

Chapter 5 – Kinetics of Rigid Bodies

Prepared by: *Ahmed M. El-Sherbeeney, PhD* – Fall 2025

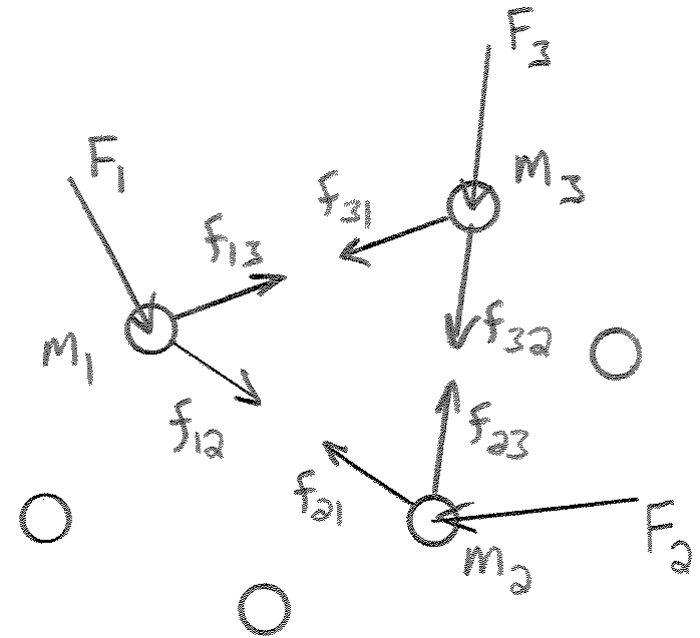
PWE for a Rigid Body

RB made up of $n \rightarrow \infty$ particles.

$$T_1 + U_{1 \rightarrow 2} = T_2$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots$$

$$= \sum_{i=1}^n \frac{1}{2} m_i v_i^2$$



$U_{1 \rightarrow 2} = \text{Work of internal and external forces}$
 $\bar{f}_{ij}$ $\bar{F}_i$

FACT: For a RB, work done by internal forces is zero.

$U_{1 \rightarrow 2}$ = work done by external forces

$$= U_{1 \rightarrow 2}^c + U_{1 \rightarrow 2}^{nc}$$

$$= V_1 - V_2 + U_{1 \rightarrow 2}^{nc}$$

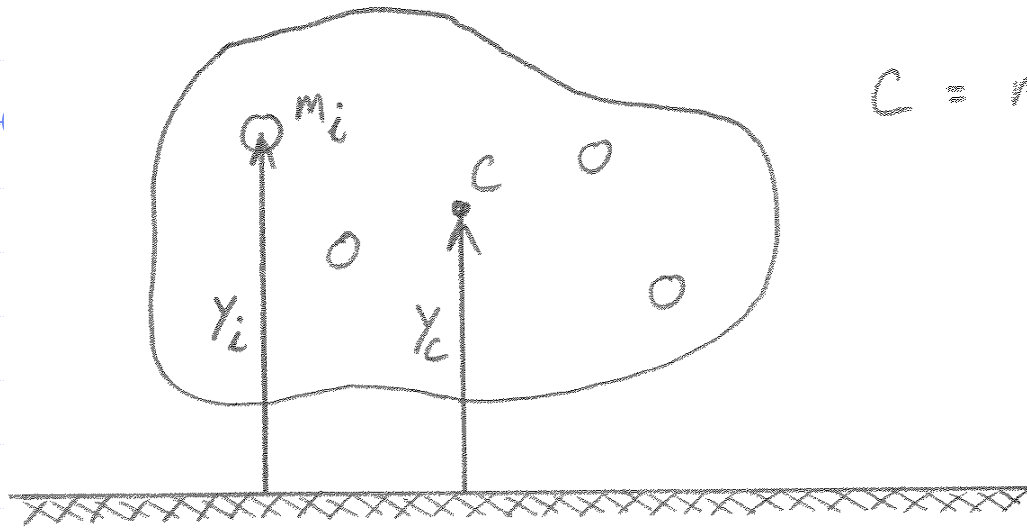
$$T_1 + V_1 + U_{1 \rightarrow 2}^{nc} = T_2 + V_2$$

Spring Potential

$$V_{\text{spring}} = \frac{1}{2} k x^2$$

x = compression/stretch
in spring

Gravitational Potential



C = mass center

$$\begin{aligned} V_{\text{weight}} &= m_1 g y_1 + m_2 g y_2 + \dots \\ &= \sum_{i=1}^{\infty} m_i g y_i = g \sum_{i=1}^{\infty} m_i y_i \end{aligned}$$

$m y_c$

$$\text{But } y_c = \frac{1}{m} \sum_{i=1}^{\infty} m_i y_i \quad \therefore$$

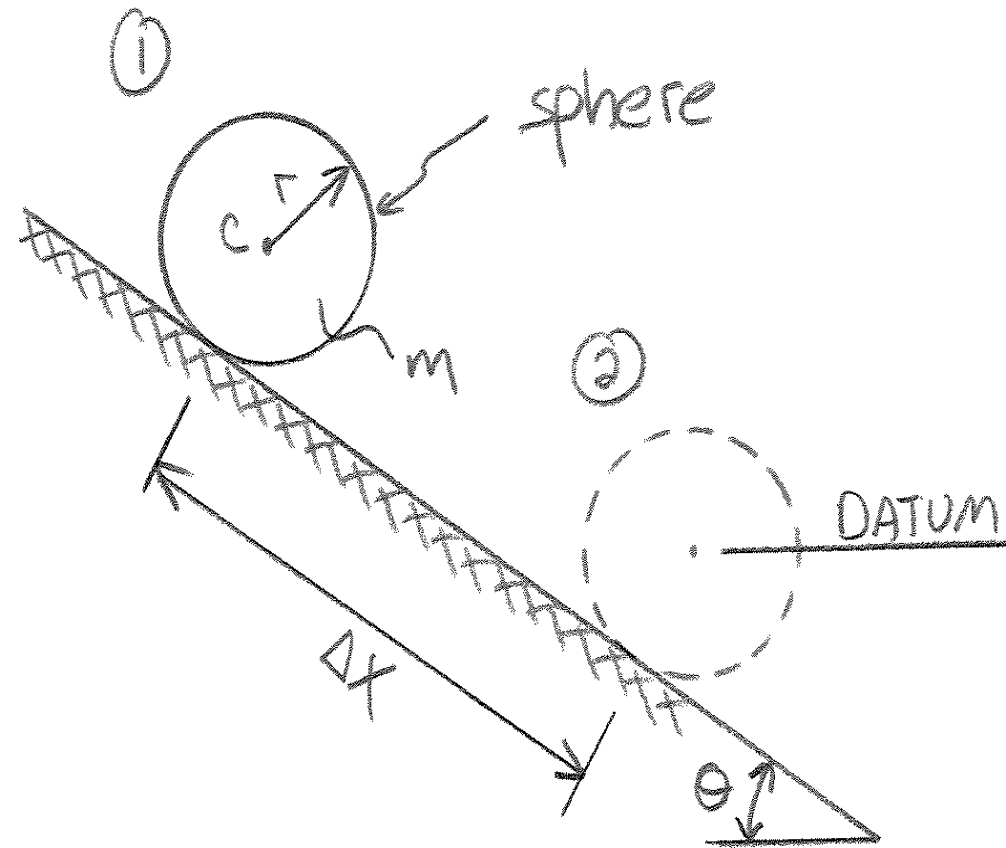
$$V_{\text{weight}} = m g y_c$$

Kinetic Energy

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$$

$$= \begin{array}{c} \text{trans.} \\ \text{KE} \end{array} + \begin{array}{c} \text{rot.} \\ \text{KE} \end{array}$$

Example:



Released from rest, rolls without slip.

Find $\bar{N}_C$ after C moves $\Delta x = 10$ ft.

Data: $\theta = 30^\circ$

Invoke $T_1 + V_1 + U_{1 \rightarrow 2}^{NC} = T_2 + V_2$

$$T_1 = \frac{1}{2} m \overset{0}{\cancel{\omega_1^2}} + \frac{1}{2} I_C \overset{0}{\cancel{\omega_1^2}} = 0$$

$$V_1 = mg \Delta X \sin \theta$$

$$U_{1 \rightarrow 2}^{NC} = 0 \quad (\text{no slip} \Rightarrow \text{friction force does no work})$$

$$T_2 = \frac{1}{2} m \omega_C^2 + \frac{1}{2} I_C \omega^2$$

$$V_2 = 0$$

Kinematic Constraint: $\omega_C = r \omega$

$$0 + mg \Delta x \sin \theta + 0 = \frac{1}{2} m v_c^2 + \frac{1}{2} \cancel{I_c} \omega^2 \quad \begin{matrix} \nearrow (v_c/r)^2 \\ \downarrow \frac{2}{5} m r^2 \end{matrix}$$

$$= \frac{1}{2} m v_c^2 + \frac{1}{5} m \cancel{r^2} \cdot \frac{v_c^2}{r^2}$$

$$= \frac{7}{10} m v_c^2$$

$$v_c = \sqrt{\frac{10}{7} g \Delta x \sin \theta}$$

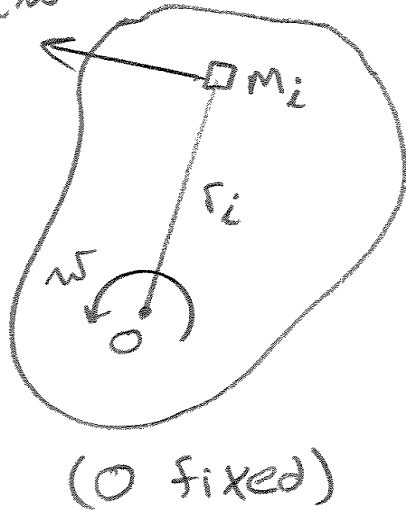
$$= \sqrt{\frac{10}{7} (32.174 \text{ ft/s}^2) (10 \text{ ft}) \sin(30^\circ)}$$

$$= 15.160 \text{ ft/s}$$

$$\therefore \bar{v}_c = 15.2 \text{ ft/s} \nearrow 30^\circ$$

Noncentroidal Rotation

$$v_i = r_i \omega$$

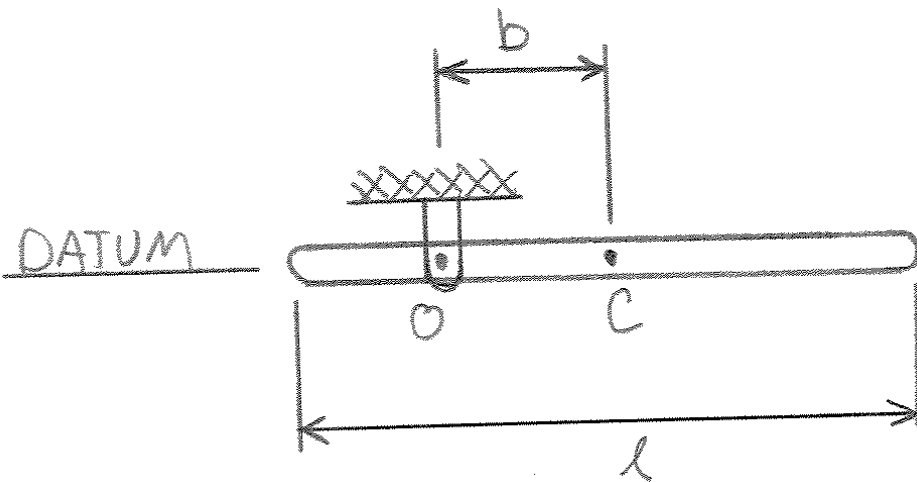


$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i v_i^2 \\ &= \frac{1}{2} \sum_i m_i (r_i \omega)^2 \\ &= \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2 \\ &= \frac{1}{2} I_O \omega^2 \end{aligned}$$

$$\therefore \boxed{T = \frac{1}{2} I_O \omega^2} \quad \text{—————} \quad (*)$$

I_O = MMI of RB about fixed pt. O

Example:



Released from rest.

Find b that maximizes ω when rod is in vertical position.

Invoke $T_1 + V_1 + U_{1 \rightarrow 2} = T_2 + V_2$

$$T_2 = \frac{1}{2} I_O \omega^2$$
$$= \frac{1}{2} \left(\frac{1}{12} m l^2 + m b^2 \right) \omega^2$$

$$V_2 = -mgb$$

Thus,

$$0 + 0 + 0 = \frac{1}{24} (ml^2 + 12mb^2) \omega^2 - mgb$$

$$\therefore \omega(b) = \sqrt{\frac{24gb}{l^2 + 12b^2}}$$

b	$\omega / \sqrt{\frac{g}{l}}$
0	0
$l/4$	1.85
$l/2$	1.73
$3l/4$	1.52
l	1.36

Maximize w by maximizing w^2 .

$$w^2 = 24gb(l^2 + 12b^2)^{-1}$$

$$\frac{d(w^2)}{db} = 24g(l^2 + 12b^2)^{-1} - 24gb(l^2 + 12b^2)^{-2} \cdot 24b$$

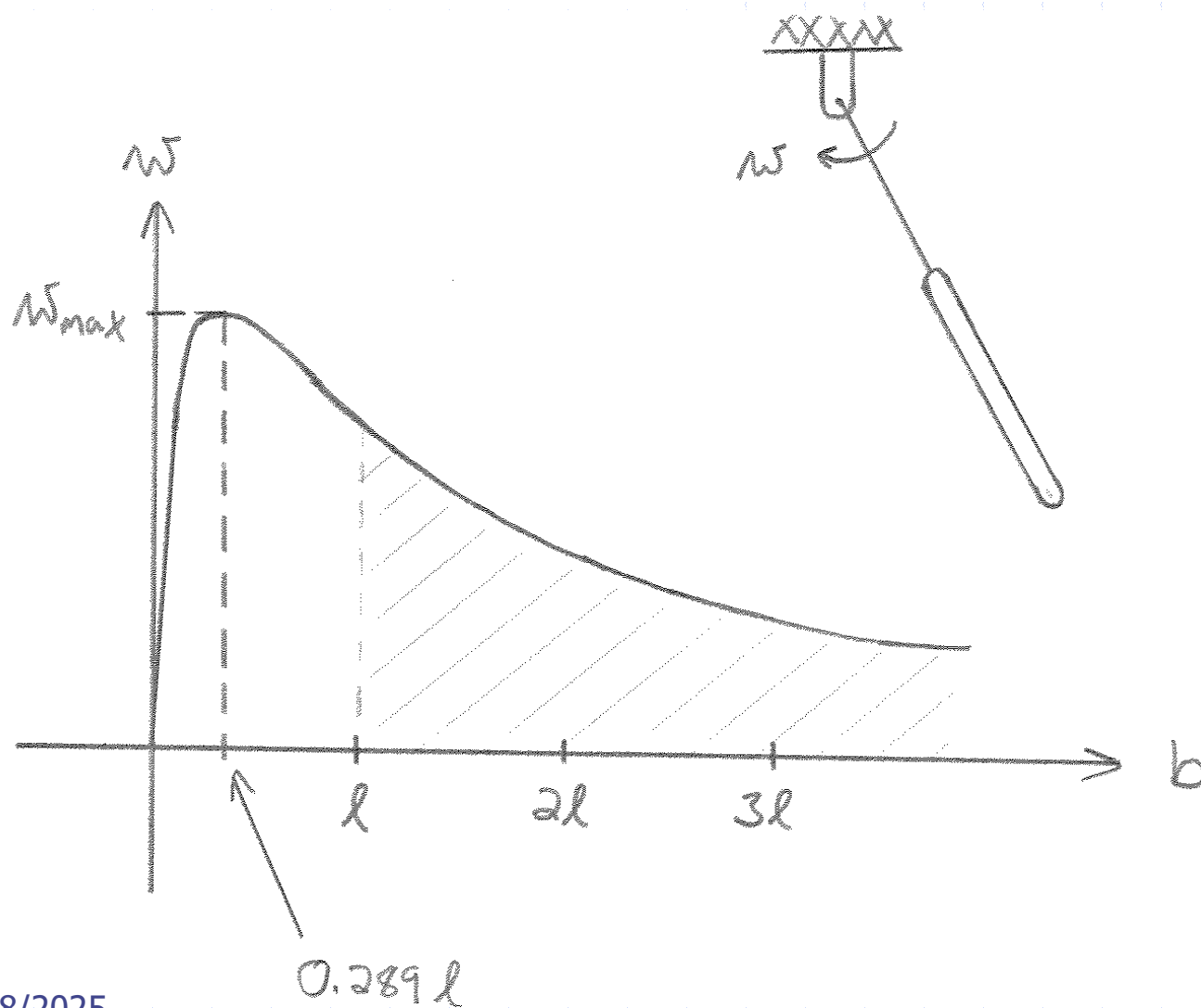
$$= \frac{\cancel{24g}(l^2 + 12b^2) - \cancel{24gb} \cdot 24b}{(l^2 + 12b^2)^2} = 0$$

$$\Rightarrow l^2 + 12b^2 - 24b^2 = l^2 - 12b^2 = 0$$

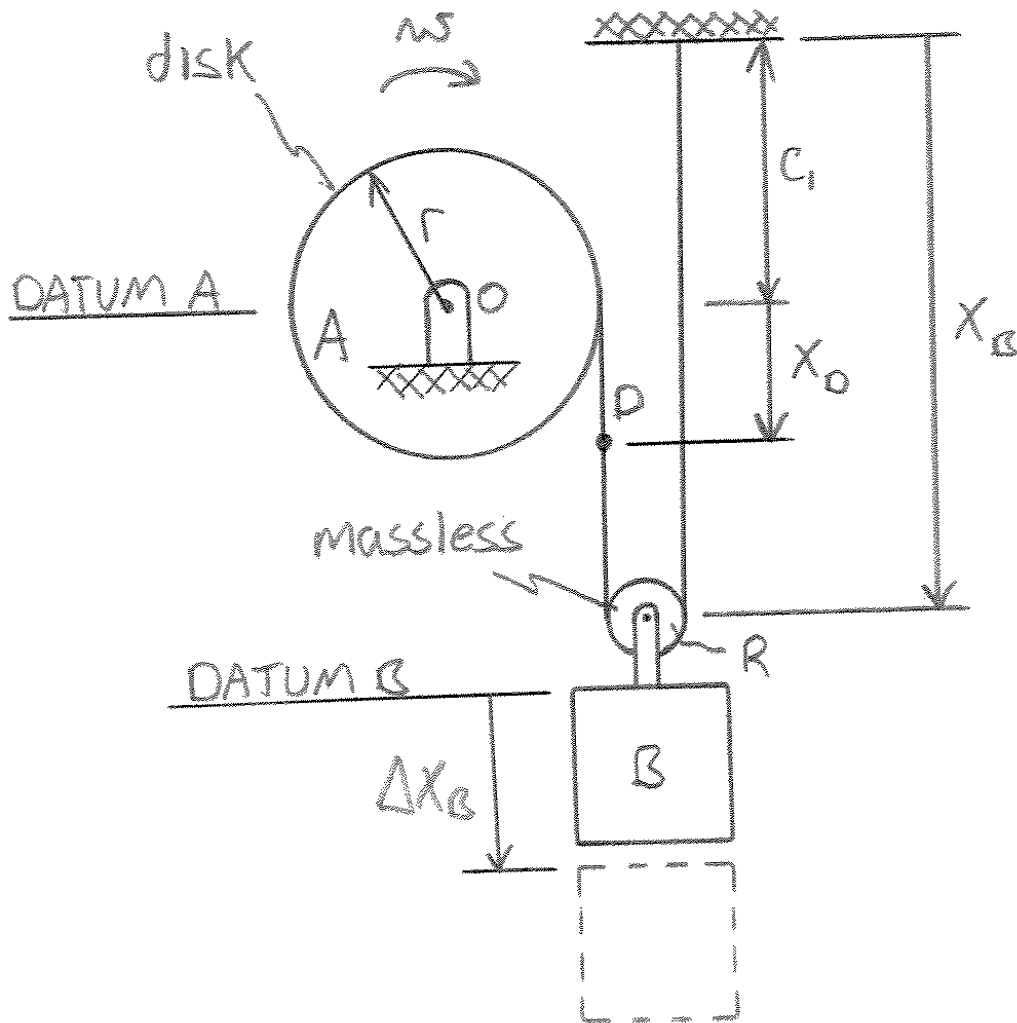
$$\Rightarrow b = \pm \sqrt{\frac{l^2}{12}} = \pm \frac{l}{2\sqrt{3}}$$

Keep positive root.

$$\therefore b = \frac{l}{2\sqrt{3}} \approx 0.289l$$



Example:



Released from rest.

Data: $m_A = 5 \text{ Kg}$
 $m_B = 2 \text{ Kg}$
 $r = 200 \text{ mm}$

Find $\bar{\omega}$ of A if:

$$\Delta x_B = 500 \text{ mm}$$

Kinematic Constraint

$$X_B + \uparrow R + (X_B - C_1 - X_D) = \text{const.}$$

$$\Rightarrow 2X_B - X_D + \underbrace{\uparrow R - C_1}_{\text{const.}} = \text{const.}$$

$$\Rightarrow 2X_B - X_D = \text{const.}$$

$$\frac{d}{dt}: 2\dot{X}_B - \dot{X}_D = 0$$

$\nwarrow$
 Γ_N

$$\Rightarrow 2\dot{X}_B = \dot{X}_D \quad \text{—————} (*)$$

Kinetic Analysis

Invoke $\cancel{T_1} + \cancel{V_1} + \cancel{U_{1 \rightarrow 2}} = T_2 + V_2$

$$T_2 = T_2^A + T_2^B$$

$$= \frac{1}{2} I_O^A \omega^2 + \frac{1}{2} m_B v_B^2$$

$$= \frac{1}{2} \cdot \frac{1}{2} m_A r^2 \omega^2 + \frac{1}{2} m_B \left(\frac{r\omega}{2} \right)^2 \quad (\text{from } *)$$

$$= \left(\frac{1}{4} m_A + \frac{1}{8} m_B \right) r^2 \omega^2$$

$$V_2 = -m_B g \Delta x_B$$

Thus

$$0 + 0 + 0 = \left(\frac{1}{4}m_A + \frac{1}{8}m_B\right)r^2\omega^2 - m_B g \Delta x_B$$

$$\omega = \sqrt{\frac{8m_B g \Delta x_B}{(2m_A + m_B)r^2}}$$

$$= \sqrt{\frac{8(2\text{kg})(9.81\text{m/s}^2)(0.50\text{m})}{[2(5\text{kg}) + (2\text{kg})](0.20\text{m})^2}} = 12.786 \text{ rad/s}$$

$$\therefore \bar{\omega} = 12.8 \text{ rad/s} \quad \downarrow$$

Overview of Kinetics

① Newton's Equations

Particle: $\vec{F} = m\vec{a}$ (NSL)

RB: $\vec{F} = m\vec{a}_c$ (NSL)

$$\vec{M}_p = I_p \vec{\alpha}$$

② Principle of Work-Energy (PWE)

$$T_1 + U_{1 \rightarrow 2} = T_2$$

$$T_1 + V_1 + U_{1 \rightarrow 2}^{nc} = T_2 + V_2$$

Particle: $T = \frac{1}{2}mv^2$

$$V_{\text{weight}} = mgy$$

$$V_{\text{spring}} = \frac{1}{2}kx^2$$

RB: $T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$

$$T = \frac{1}{2} I_o \omega^2 \quad (\text{O fixed})$$

$$V_{\text{weight}} = mgy_c$$

$$V_{\text{spring}} = \frac{1}{2} K x^2$$

Interpretation

NSL: Forces = translational resistance $\times$ linear acceleration

Moments = rotational resistance $\times$ angular acceleration

Relates: Forces, acc.

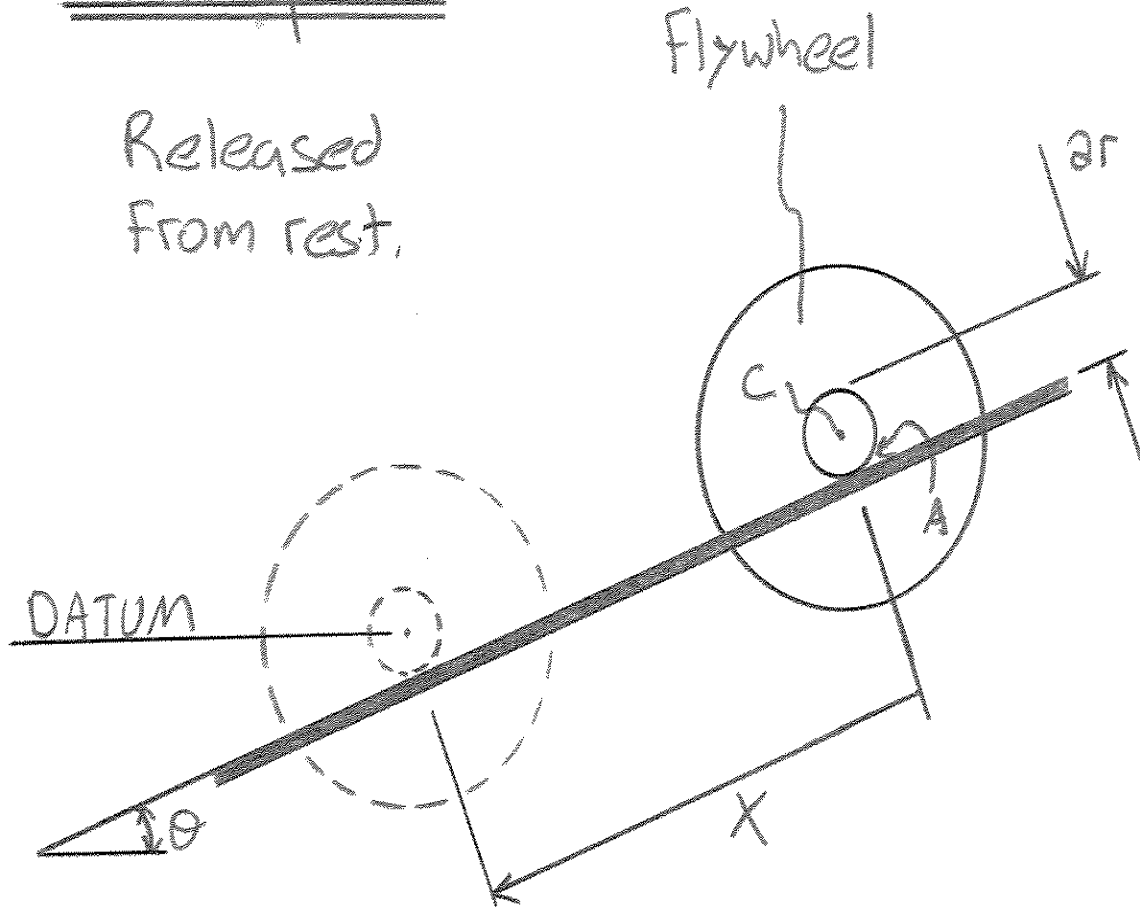
PWE:

$$\text{Initial Energy} + \text{Work Done} = \text{Final Energy}$$

Relates: speed, disp. (forces)

Example:

Released
from rest,



Radius of gyration k .

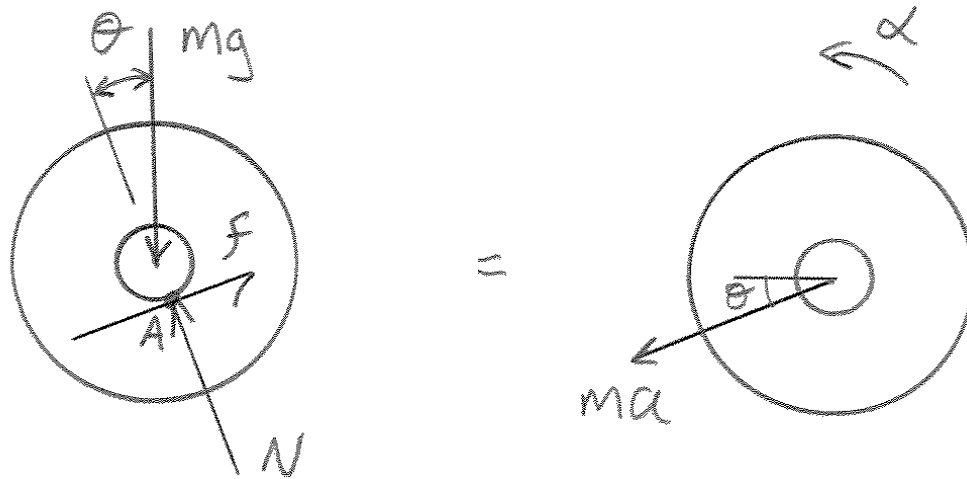
Find time t for the
flywheel to move a
dist. X down the
incline.

Assume no slip,

$$- v_c = r\omega$$

$$- a_c = r\alpha$$

NSL Formulation



$$+\uparrow \sum M_A: mgsin\theta \cdot r = \cancel{I_A \alpha} \quad \begin{matrix} mk^2 + mr^2 \\ a_c/r \end{matrix}$$

$$\Rightarrow mgsin\theta = m(k^2 + r^2) \frac{a_c}{r}$$

$$\Rightarrow a_c = \frac{r^2}{r^2 + k^2} gsin\theta \quad (\text{constant})$$

Kinematic Analysis

Invoke $x - \cancel{x_0} = \cancel{v_0}t + \frac{1}{2}\cancel{a}t^2$

$$\Rightarrow t^2 = \frac{2x}{a_c} = 2x \cdot \frac{r^2 + k^2}{gr^2 \sin \theta}$$

$$\therefore t = \sqrt{\frac{2(r^2 + k^2)x}{gr^2 \sin \theta}}$$

PWE Formulation

Invoke $\cancel{T_1} + V_1 + \cancel{U_1} = T_2 + \cancel{V_2}$

$$V_1 = mgX \sin \Theta$$

$$T_2 = \frac{1}{2} m \omega_c^2 + \frac{1}{2} I_c \omega^2$$

$$= \frac{1}{2} m \omega_c^2 + \frac{1}{2} m K^2 \left(\frac{\omega_c}{r} \right)^2$$

$$= \frac{1}{2} m \left(1 + \frac{K^2}{r^2} \right) \omega_c^2$$

$$= \frac{1}{2} m \left(\frac{r^2 + K^2}{r^2} \right) \omega_c^2$$

Thus

$$\cancel{m}gx \sin \theta = \frac{1}{2} \cancel{m} \left(\frac{r^2 + k^2}{r^2} \right) v_c^2$$

$$\Rightarrow v_c^2 = 2 \cdot \underbrace{\frac{gr^2 \sin \theta}{r^2 + k^2}}_{\text{call this } \gamma} \cdot x$$

call this γ

$$\Rightarrow v_c(x) = \sqrt{2\gamma x}$$

Kinematic Analysis

$$v_c(x) = \frac{dx}{dt} = \sqrt{2gx}$$

$$\Rightarrow dt = \frac{dx}{\sqrt{2gx}} = \frac{x^{-1/2}}{\sqrt{2g}} dx$$

$$\Rightarrow t = \frac{1}{\sqrt{2g}} \cdot \frac{x^{1/2}}{1/2} = \frac{2\sqrt{x}}{\sqrt{2g}} = \sqrt{\frac{4x}{2g}} = \sqrt{\frac{2x}{g}}$$

$$= \sqrt{2x \cdot \frac{r^2 + k^2}{gr^2 \sin \theta}}$$

$$\therefore t = \sqrt{\frac{2(r^2 + k^2)x}{gr^2 \sin \theta}}$$