

Chapter 3 – Kinetics of Particles

1. Force-Mass-Acceleration Method
2. Work and Energy Principles
3. Impulse and Momentum Methods

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1. Force-Mass-Acceleration Method



Kinetics of Particles

→ Relates the forces acting on a body to the corresponding motion.

Newton's Second Law of Motion

$$\vec{F} = m\vec{a}$$

$$\left(N = \frac{\text{kg} \cdot \text{m}}{\text{s}^2}, \quad \text{lb}_f = \frac{\text{slug} \cdot \text{ft}}{\text{s}^2} \right)$$

$$\bar{\mathbf{F}} = \frac{d}{dt} (m\bar{\mathbf{v}}) \quad (\text{a vector eqn!})$$

Linear momentum,
denoted $\bar{\mathbf{L}}$

Resultant of external
forces acting on body

$$\text{If } m = \text{const}, \quad \frac{d}{dt} (m\bar{\mathbf{v}}) = m \frac{d\bar{\mathbf{v}}}{dt} = m\bar{\mathbf{a}}$$

$\therefore$

$$\bar{\mathbf{F}} = m\bar{\mathbf{a}}$$

Equations of Motion

Rectangular Components

$$(F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) = m(a_x \hat{i} + a_y \hat{j} + a_z \hat{k})$$

Main Point:

The vector eqn $\vec{F} = m\vec{a}$

represents three independent
Scalar EOM.

$$F_x = ma_x$$

$$F_y = ma_y$$

$$F_z = ma_z$$

Other Components (plane motion)



Normal/Tangential:

$$(F_t \hat{e}_t + F_n \hat{e}_n) = m \left(\frac{dv}{dt} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n \right)$$

Radial/Transverse:

$$(F_r \hat{e}_r + F_\theta \hat{e}_\theta) = m \left((\dots) \hat{e}_r + (\dots) \hat{e}_\theta \right)$$

Sample Problem 3/1

A 75-kg man stands on a spring scale in an elevator. During the first 3 seconds of motion from rest, the tension T in the hoisting cable is 8300 N. Find the reading R of the scale in newtons during this interval and the upward velocity v of the elevator at the end of the 3 seconds. The total mass of the elevator, man, and scale is 750 kg.

Solution. The force registered by the scale and the velocity both depend on the acceleration of the elevator, which is constant during the interval for which the forces are constant. From the free-body diagram of the elevator, scale, and man taken together, the acceleration is found to be

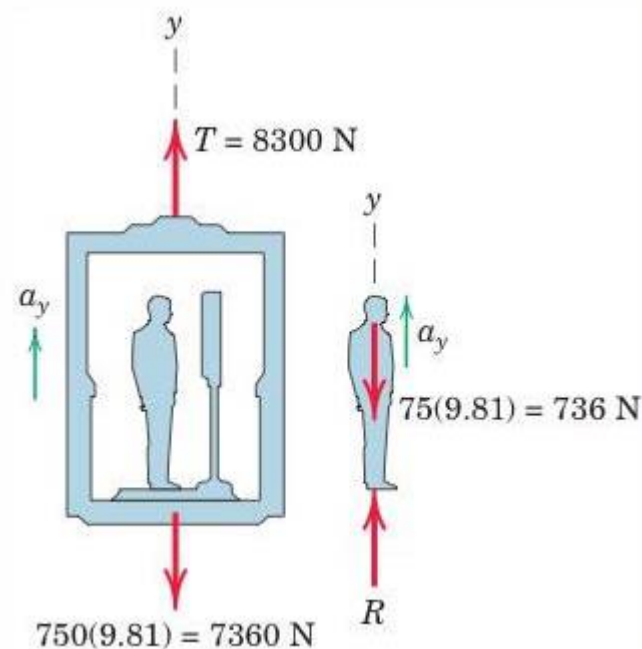
$$[\Sigma F_y = ma_y] \quad 8300 - 7360 = 750a_y \quad a_y = 1.257 \text{ m/s}^2$$

The scale reads the downward force exerted on it by the man's feet. The equal and opposite reaction R to this action is shown on the free-body diagram of the man alone together with his weight, and the equation of motion for him gives

$$[\Sigma F_y = ma_y] \quad R - 736 = 75(1.257) \quad R = 830 \text{ N} \quad \text{Ans.}$$

The velocity reached at the end of the 3 seconds is

$$[\Delta v = \int a \, dt] \quad v - 0 = \int_0^3 1.257 \, dt \quad v = 3.77 \text{ m/s} \quad \text{Ans.}$$



Helpful Hint

- ① If the scale were calibrated in kilograms it would read $830/9.81 = 84.6 \text{ kg}$ which, of course, is not his true mass since the measurement was made in a noninertial (accelerating) system. *Suggestion:* Rework this problem in U.S. customary units.

Sample Problem 3/2

A small inspection car with a mass of 200 kg runs along the fixed overhead cable and is controlled by the attached cable at A. Determine the acceleration of the car when the control cable is horizontal and under a tension $T = 2.4$ kN. Also find the total force P exerted by the supporting cable on the wheels.

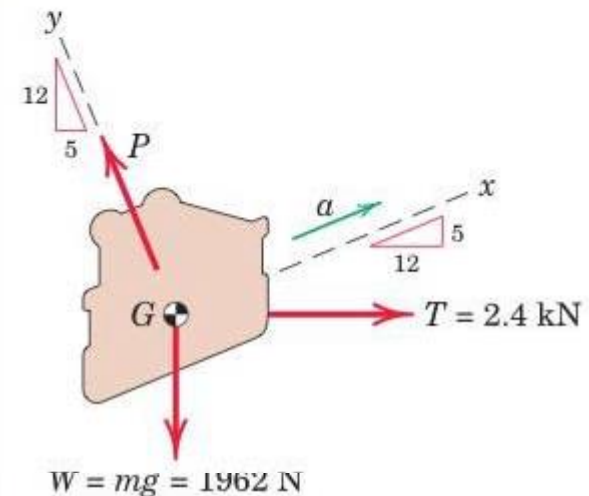
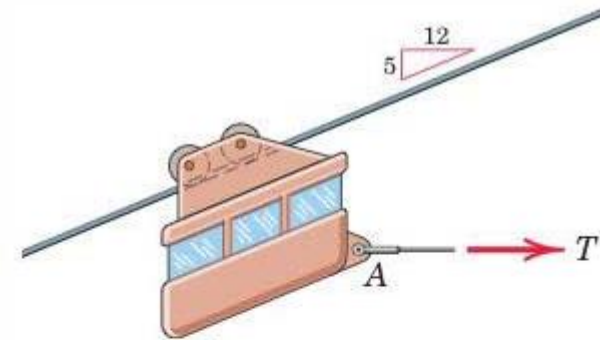
Solution. The free-body diagram of the car and wheels taken together and treated as a particle discloses the 2.4-kN tension T , the weight $W = mg = 200(9.81) = 1962$ N, and the force P exerted on the wheel assembly by the cable.

The car is in equilibrium in the y -direction since there is no acceleration in this direction. Thus,

$$[\Sigma F_y = 0] \quad P - 2.4\left(\frac{5}{13}\right) - 1962\left(\frac{12}{13}\right) = 0 \quad P = 2.73 \text{ kN} \quad \text{Ans.}$$

① In the x -direction the equation of motion gives

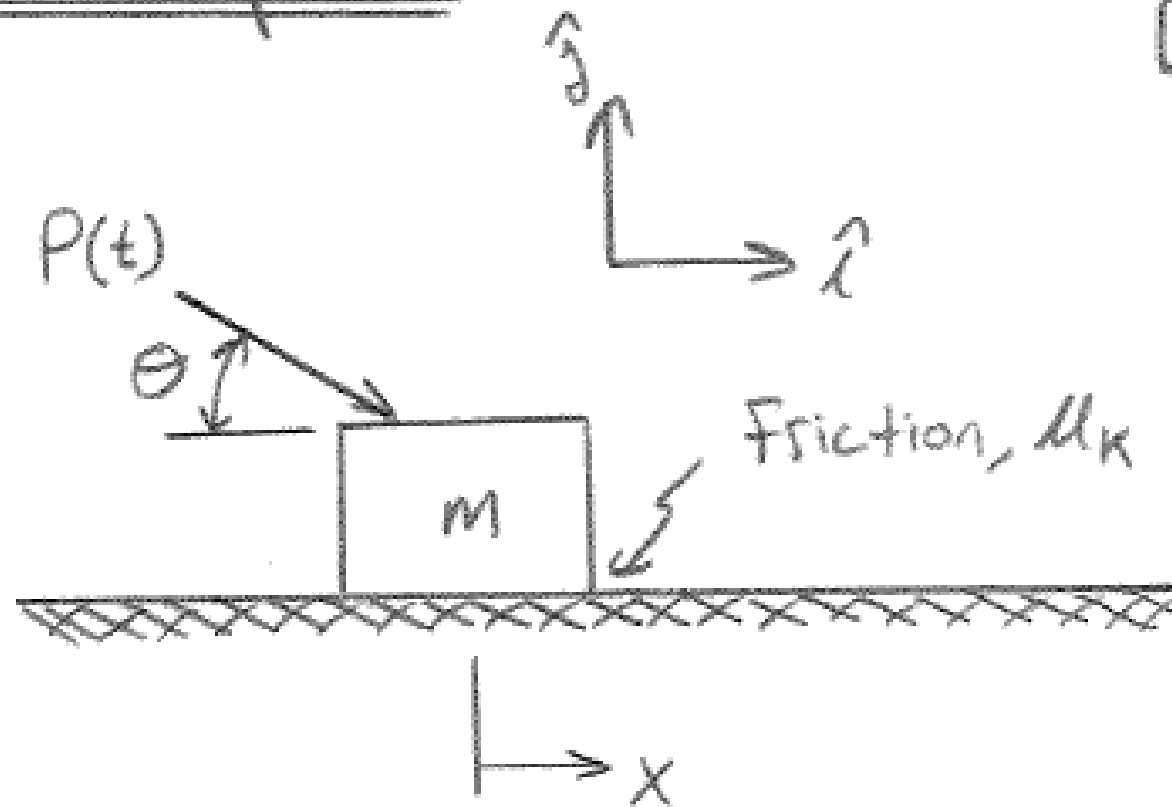
$$[\Sigma F_x = ma_x] \quad 2400\left(\frac{12}{13}\right) - 1962\left(\frac{5}{13}\right) = 200a \quad a = 7.30 \text{ m/s}^2 \quad \text{Ans.}$$



Helpful Hint

- ① By choosing our coordinate axes along and normal to the direction of the acceleration, we are able to solve the two equations independently. Would this be so if x and y were chosen as horizontal and vertical?

Example:



Data:

$$P(t) = P_0(1 + \gamma t)$$

$$P_0 = 500 \text{ N}$$

$$m = 80 \text{ kg}$$

$$\theta = 30^\circ$$

$$\mu_k = 0.25$$

$$\gamma = 0.1 \text{ s}^{-1}$$

m is initially at rest when $P(t)$ is applied @ $t=0$.

Find:

- (A) the equations of motion
- (B) $a = a(t)$
- (C) dist. traveled in $t = 2s$

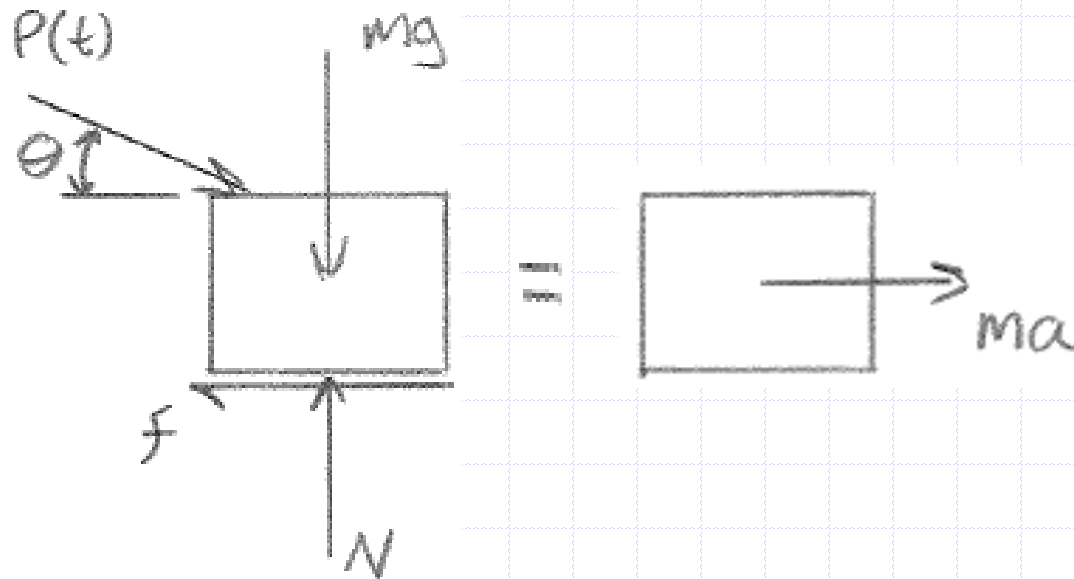
Assume motion
at $t = 0$

i.e. $P_0 \cos \theta$ is
enough to
overcome static
Friction

PART A: Find EOM.

⊕ Force Diagram, "diagrammatical" approach

FBD:



$$\Sigma (\text{external forces}) = \text{mass} \times \text{acc.}$$

$$\rightarrow \sum F: P(t) \cos \theta - f = ma \quad - (*)$$

$$+\uparrow \sum F: -mg - P(t) \sin \theta + N = 0 \quad - (**)$$

PART B: Find $a = a(t)$

Unknowns in (*), (**): f, a, N

2 eqns, 3 unknowns $\Rightarrow$ need one more independent eqn.

$$\text{Friction Law: } f = \mu_k N \quad - (***)$$

(**) $\rightarrow$ (***) to eliminate N .

Plug into (*).

$$P(t) \cos \theta - \mu_k (mg + P(t) \sin \theta) = ma$$

$$\Rightarrow ma = \cancel{P(t)} [\cos \theta - \mu_k \sin \theta] - \mu_k mg$$

$P_0(1 + \gamma t)$

$$\Rightarrow a(t) = \underbrace{\frac{P_0 (\cos \theta - \mu_k \sin \theta) - \mu_k mg}{m}}_{\text{call } a_0} + \underbrace{\frac{P_0 \gamma (\cos \theta - \mu_k \sin \theta)}{m} t}_{\text{call } a_1}$$

$$\therefore a(t) = a_0 + a_1 t$$

Where

$$a_c = \frac{(500\text{N})[\cos(30^\circ) - (0.25)\sin(30^\circ)] - (0.25)(80\text{kg})(9.81\text{m/s}^2)}{(80\text{kg})}$$
$$= 2.1789 \text{ m/s}^2$$

$$a_t = \frac{(500\text{N})(0.15^{-1})[\cos(30^\circ) - (0.25)\sin(30^\circ)]}{(80\text{kg})}$$
$$= 0.4631 \text{ m/s}^2$$

PART C: Find $X(t=2s)$

Invoke $a(t) = \frac{dv}{dt}$

$$\Rightarrow dv = (a_0 + a_1 t) dt$$

$$\Rightarrow v(t) - \cancel{v_0^0} = a_0 t + a_1 \frac{t^2}{2}$$

$$\text{But } v(t) = \frac{dX}{dt} \Rightarrow dX = (a_0 t + \frac{1}{2} a_1 t^2) dt$$

$$\Rightarrow X(t) - \cancel{X_0^0} = a_0 \frac{t^2}{2} + \frac{1}{2} a_1 \frac{t^3}{3}$$

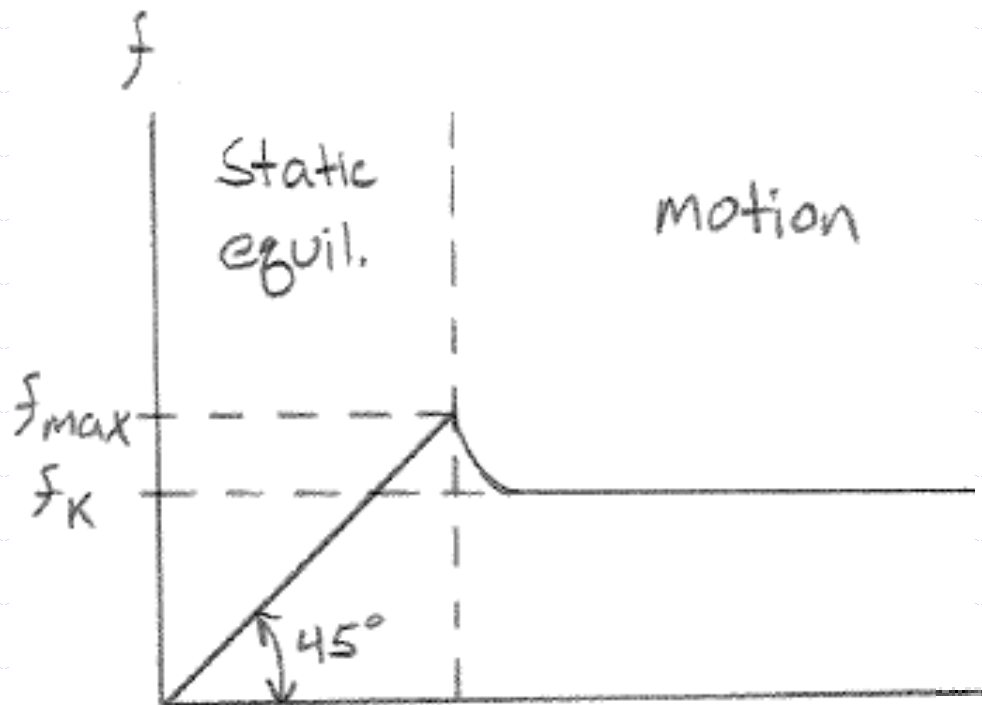
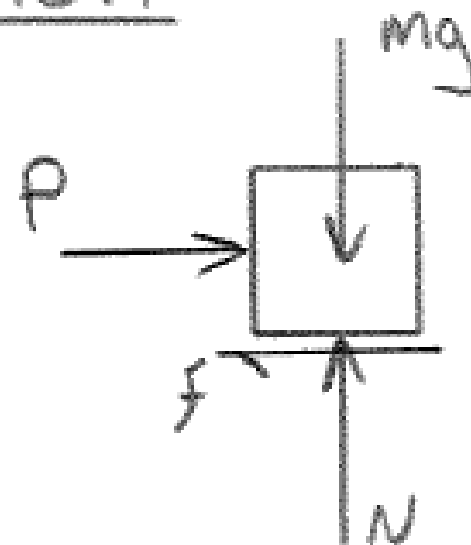
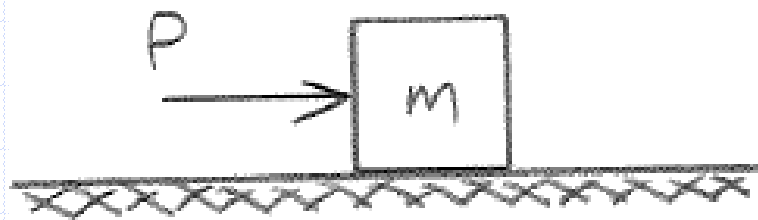
$$\Rightarrow X(t) = \frac{a_0}{2} t^2 + \frac{a_1}{6} t^3$$

Thus

$$X(t=2s) = \frac{(2.1789 \text{ m/s}^2)}{2} (2s)^2 + \frac{(0.4631 \text{ m/s}^3)}{6} (2s)^3$$

$$\therefore X(t=2s) = 4.98 \text{ m}$$

Review of Dry Friction



$$f_{max} = \mu_s N$$

$$f_K = \mu_K N \text{ (motion)}$$

P

Experimental evidence shows

$$+ f_{\max} = \mu_s N \quad (\text{impending motion})$$

$$f_k = \mu_k N \quad (\text{motion})$$

Always, $\mu_k < \mu_s$

Sample Problem 3/3

The 250-lb concrete block *A* is released from rest in the position shown and pulls the 400-lb log up the 30° ramp. If the coefficient of kinetic friction between the log and the ramp is 0.5, determine the velocity of the block as it hits the ground at *B*.

Solution. The motions of the log and the block *A* are clearly dependent. Although by now it should be evident that the acceleration of the log up the incline is half the downward acceleration of *A*, we may prove it formally. The constant total length of the cable is $L = 2s_C + s_A + \text{constant}$, where the constant accounts for the cable portions wrapped around the pulleys. Differentiating twice with respect to time gives $0 = 2\ddot{s}_C + \ddot{s}_A$, or

$$0 = 2a_C + a_A$$

We assume here that the masses of the pulleys are negligible and that they turn with negligible friction. With these assumptions the free-body diagram of the pulley *C* discloses force and moment equilibrium. Thus, the tension in the cable attached to the log is twice that applied to the block. Note that the accelerations of the log and the center of pulley *C* are identical.

The free-body diagram of the log shows the friction force $\mu_k N$ for motion up the plane. Equilibrium of the log in the *y*-direction gives

$$[\Sigma F_y = 0] \quad N - 400 \cos 30^\circ = 0 \quad N = 346 \text{ lb}$$

and its equation of motion in the *x*-direction gives

$$[\Sigma F_x = ma_x] \quad 0.5(346) - 2T + 400 \sin 30^\circ = \frac{400}{32.2} a_C$$

For the block in the positive downward direction, we have

$$[\downarrow \Sigma F = ma] \quad 250 - T = \frac{250}{32.2} a_A$$

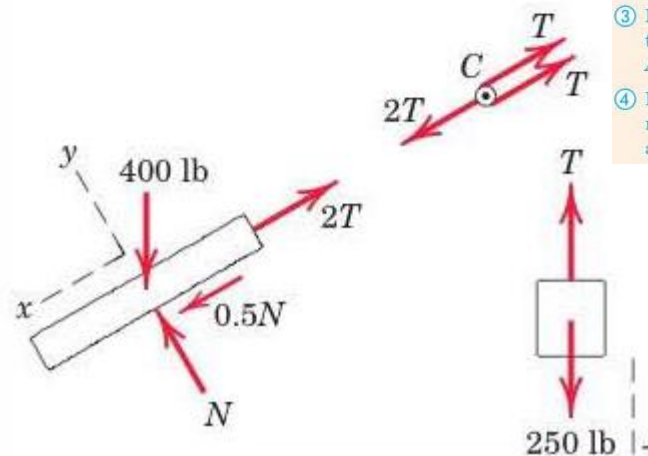
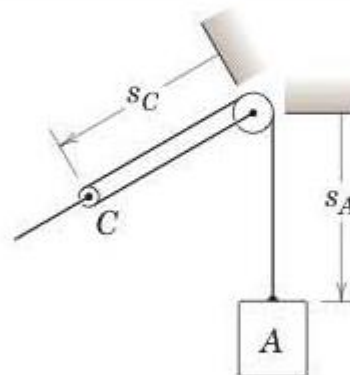
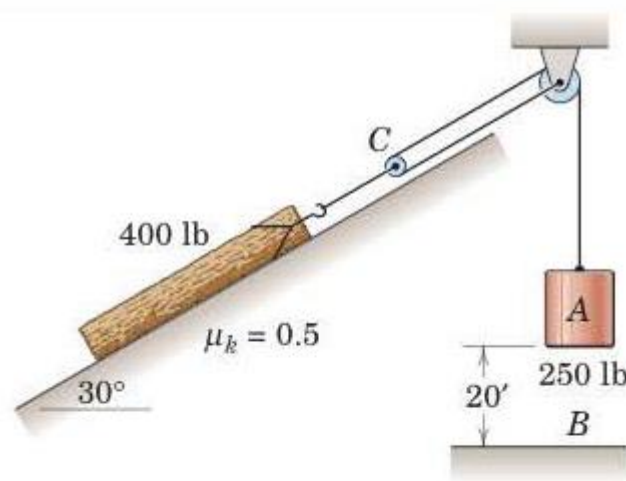
Solving the three equations in a_C , a_A , and T gives us

$$a_A = 5.83 \text{ ft/sec}^2 \quad a_C = -2.92 \text{ ft/sec}^2 \quad T = 205 \text{ lb}$$

For the 20-ft drop with constant acceleration, the block acquires a velocity

$$[v^2 = 2ax] \quad v_A = \sqrt{2(5.83)(20)} = 15.27 \text{ ft/sec}$$

Ans.



Helpful Hints

- ① The coordinates used in expressing the final kinematic constraint relationship must be consistent with those used for the kinetic equations of motion.
- ② We can verify that the log will indeed move up the ramp by calculating the force in the cable necessary to initiate motion from the equilibrium condition. This force is $2T = 0.5N + 400 \sin 30^\circ = 373 \text{ lb}$ or $T = 186.5 \text{ lb}$, which is less than the 250-lb weight of block *A*. Hence, the log will move up.
- ③ Note the serious error in assuming that $T = 250 \text{ lb}$, in which case, block *A* would not accelerate.
- ④ Because the forces on this system remain constant, the resulting accelerations also remain constant.

Sample Problem 3/4

The design model for a new ship has a mass of 10 kg and is tested in an experimental towing tank to determine its resistance to motion through the water at various speeds. The test results are plotted on the accompanying graph, and the resistance R may be closely approximated by the dashed parabolic curve shown. If the model is released when it has a speed of 2 m/s, determine the time t required for it to reduce its speed to 1 m/s and the corresponding travel distance x .

Solution. We approximate the resistance-velocity relation by $R = kv^2$ and find k by substituting $R = 8$ N and $v = 2$ m/s into the equation, which gives $k = 8/2^2 = 2$ N·s²/m². Thus, $R = 2v^2$.

The only horizontal force on the model is R , so that

$$\textcircled{1} \quad [\Sigma F_x = ma_x] \quad -R = ma_x \quad \text{or} \quad -2v^2 = 10 \frac{dv}{dt}$$

We separate the variables and integrate to obtain

$$\int_0^t dt = -5 \int_2^v \frac{dv}{v^2} \quad t = 5 \left(\frac{1}{v} - \frac{1}{2} \right) \text{ s}$$

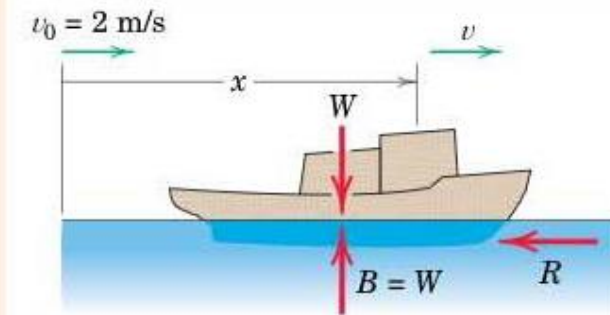
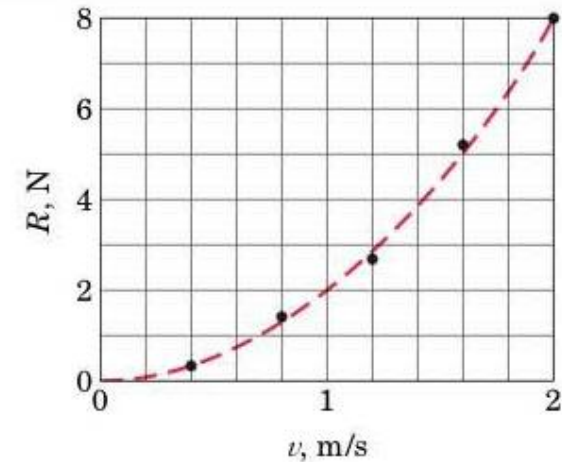
Thus, when $v = v_0/2 = 1$ m/s, the time is $t = 5 \left(\frac{1}{1} - \frac{1}{2} \right) = 2.5$ s.

Ans.

The distance traveled during the 2.5 seconds is obtained by integrating $v = dx/dt$. Thus, $v = 10/(5 + 2t)$ so that

$$\textcircled{2} \quad \int_0^x dx = \int_0^{2.5} \frac{10}{5 + 2t} dt \quad x = \frac{10}{2} \ln(5 + 2t) \Big|_0^{2.5} = 3.47 \text{ m}$$

Ans.

**Helpful Hints**

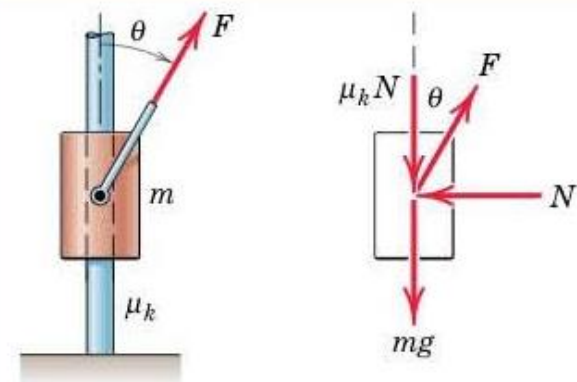
- ① Be careful to observe the minus sign for R .
- ② *Suggestion:* Express the distance x after release in terms of the velocity v and see if you agree with the resulting relation $x = 5 \ln(v_0/v)$.

(2D – Advanced)

Sample Problem 3/5

The collar of mass m slides up the vertical shaft under the action of a force F of constant magnitude but variable direction. If $\theta = kt$ where k is a constant and if the collar starts from rest with $\theta = 0$, determine the magnitude F of the force which will result in the collar coming to rest as θ reaches $\pi/2$. The coefficient of kinetic friction between the collar and shaft is μ_k .

Solution. After drawing the free-body diagram, we apply the equation of motion in the y -direction to get



Helpful Hints

- ① If θ were expressed as a function of the vertical displacement y instead of the time t , the acceleration would become a function of the displacement and we would use $v dv = a dy$.
- ② We see that the results do not depend on k , the rate at which the force changes direction.

$$\textcircled{1} \quad [\Sigma F_y = ma_y] \quad F \cos \theta - \mu_k N - mg = m \frac{dv}{dt}$$

where equilibrium in the horizontal direction requires $N = F \sin \theta$. Substituting $\theta = kt$ and integrating first between general limits give

$$\int_0^t (F \cos kt - \mu_k F \sin kt - mg) dt = m \int_0^v dv$$

which becomes

$$\frac{F}{k} [\sin kt + \mu_k (\cos kt - 1)] - mgt = mv$$

For $\theta = \pi/2$ the time becomes $t = \pi/2k$, and $v = 0$ so that

$$\textcircled{2} \quad \frac{F}{k} [1 + \mu_k (0 - 1)] - \frac{mg\pi}{2k} = 0 \quad \text{and} \quad F = \frac{mg\pi}{2(1 - \mu_k)} \quad \text{Ans.}$$

Kinetics – Curvilinear Motion

Rectangular coordinates

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = ma_y$$

where $a_x = \ddot{x}$ and $a_y = \ddot{y}$

Normal and tangential coordinates

$$\Sigma F_n = ma_n$$

$$\Sigma F_t = ma_t$$

where $a_n = \rho \dot{\beta}^2 = v^2/\rho = v \dot{\beta}$, $a_t = \dot{v}$, and $v = \rho \dot{\beta}$

Polar coordinates

$$\Sigma F_r = ma_r$$

$$\Sigma F_\theta = ma_\theta$$

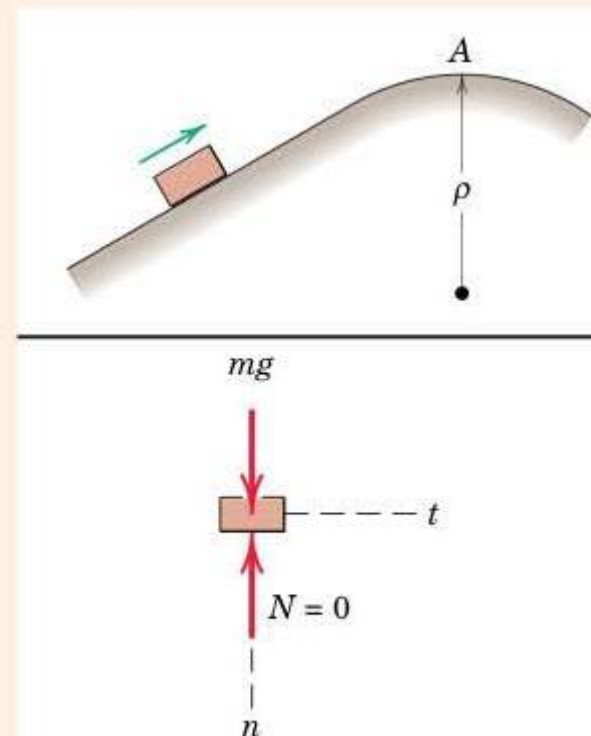
Sample Problem 3/6

Determine the maximum speed v which the sliding block may have as it passes point A without losing contact with the surface.

Solution. The condition for loss of contact is that the normal force N which the surface exerts on the block goes to zero. Summing forces in the normal direction gives

$$[\Sigma F_n = ma_n] \quad mg = m \frac{v^2}{\rho} \quad v = \sqrt{g\rho} \quad \text{Ans.}$$

If the speed at A were less than $\sqrt{g\rho}$, then an upward normal force exerted by the surface on the block would exist. In order for the block to have a speed at A which is greater than $\sqrt{g\rho}$, some type of constraint, such as a second curved surface above the block, would have to be introduced to provide additional downward force.



Sample Problem 3/7

Small objects are released from rest at A and slide down the smooth circular surface of radius R to a conveyor B . Determine the expression for the normal contact force N between the guide and each object in terms of θ and specify the correct angular velocity ω of the conveyor pulley of radius r to prevent any sliding on the belt as the objects transfer to the conveyor.

Solution. The free-body diagram of the object is shown together with the coordinate directions n and t . The normal force N depends on the n -component of the acceleration which, in turn, depends on the velocity. The velocity will be cumulative according to the tangential acceleration a_t . Hence, we will find a_t first for any general position.

$$[\Sigma F_t = ma_t] \quad mg \cos \theta = ma_t \quad a_t = g \cos \theta$$

① Now we can find the velocity by integrating

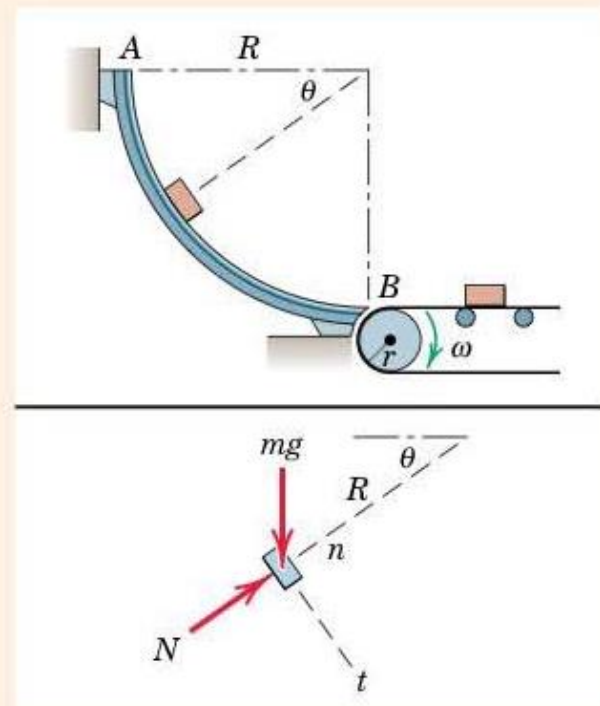
$$[v dv = a_t ds] \quad \int_0^v v dv = \int_0^\theta g \cos \theta d(R\theta) \quad v^2 = 2gR \sin \theta$$

We obtain the normal force by summing forces in the positive n -direction, which is the direction of the n -component of acceleration.

$$[\Sigma F_n = ma_n] \quad N - mg \sin \theta = m \frac{v^2}{R} \quad N = 3mg \sin \theta \quad \text{Ans.}$$

The conveyor pulley must turn at the rate $v = r\omega$ for $\theta = \pi/2$, so that

$$\omega = \sqrt{2gR}/r \quad \text{Ans.}$$



Helpful Hint

- ① It is essential here that we recognize the need to express the tangential acceleration as a function of position so that v may be found by integrating the kinematical relation $v dv = a_t ds$, in which all quantities are measured along the path.

Sample Problem 3/8

A 1500-kg car enters a section of curved road in the horizontal plane and slows down at a uniform rate from a speed of 100 km/h at A to a speed of 50 km/h as it passes C. The radius of curvature ρ of the road at A is 400 m and at C is 80 m. Determine the total horizontal force exerted by the road on the tires at positions A, B, and C. Point B is the inflection point where the curvature changes direction.

Solution. The car will be treated as a particle so that the effect of all forces exerted by the road on the tires will be treated as a single force. Since the motion is described along the direction of the road, normal and tangential coordinates will be used to specify the acceleration of the car. We will then determine the forces from the accelerations.

The constant tangential acceleration is in the negative t -direction, and its magnitude is given by

$$\textcircled{1} \quad [v_C^2 = v_A^2 + 2a_t \Delta s] \quad a_t = \left| \frac{(50/3.6)^2 - (100/3.6)^2}{2(200)} \right| = 1.447 \text{ m/s}^2$$

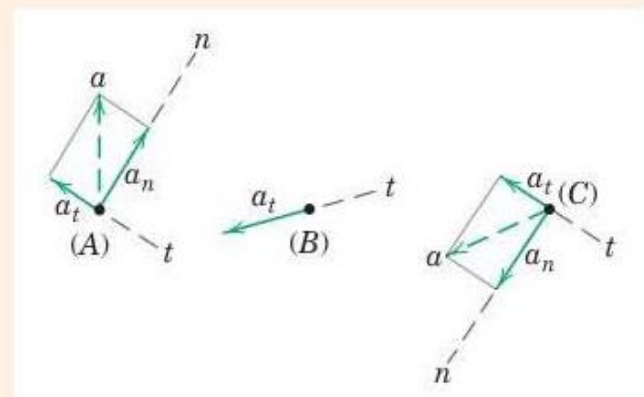
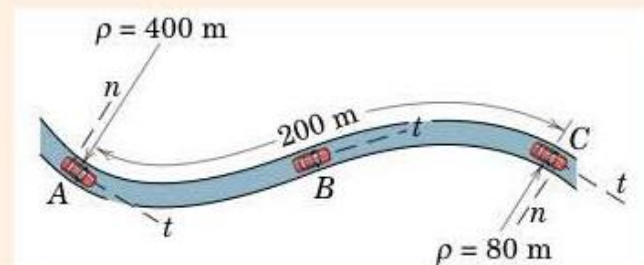
The normal components of acceleration at A, B, and C are

$$\textcircled{2} \quad [a_n = v^2/\rho]$$

At A, $a_n = \frac{(100/3.6)^2}{400} = 1.929 \text{ m/s}^2$

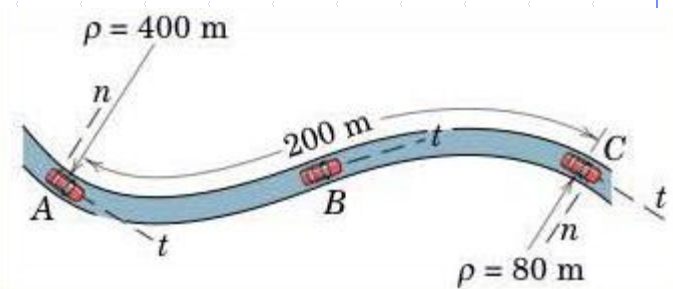
At B, $a_n = 0$

At C, $a_n = \frac{(50/3.6)^2}{80} = 2.41 \text{ m/s}^2$



Helpful Hints

- ① Recognize the numerical value of the conversion factor from km/h to m/s as 1000/3600 or 1/3.6.
- ② Note that a_n is always directed toward the center of curvature.



Application of Newton's second law in both the n - and t -directions to the free-body diagrams of the car gives

$$[\Sigma F_t = ma_t]$$

$$F_t = 1500(1.447) = 2170 \text{ N}$$

③ $[\Sigma F_n = ma_n]$

At A, $F_n = 1500(1.929) = 2890 \text{ N}$

At B, $F_n = 0$

At C, $F_n = 1500(2.41) = 3620 \text{ N}$

Thus, the total horizontal force acting on the tires becomes

At A, $F = \sqrt{F_n^2 + F_t^2} = \sqrt{(2890)^2 + (2170)^2} = 3620 \text{ N}$

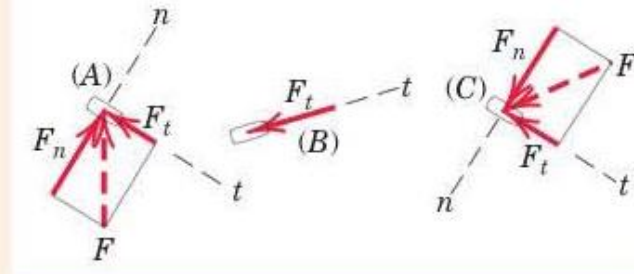
Ans.

At B, $F = F_t = 2170 \text{ N}$

Ans.

At C, $F = \sqrt{F_n^2 + F_t^2} = \sqrt{(3620)^2 + (2170)^2} = 4220 \text{ N}$

Ans.



③ Note that the direction of F_n must agree with that of a_n .

④ The angle made by $\mathbf{a}$ and $\mathbf{F}$ with the direction of the path can be computed if desired.

Sample Problem 3/10

Tube A rotates about the vertical O -axis with a constant angular rate $\dot{\theta} = \omega$ and contains a small cylindrical plug B of mass m whose radial position is controlled by the cord which passes freely through the tube and shaft and is wound around the drum of radius b . Determine the tension T in the cord and the horizontal component F_θ of force exerted by the tube on the plug if the constant angular rate of rotation of the drum is ω_0 first in the direction for case (a) and second in the direction for case (b). Neglect friction.

Solution. With r a variable, we use the polar-coordinate form of the equations of motion, Eqs. 3/8. The free-body diagram of B is shown in the horizontal plane and discloses only T and F_θ . The equations of motion are

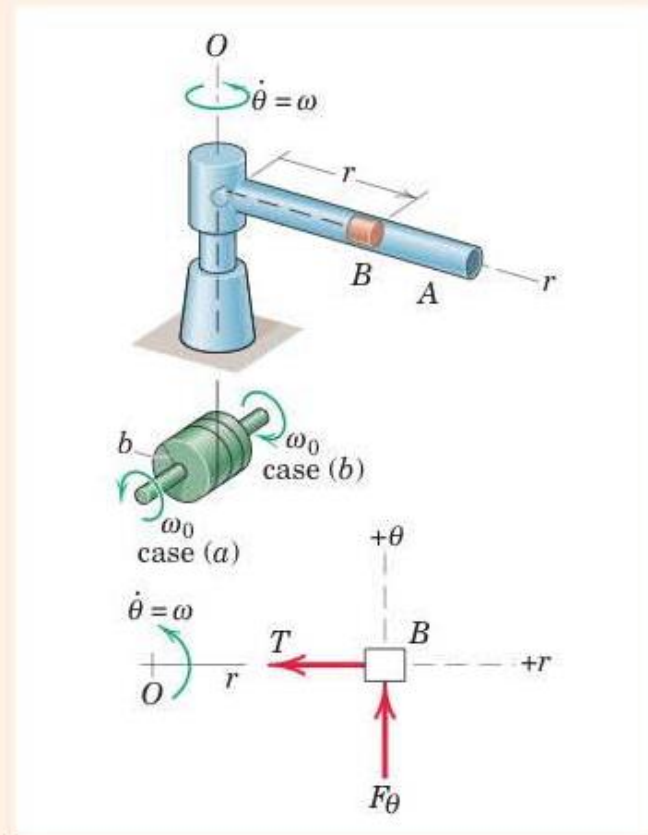
$$\begin{aligned} [\Sigma F_r = ma_r] \quad & -T = m(\ddot{r} - r\dot{\theta}^2) \\ [\Sigma F_\theta = ma_\theta] \quad & F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{aligned}$$

Case (a). With $\dot{r} = +b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \quad F_\theta = 2mb\omega_0\omega \quad \text{Ans.}$$

① **Case (b).** With $\dot{r} = -b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \quad F_\theta = -2mb\omega_0\omega \quad \text{Ans.}$$



Helpful Hint

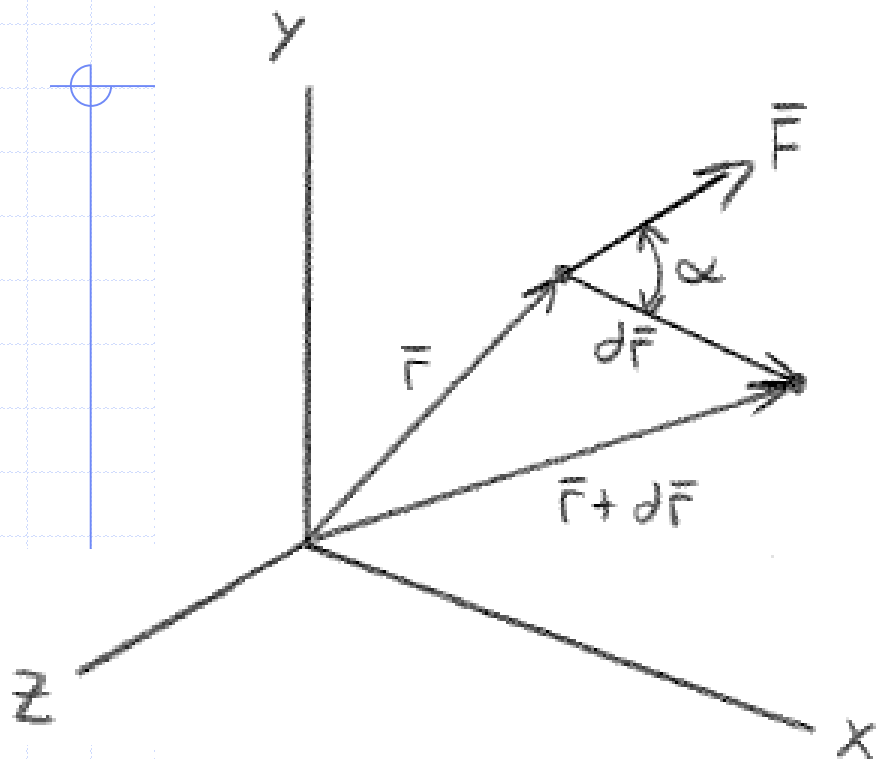
- ① The minus sign shows that F_θ is in the direction opposite to that shown on the free-body diagram.

2. Work and Energy Principles



Energy and Momentum Methods

Work of a Force



Letting $|\vec{r}| = r$, $|d\vec{r}| = dr$

$$dU = F dr \cos \alpha$$

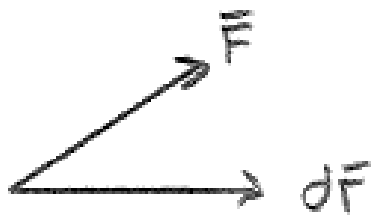
we define

$$dU = \vec{F} \cdot d\vec{r}$$

Differential
Work of $\vec{F}$

$$\left(\begin{array}{l} N \cdot m = J \\ ft \cdot lb_f \end{array} \right)$$

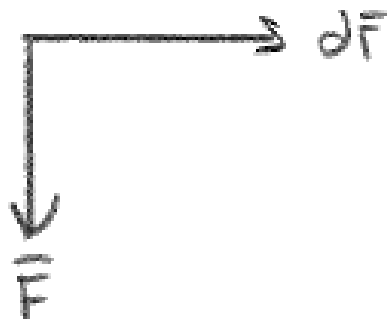
Ex



$$dU > 0$$



$$dU < 0$$



$$dU = 0$$

Comments:

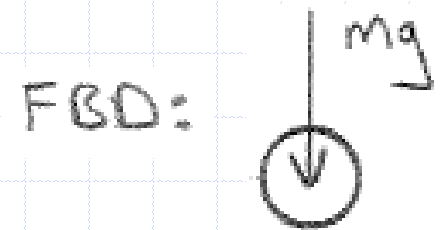
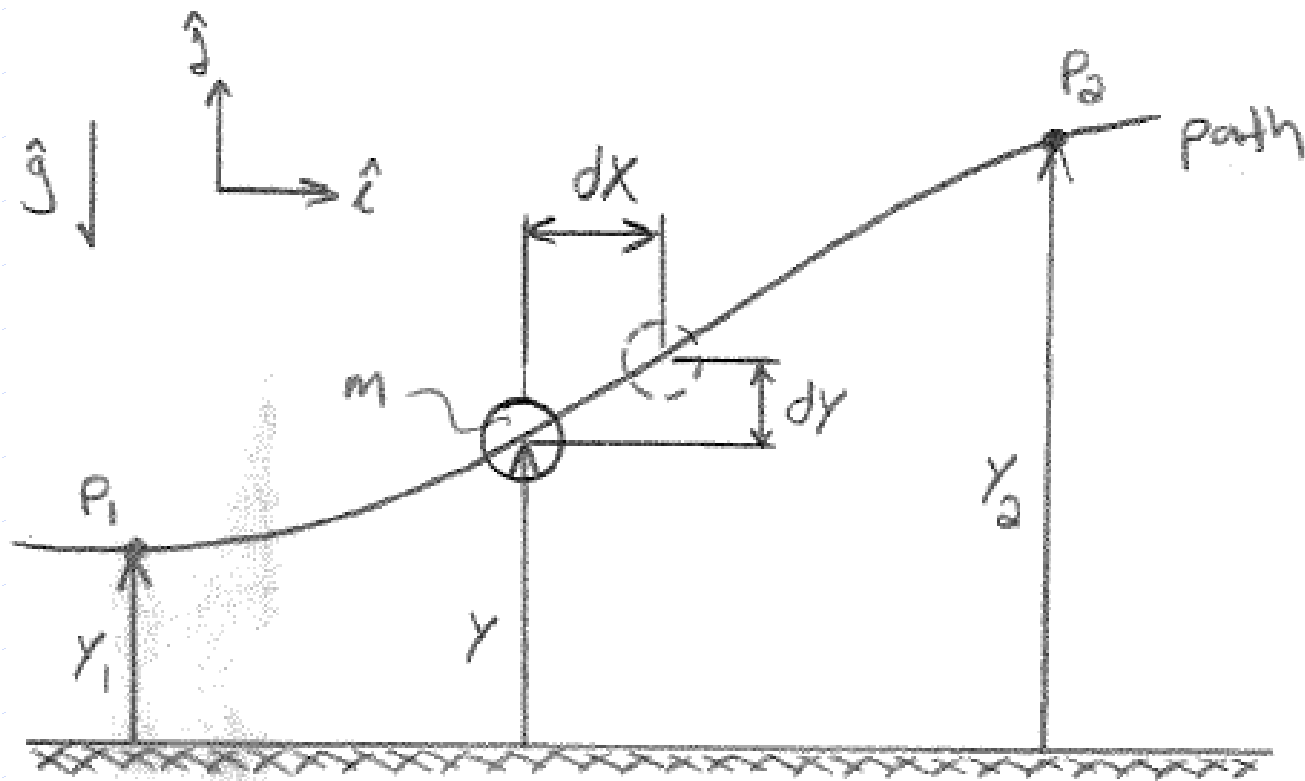
- dU gives a measure of how parallel forces and displacements are.
- Only that component of $\vec{F}$ tangent to path of motion does work.

Over a finite path from P_1 to P_2

$$U_{1 \rightarrow 2} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

Work

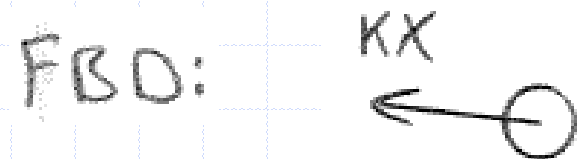
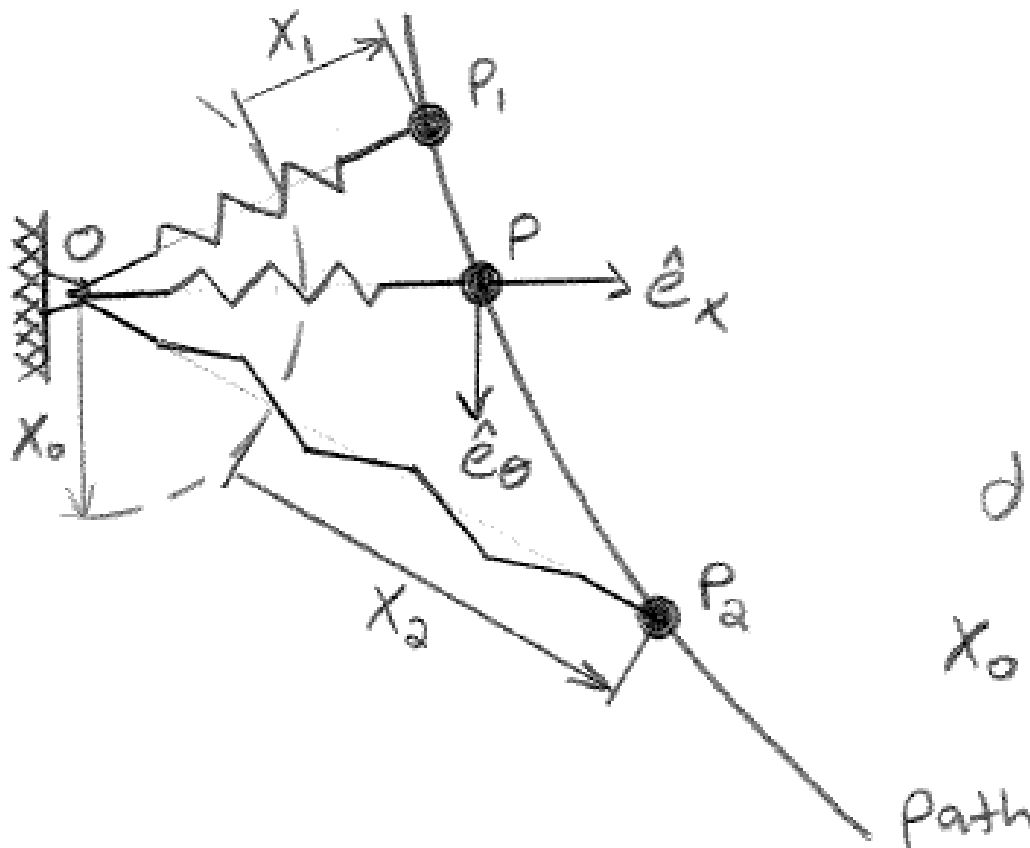
Work of Force due to Gravity: $P_1 \rightarrow P_2$



$$U_{1 \rightarrow 2} = -mg(y_2 - y_1) = -mg \Delta y$$

Work of a Spring Force: $P_1 \rightarrow P_2$

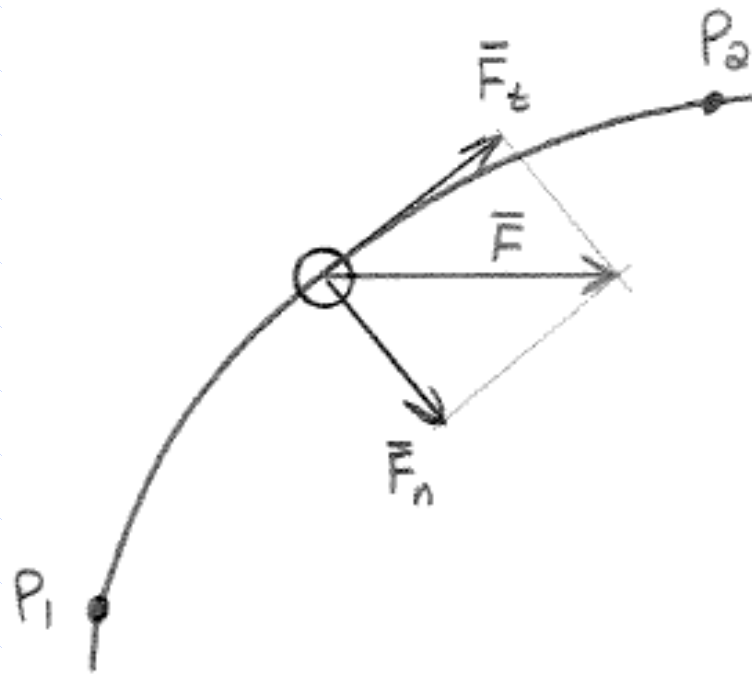
$$U_{1 \rightarrow 2} = \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2$$



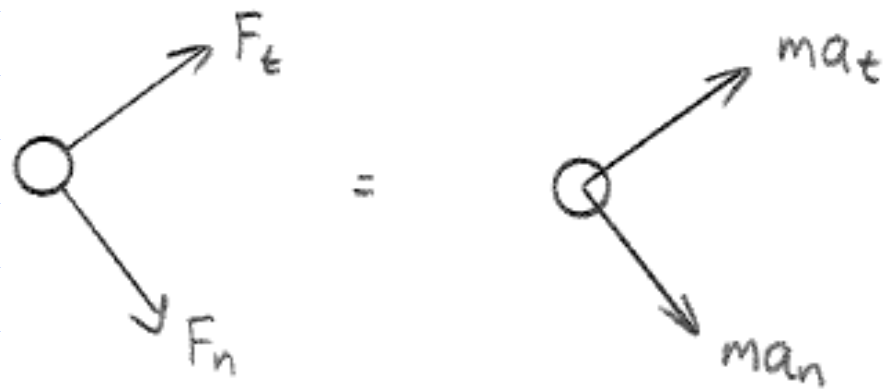
$$dU = -Kx dx$$

x_0 = undeformed length

Kinetic Energy of a Particle. Principle of Work-Energy.



FBD:



Consider only tangential dynamics.

$$F_t = ma_t$$

$$F_t = ma_t$$

$$= m \frac{dv}{dt} = m \frac{dv}{ds} \frac{ds}{dt} \overset{v}{=} = m v \frac{dv}{ds}$$

$$\Rightarrow F_t ds = m v dv \quad (*)$$

Integrate (*) from P_1 to P_2 , where:

$$@ P_1: s = s_1, \quad v = v_1$$

$$@ P_2: s = s_2, \quad v = v_2$$

$$\int_{s_1}^{s_2} F_t ds = m \int_{v_1}^{v_2} v dv = m \left(\frac{v_2^2}{2} - \frac{v_1^2}{2} \right)$$

$$\Rightarrow \int_{s_1}^{s_2} F_t ds = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \quad \text{--- (**)}$$

We define $T = \frac{1}{2} m v^2$ Kinetic Energy

Since $\vec{v} \cdot \vec{v} = |\vec{v}|^2 = v^2$, can also write

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

Finally, note

$$\int_{s_1}^{s_2} F_t ds = U_1 \rightarrow 2$$

Thus from (**),

$$U_{1 \rightarrow 2} = T_2 - T_1$$

$\Rightarrow$

$$T_1 + U_{1 \rightarrow 2} = T_2$$

Principle of
Work-Energy

Sample Problem 3/11

Calculate the velocity v of the 50-kg crate when it reaches the bottom of the chute at B if it is given an initial velocity of 4 m/s down the chute at A . The coefficient of kinetic friction is 0.30.

Solution. The free-body diagram of the crate is drawn and includes the normal force R and the kinetic friction force F calculated in the usual manner. The work done by the weight is positive, whereas that done by the friction force is negative. The total work done on the crate during the motion is

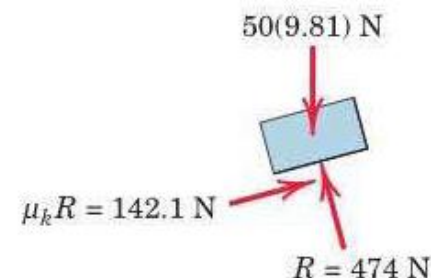
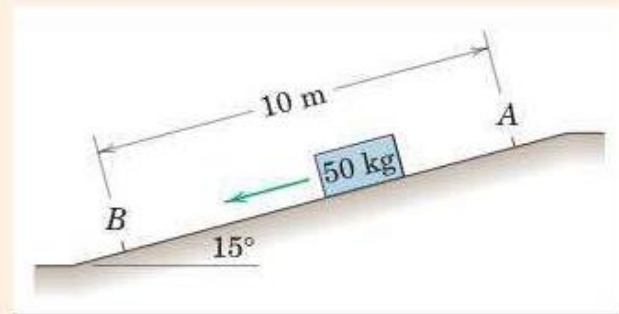
$$\textcircled{1} [U = Fs] \quad U_{1-2} = 50(9.81)(10 \sin 15^\circ) - 142.1(10) = -151.9 \text{ J}$$

The work-energy equation gives

$$\begin{aligned} [T_1 + U_{1-2} = T_2] \quad \frac{1}{2}mv_1^2 + U_{1-2} &= \frac{1}{2}mv_2^2 \\ \frac{1}{2}(50)(4)^2 - 151.9 &= \frac{1}{2}(50)v_2^2 \\ v_2 &= 3.15 \text{ m/s} \end{aligned}$$

Since the net work done is negative, we obtain a decrease in the kinetic energy.

Ans.



Helpful Hint

- $\textcircled{1}$ The work due to the weight depends only on the *vertical* distance traveled.

Sample Problem 3/12

The flatbed truck, which carries an 80-kg crate, starts from rest and attains a speed of 72 km/h in a distance of 75 m on a level road with constant acceleration. Calculate the work done by the friction force acting on the crate during this interval if the static and kinetic coefficients of friction between the crate and the truck bed are (a) 0.30 and 0.28, respectively, or (b) 0.25 and 0.20, respectively.

Solution. If the crate does not slip on the bed, its acceleration will be that of the truck, which is

$$[v^2 = 2as] \quad a = \frac{v^2}{2s} = \frac{(72/3.6)^2}{2(75)} = 2.67 \text{ m/s}^2$$

Case (a). This acceleration requires a friction force on the block of

$$[F = ma] \quad F = 80(2.67) = 213 \text{ N}$$

which is less than the maximum possible value of $\mu_s N = 0.30(80)(9.81) = 235 \text{ N}$. Therefore, the crate does not slip and the work done by the actual static friction force of 213 N is

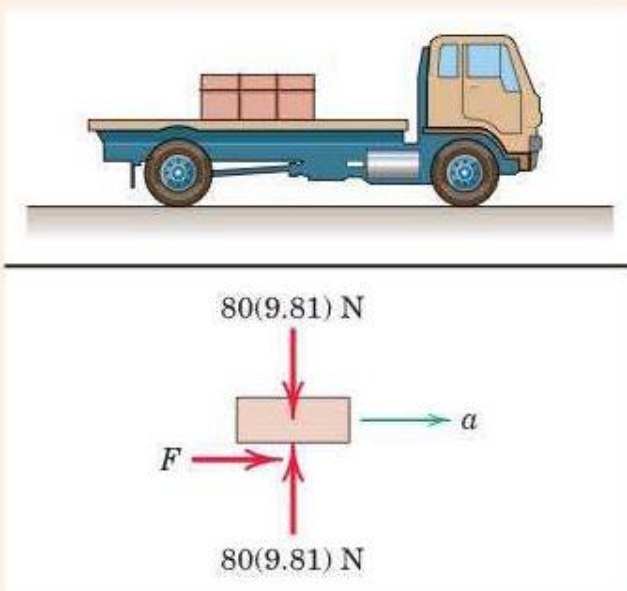
$$\textcircled{1} [U = Fs] \quad U_{1-2} = 213(75) = 16\,000 \text{ J} \quad \text{or} \quad 16 \text{ kJ} \quad \text{Ans.}$$

Case (b). For $\mu_s = 0.25$, the maximum possible friction force is $0.25(80)(9.81) = 196.2 \text{ N}$, which is slightly less than the value of 213 N required for no slipping. Therefore, we conclude that the crate slips, and the friction force is governed by the kinetic coefficient and is $F = 0.20(80)(9.81) = 157.0 \text{ N}$. The acceleration becomes

$$[F = ma] \quad a = F/m = 157.0/80 = 1.962 \text{ m/s}^2$$

The distances traveled by the crate and the truck are in proportion to their accelerations. Thus, the crate has a displacement of $(1.962/2.67)75 = 55.2 \text{ m}$, and the work done by kinetic friction is

$$\textcircled{2} [U = Fs] \quad U_{1-2} = 157.0(55.2) = 8660 \text{ J} \quad \text{or} \quad 8.66 \text{ kJ} \quad \text{Ans.}$$



Helpful Hints

- $\textcircled{1}$ We note that static friction forces do no work when the contacting surfaces are both at rest. When they are in motion, however, as in this problem, the static friction force acting on the crate does positive work and that acting on the truck bed does negative work.
- $\textcircled{2}$ This problem shows that a kinetic friction force can do positive work when the surface which supports the object and generates the friction force is in motion. If the supporting surface is at rest, then the kinetic friction force acting on the moving part always does negative work.

Power – Efficiency

Accordingly, the power P developed by a force $\mathbf{F}$ which does an amount of work U is $P = dU/dt = \mathbf{F} \cdot d\mathbf{r}/dt$. Because $d\mathbf{r}/dt$ is the velocity $\mathbf{v}$ of the point of application of the force, we have

$$P = \mathbf{F} \cdot \mathbf{v} \quad (3/16)$$

Power is clearly a scalar quantity, and in SI it has the units of $\text{N} \cdot \text{m}/\text{s} = \text{J}/\text{s}$. The special unit for power is the *watt* (W), which equals one joule per second (J/s). In U.S. customary units, the unit for mechanical power is the *horsepower* (hp). These units and their numerical equivalences are

$$1 \text{ W} = 1 \text{ J/s}$$

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb}/\text{sec} = 33,000 \text{ ft} \cdot \text{lb}/\text{min}$$

$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

Power – Efficiency (cont.)

Efficiency

The ratio of the work done *by* a machine to the work done *on* the machine during the same time interval is called the *mechanical efficiency* e_m of the machine. This definition assumes that the machine operates uniformly so that there is no accumulation or depletion of energy within it. Efficiency is always less than unity since every device operates with some loss of energy and since energy cannot be created within the machine. In mechanical devices which involve moving parts, there will always be some loss of energy due to the negative work of kinetic friction forces. This work is converted to heat energy which, in turn, is dissipated to the surroundings. The mechanical efficiency at any instant of time may be expressed in terms of mechanical power P by

$$e_m = \frac{P_{\text{output}}}{P_{\text{input}}} \quad (3/17)$$

Sample Problem 3/14

The power winch A hoists the 800-lb log up the 30° incline at a constant speed of 4 ft/sec. If the power output of the winch is 6 hp, compute the coefficient of kinetic friction μ_k between the log and the incline. If the power is suddenly increased to 8 hp, what is the corresponding instantaneous acceleration a of the log?

Solution. From the free-body diagram of the log, we get $N = 800 \cos 30^\circ = 693$ lb, and the kinetic friction force becomes $693\mu_k$. For constant speed, the forces are in equilibrium so that

$$[\Sigma F_x = 0] \quad T - 693\mu_k - 800 \sin 30^\circ = 0 \quad T = 693\mu_k + 400$$

The power output of the winch gives the tension in the cable

$$\textcircled{1} [P = Tv] \quad T = P/v = 6(550)/4 = 825 \text{ lb}$$

Substituting T gives

$$825 = 693\mu_k + 400 \quad \mu_k = 0.613$$

Ans.

When the power is increased, the tension momentarily becomes

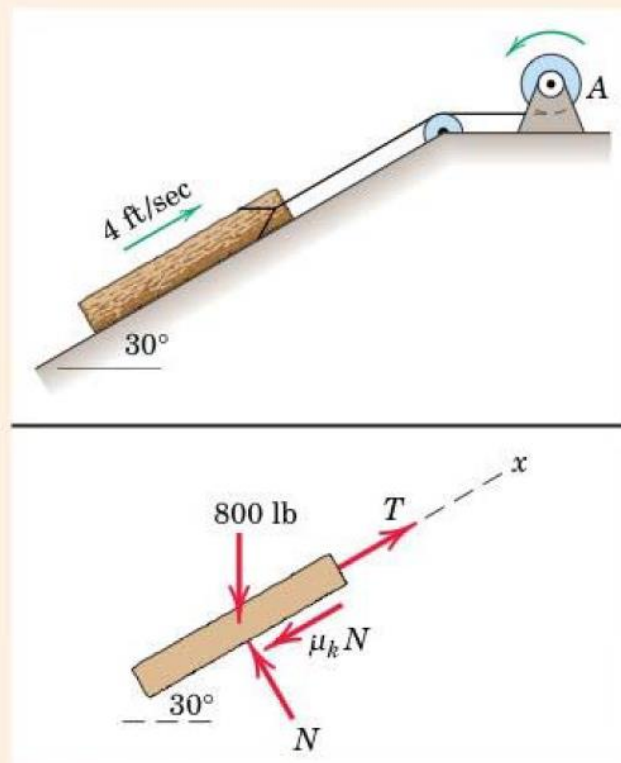
$$[P = Tv] \quad T = P/v = 8(550)/4 = 1100 \text{ lb}$$

and the corresponding acceleration is given by

$$[\Sigma F_x = ma_x] \quad 1100 - 693(0.613) - 800 \sin 30^\circ = \frac{800}{32.2} a$$

$$\textcircled{2} \quad a = 11.07 \text{ ft/sec}^2$$

Ans.

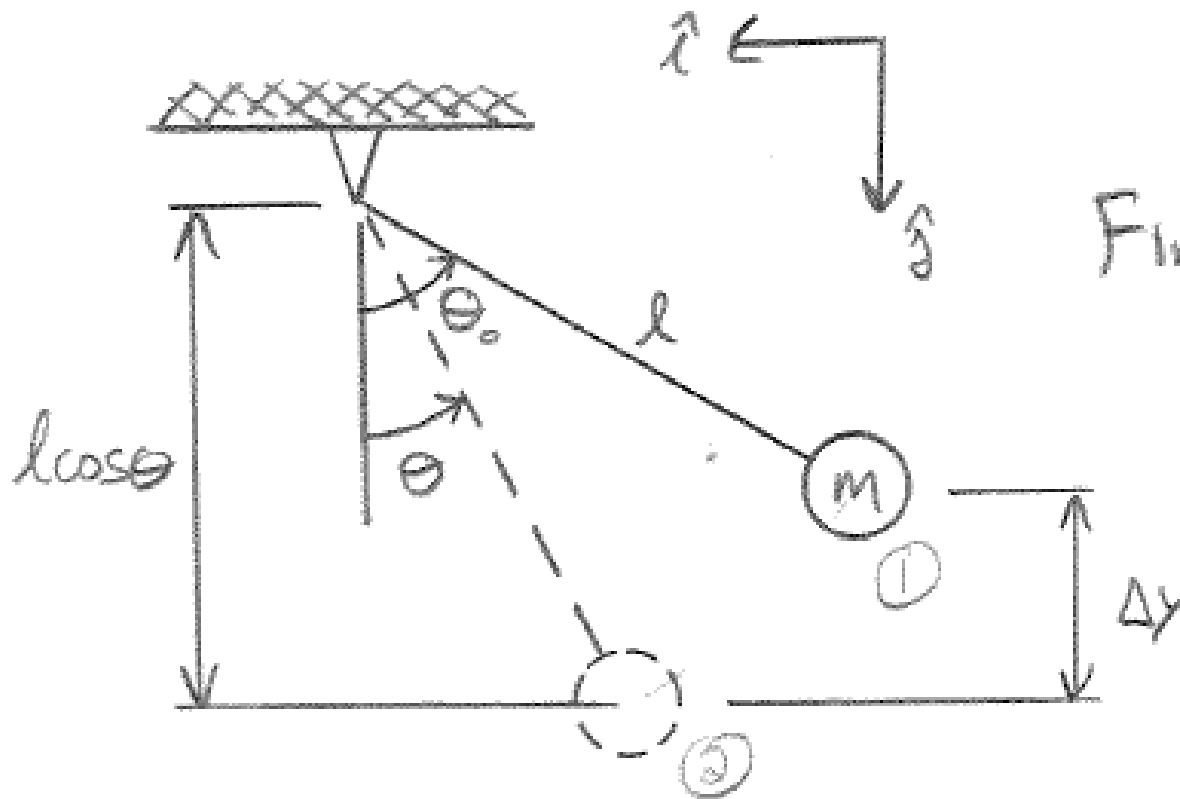


Helpful Hints

- ① Note the conversion from horsepower to ft-lb/sec.
- ② As the speed increases, the acceleration will drop until the speed stabilizes at a value higher than 4 ft/sec.

Example:

m is released @ ① from $\theta = \theta_0$
with $v_0 = 0$.



Find $v = v(\theta)$

Invoke PWE: $T_1 + U_{1 \rightarrow 2} = T_2$

$$T_1 = \frac{1}{2} m \cancel{v_1^2} = 0$$

$v_0 = 0$

$$T_2 = \frac{1}{2} m \cancel{v_2^2} = \frac{1}{2} m v^2$$

$$U_{1 \rightarrow 2} = U_{1 \rightarrow 2}^{mg} + \cancel{U_{1 \rightarrow 2}^T}$$

$$U_{1 \rightarrow 2}^{mg} = mg(y_2 - y_1) = mg \Delta y$$



where $\Delta y = l \cos \theta - l \cos \theta_0$

$$\therefore U_{1 \rightarrow 2} = mgl(\cos \theta - \cos \theta_0)$$

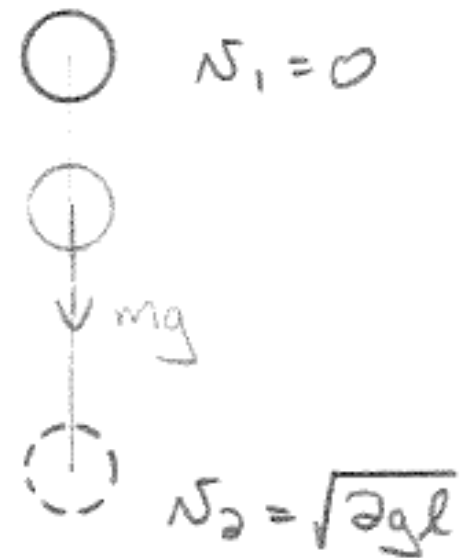
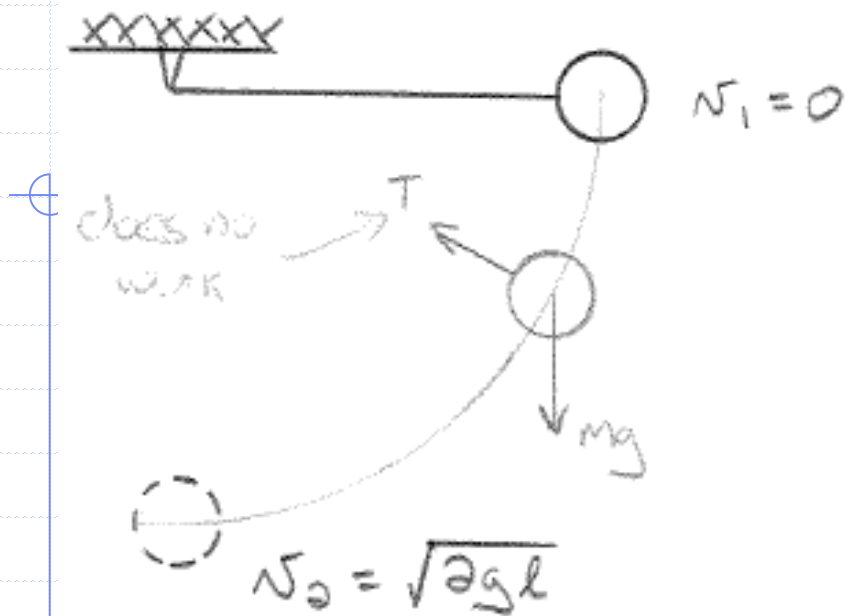
PWE gives $0 + mgl(\cos \theta - \cos \theta_0) = \frac{1}{2}mv^2$

$$\therefore v(\theta) = \sqrt{2gl(\cos \theta - \cos \theta_0)}$$

Consider $\theta_0 = \frac{\pi}{2}$, $\theta = 0$

Then $v = \sqrt{2gl[\cos(0) - \cos(\pi/2)]} = \sqrt{2gl}$

→ Same as if m simply dropped!
why?

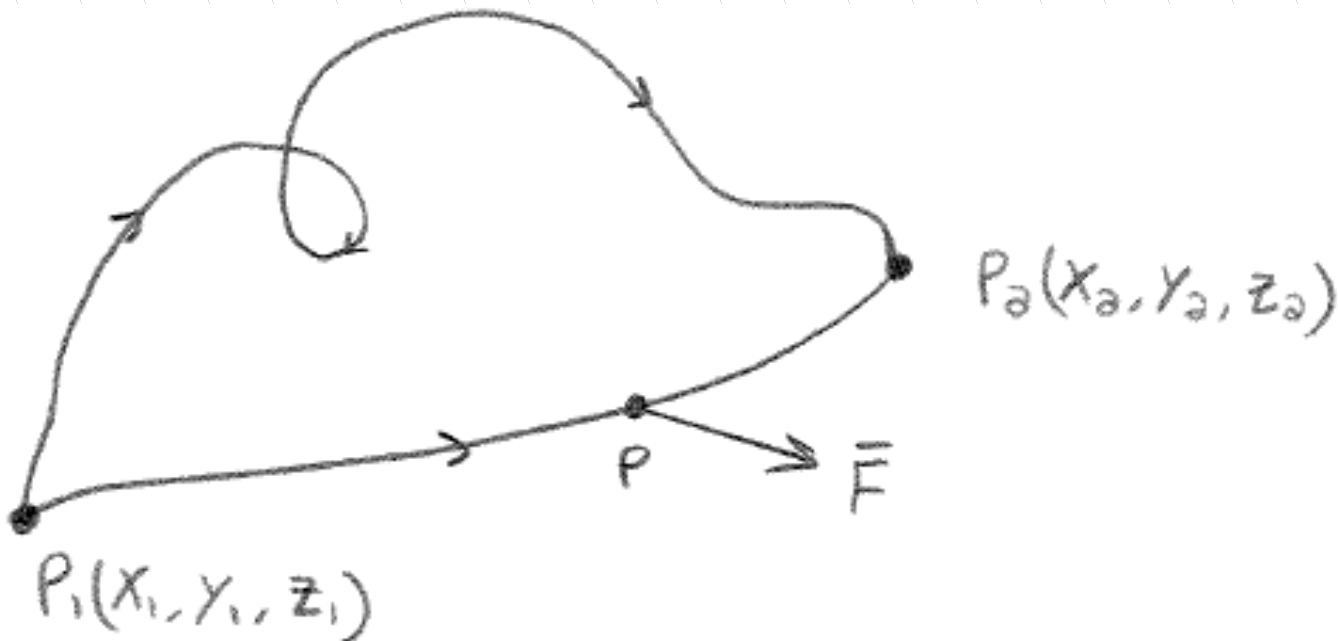


$$T_1 + U_{1 \rightarrow 2} = T_2$$

Same both cases

Conservative Forces

A force $\vec{F}$ acting on a particle P is said to be conservative if its work $U_{1 \rightarrow 2}$ is independent of the path followed by P in moving from P_1 to P_2 .



FACT: If $\vec{F}$ is conservative, then

$$U_{1 \rightarrow 2} = \int_1^2 \vec{F} \cdot d\vec{r} = V(\underset{\textcircled{1}}{x_1, y_1, z_1}) - V(\underset{\textcircled{2}}{x_2, y_2, z_2})$$

V is called a potential function.

Potential Energy

Recall work of force due to weight: ($F = mg$)

$$\begin{aligned} U_{1 \rightarrow 2}^{\text{weight}} &= -mg(y_2 - y_1) \\ &= mgy_1 - mgy_2 \end{aligned}$$

Path independent $\Rightarrow F = mg$ is conservative

$$\therefore \boxed{V_{\text{weight}} = mgy}$$

Recall work of a spring Force: ($F = Kx$)

$$U_{1 \rightarrow 2}^{\text{spring}} = \frac{1}{2} K x_1^2 - \frac{1}{2} K x_2^2$$

Path independent $\Rightarrow F = Kx$ is conservative

$$\therefore \boxed{V_{\text{spring}} = \frac{1}{2} K x^2}$$

Nonconservative Forces

A force $\vec{F}$ is nonconservative if it is not conservative.

↳ are path dependent

↳ cannot be expressed as a change in potential energy

Examples:

- friction forces $f = \mu N$

Principle of Work - Energy Revisited

Recall $T_1 + U_{1 \rightarrow 2} = T_2$

Separate work done by conservative and nonconservative forces:

Let $\bar{F} = \bar{F}^c + \bar{F}^{NC}$

Thus $T_1 + V_1 - V_2 + U_{1 \rightarrow 2}^{NC} = T_2$

$$\Rightarrow \boxed{T_1 + V_1 + U_{1 \rightarrow 2}^{NC} = T_2 + V_2}$$

$\therefore U_{1 \rightarrow 2}^{NC} = 0 \Rightarrow$ Conservation of Energy

Datums for Potential Energy

Consider PWE:

$$T_1 + V_1 + U_{1 \rightarrow 2}^{NC} = T_2 + V_2$$

$$\Rightarrow U_{1 \rightarrow 2}^{NC} = \cancel{T_2 - T_1} + \cancel{V_2 - V_1}$$

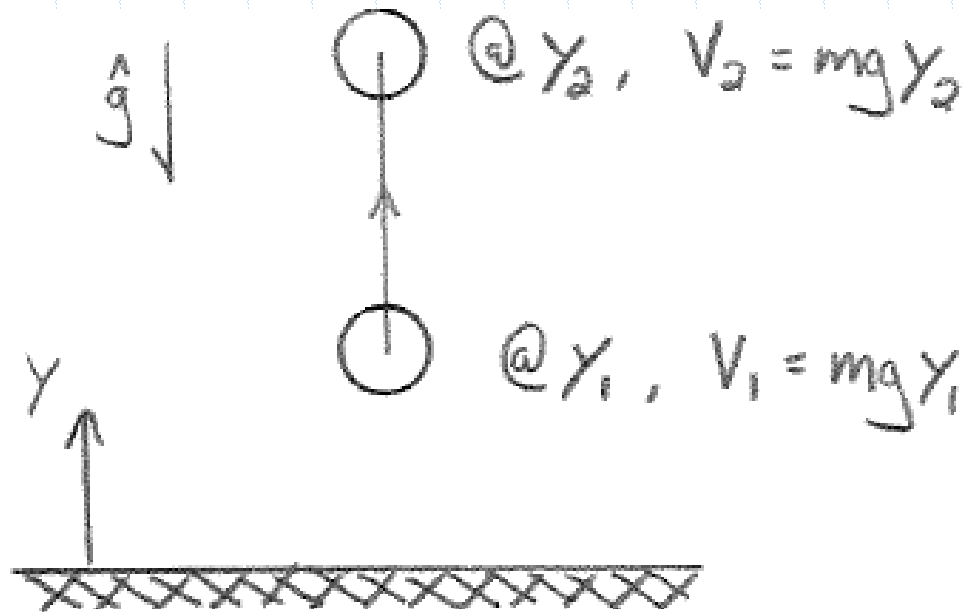
$\Delta T \qquad \Delta V$

$\therefore$ only changes in energy are important.

FACT:

If a conservative force is constant,
can set $V = 0$ at an arbitrary datum.

ΔV due to Weight



$$\Delta V = V_2 - V_1 = mgy_2 - mgy_1 = \int_{y_1}^{y_2} mg \, dy$$

Only dist. from y_1 to y_2 matters!

EX] Set datum at y_1

$$@ y_1: \quad V_1 = 0 \quad (\text{at datum})$$

$$@ y_2: \quad V_2 = mg \cdot (\text{dist. between } y_1 \text{ and } y_2) \\ = mg(y_2 - y_1)$$

$$\therefore \Delta V = V_2 - \cancel{V_1}^{\circ} = mg(y_2 - y_1) \quad \checkmark$$

EX Set datum at y_2

@ y_2 : $V_2 = 0$ (at datum)

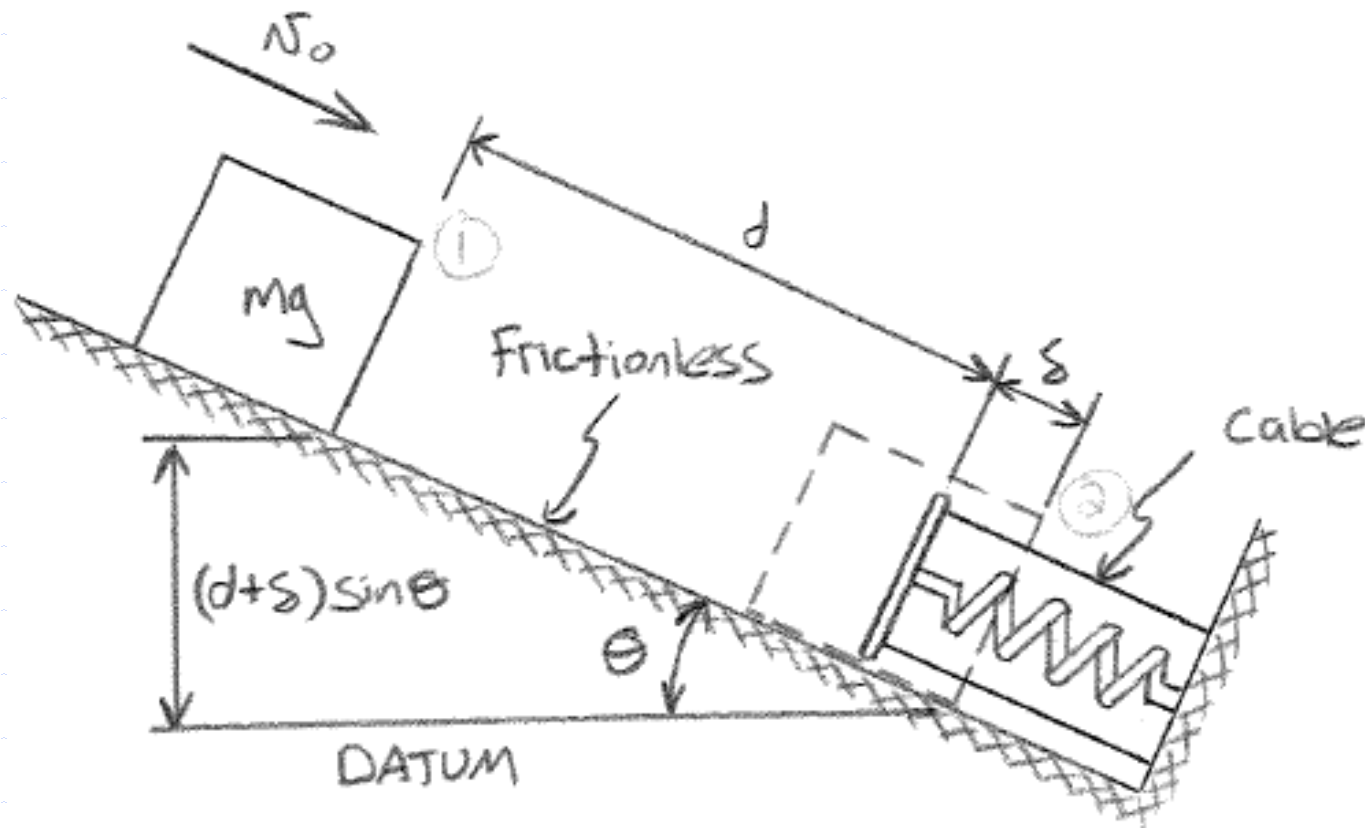
@ y_1 : $V_1 = -mg \cdot (\text{dist. between } y_1 \text{ and } y_2)$
 $= -mg(y_2 - y_1)$

$\therefore \Delta V = \overset{\circ}{V_2} - V_1 = +mg(y_2 - y_1)$

Key: Above datum $\rightarrow V$ pos.

Below datum $\rightarrow V$ neg.

Example:



Neglect Friction

Spring is pre-deformed by $X_0 = 6 \text{ in}$

Find maximum additional deformation s in the spring.

Data: $V_0 = 8 \frac{\text{ft}}{\text{s}}$ (initial speed @ state ①)

$$mg = 200 \text{ lbf}$$

$$\theta = 20^\circ$$

$$K = 125 \text{ lbf/in (spring constant)}$$

$$d = 25 \text{ ft}$$

Invoke PWE with the block as the system.

$$T_1 + V_1 + U_{1 \rightarrow 2}^{NC} = T_2 + V_2 \quad \text{--- (*)}$$

$$T_1 = \frac{1}{2} m v_0^2 \quad (v_1 = v_0 \text{ @ } \textcircled{1})$$

$$V_1 = mg(d+s) \sin \theta$$

$$U_{1 \rightarrow 2}^{NC} = 0 \quad (\text{all acting forces are conservative})$$

$$T_2 = 0 \quad (v_2 = 0 \text{ @ max deformation})$$

$$V_2 = \underbrace{\frac{1}{2} K (x_0 + s)^2}_{\text{"total deformed" energy}} - \underbrace{\frac{1}{2} K x_0^2}_{\text{"pre-deformed" energy}}$$

"total deformed" energy

"pre-deformed" energy

Plug into (*) ...

$$\frac{1}{2}mv_0^2 + mg(d+s)\sin\theta + 0 = 0 + \frac{1}{2}K(x_0+s)^2 - \frac{1}{2}Kx_0^2$$

└→ quadratic in s

Put in the form

$$(\quad) s^2 + (\quad) s + (\quad) = 0$$

solve

$$s_1 = 1.204 \text{ ft}$$

$$s_2 = -2.113 \text{ ft}$$

$$\therefore s = 1.204 \text{ ft}$$

Sample Problem 3/16

The 6-lb slider is released from rest at position 1 and slides with negligible friction in a vertical plane along the circular rod. The attached spring has a stiffness of 2 lb/in. and has an unstretched length of 24 in. Determine the velocity of the slider as it passes position 2.

Solution. The work done by the weight and the spring force on the slider will be treated using potential-energy methods. The reaction of the rod on the slider is normal to the motion and does no work. Hence, $U'_{1-2} = 0$. We define the datum to be at the level of position 1, so that the gravitational potential energies are

①

$$V_1 = 0$$

$$V_2 = -mgh = -6\left(\frac{24}{12}\right) = -12 \text{ ft}\cdot\text{lb}$$

The initial and final elastic (spring) potential energies are

$$V_1 = \frac{1}{2}kx_1^2 = \frac{1}{2}(2)(12)\left(\frac{24}{12}\right)^2 = 48 \text{ ft}\cdot\text{lb}$$

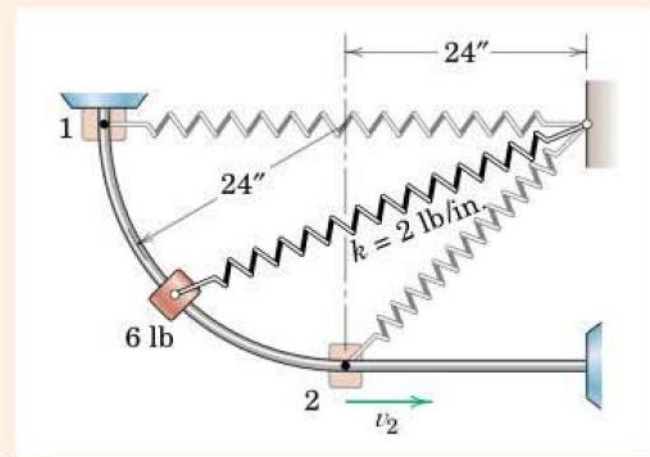
$$V_2 = \frac{1}{2}kx_2^2 = \frac{1}{2}(2)(12)\left(\frac{24\sqrt{2}}{12} - \frac{24}{12}\right)^2 = 8.24 \text{ ft}\cdot\text{lb}$$

Substitution into the alternative work-energy equation yields

$$[T_1 + V_1 + U'_{1-2} = T_2 + V_2] \quad 0 + 48 + 0 = \frac{1}{2}\left(\frac{6}{32.2}\right)v_2^2 - 12 + 8.24$$

$$v_2 = 23.6 \text{ ft/sec}$$

Ans.



Helpful Hint

- ① Note that if we evaluated the work done by the spring force acting on the slider by means of the integral $\int \mathbf{F} \cdot d\mathbf{r}$, it would necessitate a lengthy computation to account for the change in the magnitude of the force, along with the change in the angle between the force and the tangent to the path. Note further that v_2 depends only on the end conditions of the motion and does not require knowledge of the shape of the path.

3. Impulse and Momentum Methods



Principle of Impulse - Momentum

We have already seen,

$$\vec{F} = m\vec{a} \quad , \quad \int (\quad) ds \quad \rightarrow \quad \text{Principle of Work - Energy}$$

Now, $\int (\quad) dt \quad \rightarrow \quad \text{Principle of Impulse - Momentum}$

$$\vec{F} = \frac{d}{dt} (m\vec{v})$$

↑
linear momentum

Sample Problem 3/20

A 2-lb particle moves in the vertical y - z plane (z up, y horizontal) under the action of its weight and a force $\mathbf{F}$ which varies with time. The linear momentum of the particle in pound-seconds is given by the expression $\mathbf{G} = \frac{3}{2}(t^2 + 3)\mathbf{j} - \frac{2}{3}(t^3 - 4)\mathbf{k}$, where t is the time in seconds. Determine $\mathbf{F}$ and its magnitude for the instant when $t = 2$ sec.

Solution. The weight expressed as a vector is $-2\mathbf{k}$ lb. Thus, the force-momentum equation becomes

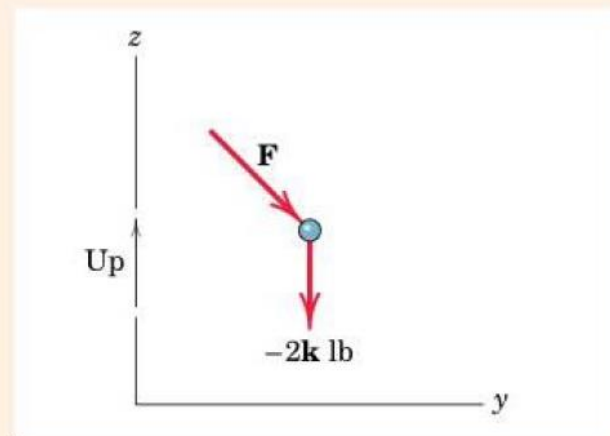
$$\begin{aligned} \textcircled{1} \quad [\Sigma \mathbf{F} = \dot{\mathbf{G}}] \quad \mathbf{F} - 2\mathbf{k} &= \frac{d}{dt} \left[\frac{3}{2}(t^2 + 3)\mathbf{j} - \frac{2}{3}(t^3 - 4)\mathbf{k} \right] \\ &= 3t\mathbf{j} - 2t^2\mathbf{k} \end{aligned}$$

$$\text{For } t = 2 \text{ sec,} \quad \mathbf{F} = 2\mathbf{k} + 3(2)\mathbf{j} - 2(2^2)\mathbf{k} = 6\mathbf{j} - 6\mathbf{k} \text{ lb}$$

Ans.

$$\text{Thus,} \quad F = \sqrt{6^2 + 6^2} = 6\sqrt{2} \text{ lb}$$

Ans.



Helpful Hint

- ① Don't forget that $\Sigma \mathbf{F}$ includes *all* external forces acting on the particle, including the weight.

$$\Rightarrow \bar{F} dt = d(m\bar{v})$$

$$\Rightarrow \int_{t_1}^{t_2} \bar{F} dt = \int_{m\bar{v}_1}^{m\bar{v}_2} d(m\bar{v})$$

$$= m\bar{v}_2 - m\bar{v}_1$$

Define

$$\text{Imp}_{1 \rightarrow 2} = \int_{t_1}^{t_2} \bar{F} dt$$

Linear
Impulse

Then $\text{Imp}_{1 \rightarrow 2} = m\bar{v}_2 - m\bar{v}_1$

$$m\vec{v}_1 + \text{Imp}_{1 \rightarrow 2} = m\vec{v}_2$$

Principle of
Impulse - Momentum

$\text{Imp}_{1 \rightarrow 2} = 0$
(In some direction) $\Rightarrow$ Conservation of Momentum
(In that direction)

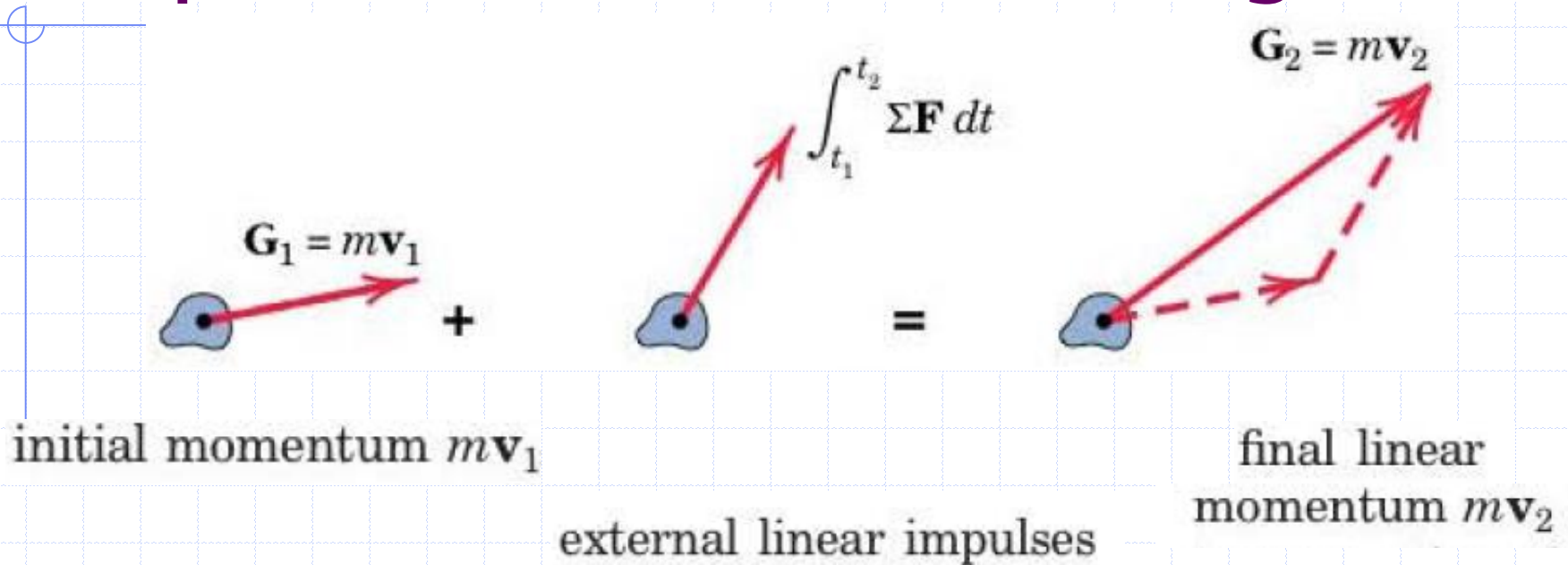
Average Impulsive Force

$$\bar{F}_{AVE} = \frac{1}{\Delta t} \int_{t_1}^{t_2} \bar{F} dt = \frac{1}{\Delta t} \text{Imp}_{1 \rightarrow 2}$$

$\Rightarrow$

$$\text{Imp}_{1 \rightarrow 2} = \bar{F}_{AVE} \Delta t$$

Impulse – Momentum Diagram



Example:

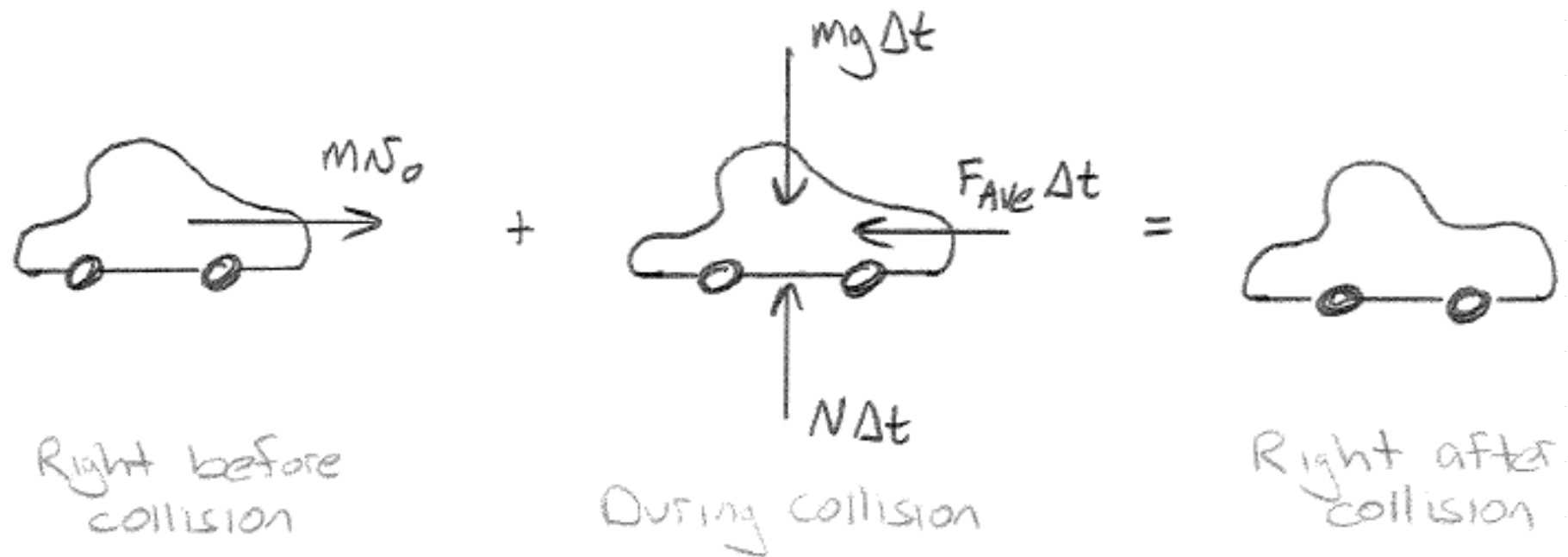


Data: $mg = 3000 \text{ lbf}$

$$v_0 = 2.5 \text{ mph} = 3.667 \text{ ft/s}$$

Car hits wall, comes to rest in $\Delta t = 75 \text{ ms}$.

Find F_{ave} between wall and car bumper during collision.

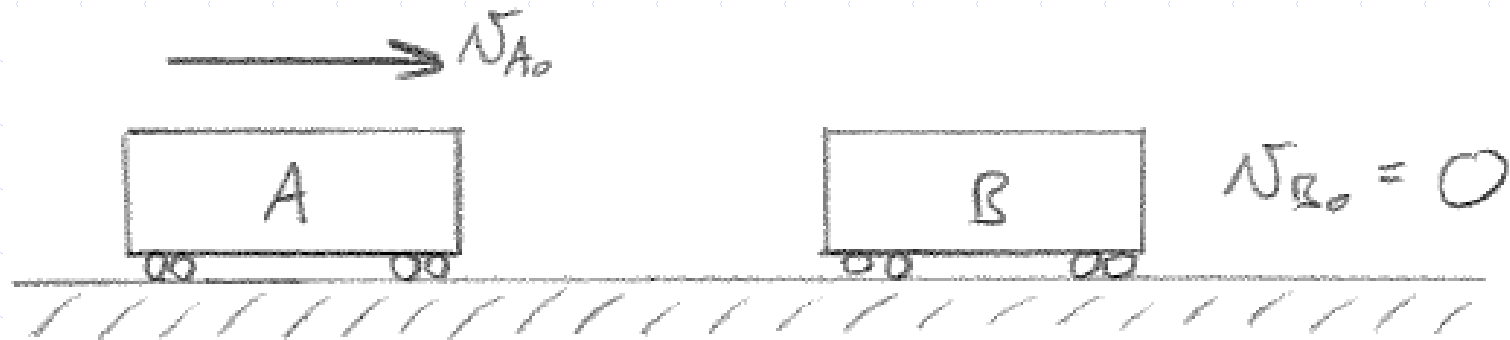


$$\overset{+}{\rightarrow} \Sigma \text{Mom: } mN_0 - F_{\text{Ave}} \Delta t = 0$$

$$\Rightarrow F_{\text{Ave}} = \frac{mN_0}{\Delta t} = \frac{mgN_0}{g\Delta t}$$

$$F_{\text{Ave}} = \frac{(3000 \text{ lbf})(3.667 \text{ ft/s})}{(32.174 \text{ ft/s}^2)(0.075 \text{ s})} = 4558.55 \text{ lbf}$$

Example:



A hits B and they become coupled.

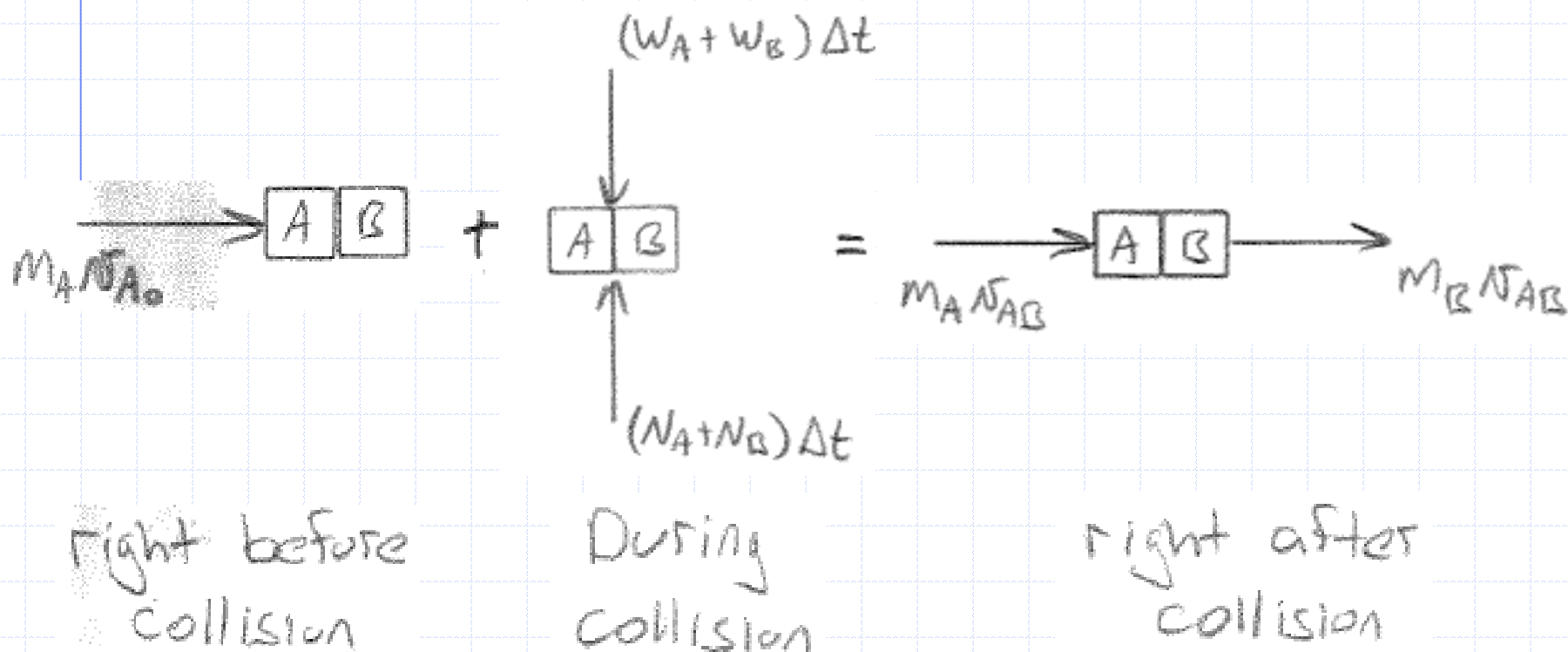
Data: $m_A = 45000 \text{ Kg}$

$$m_B = 35000 \text{ Kg}$$

$$v_{A0} = 3 \frac{\text{km}}{\text{hr}} = 0.8333 \text{ m/s}$$

Determine:

- (A) Final velocity of the coupled cars
- (B) Magnitude of Ave. impulsive force acting on each car if coupling takes 0.3s.



$$\rightarrow \oplus \sum \text{Mom: } m_A v_{A0} + 0 = (m_A + m_B) v_{AB}$$

$$\Rightarrow v_{AB} = \frac{m_A}{m_A + m_B} v_{A0}$$

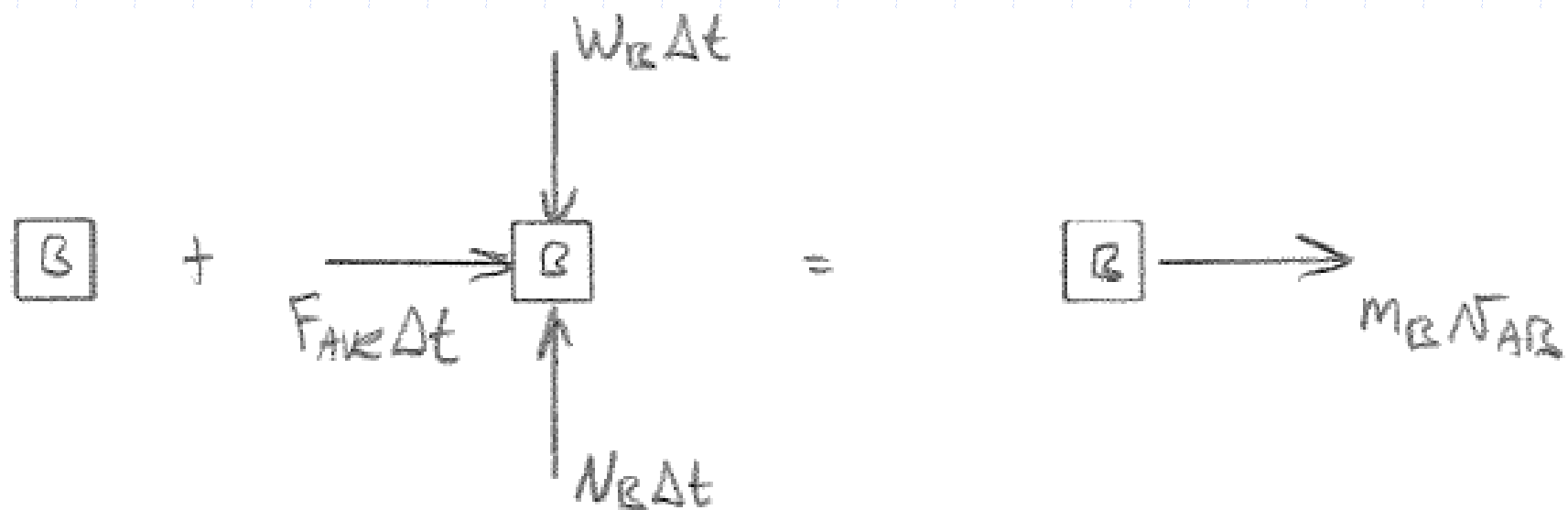
$$v_{AB} = \frac{(45000 \text{ kg})(0.8333 \text{ m/s})}{(45000 \text{ kg}) + (25000 \text{ kg})}$$

$$= 0.5357 \text{ m/s}$$

$$= 1.93 \text{ km/hr}$$

$$\therefore \bar{v}_{AB} = 1.93 \text{ km/hr} \rightarrow$$

PART B: Invoke PIM with B as system.



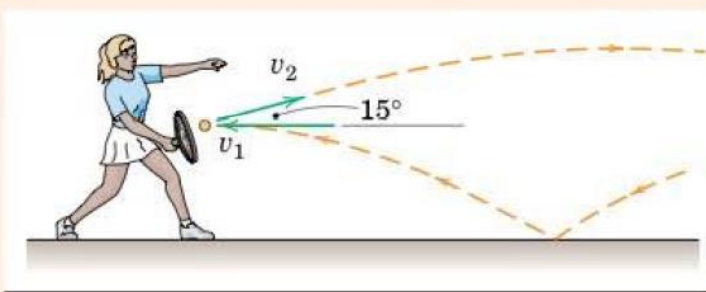
$$\rightarrow \oplus \sum \text{Mom: } 0 + F_{AKE} \Delta t = m_B \Delta v_B$$

$$\Rightarrow F_{AKE} = \frac{m_B \Delta v_B}{\Delta t}$$

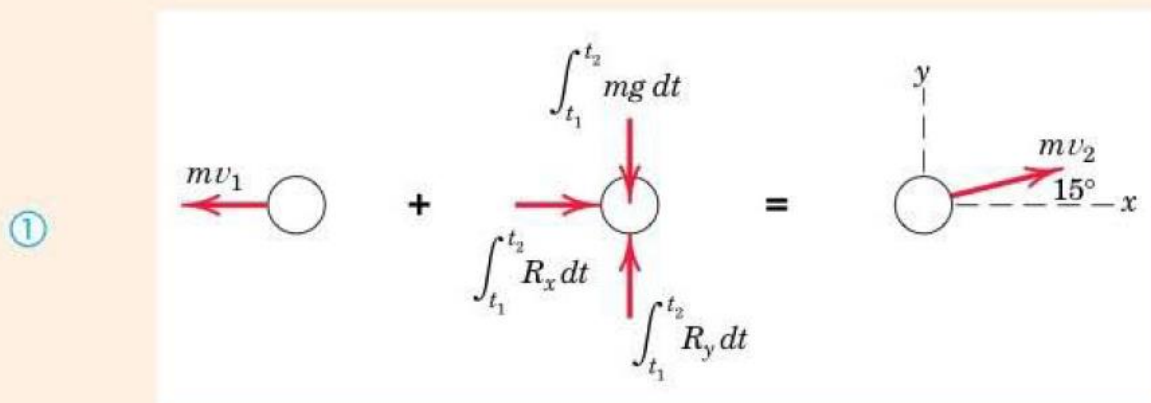
$$F_{AKE} = \frac{(25000 \text{ kg})(0.5357 \text{ m/s})}{(0.3 \text{ s})} \left(\frac{1 \text{ kN}}{1000 \text{ N}} \right) = 44.643 \text{ kN}$$

Sample Problem 3/19

A tennis player strikes the tennis ball with her racket when the ball is at the uppermost point of its trajectory as shown. The horizontal velocity of the ball just before impact with the racket is $v_1 = 50$ ft/sec and just after impact its velocity is $v_2 = 70$ ft/sec directed at the 15° angle as shown. If the 4-oz ball is in contact with the racket for 0.02 sec, determine the magnitude of the average force $\mathbf{R}$ exerted by the racket on the ball. Also determine the angle β made by $\mathbf{R}$ with the horizontal.



Solution. We construct the impulse-momentum diagrams for the ball as follows:



Helpful Hints

- ① Recall that for the impulse-momentum diagrams, initial linear momentum goes in the first diagram, all external linear impulses go in the second diagram, and final linear momentum goes in the third diagram.

$$\textcircled{2} [m(v_x)_1 + \int_{t_1}^{t_2} \Sigma F_x dt = m(v_x)_2] \quad -\frac{4/16}{32.2}(50) + R_x(0.02) = \frac{4/16}{32.2}(70 \cos 15^\circ)$$

$$[m(v_y)_1 + \int_{t_1}^{t_2} \Sigma F_y dt = m(v_y)_2]$$

$$\frac{4/16}{32.2}(0) + R_y(0.02) - (4/16)(0.02) = \frac{4/16}{32.2}(70 \sin 15^\circ)$$

We can now solve for the impact forces as

$$R_x = 45.7 \text{ lb}$$

$$R_y = 7.28 \text{ lb}$$

We note that the impact force $R_y = 7.28 \text{ lb}$ is considerably larger than the 0.25-lb weight of the ball. Thus, the weight mg , a nonimpulsive force, could have been neglected as small in comparison with R_y . Had we neglected the weight, the computed value of R_y would have been 7.03 lb.

We now determine the magnitude and direction of $\mathbf{R}$ as

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{45.7^2 + 7.28^2} = 46.2 \text{ lb} \quad \text{Ans.}$$

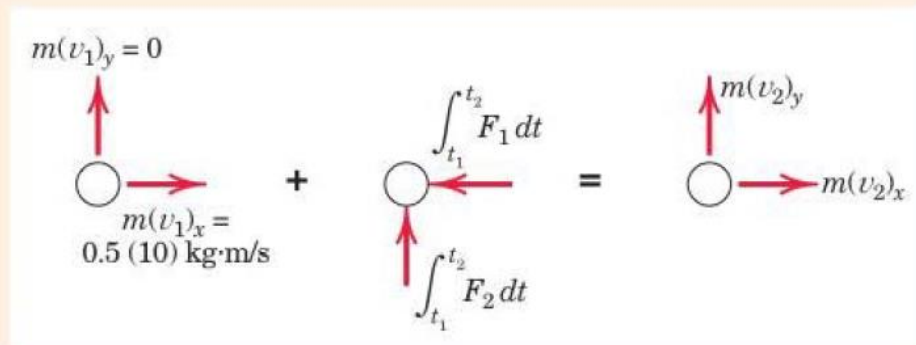
$$\beta = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{7.28}{45.7} = 9.06^\circ \quad \text{Ans.}$$

$\textcircled{2}$ For the linear impulse $\int_{t_1}^{t_2} R_x dt$, the average impact force R_x is a constant, so that it can be brought outside the integral sign, resulting in $R_x \int_{t_1}^{t_2} dt = R_x(t_2 - t_1) = R_x \Delta t$. The linear impulse in the y -direction has been similarly treated.

Sample Problem 3/21

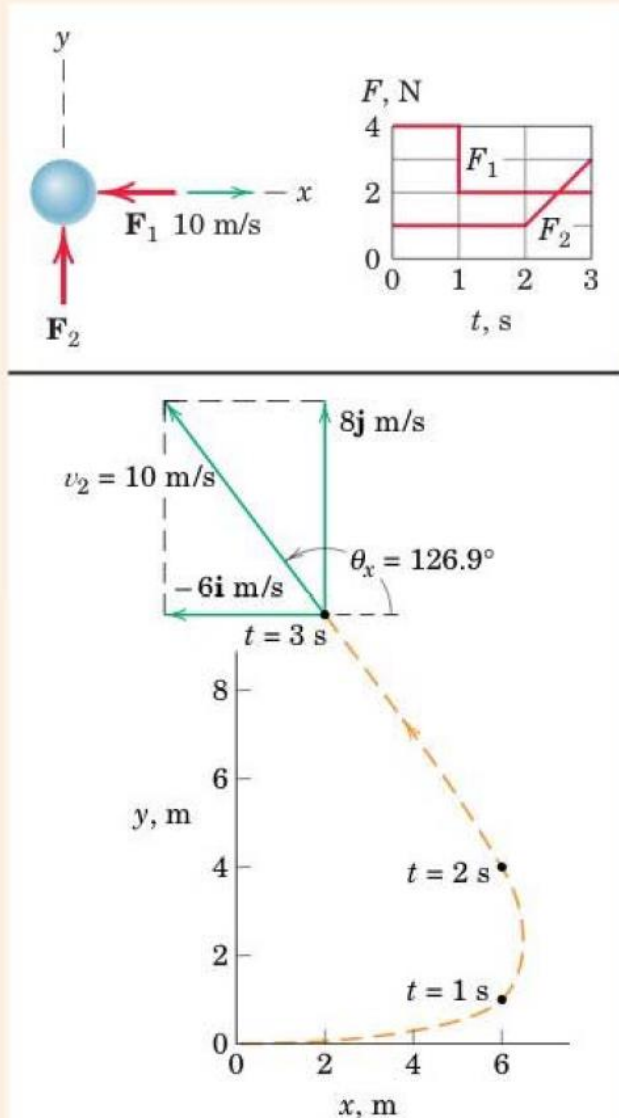
A particle with a mass of 0.5 kg has a velocity of 10 m/s in the x -direction at time $t = 0$. Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the particle, and their magnitudes change with time according to the graphical schedule shown. Determine the velocity $\mathbf{v}_2$ of the particle at the end of the 3-s interval. The motion occurs in the horizontal x - y plane.

Solution. First, we construct the impulse-momentum diagrams as shown.



Then the impulse-momentum equations follow as

$$\begin{aligned} \textcircled{1} \quad [m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x dt &= m(v_2)_x] & 0.5(10) - [4(1) + 2(3 - 1)] &= 0.5(v_2)_x \\ & & (v_2)_x &= -6 \text{ m/s} \\ [m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y dt &= m(v_2)_y] & 0.5(0) + [1(2) + 2(3 - 2)] &= 0.5(v_2)_y \\ & & (v_2)_y &= 8 \text{ m/s} \end{aligned}$$



Thus,

$$\mathbf{v}_2 = -6\mathbf{i} + 8\mathbf{j} \text{ m/s} \quad \text{and} \quad v_2 = \sqrt{6^2 + 8^2} = 10 \text{ m/s}$$

$$\theta_x = \tan^{-1} \frac{8}{-6} = 126.9^\circ$$

Ans.

Although not called for, the path of the particle for the first 3 seconds is plotted in the figure. The velocity at $t = 3$ s is shown together with its components.

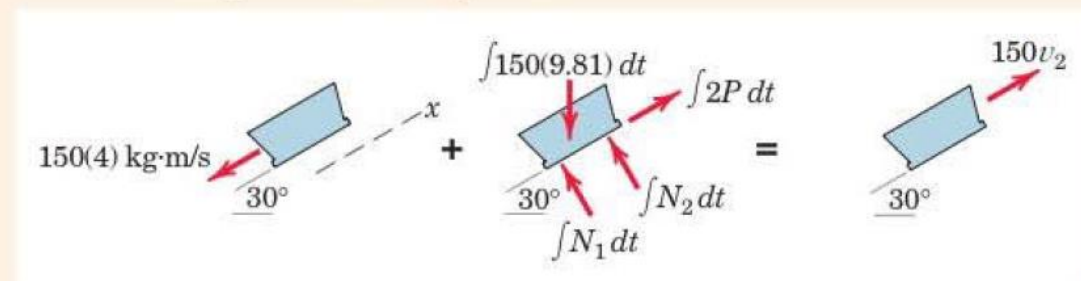
Helpful Hint

- ① The impulse in each direction is the corresponding area under the force-time graph. Note that F_1 is in the negative x -direction, so its impulse is negative.

Sample Problem 3/22

The loaded 150-kg skip is rolling down the incline at 4 m/s when a force P is applied to the cable as shown at time $t = 0$. The force P is increased uniformly with the time until it reaches 600 N at $t = 4$ s, after which time it remains constant at this value. Calculate (a) the time t' at which the skip reverses its direction and (b) the velocity v of the skip at $t = 8$ s. Treat the skip as a particle.

Solution. The stated variation of P with the time is plotted, and the impulse-momentum diagrams of the skip are drawn.



Part (a). The skip reverses direction when its velocity becomes zero. We will assume that this condition occurs at $t = 4 + \Delta t$ s. The impulse-momentum equation applied consistently in the positive x -direction gives

$$m(v_1)_x + \int \Sigma F_x dt = m(v_2)_x$$

$$\textcircled{1} \quad 150(-4) + \frac{1}{2}(4)(2)(600) + 2(600)\Delta t - 150(9.81) \sin 30^\circ(4 + \Delta t) = 150(0)$$

$$\Delta t = 2.46 \text{ s} \quad t' = 4 + 2.46 = 6.46 \text{ s} \quad \text{Ans.}$$

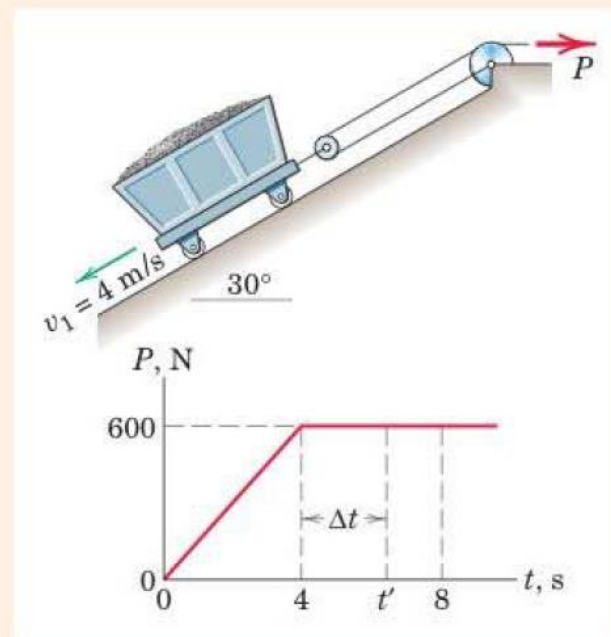
Part (b). Applying the momentum equation to the entire 8-s interval gives

$$m(v_1)_x + \int \Sigma F_x dt = m(v_2)_x$$

$$150(-4) + \frac{1}{2}(4)(2)(600) + 4(2)(600) - 150(9.81) \sin 30^\circ(8) = 150(v_2)_x$$

$$(v_2)_x = 4.76 \text{ m/s} \quad \text{Ans.}$$

The same result is obtained by analyzing the interval from t' to 8 s.



Helpful Hint

- ① The impulse-momentum diagram keeps us from making the error of using the impulse of P rather than $2P$ or of forgetting the impulse of the component of the weight. The first term in the linear impulse is the triangular area of the P - t relation for the first 4 s, doubled for the force of $2P$.

Sample Problem 3/23

The 50-g bullet traveling at 600 m/s strikes the 4-kg block centrally and is embedded within it. If the block slides on a smooth horizontal plane with a velocity of 12 m/s in the direction shown prior to impact, determine the velocity $\mathbf{v}_2$ of the block and embedded bullet immediately after impact.

Solution. Since the force of impact is internal to the system composed of the block and bullet and since there are no other external forces acting on the system in the plane of motion, it follows that the linear momentum of the system is conserved. Thus,

$$\textcircled{1} [\mathbf{G}_1 = \mathbf{G}_2] \quad 0.050(600\mathbf{j}) + 4(12)(\cos 30^\circ\mathbf{i} + \sin 30^\circ\mathbf{j}) = (4 + 0.050)\mathbf{v}_2$$

$$\mathbf{v}_2 = 10.26\mathbf{i} + 13.33\mathbf{j} \text{ m/s}$$

Ans.

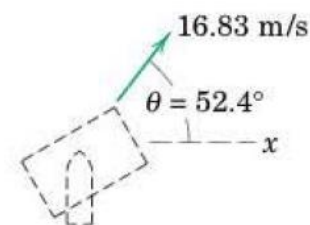
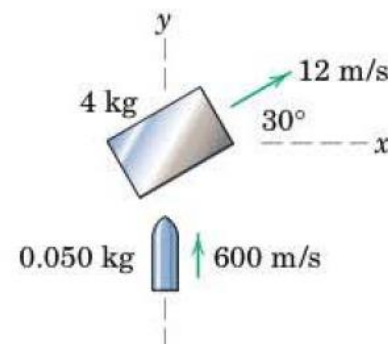
The final velocity and its direction are given by

$$[v = \sqrt{v_x^2 + v_y^2}] \quad v_2 = \sqrt{(10.26)^2 + (13.33)^2} = 16.83 \text{ m/s}$$

Ans.

$$[\tan \theta = v_y/v_x] \quad \tan \theta = \frac{13.33}{10.26} = 1.299 \quad \theta = 52.4^\circ$$

Ans.



Helpful Hint

- ① Working with the vector form of the principle of conservation of linear momentum is clearly equivalent to working with the component form.