# Chapter 5 – Kinetics of Rigid Bodies

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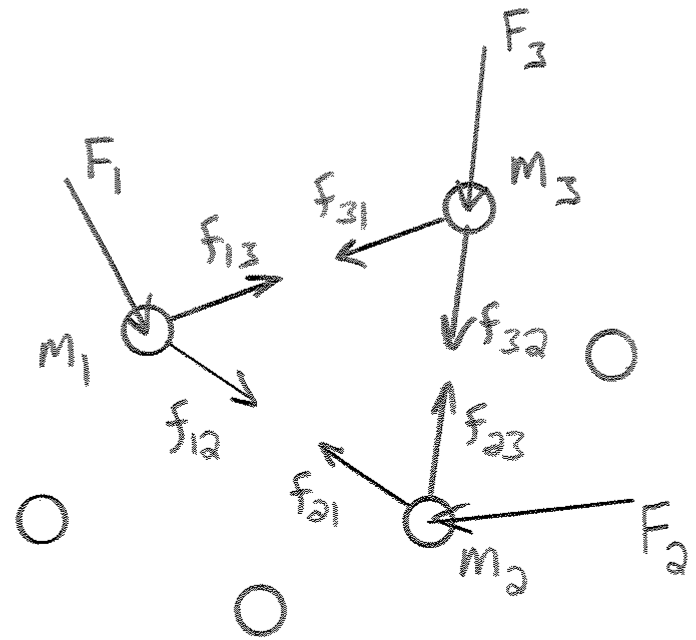
# PWE for a Rigid Body

RB made up of $n \to \infty$ particles.

$$T_1 + U_{1\to 2} = T_2$$




$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots$$

$$= \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

$U_{1\to 2}$ = work of internal and external forces
$\bar{f}_{ij}$ $\bar{F}_i$

FACT: For a RB, work done by internal forces is zero.

$U_{1\to 2}$ = work done by external forces

$$= U^{C}_{1\to 2} + U^{NC}_{1\to 2}$$

$$= V_1 - V_2 + U^{NC}_{1\to 2}$$

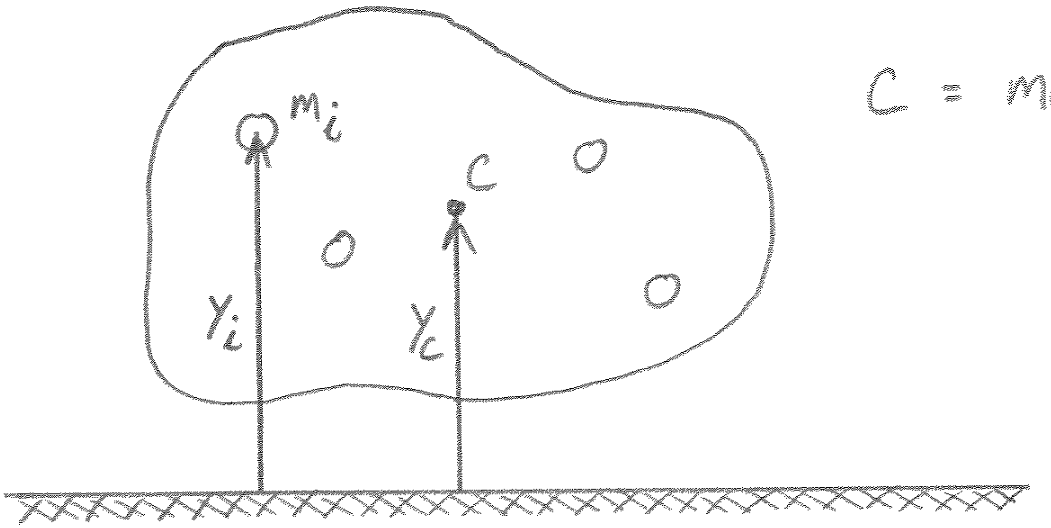
$$\boxed{T_1 + V_1 + U^{NC}_{1\to 2} = T_2 + V_2}$$

## Spring Potential

$$\boxed{V_{spring} = \frac{1}{2} k x^2}$$

$x$ = compression/stretch in spring

# Gravitational Potential




$C$ = mass center

$$V_{weight} = m_1 g y_1 + m_2 g y_2 + \cdots$$

$$= \sum_{i=1}^{\infty} m_i g y_i = g \cancelto{m y_c}{\sum_{i=1}^{\infty} m_i y_i}$$

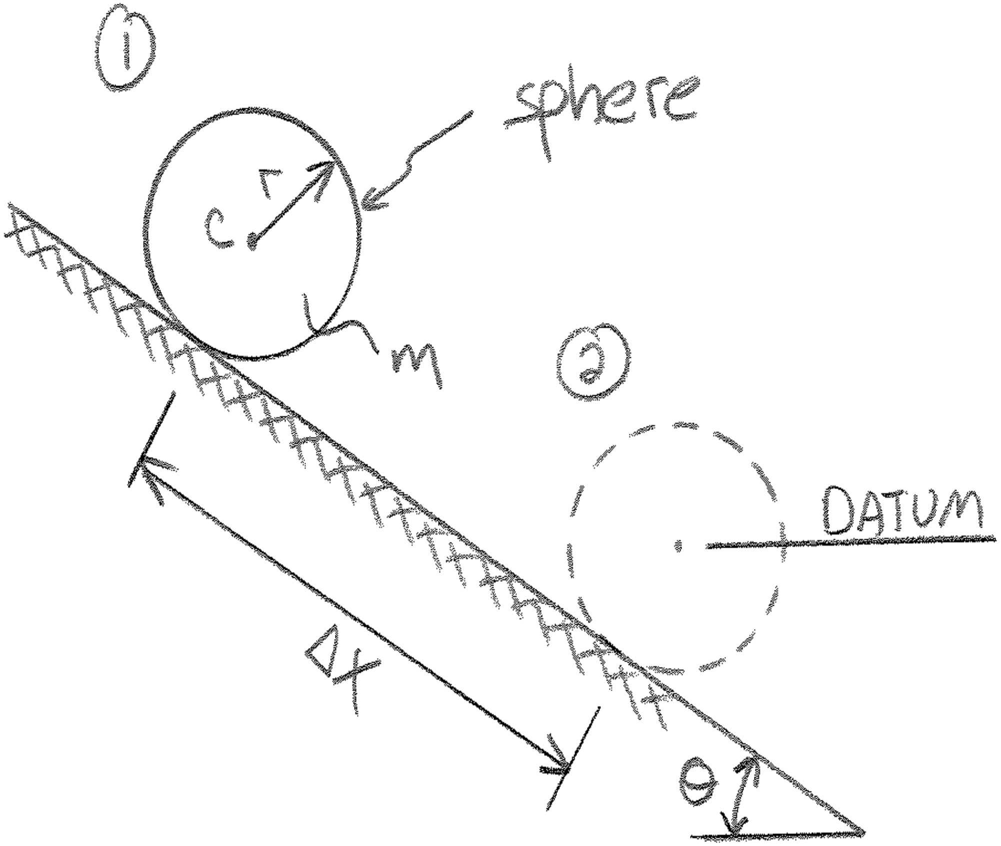
But $\quad y_c = \frac{1}{m} \sum_{i=1}^{\infty} m_i y_i \qquad \therefore \qquad \boxed{V_{weight} = m g y_c}$

# Kinetic Energy

$$\boxed{T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2}$$

$$= \begin{matrix}\text{trans.}\\ \text{KE}\end{matrix} + \begin{matrix}\text{rot.}\\ \text{KE}\end{matrix}$$

# Example:




Released from rest, rolls without slip.

Find $\bar{v}_c$ after c moves $\Delta x = 10$ ft.

Data: $\theta = 30°$

Invoke $T_1 + V_1 + U_{1\to2}^{NC} = T_2 + V_2$

$T_1 = \frac{1}{2} m \cancelto{0}{v_1^2} + \frac{1}{2} I_c \cancelto{0}{\omega_1^2} = 0$

$V_1 = mg \Delta X \sin\theta$

$U_{1\to2}^{NC} = 0$ (no slip $\Rightarrow$ friction force does no work)

$T_2 = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$

$V_2 = 0$

Kinematic Constraint: $v_c = r\omega$

$$0 + mg\Delta x\sin\theta + 0 = \frac{1}{2}mv_c^2 + \frac{1}{2}\underbrace{\cancel{I_c}}_{\frac{2}{5}mr^2}\overbrace{\cancel{\omega^2}}^{(v_c/r)^2}$$

$$= \frac{1}{2}mv_c^2 + \frac{1}{5}m\cancel{r^2}\cdot\frac{v_c^2}{\cancel{r^2}}$$

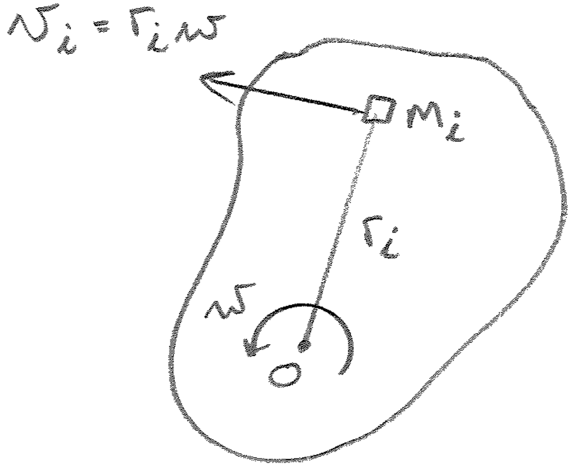
$$= \frac{7}{10}mv_c^2$$

$$v_c = \sqrt{\frac{10}{7}g\Delta x\sin\theta}$$

$$= \sqrt{\frac{10}{7}(32.174\ ft/s^2)(10\ ft)\sin(30^\circ)}$$

$$= 15.160\ ft/s \qquad \therefore \quad \bar{v}_c = 15.2\ ft/s \searrow 30^\circ$$

# Noncentroidal Rotation



(O fixed)

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

$$= \frac{1}{2}\sum_i m_i (r_i\omega)^2$$

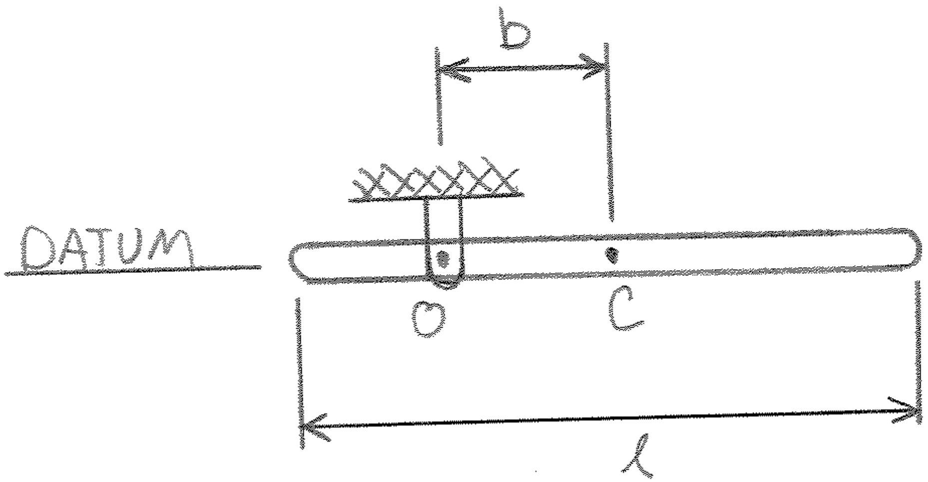
$$= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$= \frac{1}{2} I_o \omega^2$$

$$\therefore \quad \boxed{T = \frac{1}{2} I_o \omega^2} \qquad \text{———————} \qquad (*)$$

$I_o$ = MMI of RB about fixed pt. O

Example:




Released from rest.

Find b that maximizes $\omega$ when rod is in vertical position.

Invoke $\cancel{T_1}^{0} + \cancel{V_1}^{0} + \cancel{U^{NC}_{1\to 2}}^{0} = T_2 + V_2$

$$T_2 = \frac{1}{2} I_O \omega^2$$

$$= \frac{1}{2}\left(\frac{1}{12} m l^2 + m b^2\right)\omega^2$$

$$V_2 = -mgb$$

Thus,

$$0 + 0 + 0 = \frac{1}{24}\left(\cancel{m}\ell^2 + 12\cancel{m}b^2\right)\omega^2 - \cancel{m}gb$$

$$\therefore \quad \omega(b) = \sqrt{\frac{24gb}{\ell^2 + 12b^2}}$$

| $b$ | $\omega / \sqrt{\frac{g}{\ell}}$ |
|---|---|
| $0$ | $0$ |
| $\ell/4$ | $1.85$ |
| $\ell/2$ | $1.73$ |
| $3\ell/4$ | $1.52$ |
| $\ell$ | $1.36$ |

Maximize $\omega$ by maximizing $\omega^2$.

$$\omega^2 = 24gb\left(\ell^2 + 12b^2\right)^{-1}$$

$$\frac{d(\omega^2)}{db} = 24g\left(\ell^2+12b^2\right)^{-1} - 24gb\left(\ell^2+12b^2\right)^{-2}\cdot 24b$$

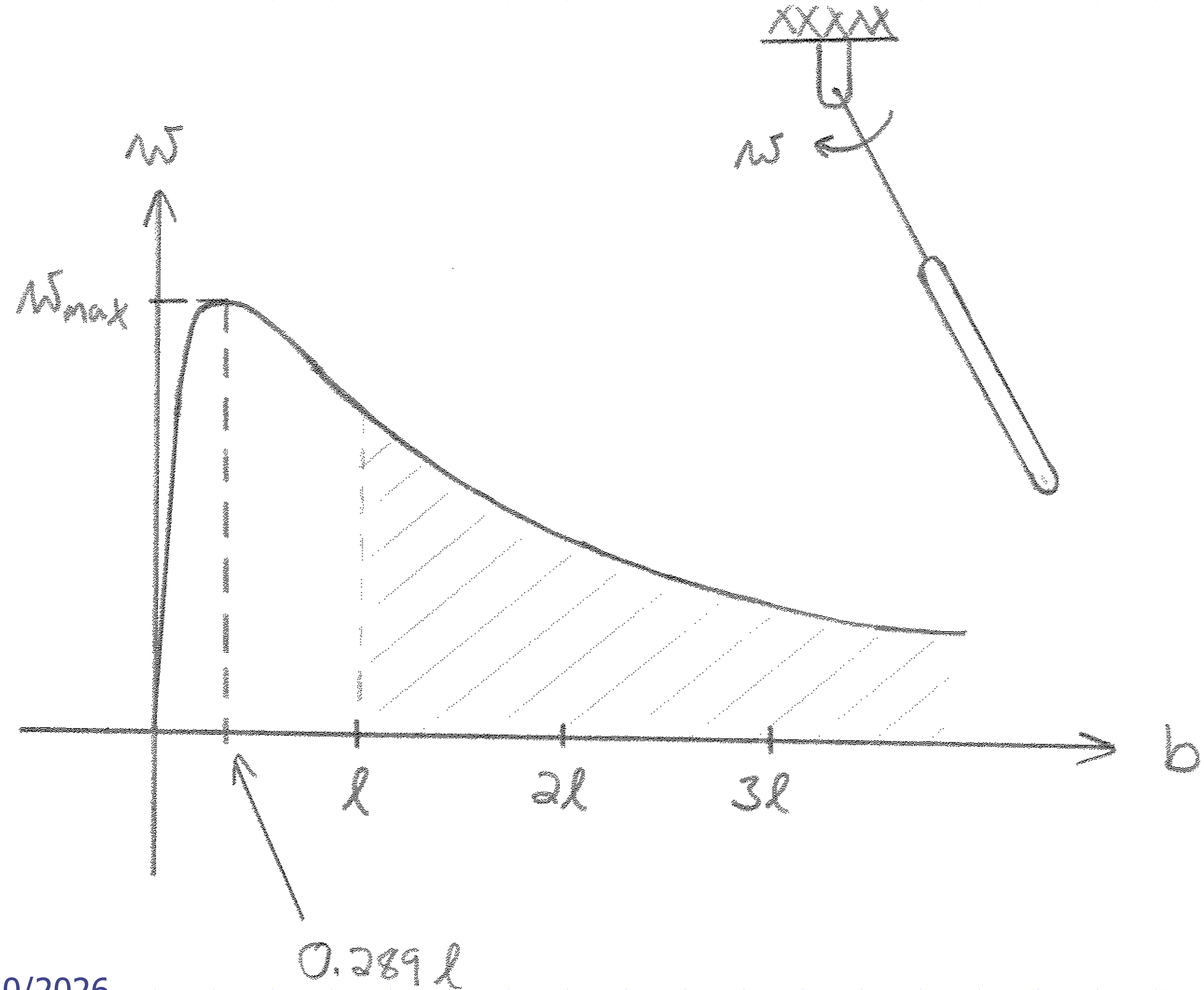
$$= \frac{\cancel{24g}\left(\ell^2+12b^2\right) - \cancel{24g}b\cdot 24b}{\left(\ell^2+12b^2\right)^2} = 0$$

$$\Longrightarrow \quad \ell^2 + 12b^2 - 24b^2 = \ell^2 - 12b^2 = 0$$

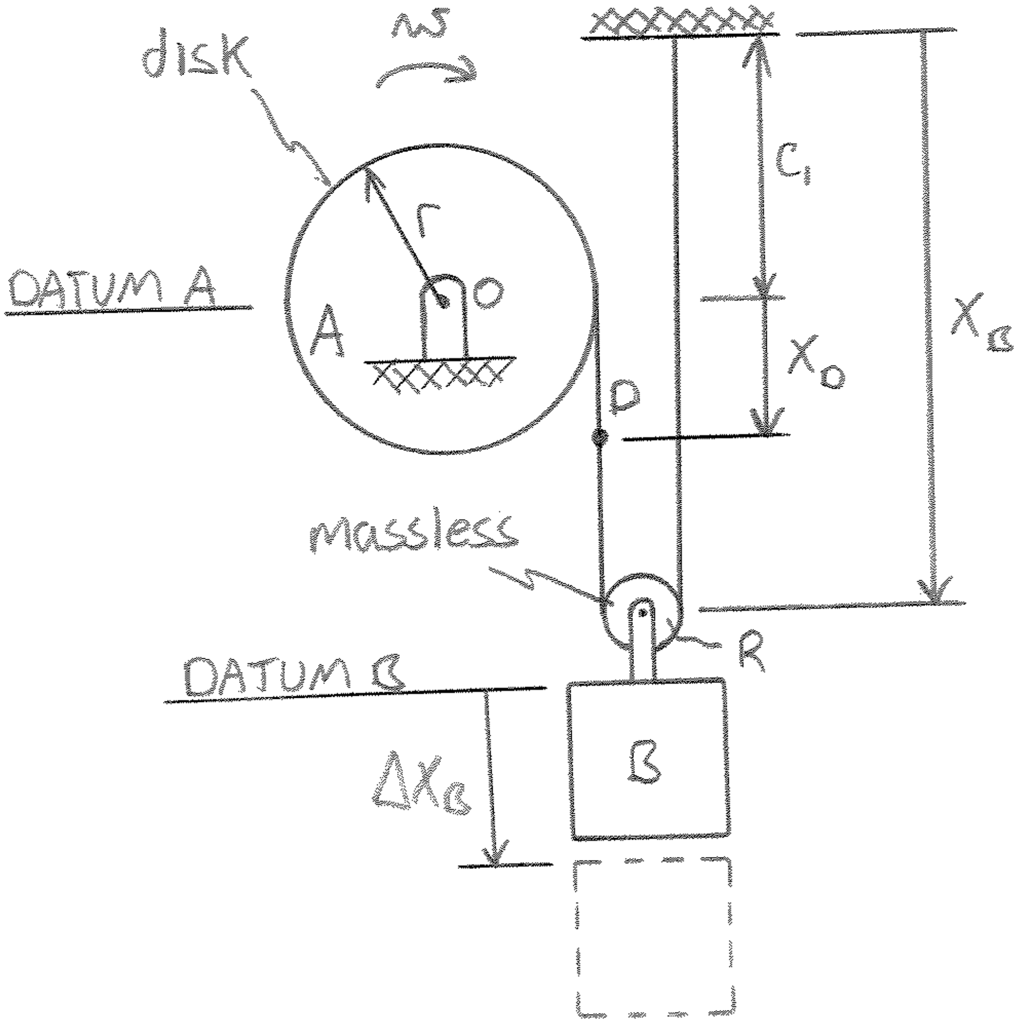
$$\Longrightarrow \quad b = \pm\sqrt{\frac{\ell^2}{12}} = \pm\frac{\ell}{2\sqrt{3}}$$

Keep positive root.

$$\therefore \quad b = \frac{\ell}{2\sqrt{3}} \approx 0.289\,\ell$$



## Example:




Released from rest.

Data: $m_A = 5\,Kg$

$m_B = 2\,Kg$

$r = 200\,mm$

Find $\bar{\omega}$ of A if:

$\Delta X_B = 500\,mm$

## Kinematic Constraint

$$X_B + \pi R + (X_B - C_1 - X_D) = \text{const.}$$

$$\Rightarrow \quad 2X_B - X_D + \underbrace{\pi R - C_1} = \text{const.}$$

$$\Rightarrow \quad 2X_B - X_D = \text{const.}$$

$$\frac{\partial}{\partial t}: \quad 2v_B - \underset{r\omega}{\cancel{v_D}} = 0$$

$$\Rightarrow \quad 2v_B = r\omega \quad \text{———} \quad (*)$$

## Kinetic Analysis

Invoke $\cancel{T_1}^{0} + \cancel{V_1}^{0} + \cancel{U_{1\to 2}^{NC}}^{0} = T_2 + V_2$

$$T_2 = T_2^A + T_2^B$$

$$= \frac{1}{2} I_0^A \omega^2 + \frac{1}{2} m_B v_B^2$$

$$= \frac{1}{2} \cdot \frac{1}{2} m_A r^2 \omega^2 + \frac{1}{2} m_B \left(\frac{r\omega}{2}\right)^2 \qquad \text{(from (*))}$$

$$= \left(\frac{1}{4} m_A + \frac{1}{8} m_B\right) r^2 \omega^2$$

$$V_2 = -m_B g \Delta x_B$$

Thus

$$0 + 0 + 0 = \left(\frac{1}{4}m_A + \frac{1}{8}m_B\right)r^2\omega^2 - m_B g \Delta x_B$$

$$\omega = \sqrt{\frac{8 m_B g \Delta x_B}{(2m_A + m_B)r^2}}$$

$$= \sqrt{\frac{8(2\,\text{kg})(9.81\,\text{m/s}^2)(0.50\,\text{m})}{[2(5\,\text{kg}) + (2\,\text{kg})](0.20\,\text{m})^2}} = 12.786\ \text{rad/s}$$

$$\therefore \quad \bar{\omega} = 12.8\ \text{rad/s} \ \circlearrowright$$

# Overview of Kinetics

## ① Newton's Equations

Particle: $\bar{F} = m\bar{a}$ (NSL)

RB: $\bar{F} = m\bar{a}_c$ (NSL)

$\bar{M}_P = I_P \bar{\alpha}$

# ② Principle of Work-Energy (PWE)

$$T_1 + U_{1\to 2} = T_2$$

$$T_1 + V_1 + U_{1\to 2}^{NC} = T_2 + V_2$$

Particle: $T = \frac{1}{2} m v^2$

$$V_{weight} = mgy$$

$$V_{spring} = \frac{1}{2} k x^2$$

RB:

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$$

$$T = \frac{1}{2} I_o \omega^2 \quad (O \text{ fixed})$$

$$V_{weight} = m g y_c$$

$$V_{spring} = \frac{1}{2} K x^2$$

## Interpretation

**NSL:** Forces = translational resistance x linear acceleration

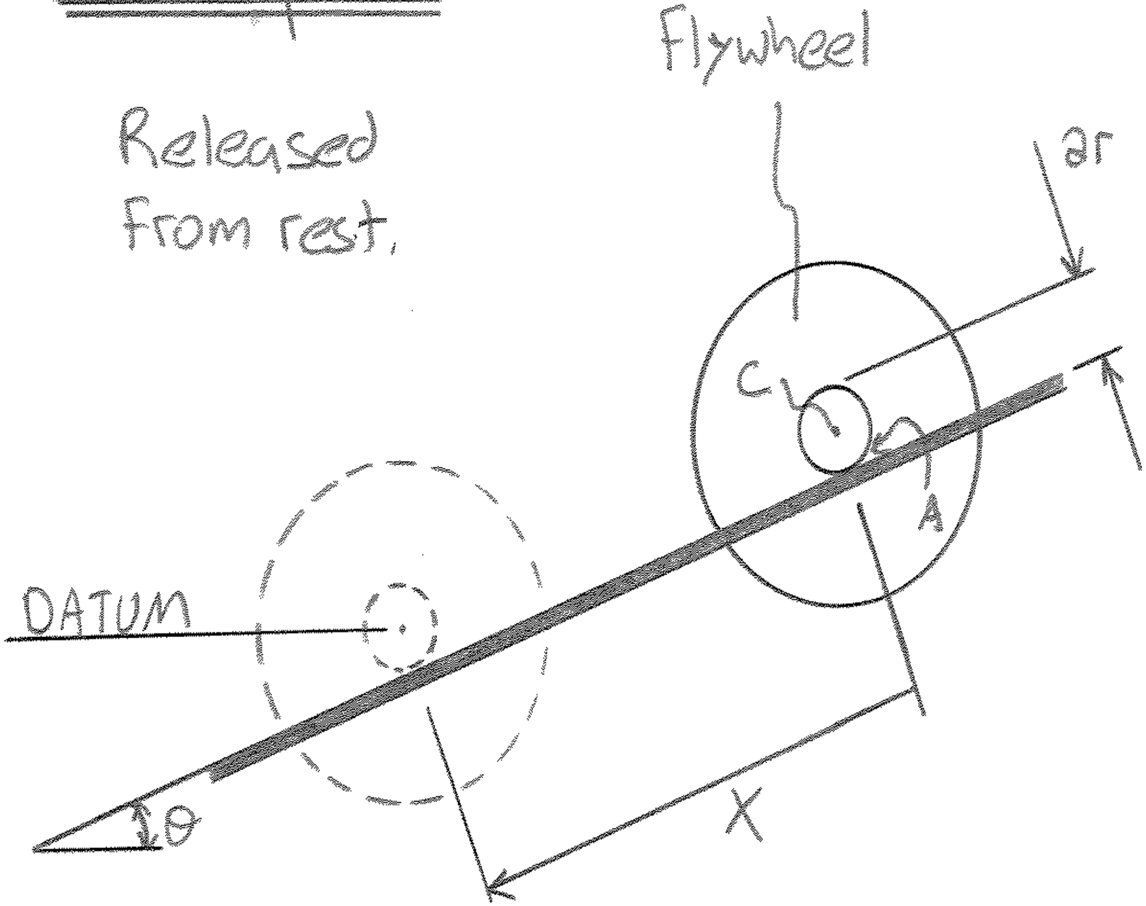
Moments = rotational resistance x angular acceleration

**Relates: forces, acc.**

**PWE:** Initial Energy + Work Done = Final Energy

Relates: speed, disp. (Forces)

## Example:



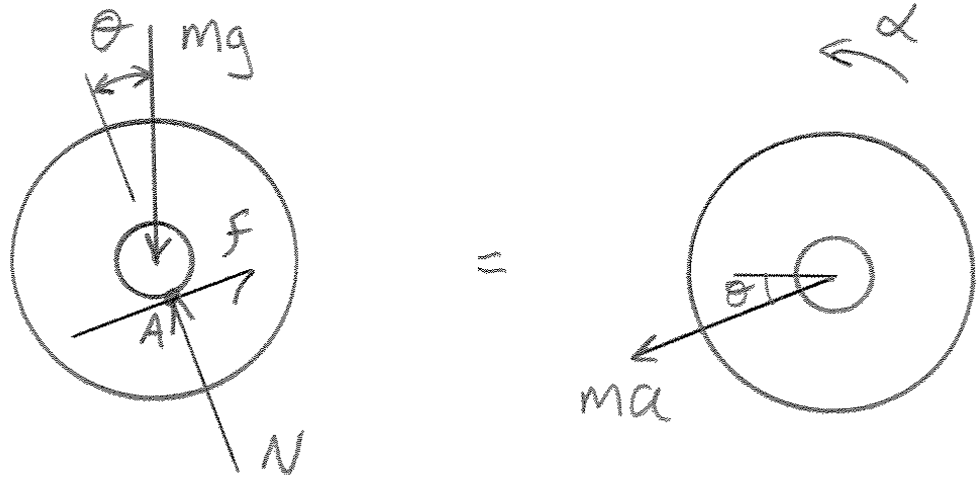

Radius of gyration k.

Find time t for the flywheel to move a dist. X down the incline.

Assume no slip.

- $v_c = r\omega$
- $a_c = r\alpha$

# NSL Formulation




$$+\circlearrowleft \ \Sigma M_A: \quad mg\sin\theta\cdot r = \overset{mk^2+mr^2}{\cancel{I_A}}\ \underset{a_c/r}{\cancel{\alpha}}$$

$$\Rightarrow \quad \cancel{m}gr\sin\theta = \cancel{m}(K^2+r^2)\frac{a_c}{r}$$

$$\Rightarrow \quad a_c = \frac{r^2}{r^2+K^2}\, g\sin\theta \qquad \text{(constant)}$$

# Kinematic Analysis

Invoke $\quad x - \cancel{x_0}^{\,0} = \cancel{v_0}^{\,0} t + \frac{1}{2}\cancel{a}^{\,a_c} t^2$

$$\Rightarrow \quad t^2 = \frac{2x}{a_c} = 2x \cdot \frac{r^2 + k^2}{g r^2 \sin\theta}$$

$$\therefore \quad t = \sqrt{\frac{2(r^2+k^2)x}{g r^2 \sin\theta}}$$

# PWE Formulation

Invoke $\cancel{T_1}^{0} + V_1 + \cancel{U_{1\to 2}^{NC}}^{0} = T_2 + \cancel{V_2}^{0}$

$$V_1 = mgx\sin\theta$$

$$T_2 = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2$$

$$= \frac{1}{2}mv_c^2 + \frac{1}{2}mk^2\left(\frac{v_c}{r}\right)^2$$

$$= \frac{1}{2}m\left(1 + \frac{k^2}{r^2}\right)v_c^2$$

$$= \frac{1}{2}m\left(\frac{r^2 + k^2}{r^2}\right)v_c^2$$

Thus

$$\cancel{m}gx\sin\theta = \frac{1}{2}\cancel{m}\left(\frac{r^2+K^2}{r^2}\right)v_c^2$$

$$\Rightarrow \quad v_c^2 = 2\cdot\underbrace{\frac{gr^2\sin\theta}{r^2+K^2}}_{\text{Call this } \gamma}\cdot x$$

$$\Rightarrow \quad v_c(x) = \sqrt{2\gamma x}$$

# Kinematic Analysis

$$v_c(x) = \frac{dx}{dt} = \sqrt{2\gamma x}$$

$$\Rightarrow \quad dt = \frac{dx}{\sqrt{2\gamma x}} = \frac{x^{-1/2}}{\sqrt{2\gamma}}\,dx$$

$$\Rightarrow \quad t = \frac{1}{\sqrt{2\gamma}} \cdot \frac{x^{1/2}}{1/2} = \frac{2\sqrt{x}}{\sqrt{2\gamma}} = \sqrt{\frac{4x}{2\gamma}} = \sqrt{\frac{2x}{\gamma}}$$

$$= \sqrt{2x \cdot \frac{r^2+k^2}{g r^2 \sin\theta}}$$

$$\therefore \quad t = \sqrt{\frac{2(r^2+k^2)x}{g r^2 \sin\theta}}$$