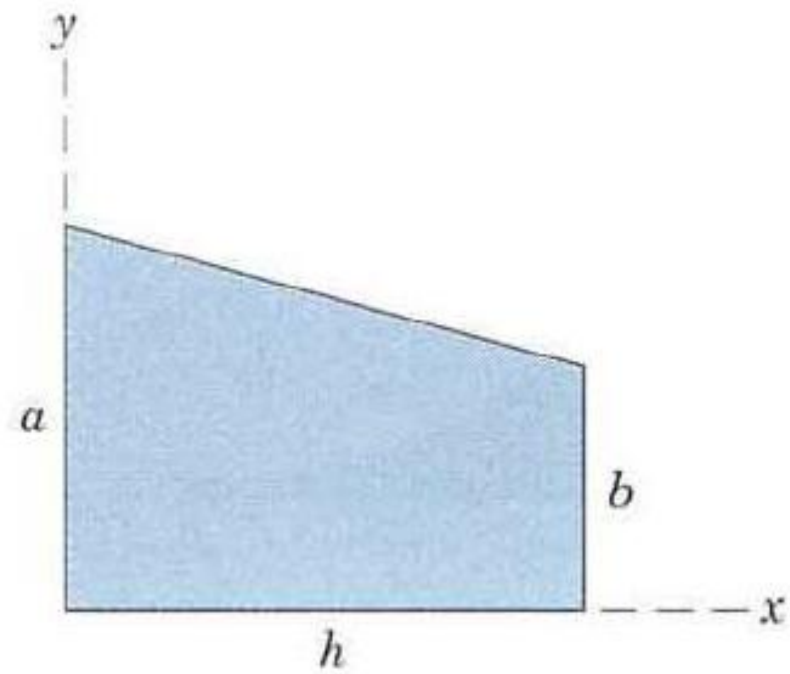
$$\bar{y} = \frac{\int y\,dA}{A} = \frac{\frac{2}{3\sqrt{3}}(h^3 - a^3)}{\frac{1}{\sqrt{3}}(h^2 - a^2)} = \boxed{\frac{2}{3} \cdot \frac{h^3 - a^3}{h^2 - a^2}}.$$

measured upward from the x-axis.

Checks

- Special case $a = 0$ (full isosceles triangle): $\bar{y} = \frac{2}{3}\frac{h^2}{h} = \boxed{\frac{2h}{3}}$.

Determine the x- and y-coordinates of the centroid of the trapezoidal area.



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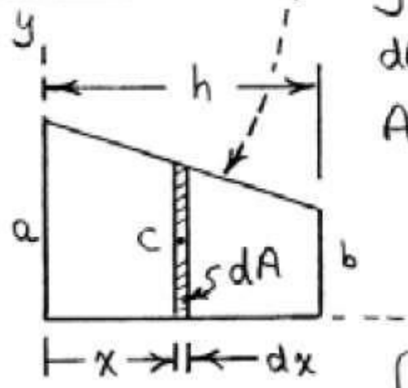


$$y = \left(\frac{b-a}{h}\right)x + a$$

$$dA = y\,dx$$

$$A = \int dA = \int_0^h \left[\left(\frac{b-a}{h}\right)x + a\right]dx$$

$$= \left[\frac{b-a}{h}\frac{x^2}{2} + ax\right]_0^h = \frac{h}{2}(a+b)$$

$$\int x_c\,dA = \int_0^h \left[\left(\frac{b-a}{h}\right)x^2 + ax\right]dx$$

$$= \left[\frac{b-a}{h}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^h = h^2\left(\frac{b}{3} + \frac{a}{6}\right)$$

$$\int y_c\,dA = \int \frac{y}{2}\,y\,dx = \frac{1}{2}\int_0^h y^2\,dx$$

$$= \frac{1}{2}\int_0^h \left[\left(\frac{b-a}{h}\right)^2 x^2 + 2\left(\frac{b-a}{h}\right)ax + a^2\right]dx$$

$$= \frac{1}{2}\left[\left(\frac{b-a}{h}\right)^2\frac{x^3}{3} + 2\left(\frac{b-a}{h}\right)a\frac{x^2}{2} + a^2x\right]_0^h$$

$$= \frac{h}{6}\left[a^2 + ab + b^2\right]$$

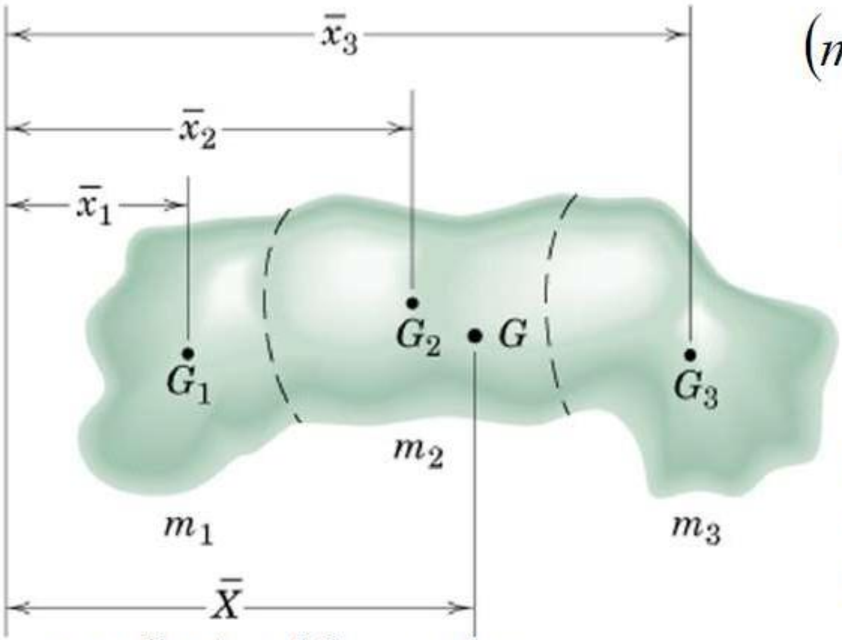
$$\bar{x} = \frac{\int x_c\,dA}{A} = \frac{h^2\left(\frac{b}{3} + \frac{a}{6}\right)}{\frac{h}{2}(a+b)} = \frac{h(a+2b)}{3(a+b)}$$

$$\bar{y} = \frac{\int y_c\,dA}{A} = \frac{\frac{h}{6}(a^2+ab+b^2)}{\frac{h}{2}(a+b)} = \frac{(a^2+ab+b^2)}{3(a+b)}$$

Center of Mass and Centroids

Composite Bodies and Figures

- Divide bodies or figures into several parts such that their mass centers can be conveniently determined
→ Use Principle of Moment for all finite elements of the body



x-coordinate of the center of mass of the whole

$$(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$$

Mass Center Coordinates can be written as:

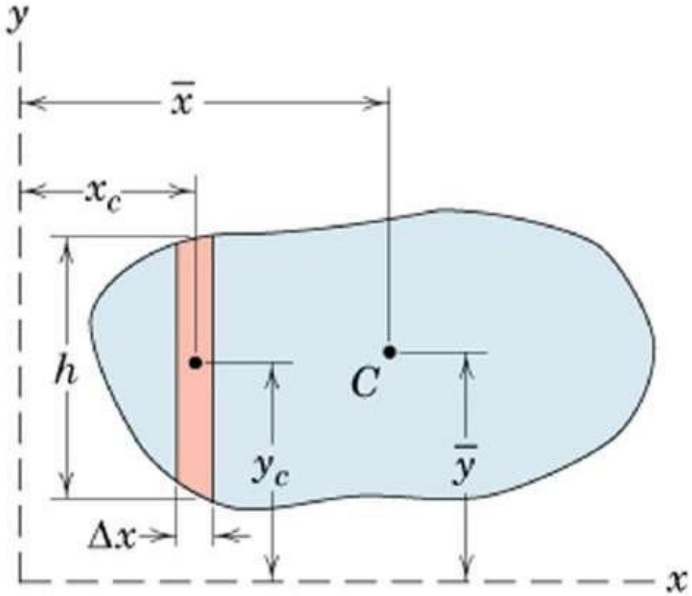
$$\bar{X} = \frac{\sum m\bar{x}}{\sum m} \quad \bar{Y} = \frac{\sum m\bar{y}}{\sum m} \quad \bar{Z} = \frac{\sum m\bar{z}}{\sum m}$$

m's can be replaced by *L*'s, *A*'s, and *V*'s for lines, areas, and volumes

Centroid of Composite Body/Figure

Irregular area :: Integration vs Approximate Summation

- Area/volume boundary cannot be expressed analytically
- **Approximate summation** instead of integration



Divide the area into several strips
Area of each strip = $h\Delta x$
Moment of this area about x- and y-axis
= $(h\Delta x)y_c$ *and* $(h\Delta x)x_c$
→Sum of moments for all strips divided by the total area will give corresponding coordinate of the centroid

$$\bar{x} = \frac{\sum A x_c}{\sum A} \qquad \bar{y} = \frac{\sum A y_c}{\sum A}$$

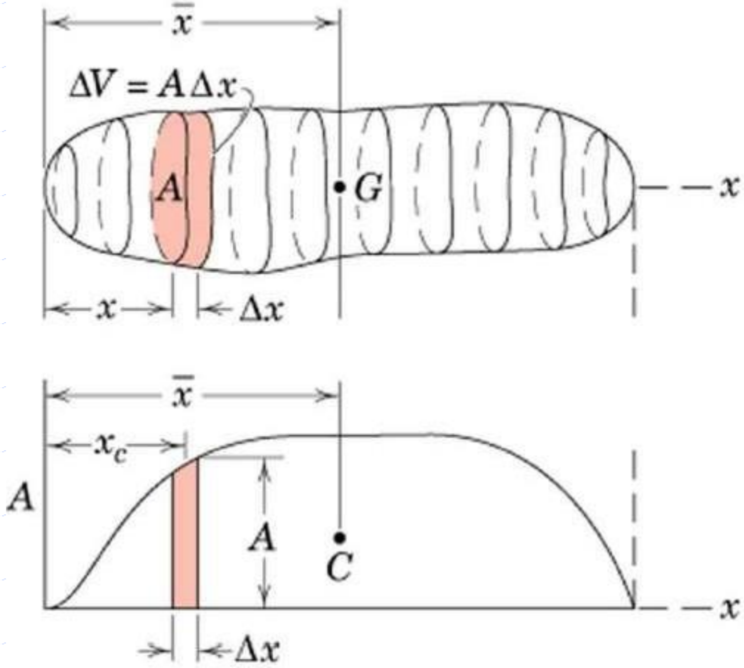
Accuracy may be improved by reducing the thickness of the strip

Centroid of Composite Body/Figure

Irregular volume :: **Integration** vs **Approximate Summation**

- Reduce the problem to one of locating the centroid of area

- **Approximate summation** instead of integration



Divide the area into several strips
Volume of each strip = $A\Delta x$
Plot all such A against x.
→ Area under the plotted curve represents volume of whole body and the x-coordinate of the centroid of the area under the curve is given by:

$$\bar{x} = \frac{\sum (A\Delta x) x_c}{\sum A\Delta x} \Rightarrow \bar{x} = \frac{\sum V x_c}{\sum V}$$

Accuracy may be improved by reducing the width of the strip

Example on Centroid of Composite Figure

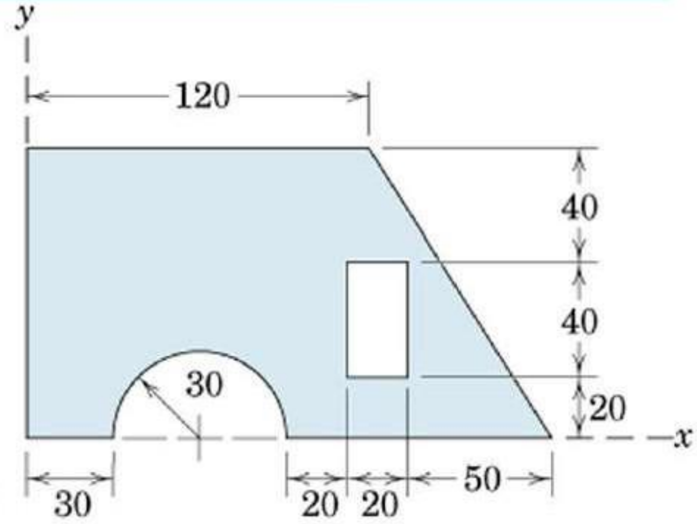
Locate the centroid of the shaded area

Solution: Divide the area into four elementary shapes: Total Area = $A_1 + A_2 - A_3 - A_4$

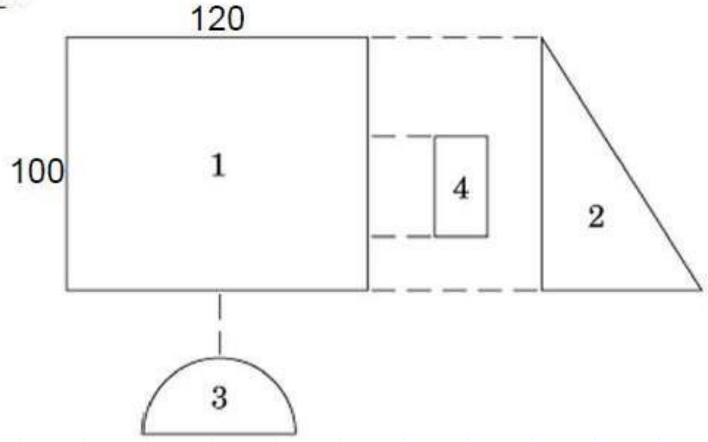
PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959\ 000}{12\ 790} = 75.0 \text{ mm}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650\ 000}{12\ 790} = 50.8 \text{ mm}$$

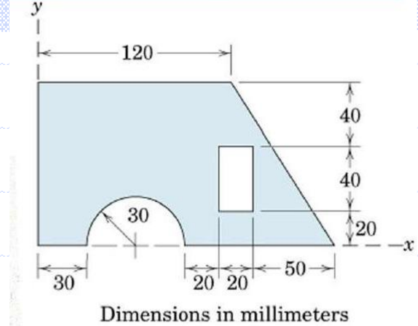


Dimensions in millimeters



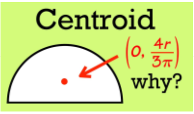
More detailed solution:

Locate the centroid of the shaded area



Parts (origin at left-bottom corner)

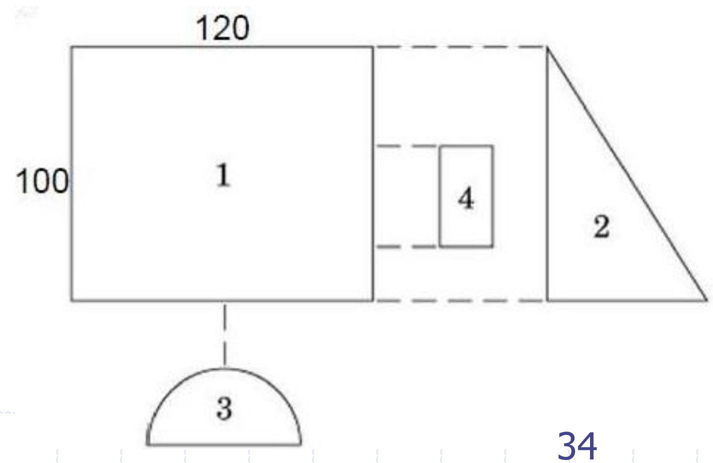
Part	Description	A_i (mm²)	x_i (mm)	y_i (mm)	$A_i x_i$ (mm³)	$A_i y_i$ (mm³)
1	Rectangle 120×100	12000	60	50	720000	600000
2	Right triangle, base 60, height 100	3000	140	33.333	420000	100000
3	Semicircle **hole**, $r = 30$	$-450\pi = -1413.717$	60	$\frac{4r}{3\pi} = 12.732$	$-84\,823.002$	$-18\,000$
4	Rectangle **hole** 20×40	-800	120	40	$-96\,000$	$-32\,000$
	Totals	**12 786.283**			**959 176.998**	**650 000.000**



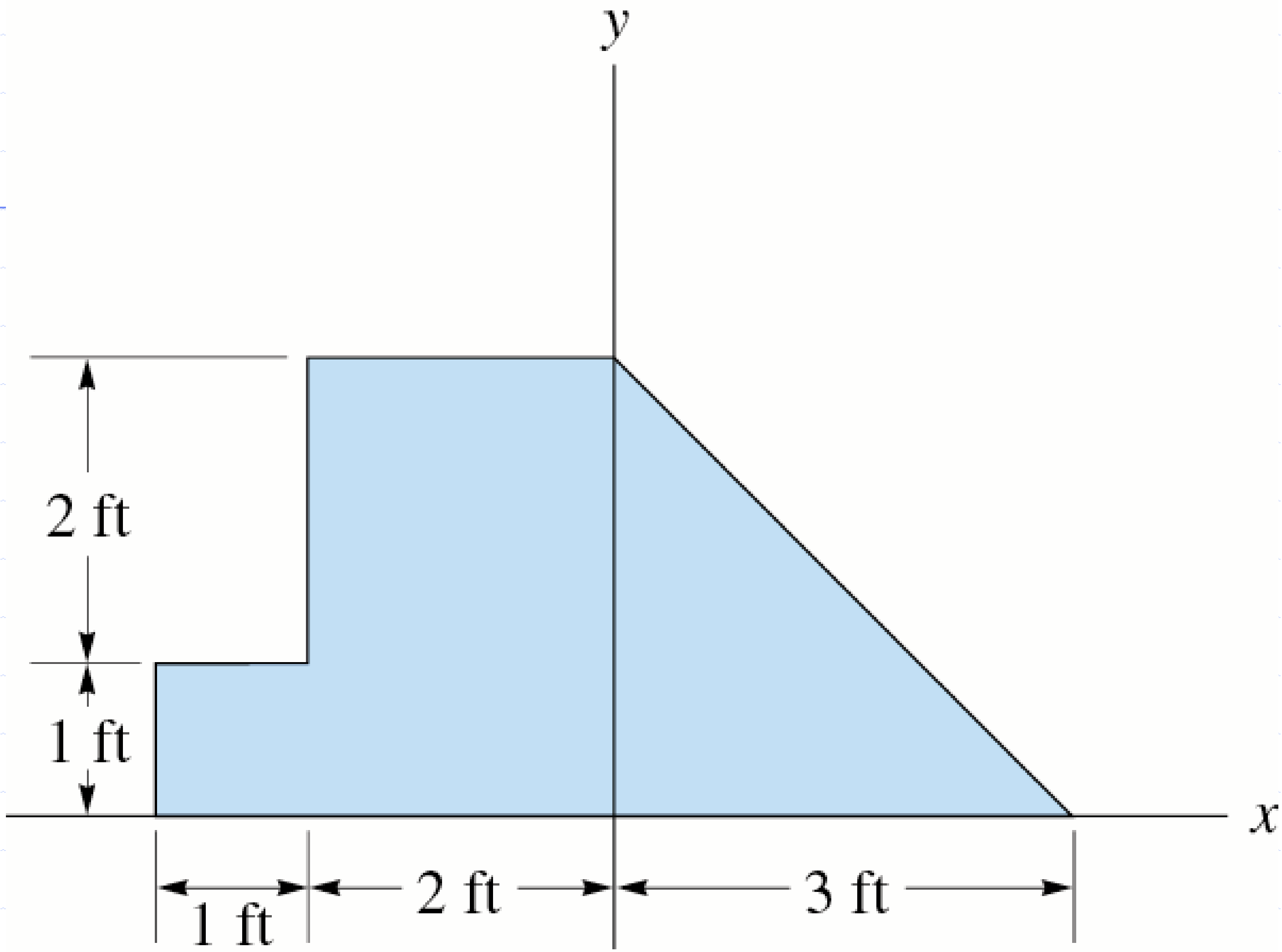
Centroid:

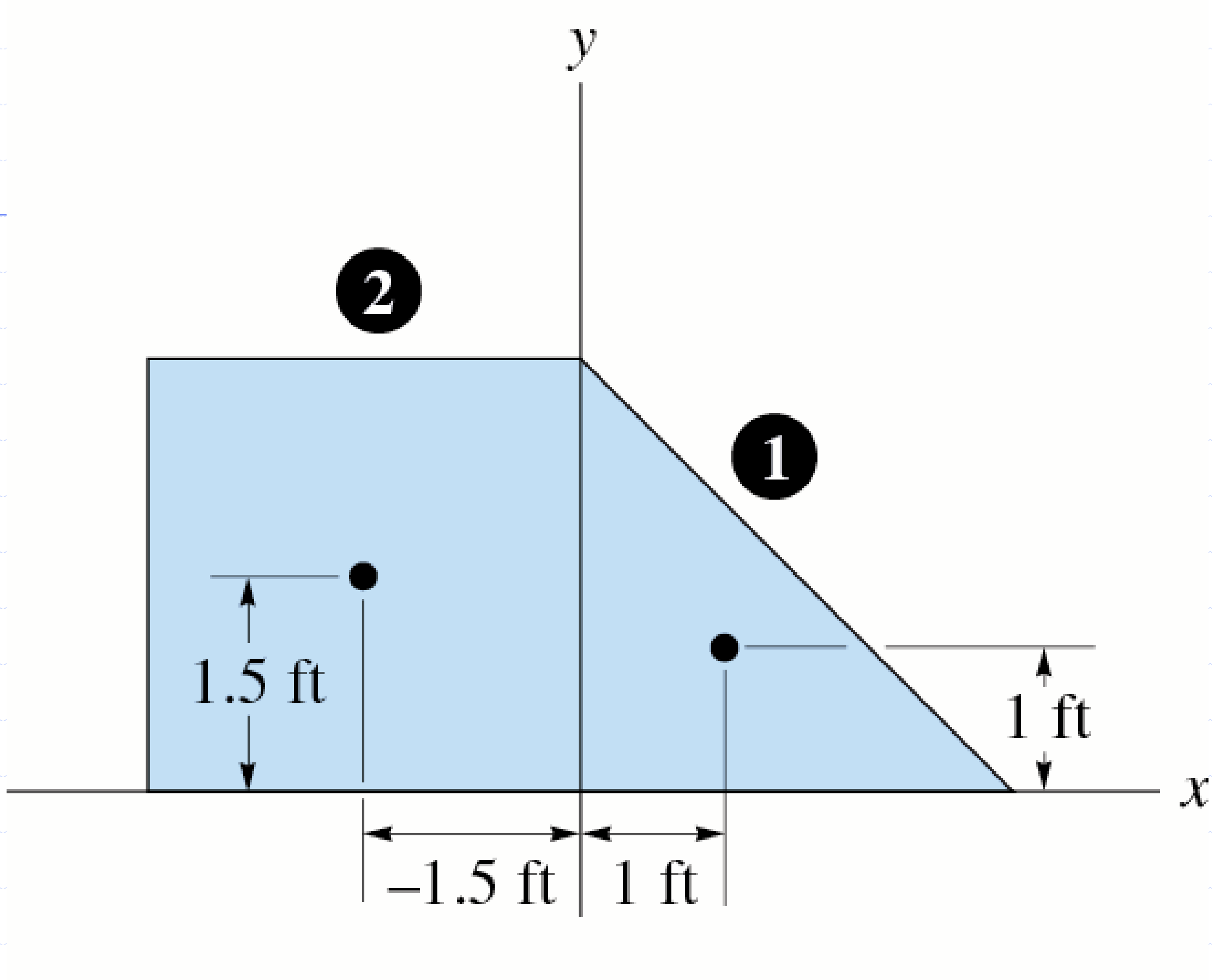
$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{959\,176.998}{12\,786.283} = \boxed{75.02 \text{ mm}},$$

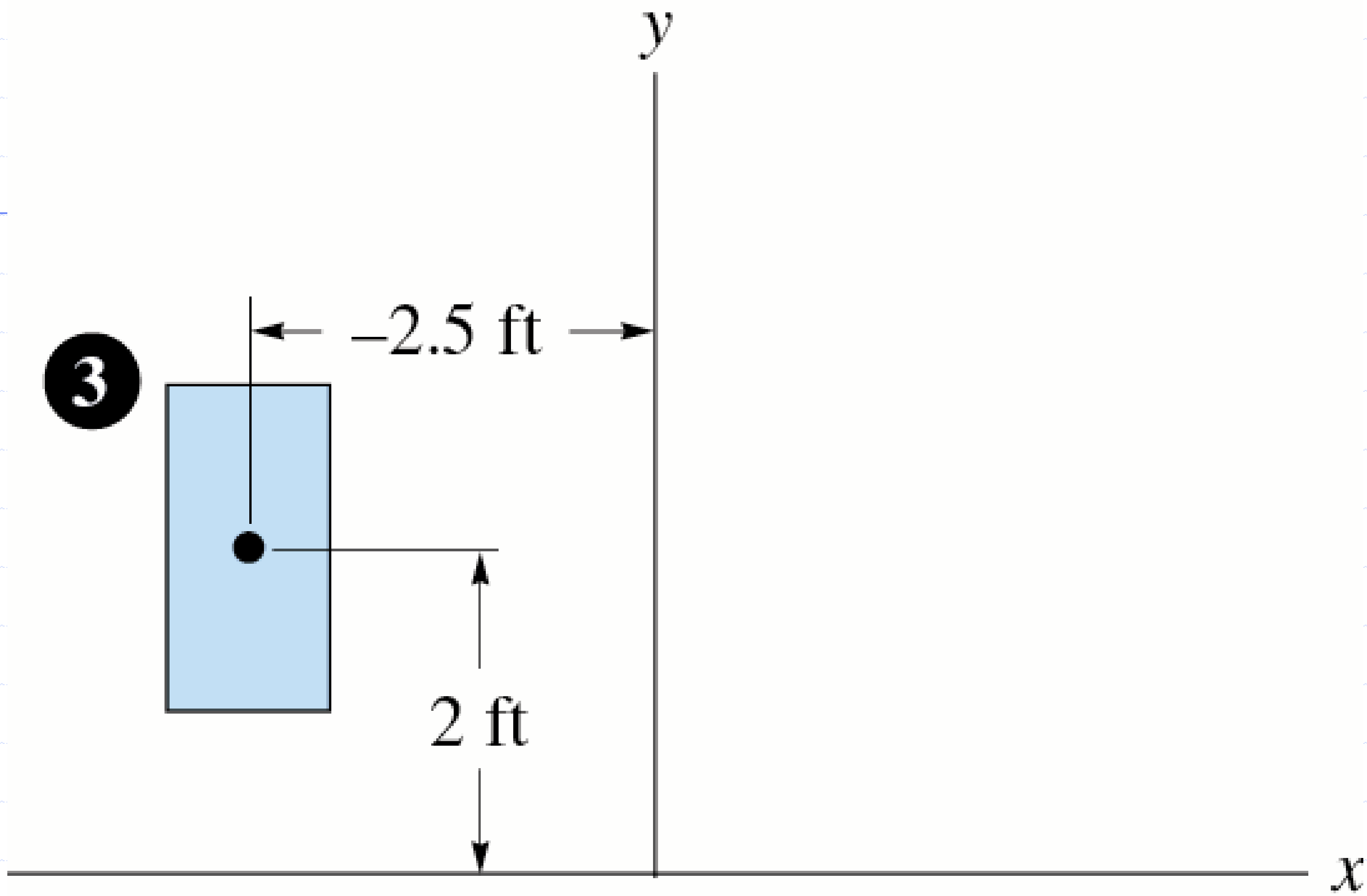
$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{650\,000.000}{12\,786.283} = \boxed{50.84 \text{ mm}}.$$

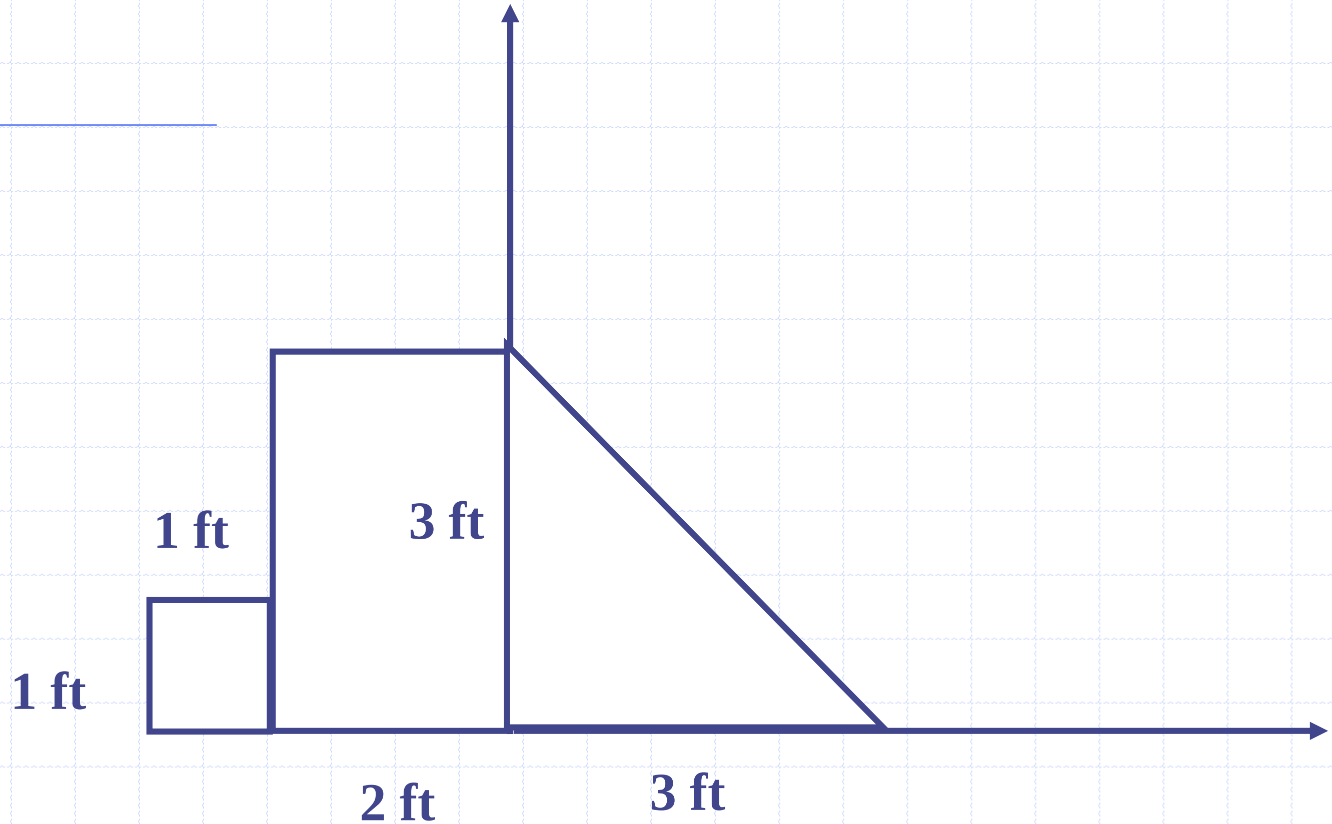


Example 2:

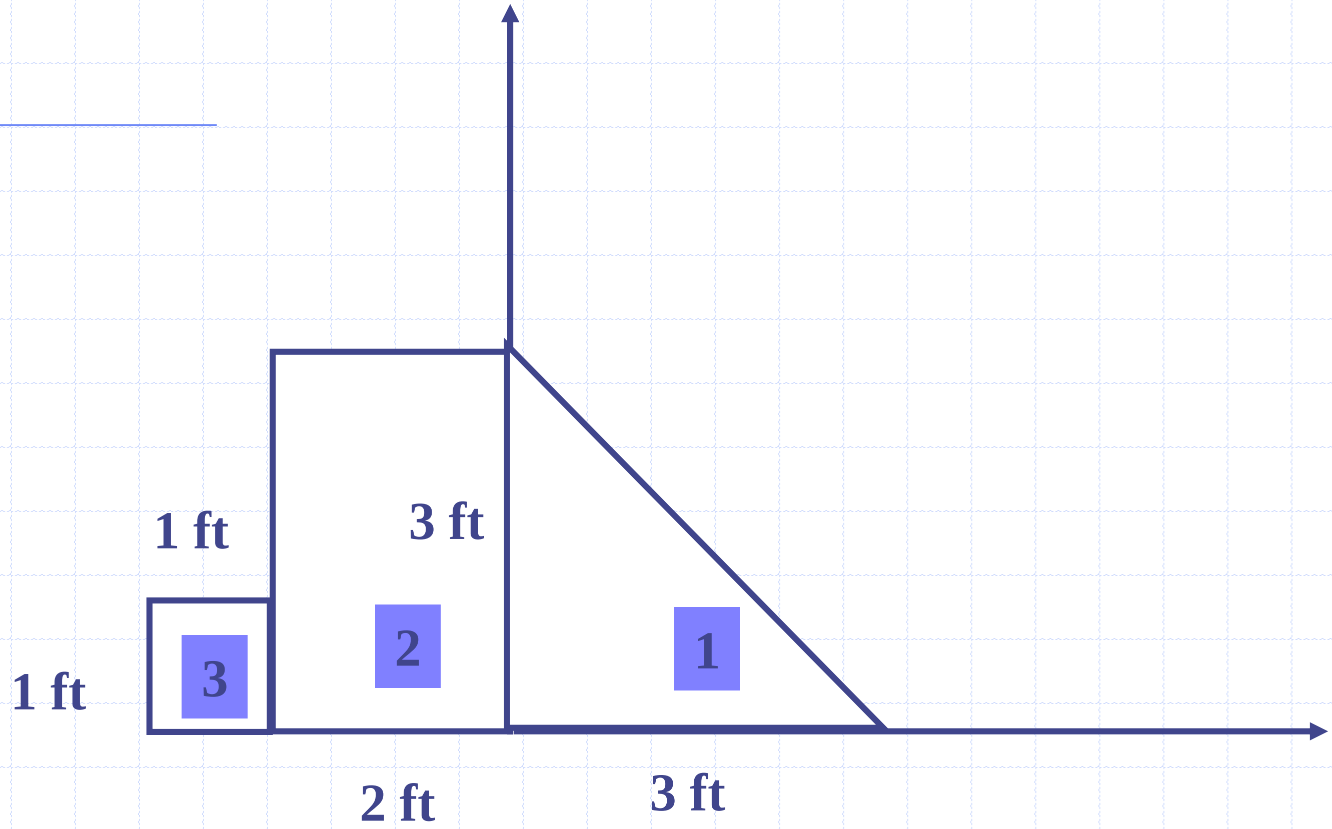



y
2
1
1.5 ft
1 ft
x
–1.5 ft
1 ft


y
3
–2.5 ft
2 ft
x



Locate Centroid of the Composite Area

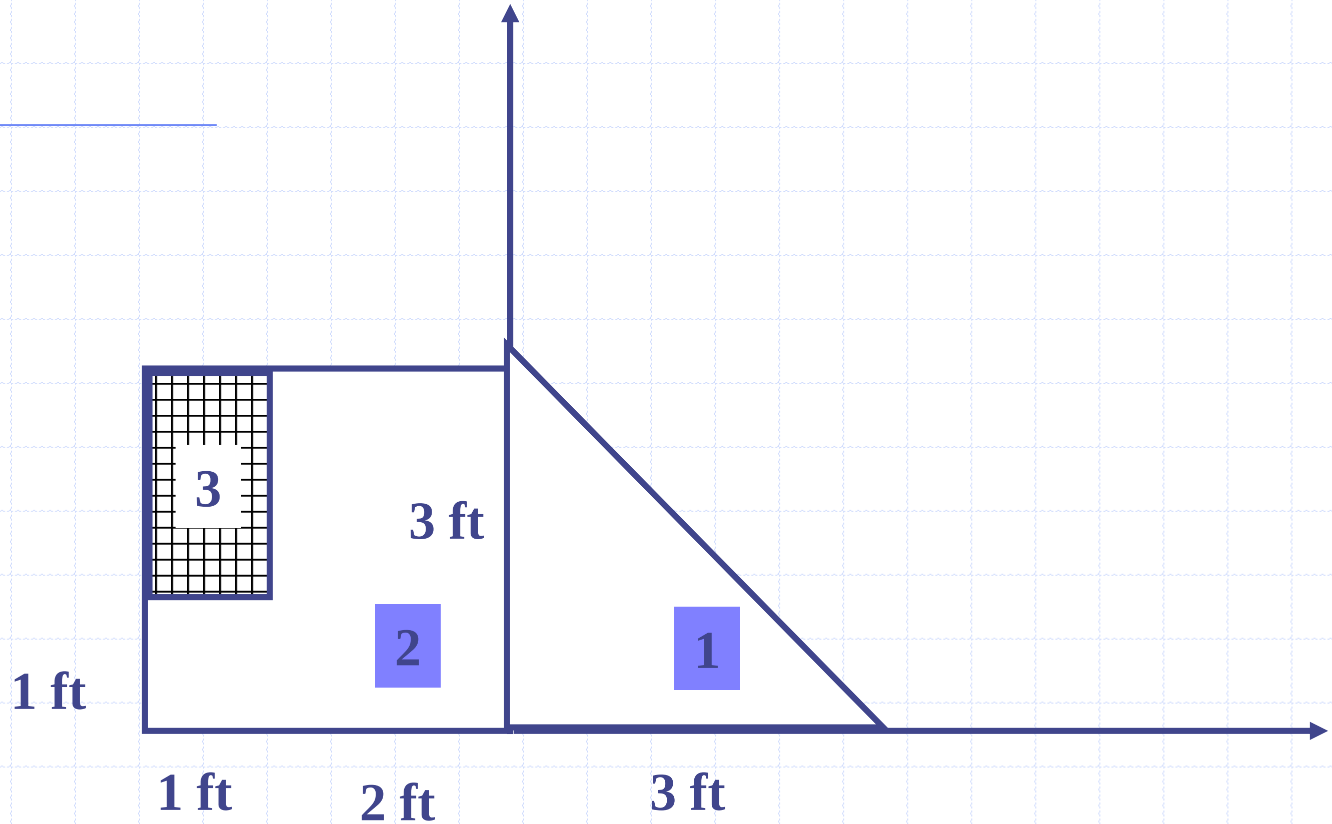

1 ft
3 ft
1 ft
3
2
1
2 ft
3 ft

Segment	A (ft^2)	x	y	xA	yA
1	4.5	1	1	4.5	4.5
2	6	-1	1.5	-6	9
3	1	-2.5	0.5	-2.5	0.5

Σ A = 11.5 Σ xA = -4 Σ xA = 14

$$\bar{x} = \frac{\sum \tilde{x}A}{\sum A} = \frac{-4}{11.5} = -0.348\,\text{ft}$$

$$\bar{y} = \frac{\sum \tilde{y}A}{\sum A} = \frac{14}{11.5} = 1.22\,\text{ft}$$


3
3 ft
2
1
1 ft
1 ft
2 ft
3 ft

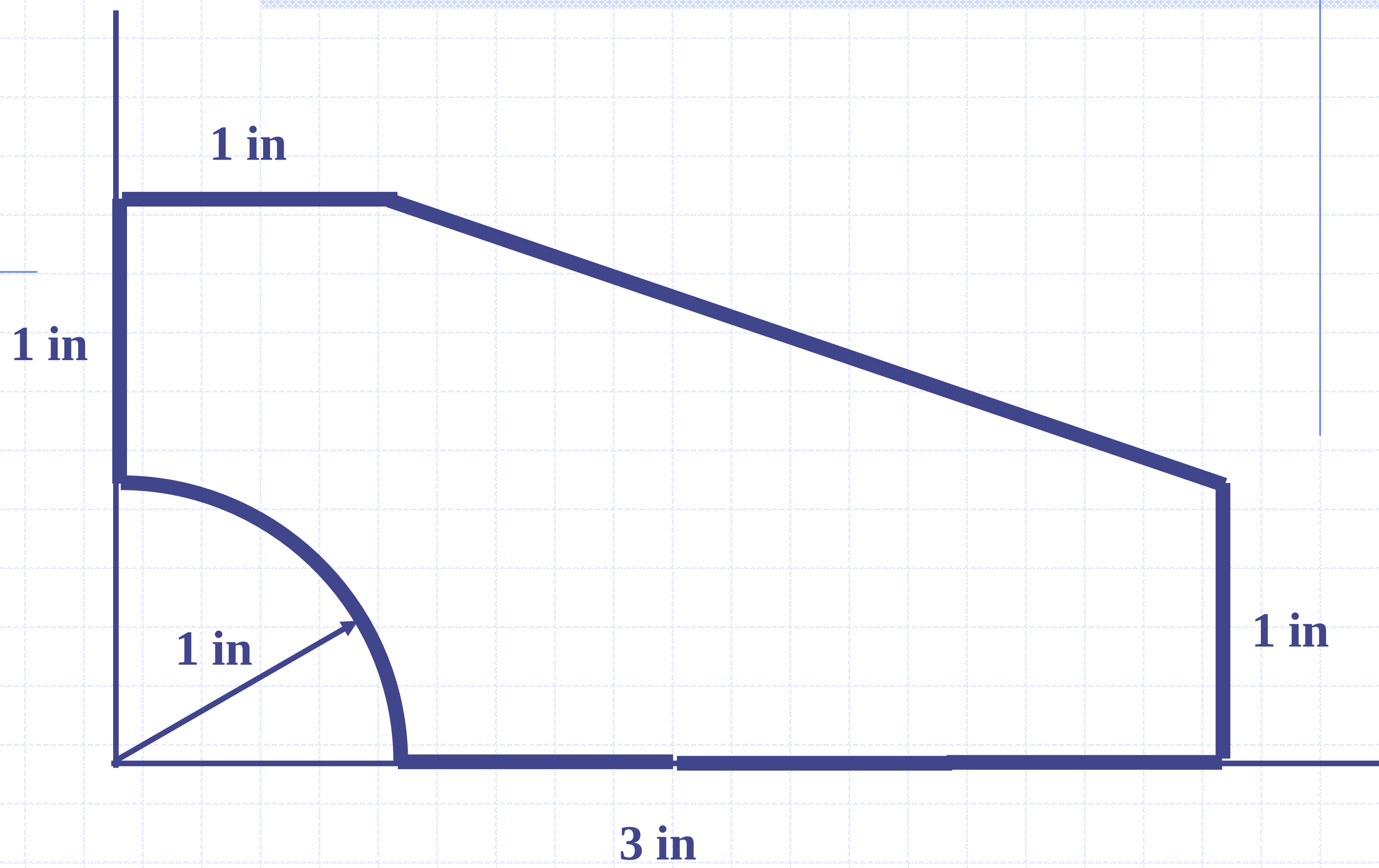
Segement	A (ft^2)	x	y	xA	yA
1	4.5	1	1	4.5	4.5
2	9	-1.5	1.5	-13.5	13.5
3	-2.5	-2.5	2	5	-4

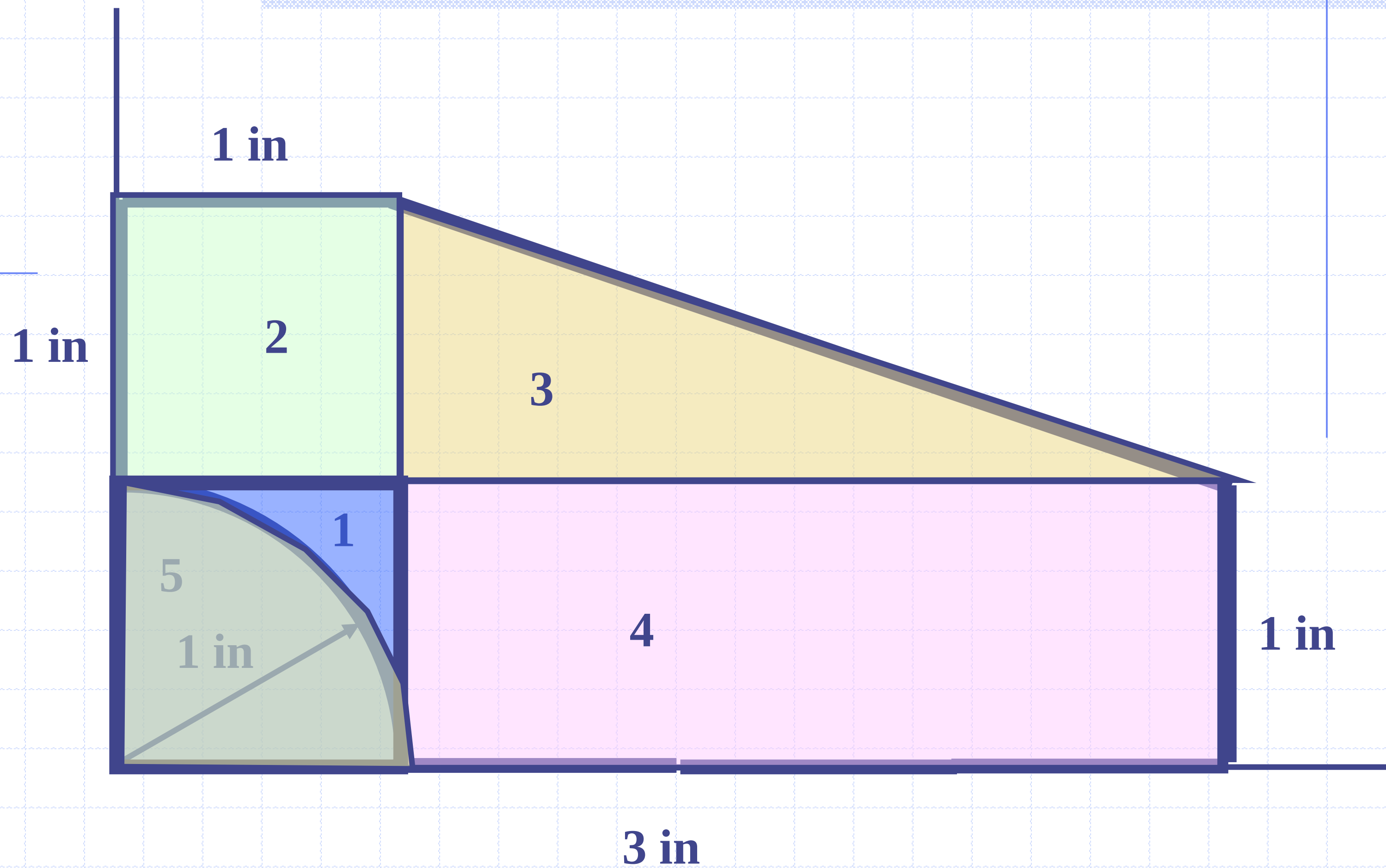
Σ A = 11.5 Σ xA = -4 Σ xA = 14

$$\bar{x} = \frac{\sum \tilde{x}A}{\sum A} = \frac{-4}{11.5} = -0.348\ \text{ft}$$

$$\bar{y} = \frac{\sum \tilde{y}A}{\sum A} = \frac{14}{11.5} = 1.22\ \text{ft}$$

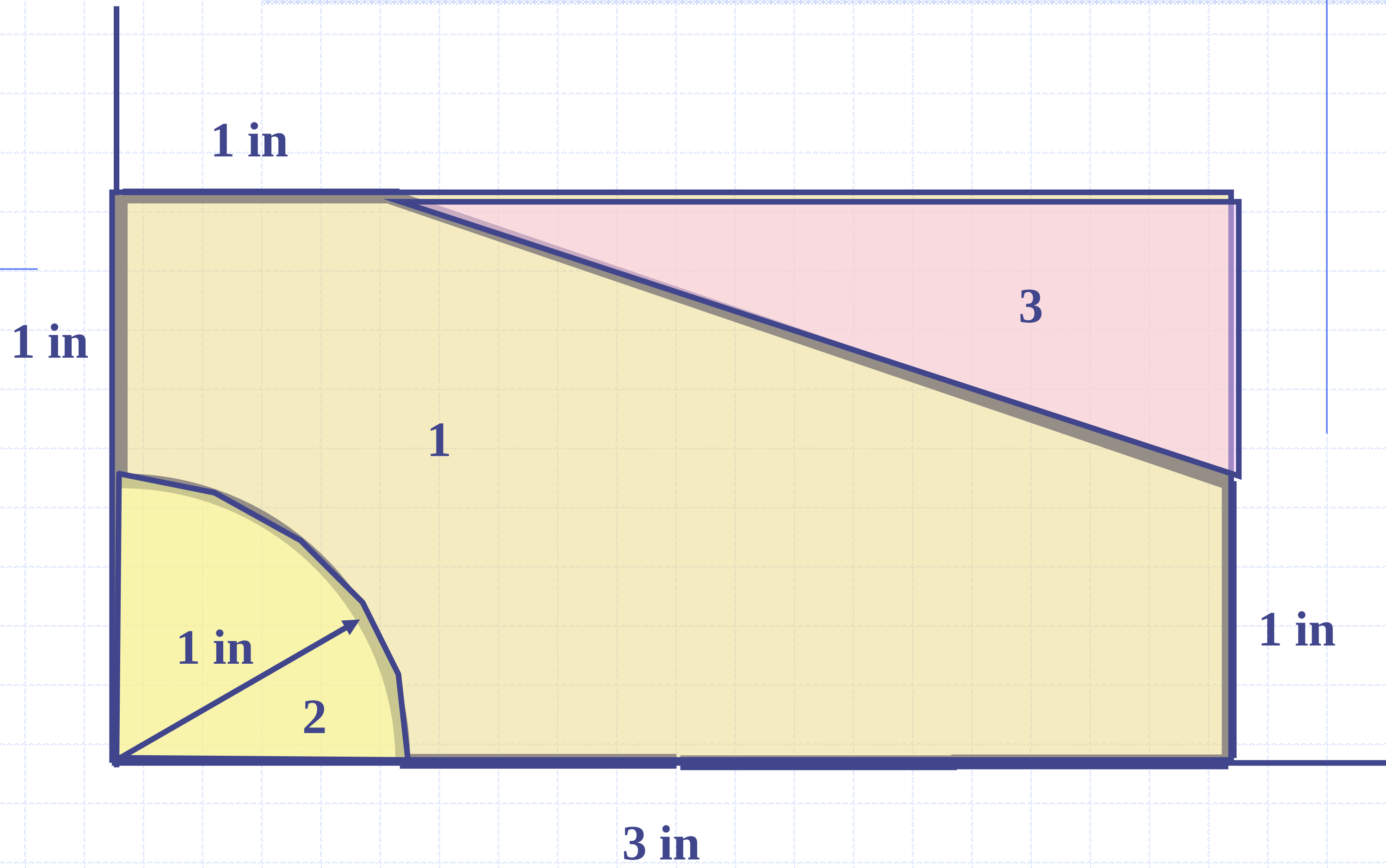
Example 3:





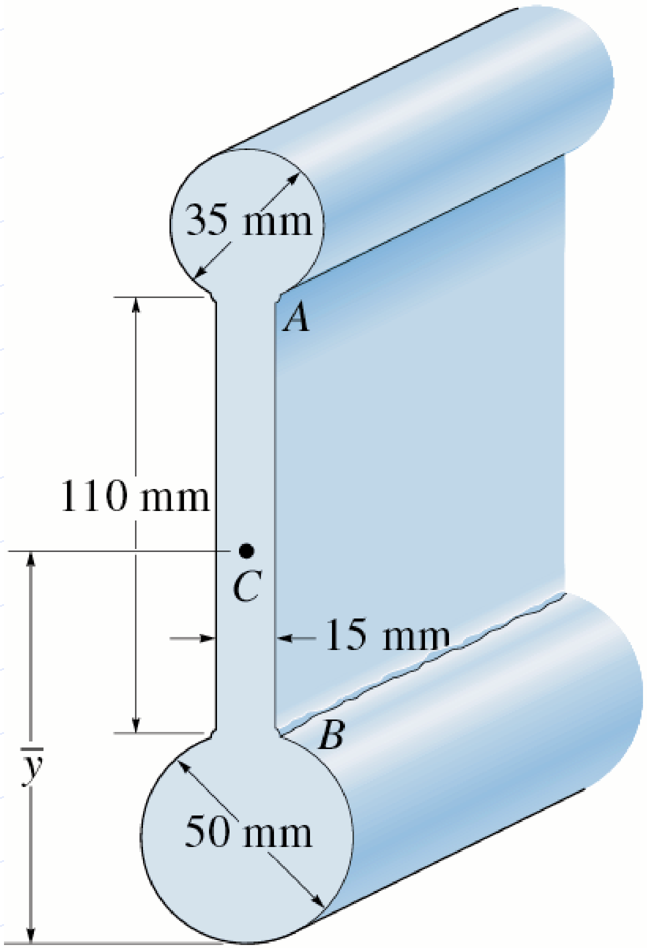
Break into sub-areas

Segment	Area	x	y	xA	yA
1.00000	1.00000	0.50000	0.50000	0.50000	0.50000
2.00000	1.00000	0.50000	1.50000	0.50000	1.50000
3.00000	1.50000	2.00000	1.33333	3.00000	2.00000
4.00000	3.00000	2.50000	0.50000	7.50000	1.50000
5.00000	-0.78540	0.42441	0.42441	-0.33333	-0.33333
	5.71460			11.16667	5.16667
	x=	1.95406			
	y=	0.90412			

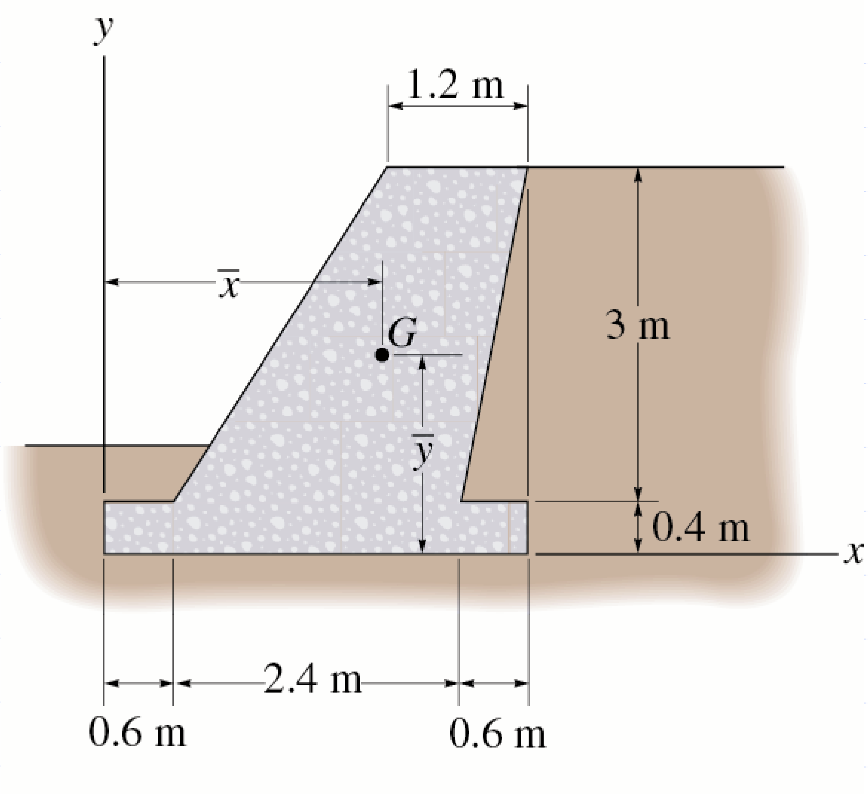

1 in
1 in
3
1
1 in
2
1 in
3 in

Segment	Area	x	y	xA	yA
1	8	2	1	16	8
2	-0.7854	0.424413	0.424413	-0.33333	-0.33333
3	-1.5	3	1.666667	-4.5	-2.5
	5.714602			11.16667	5.166667
	x=	1.954059			
	y=	0.904117			

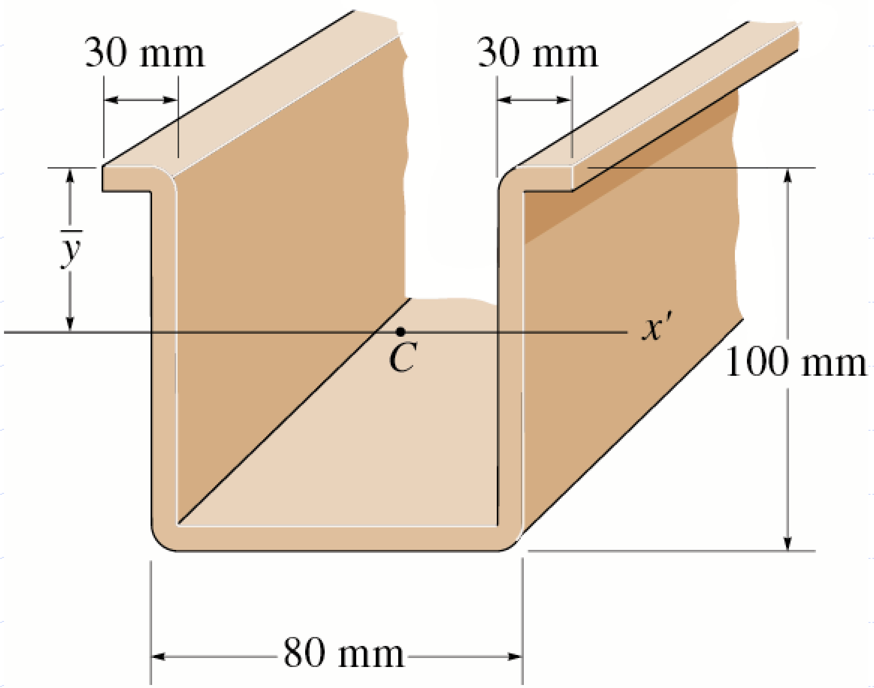
Additional Problems



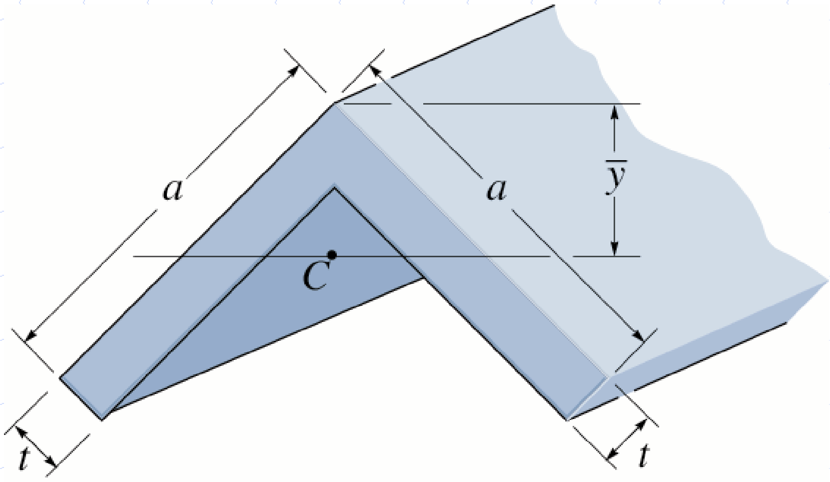
Additional Problems



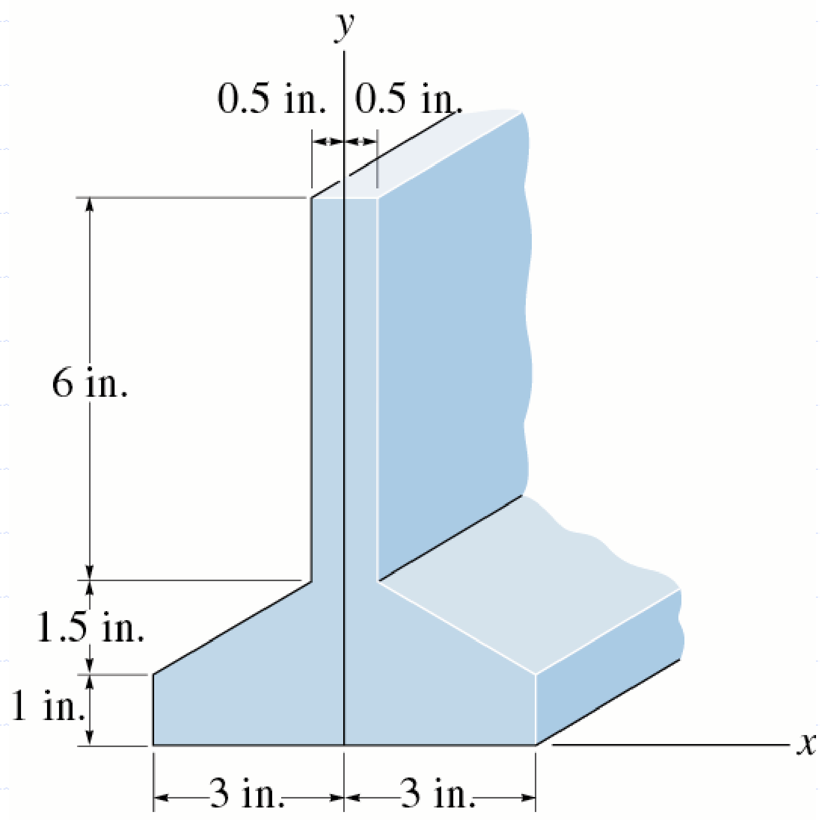
Additional Problems



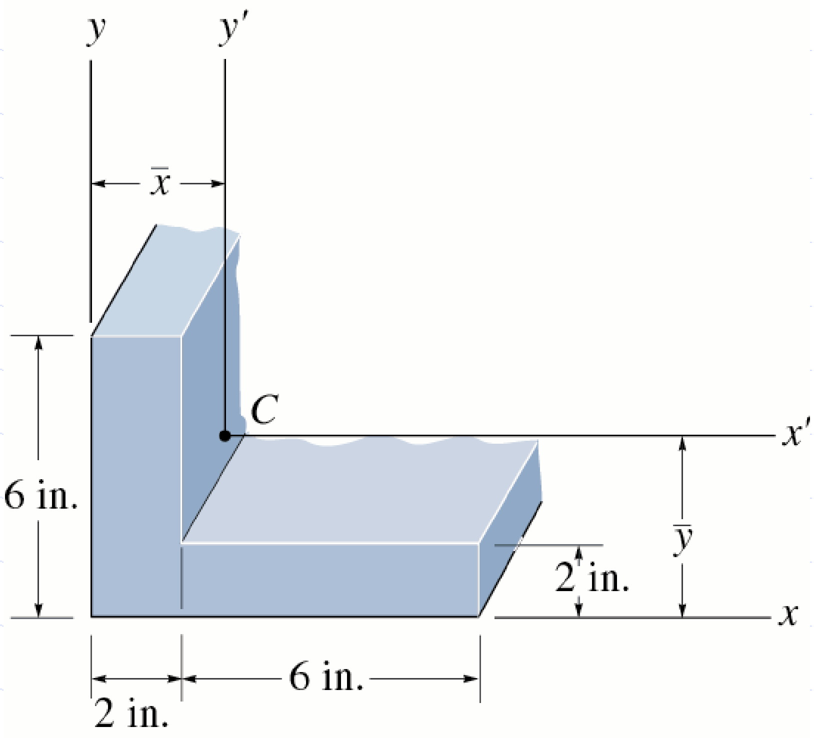
Additional Problems



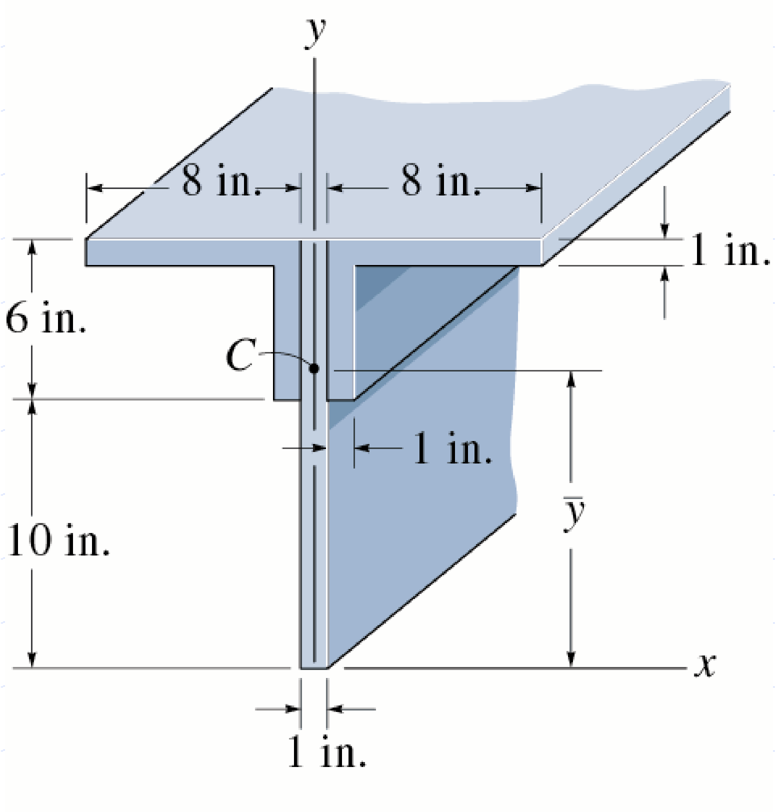
Additional Problems



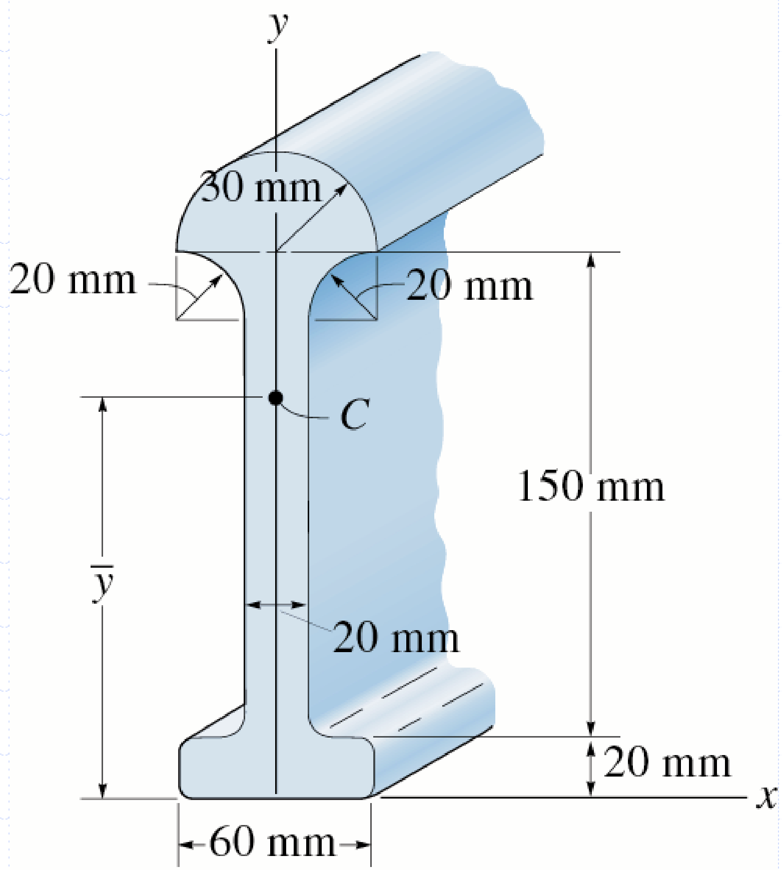
Additional Problems



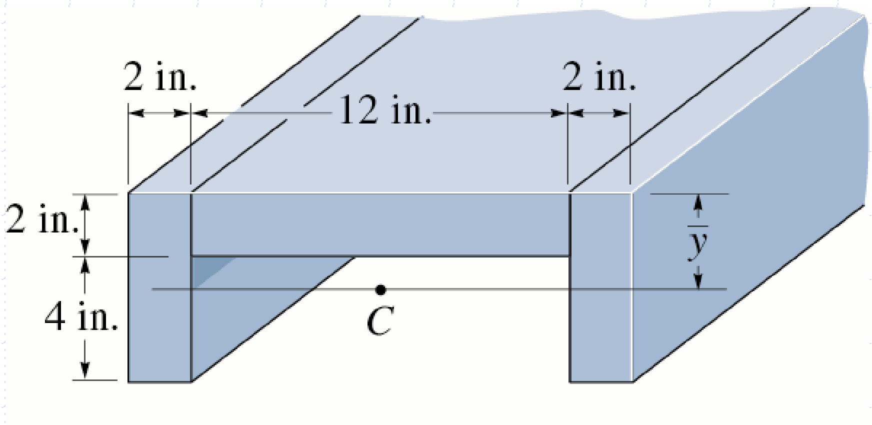
Additional Problems



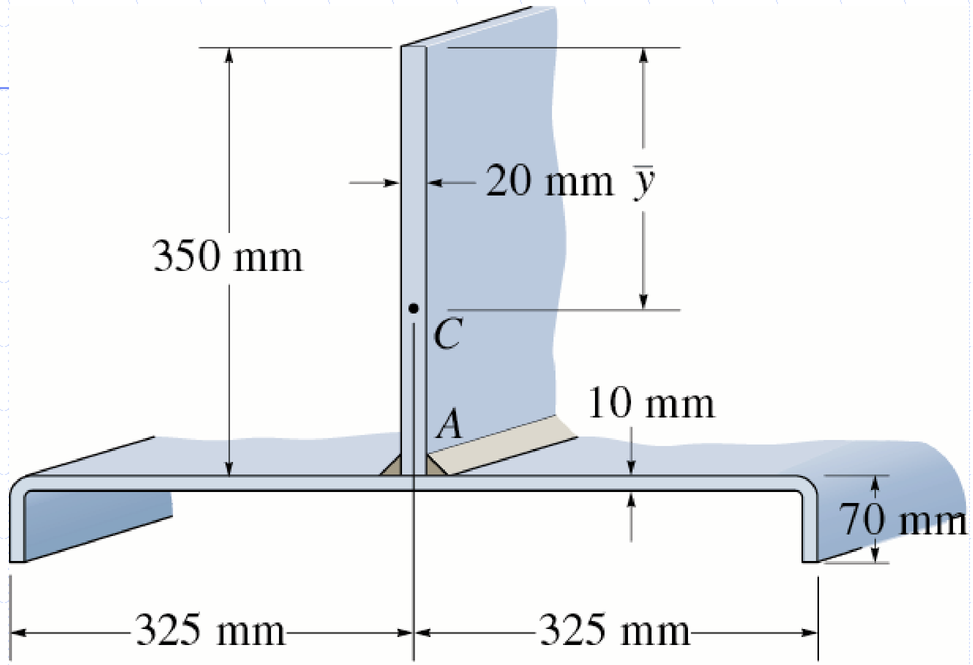
Additional Problems



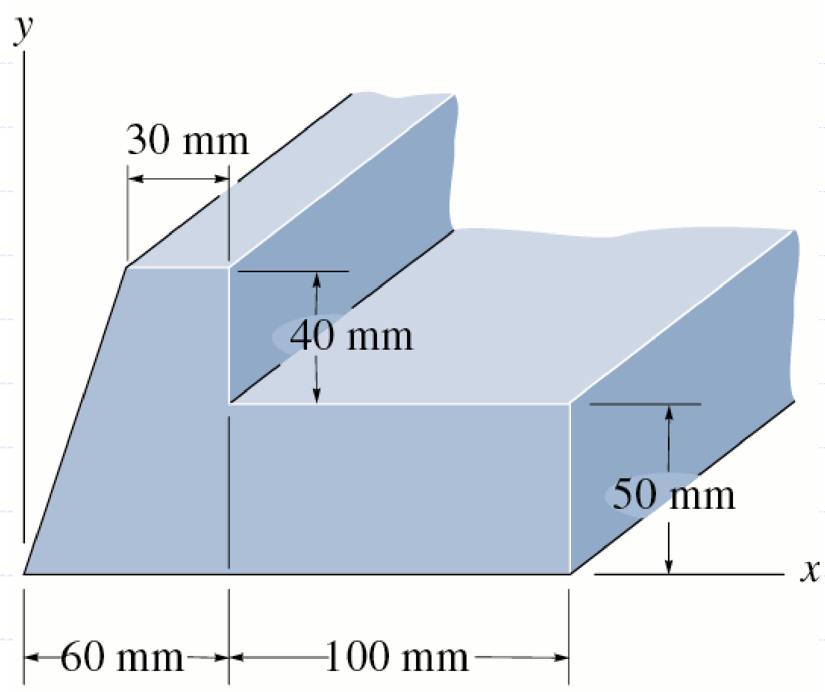
Additional Problems



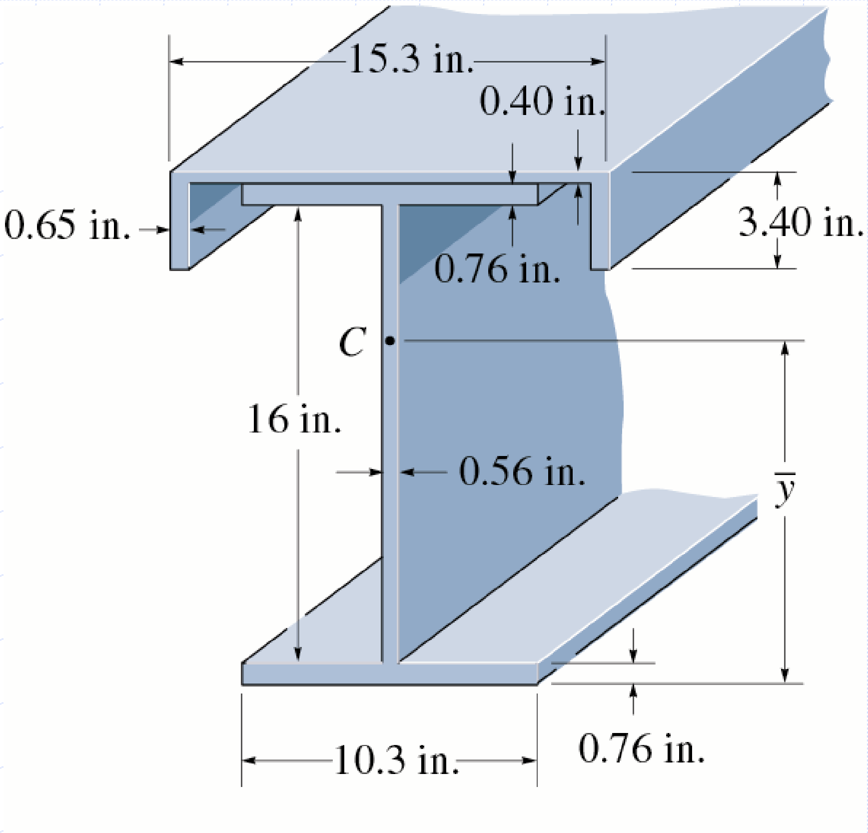
Additional Problems



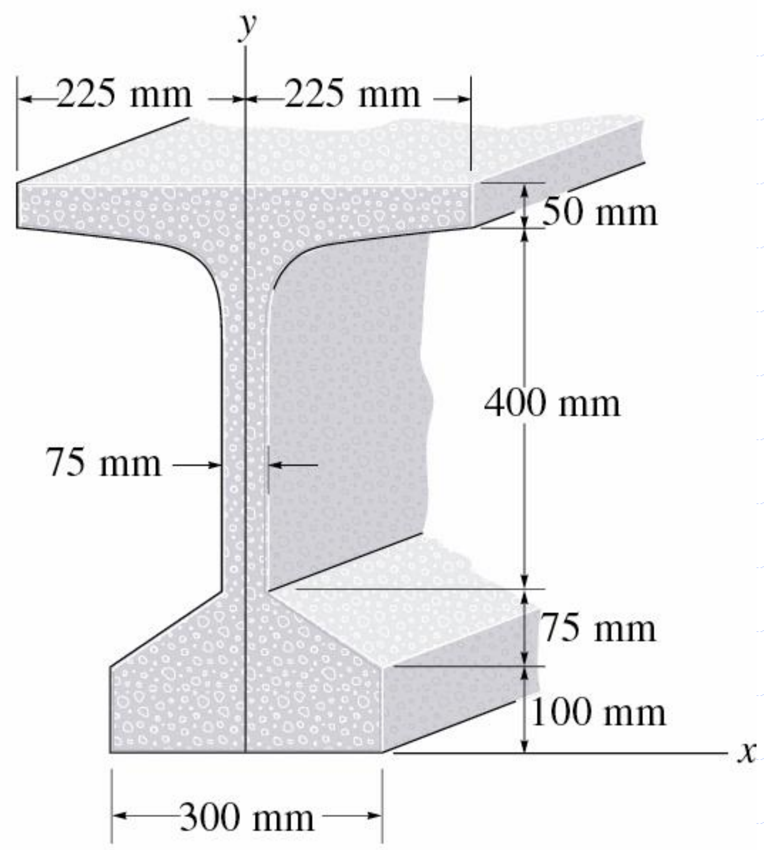
Additional Problems



Additional Problems



Additional Problems



Additional Problems

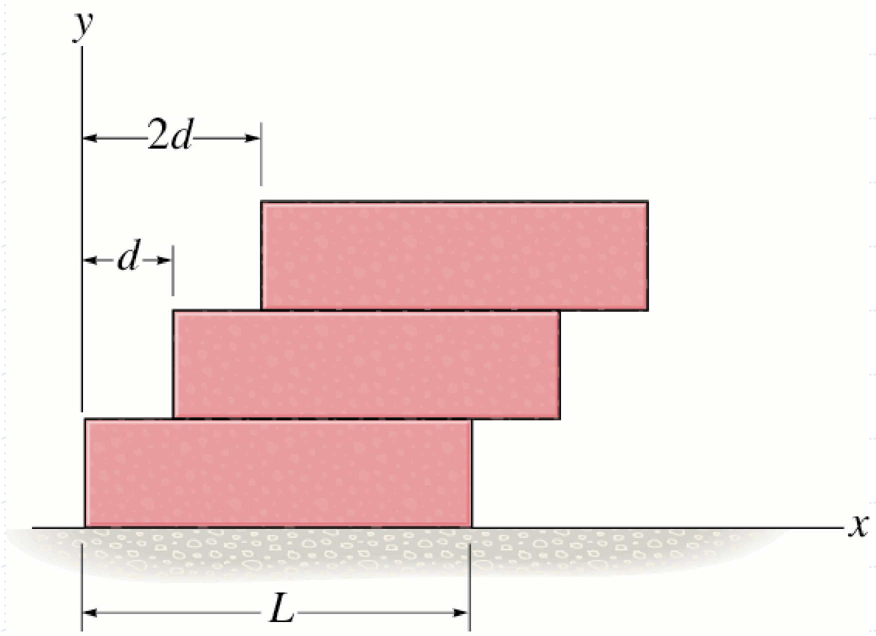
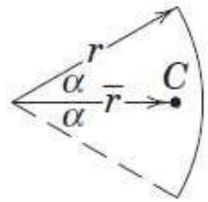
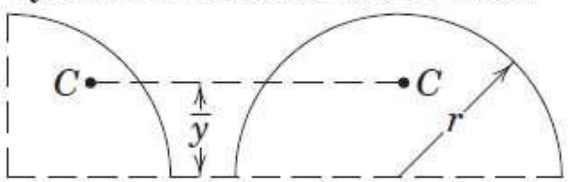
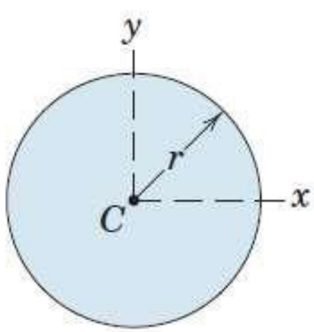


TABLE D/3 PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin\alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$

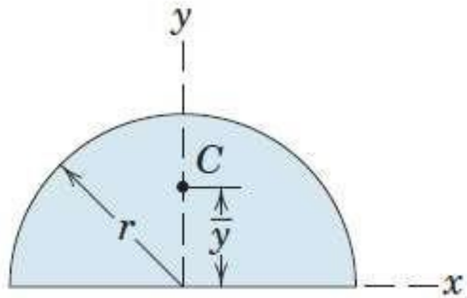
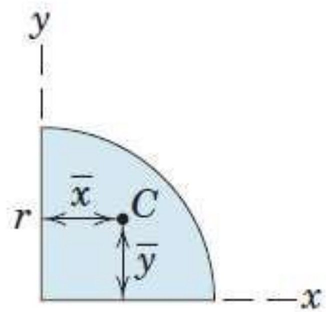
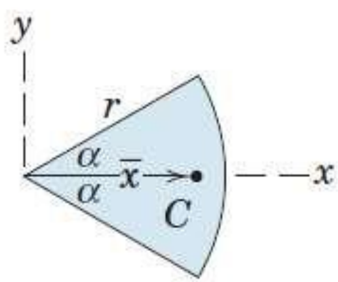
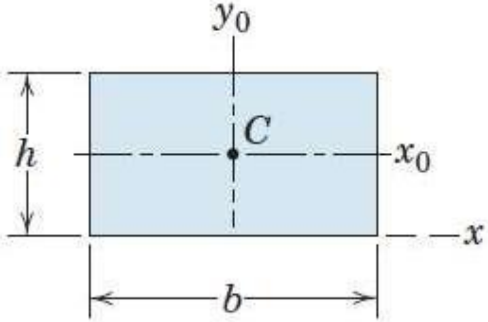
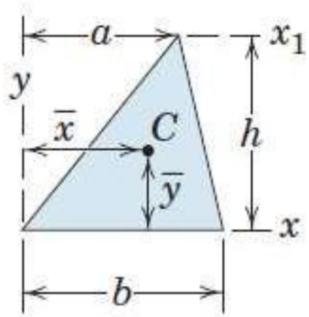
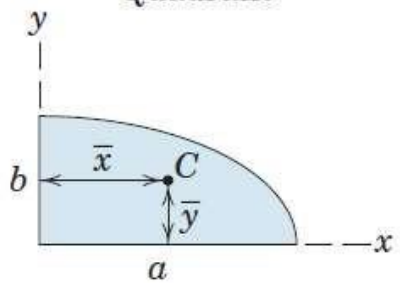
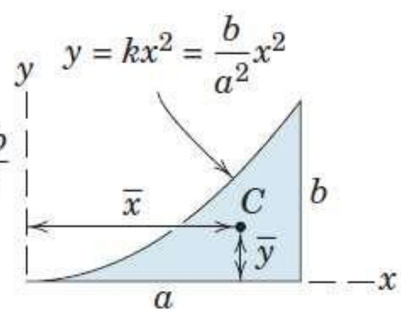
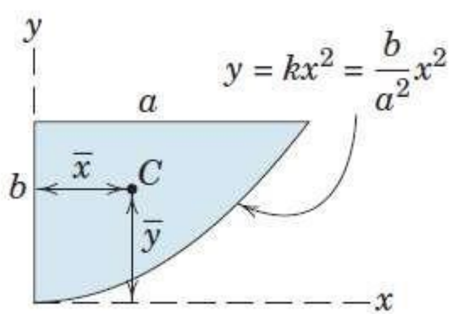
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3} \dfrac{r \sin\alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}\left(\alpha - \dfrac{1}{2}\sin 2\alpha\right)$ $I_y = \dfrac{r^4}{4}\left(\alpha + \dfrac{1}{2}\sin 2\alpha\right)$ $I_z = \dfrac{1}{2} r^4 \alpha$

TABLE D/3 PROPERTIES OF PLANE FIGURES *Continued*

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$

Area of Elliptical Quadrant 	$\bar{x} = \dfrac{4a}{3\pi}$ $\bar{y} = \dfrac{4b}{3\pi}$	$I_x = \dfrac{\pi ab^3}{16}, \quad \bar{I}_x = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) ab^3$ $I_y = \dfrac{\pi a^3 b}{16}, \quad \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) a^3 b$ $I_z = \dfrac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \dfrac{ab}{3}$ 	$\bar{x} = \dfrac{3a}{4}$ $\bar{y} = \dfrac{3b}{10}$	$I_x = \dfrac{ab^3}{21}$ $I_y = \dfrac{a^3 b}{5}$ $I_z = ab\left(\dfrac{a^3}{5} + \dfrac{b^2}{21}\right)$
Parabolic Area Area $A = \dfrac{2ab}{3}$ 	$\bar{x} = \dfrac{3a}{8}$ $\bar{y} = \dfrac{3b}{5}$	$I_x = \dfrac{2ab^3}{7}$ $I_y = \dfrac{2a^3 b}{15}$ $I_z = 2ab\left(\dfrac{a^2}{15} + \dfrac{b^2}{7}\right)$