Chapter 3 – Kinetics of Particles

1. Force-Mass-Acceleration Method
2. Work and Energy Principles
3. Impulse and Momentum Methods
4. Additional Problems

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1. Force-Mass-Acceleration Method

Kinetics of Particles

→ Relates the forces acting on a body to the corresponding motion.

Newton's Second Law of Motion

$$\bar{F} = m\bar{a}$$

$$\left(N = \frac{kg \cdot m}{s^2}, \; lb_f = \frac{slug \cdot ft}{s^2} \right)$$

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$ (a vector eqn!)

Linear momentum, denoted $\bar{L}$

Resultant of external forces acting on body

If $m = \text{const}$, $\frac{d}{dt}(m\bar{v}) = m\frac{d\bar{v}}{dt} = m\bar{a}$

$\therefore$

$$\boxed{\bar{F} = m\bar{a}}$$

Equations of Motion

Rectangular Components

$$(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) = m(a_x\hat{i} + a_y\hat{j} + a_z\hat{k})$$

Main Point:

The vector eqn $\bar{F} = m\bar{a}$ represents three independent Scalar EOM.

$$F_x = ma_x$$

$$F_y = ma_y$$

$$F_z = ma_z$$

Other Components (plane motion)

Normal/Tangential:

$$(F_t \hat{e}_t + F_n \hat{e}_n) = m\left(\frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n\right)$$

Radial/Transverse:

$$(F_r \hat{e}_r + F_\theta \hat{e}_\theta) = m\left((\cdots)\hat{e}_r + (\cdots)\hat{e}_\theta\right)$$

Sample Problem 3/1

A 75-kg man stands on a spring scale in an elevator. During the first 3 seconds of motion from rest, the tension T in the hoisting cable is 8300 N. Find the reading R of the scale in newtons during this interval and the upward velocity v of the elevator at the end of the 3 seconds. The total mass of the elevator, man, and scale is 750 kg.

Solution. The force registered by the scale and the velocity both depend on the acceleration of the elevator, which is constant during the interval for which the forces are constant. From the free-body diagram of the elevator, scale, and man taken together, the acceleration is found to be

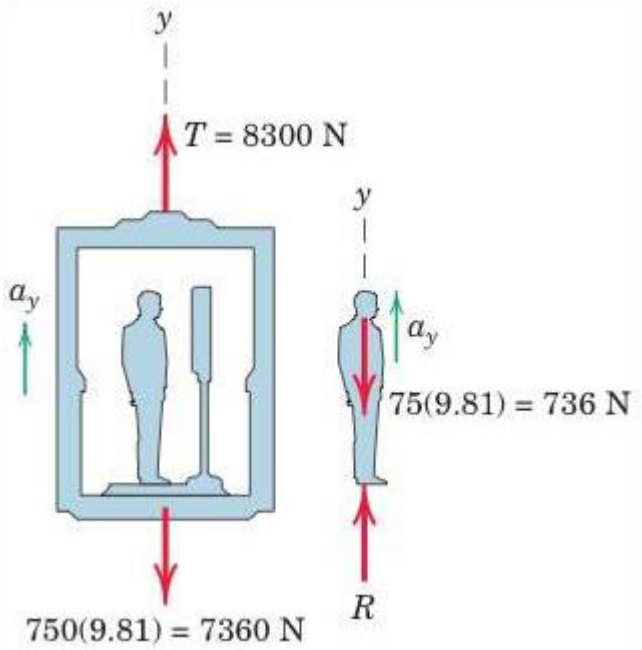
$$[\Sigma F_y = ma_y] \qquad 8300 - 7360 = 750a_y \qquad a_y = 1.257 \text{ m/s}^2$$

The scale reads the downward force exerted on it by the man's feet. The equal and opposite reaction R to this action is shown on the free-body diagram of the man alone together with his weight, and the equation of motion for him gives

$$[\Sigma F_y = ma_y] \qquad R - 736 = 75(1.257) \qquad R = 830 \text{ N} \qquad \textit{Ans.}$$

The velocity reached at the end of the 3 seconds is

$$[\Delta v = \int a\,dt] \qquad v - 0 = \int_0^3 1.257\,dt \qquad v = 3.77 \text{ m/s} \qquad \textit{Ans.}$$

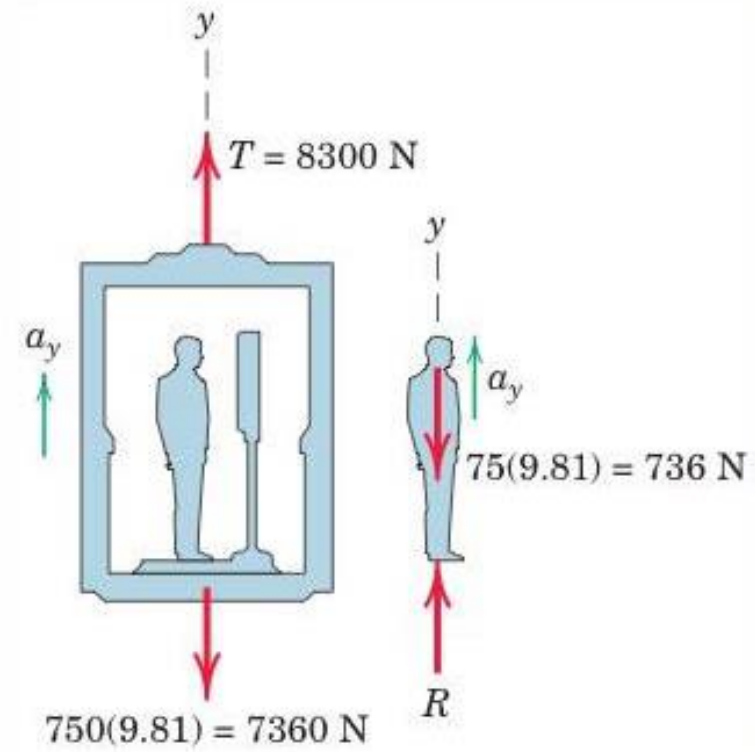


Helpful Hint

① If the scale were calibrated in kilograms it would read 830/9.81 = 84.6 kg which, of course, is not his true mass since the measurement was made in a noninertial (accelerating) system. *Suggestion:* Rework this problem in U.S. customary units.

Given

- Man's mass: $m = 75\,\text{kg}$.
- Total mass of elevator, man, and scale: $M = 750\,\text{kg}$.
- Constant cable tension: $T = 8300\,\text{N}$ during the first $3\,\text{s}$.
- The elevator starts from rest: $v_0 = 0$.
- Gravitational acceleration: $g = 9.81\,\text{m/s}^2$.



Required

① Scale reading R, in newtons, during this interval.

② Upward velocity v at $t = 3\,\text{s}$.

Strategy

Choose **upward as** $+y$ and apply Newton's second law:

$$\boxed{\sum F_y = ma_y}.$$

1. **Complete assembly:** Find a_y from the external cable tension and total weight.
2. **Man alone:** Use the same a_y to determine the supporting force R.
3. **Kinematics:** Use constant acceleration to determine the final velocity.

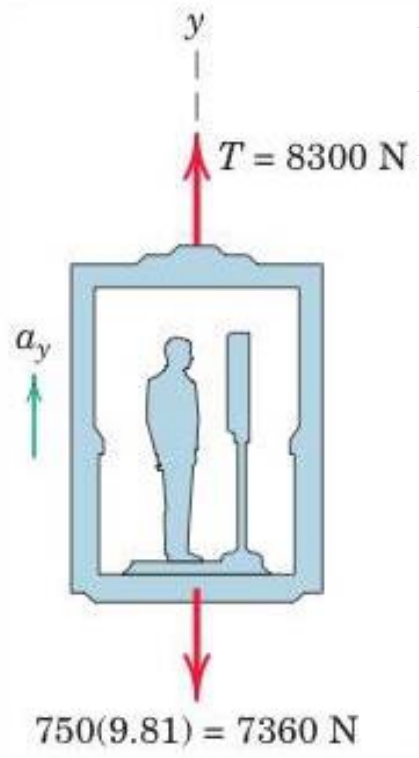
Key distinction: The scale measures the contact force, or **apparent weight**, which need not equal mg.

① Isolate the elevator, man, and scale together

The external vertical forces are:

- Cable tension T, upward.
- Total weight Mg, downward.

The contact forces between the man, scale, and elevator are **internal to this system** and do not appear on its free-body diagram.



② Apply Newton's second law

$$\sum F_y = Ma_y: \qquad \underbrace{T}_{\text{upward}} - \underbrace{Mg}_{\text{downward}} = Ma_y.$$

Substitute:

$$8300 - 750(9.81) = 750a_y$$

$$8300 - 7357.5 = 750a_y.$$

Therefore,

$$\boxed{a_y = \frac{942.5}{750} = 1.257\,\text{m/s}^2 \quad \text{upward}}$$

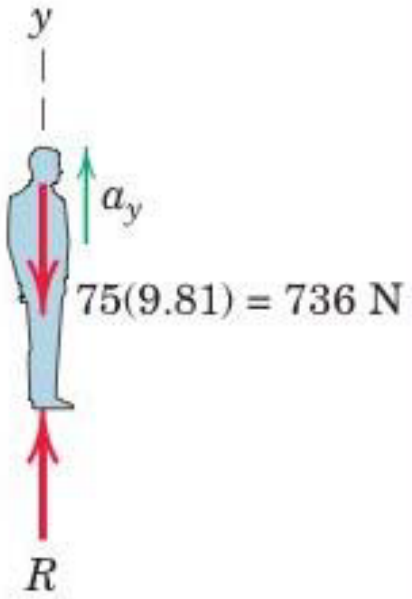
Interpretation: The cable tension exceeds the total weight, producing upward acceleration. Since both forces are constant, the acceleration is constant throughout the $3\,\text{s}$ interval.

① Draw the man's free-body diagram

The forces on the man are:

- Scale reaction R, upward.
- Weight $mg = 75(9.81) = 735.75\,\text{N}$, downward.

The man remains at rest relative to the elevator, so he has the **same upward acceleration**.



② Apply Newton's second law

$$\sum F_y = ma_y: \qquad R - mg = ma_y.$$

Rearrange:

$$R = m(g + a_y).$$

Using the unrounded acceleration,

$$R = 75\left(9.81 + \frac{942.5}{750}\right)$$

$$\boxed{R = 830\,\text{N}}.$$

By Newton's third law, the man pushes downward on the scale with this same magnitude—the **scale reading**.

③ Calculate the elevator's velocity after $3\,\text{s}$

For constant acceleration,

$$v = v_0 + a_y t.$$

$$v = 0 + \left(\frac{942.5}{750}\right)(3)$$

$$\boxed{v = 3.77\,\text{m/s} \quad \text{upward}}$$

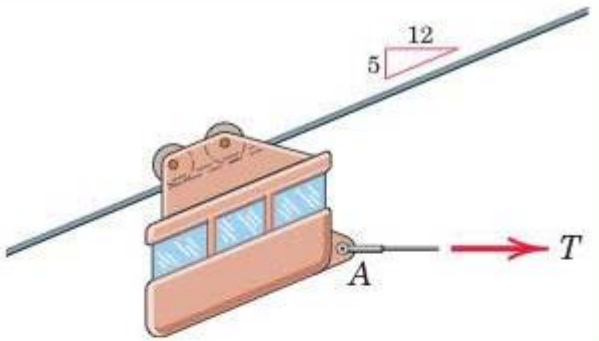
Helpful Hint

① If the scale were calibrated in kilograms it would read 830/9.81 = 84.6 kg which, of course, is not his true mass since the measurement was made in a noninertial (accelerating) system. *Suggestion:* Rework this problem in U.S. customary units.

Check: $R = 830\,\text{N} > mg = 735.75\,\text{N}$. The man's apparent weight increases because he accelerates upward; his mass remains $75\,\text{kg}$.

Sample Problem 3/2

A small inspection car with a mass of 200 kg runs along the fixed overhead cable and is controlled by the attached cable at A. Determine the acceleration of the car when the control cable is horizontal and under a tension $T = 2.4$ kN. Also find the total force P exerted by the supporting cable on the wheels.



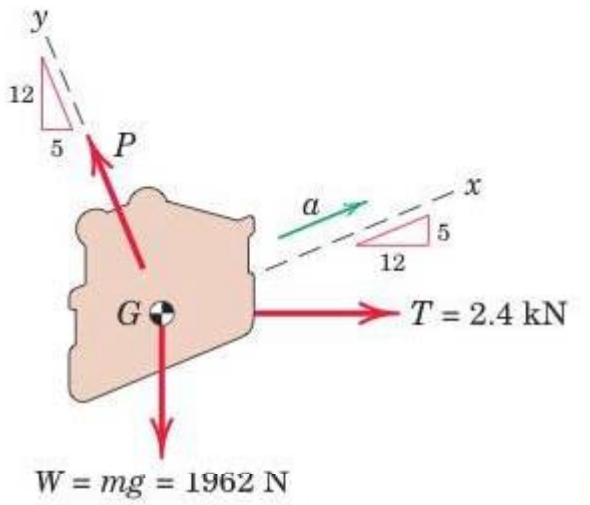
Solution. The free-body diagram of the car and wheels taken together and treated as a particle discloses the 2.4-kN tension T, the weight $W = mg = 200(9.81) = 1962$ N, and the force P exerted on the wheel assembly by the cable.

The car is in equilibrium in the y-direction since there is no acceleration in this direction. Thus,

$$[\Sigma F_y = 0] \qquad P - 2.4\left(\tfrac{5}{13}\right) - 1.962\left(\tfrac{12}{13}\right) = 0 \qquad P = 2.73 \text{ kN} \qquad \textit{Ans.}$$

① In the x-direction the equation of motion gives

$$[\Sigma F_x = ma_x] \qquad 2400\left(\tfrac{12}{13}\right) - 1962\left(\tfrac{5}{13}\right) = 200a \qquad a = 7.30 \text{ m/s}^2 \qquad \textit{Ans.}$$



Helpful Hint

① By choosing our coordinate axes along and normal to the direction of the acceleration, we are able to solve the two equations independently. Would this be so if x and y were chosen as horizontal and vertical?

Given

- Car mass: $m = 200\,\text{kg}$.
- Horizontal control-cable tension:

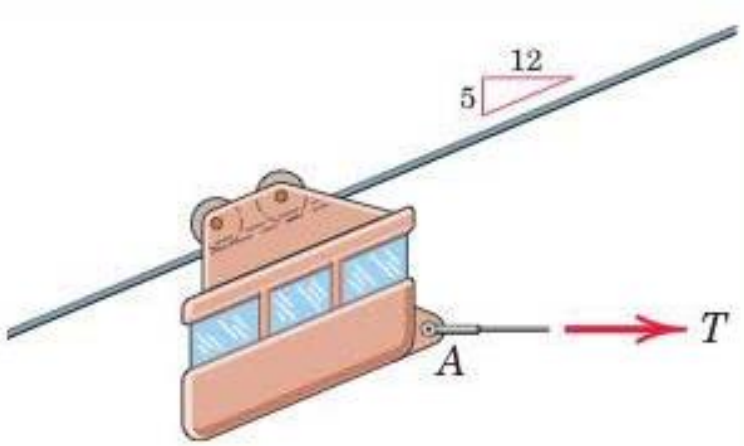
$$T = 2.4\,\text{kN} = 2400\,\text{N}.$$

- The fixed supporting cable has a slope of 5 vertical to 12 horizontal.
- Take $g = 9.81\,\text{m/s}^2$.
- Neglect resistance to rolling; the supporting force acts perpendicular to the cable.

Required

① Acceleration a of the car.

② Total force P exerted by the supporting cable on the wheels.



Strategy

Treat the car and wheels together as a particle. Choose:

- $+x$: along the supporting cable, uphill.
- $+y$: perpendicular to the cable, upward and left.

Because the car follows a **fixed straight path**,

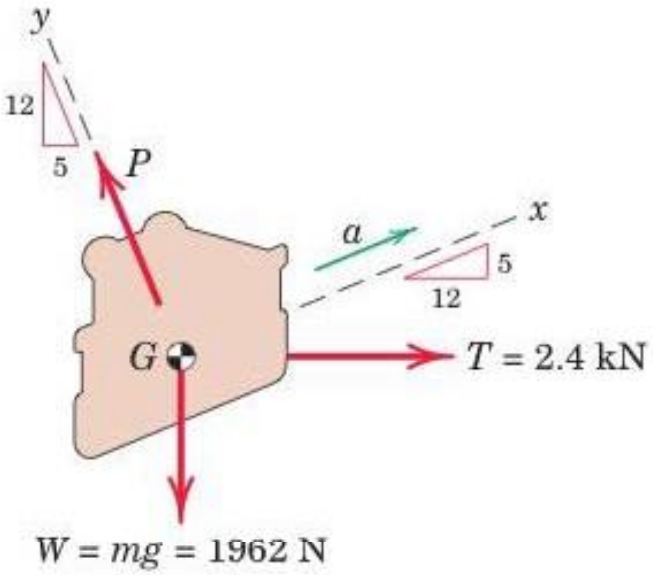
$$\boxed{a_x = a, \qquad a_y = 0}.$$

Apply Newton's second law:

$$\boxed{\sum F_y = 0 \longrightarrow P}, \qquad \boxed{\sum F_x = ma \longrightarrow a}.$$

① Identify the three external forces

- $T = 2400\,\text{N}$, horizontally right.
- $W = mg = 200(9.81) = 1962\,\text{N}$, vertically downward.
- P, perpendicular to the supporting cable, along $+y$.



② Use the slope triangle

Let α be the cable's inclination above horizontal:

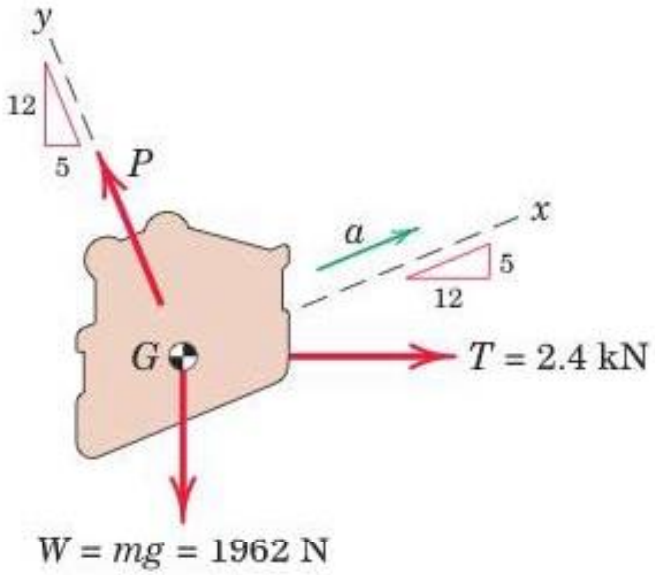
$$\sqrt{5^2 + 12^2} = 13 \quad \Rightarrow \quad \boxed{\sin\alpha = \frac{5}{13}, \qquad \cos\alpha = \frac{12}{13}}.$$

③ Resolve along the selected axes

Force	x-component: uphill positive	y-component: outward normal positive
T	$+T\cos\alpha = +2400\left(\frac{12}{13}\right)$	$-T\sin\alpha = -2400\left(\frac{5}{13}\right)$
W	$-W\sin\alpha = -1962\left(\frac{5}{13}\right)$	$-W\cos\alpha = -1962\left(\frac{12}{13}\right)$
P	0	$+P$

Interpretation: Both tension and weight have components opposite $+y$, balanced by P. Along the cable, tension acts uphill while the weight component acts downhill.

Find the supporting force and acceleration



① Perpendicular to the cable: Find P

Since $a_y = 0$,

$$\sum F_y = ma_y = 0 :$$

$$P - 2400\left(\frac{5}{13}\right) - 1962\left(\frac{12}{13}\right) = 0.$$

Therefore,

$$P = 923.1 + 1811.1 = 2734.2\,\text{N}$$

$$\boxed{P = 2.73\,\text{kN}}.$$

② Along the cable: Find a

$$\sum F_x = ma :$$

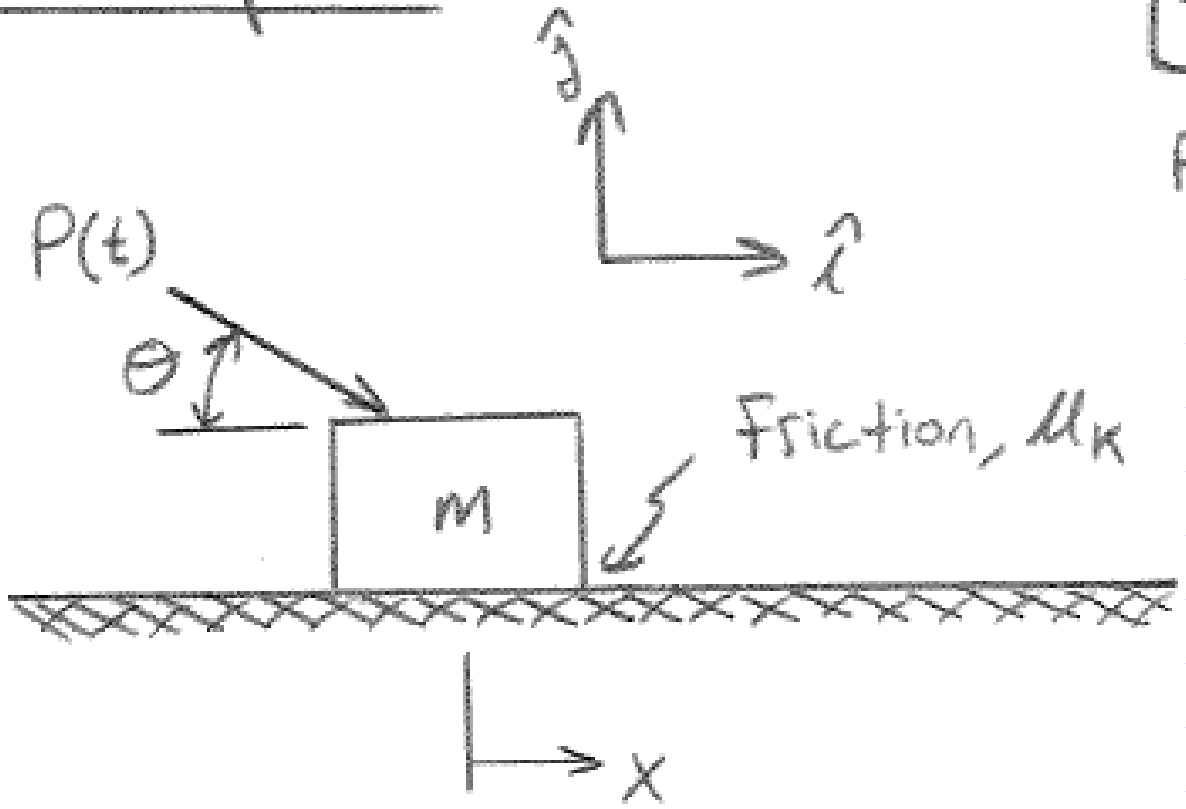
$$\underbrace{2400\left(\frac{12}{13}\right)}_{\text{uphill pull}} - \underbrace{1962\left(\frac{5}{13}\right)}_{\text{downhill weight component}} = 200a.$$

$$2215.4 - 754.6 = 200a$$

$$\boxed{a = 7.30\,\text{m/s}^2 \quad \text{uphill along the cable}}.$$

Teaching takeaway: The car has **zero acceleration perpendicular to the cable**, but it is not in overall equilibrium. Choosing axes along and perpendicular to its path lets us solve for P and a independently.

Example:



Data:

$P(t) = P_o(1 + \gamma t)$

$P_o = 500\ N$

$m = 80\ kg$

$\theta = 30°$

$\mu_K = 0.25$

$\gamma = 0.1\ s^{-1}$

m is initially at rest when P(t) is applied @ t=0.

Find: (A) the equations of motion

(B) $a = a(t)$

(C) dist. traveled in $t = 2s$

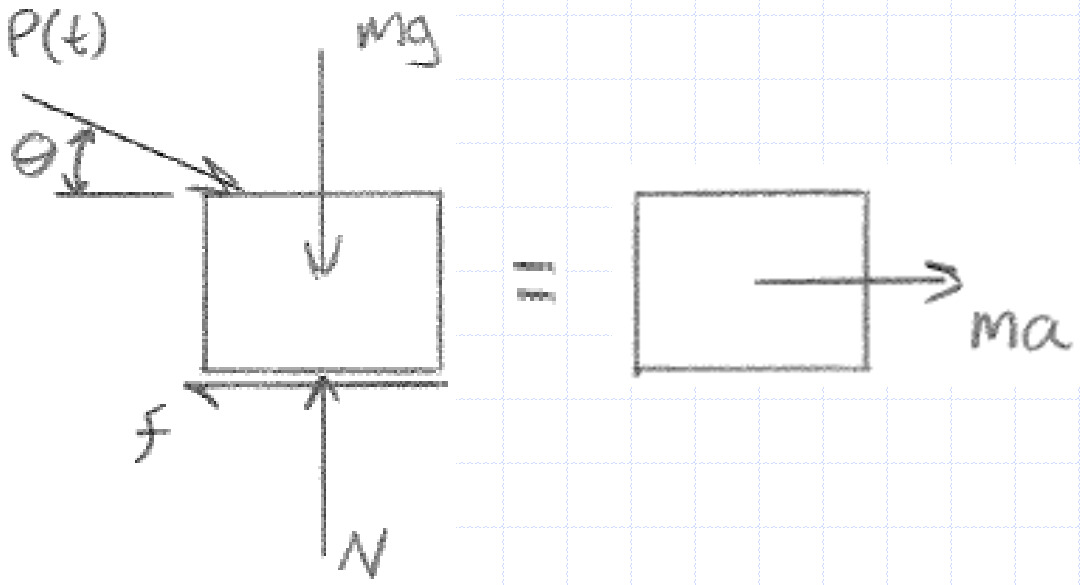
Assume motion at $t = 0$

i.e. $P_0 \cos\theta$ is enough to overcome static Friction

PART A: Find EOM.

- Force Diagram, "diagrammatical" approach

FBD:



$$\sum \begin{pmatrix} \text{external} \\ \text{forces} \end{pmatrix} = \text{Mass} \times \text{acc.}$$

$\xrightarrow{+} \sum F: \quad P(t)\cos\theta - f = ma \qquad -- \quad (*)$

$+\uparrow \sum F: \quad -mg - P(t)\sin\theta + N = 0 \qquad -- \quad (**)$

PART B: Find $a = a(t)$

unknowns in (*), (**): f, a, N

2 eqns, 3 unknowns $\Longrightarrow$ need one more independent eqn.

Friction Law: $f = \mu_K N \qquad -- \quad (***)$

(**) → (***) to eliminate N.

Plug into (*).

$$P(t)\cos\theta - \mu_K\left(mg + P(t)\sin\theta\right) = ma$$

$$\Rightarrow \quad ma = P(t)\left[\cos\theta - \mu_K\sin\theta\right] - \mu_K mg$$

$$P_0(1+\gamma t)$$

$$\Rightarrow$$

$$a(t) = \underbrace{\frac{P_0(\cos\theta - \mu_K\sin\theta) - \mu_K mg}{m}}_{\text{call } a_0} + \underbrace{\frac{P_0\gamma(\cos\theta - \mu_K\sin\theta)}{m}}_{\text{call } a_1} t$$

$$\therefore \quad a(t) = a_0 + a_1 t$$

Where

$$a_0 = \frac{(500N)[\cos(30^\circ) - (0.25)\sin(30^\circ)] - (0.25)(80Kg)(9.81 m/s^2)}{(80Kg)}$$

$$= 2.1789 \ m/s^2$$

$$a_1 = \frac{(500N)(0.15^{-1})[\cos(30^\circ) - (0.25)\sin(30^\circ)]}{(80Kg)}$$

$$= 0.4631 \ m/s^3$$

PART C: Find $X(t=2s)$

Invoke $a(t) = \frac{dv}{dt}$

$\Rightarrow \quad dv = (a_0 + a_1 t)\,dt$

$\Rightarrow \quad v(t) - \cancel{v_0}^{\,0} = a_0 t + a_1 \frac{t^2}{2}$

But $v(t) = \frac{dx}{dt}$ $\Rightarrow \quad dx = (a_0 t + \frac{1}{2} a_1 t^2)\,dt$

$\Rightarrow \quad x(t) - \cancel{x_0}^{\,0} = a_0 \frac{t^2}{2} + \frac{1}{2} a_1 \frac{t^3}{3}$

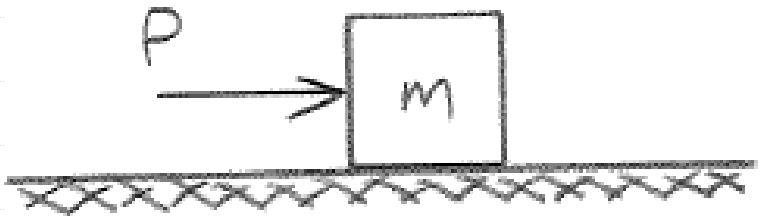
$\Rightarrow \quad x(t) = \frac{a_0}{2} t^2 + \frac{a_1}{6} t^3$

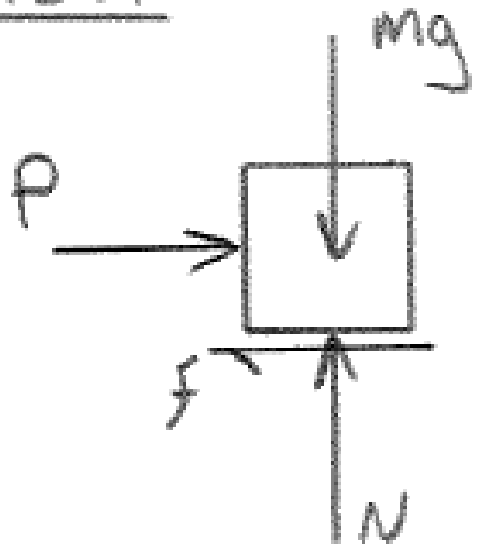
Thus

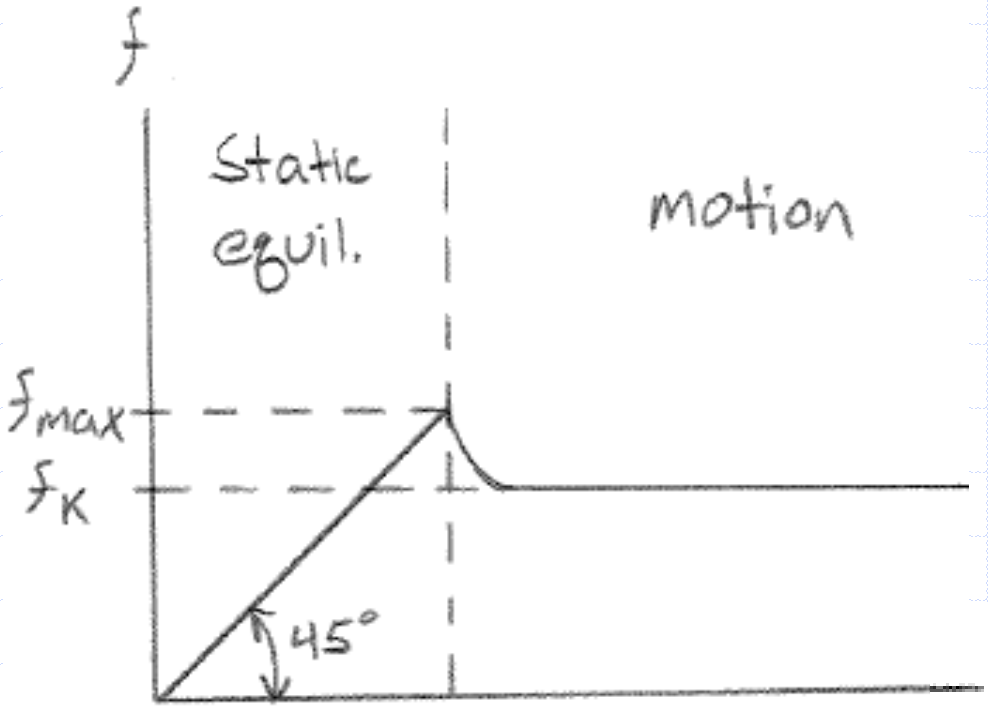
$$x(t=2s) = \frac{(2.1789\ m/s^2)}{2}(2s)^2 + \frac{(0.4631\ m/s^3)}{6}(2s)^3$$

$$\therefore \quad x(t=2s) = 4.98\ m$$

Review of Dry Friction







$$f_{max} = \mu_s N$$

$$f_K = \mu_K N \quad \text{(motion)}$$

Experimental evidence shows

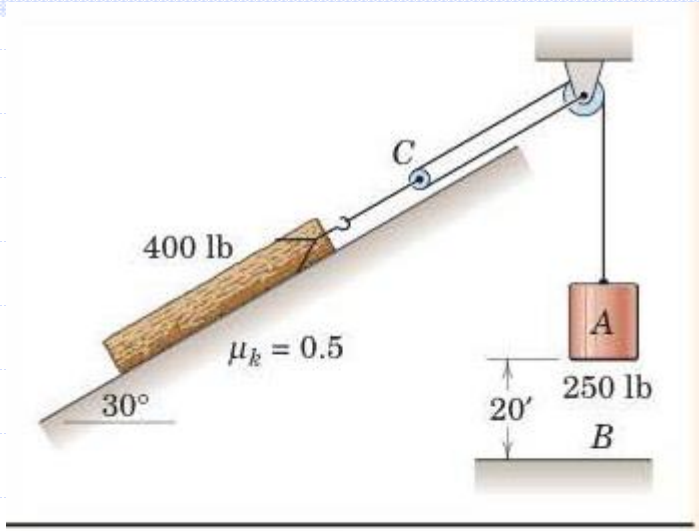
$f_{max} = \mu_s N$ (impending motion)

$f_K = \mu_K N$ (motion)

Always, $\mu_K < \mu_s$

Sample Problem 3/3

The 250-lb concrete block A is released from rest in the position shown and pulls the 400-lb log up the 30° ramp. If the coefficient of kinetic friction between the log and the ramp is 0.5, determine the velocity of the block as it hits the ground at B.



Solution. The motions of the log and the block A are clearly dependent. Although by now it should be evident that the acceleration of the log up the incline is half the downward acceleration of A, we may prove it formally. The constant total length of the cable is $L = 2s_C + s_A + \text{constant}$, where the constant accounts for the cable portions wrapped around the pulleys. Differentiating twice with respect to time gives $0 = 2\ddot{s}_C + \ddot{s}_A$, or ①

$$0 = 2a_C + a_A$$

We assume here that the masses of the pulleys are negligible and that they turn with negligible friction. With these assumptions the free-body diagram of the pulley C discloses force and moment equilibrium. Thus, the tension in the cable attached to the log is twice that applied to the block. Note that the accelerations of the log and the center of pulley C are identical.

The free-body diagram of the log shows the friction force $\mu_k N$ for motion up the plane. Equilibrium of the log in the y-direction gives

② $[\Sigma F_y = 0]$ $\qquad N - 400 \cos 30° = 0 \qquad N = 346 \text{ lb}$

and its equation of motion in the x-direction gives

$$[\Sigma F_x = ma_x] \qquad 0.5(346) - 2T + 400 \sin 30° = \frac{400}{32.2} a_C$$

For the block in the positive downward direction, we have

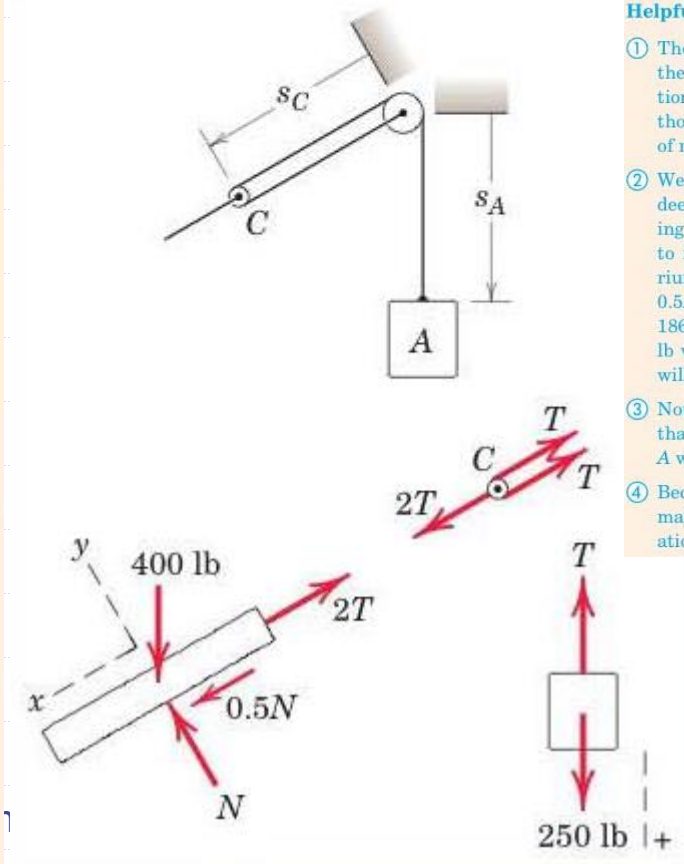
③ $[+\downarrow \Sigma F = ma]$ $\qquad 250 - T = \frac{250}{32.2} a_A$

Solving the three equations in a_C, a_A, and T gives us

$$a_A = 5.83 \text{ ft/sec}^2 \qquad a_C = -2.92 \text{ ft/sec}^2 \qquad T = 205 \text{ lb}$$

④ For the 20-ft drop with constant acceleration, the block acquires a velocity

$$[v^2 = 2ax] \qquad v_A = \sqrt{2(5.83)(20)} = 15.27 \text{ ft/sec} \qquad \textit{Ans.}$$



Helpful Hints

① The coordinates used in expressing the final kinematic constraint relationship must be consistent with those used for the kinetic equations of motion.

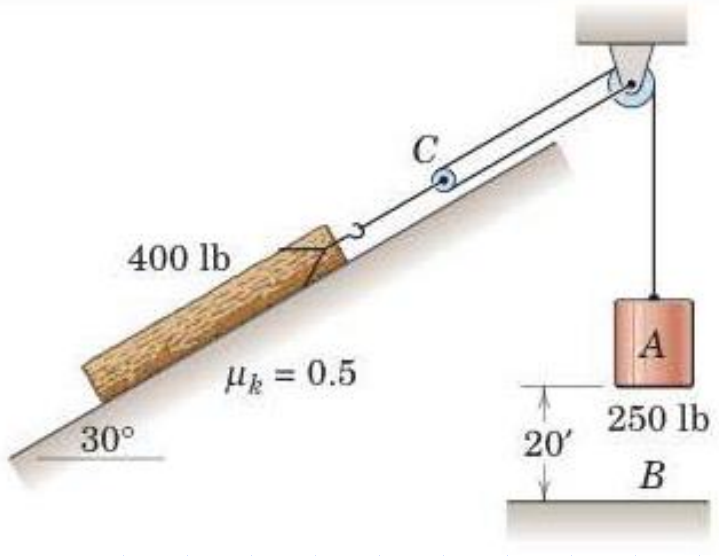
② We can verify that the log will indeed move up the ramp by calculating the force in the cable necessary to initiate motion from the equilibrium condition. This force is $2T = 0.5N + 400 \sin 30° = 373$ lb or $T = 186.5$ lb, which is less than the 250-lb weight of block A. Hence, the log will move up.

③ Note the serious error in assuming that $T = 250$ lb, in which case, block A would not accelerate.

④ Because the forces on this system remain constant, the resulting accelerations also remain constant.

Given

- Block weight: $W_A = 250\,\text{lb}$.
- Log weight: $W_L = 400\,\text{lb}$.
- Ramp inclination: 30°.
- Coefficient of kinetic friction: $\mu_k = 0.5$.
- Block A starts from rest and drops $20\,\text{ft}$.
- Assume massless, frictionless pulleys and an inextensible cable.
- Use $g = 32.2\,\text{ft/s}^2$.



Required

Find the velocity of block A immediately before it reaches the ground at B.

Strategy

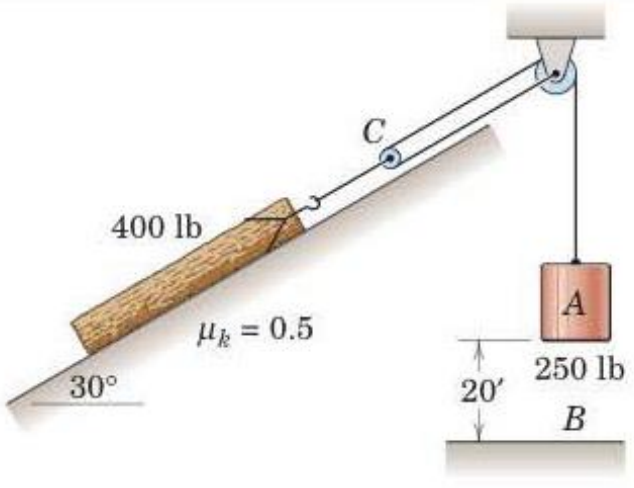
① Use constant cable length to relate the accelerations.
② Analyze pulley C to relate the cable forces.
③ Apply Newton's second law separately to the log and block.
④ Use constant-acceleration kinematics over the $20\,\text{ft}$ drop.

Units reminder

The given values are **weights**, so the masses are

$$\boxed{m_A = \frac{250}{32.2}, \qquad m_L = \frac{400}{32.2} \quad \text{slugs}}.$$

Relate the lengths, velocities, and accelerations



① Define coordinates consistent with the cable geometry

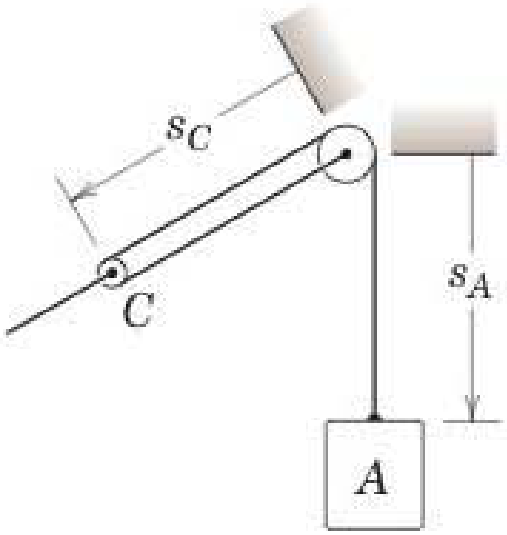
- s_A: measured vertically **downward** from a fixed reference.
- s_C: measured **down the ramp** from a fixed reference to pulley C.

The log and pulley C move together.

② Write the constant cable length

Two inclined cable portions change with s_C; one vertical portion changes with s_A:

$$L = \underbrace{2s_C}_{\text{two inclined portions}} + \underbrace{s_A}_{\text{vertical portion}} + K,$$



where K contains the constant lengths.

③ Differentiate with respect to time

Since L and K are constant,

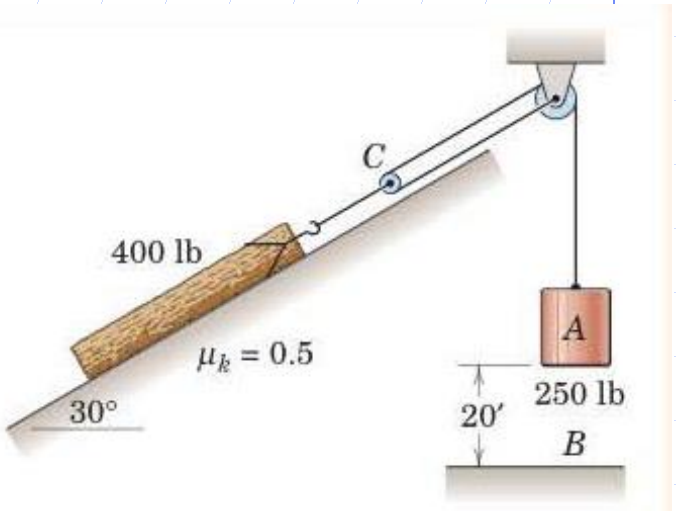
$$0 = 2\dot{s}_C + \dot{s}_A \quad \Rightarrow \quad \boxed{v_C = -\frac{v_A}{2}}.$$

Differentiate again:

$$0 = 2a_C + a_A \quad \Rightarrow \quad \boxed{a_C = -\frac{a_A}{2}}.$$

Interpretation: When block A accelerates downward, the log accelerates **up the ramp at half that magnitude**. Its acceleration a_C is negative because our positive direction is down the ramp.

Pulley C and log: Determine the force equation



① Analyze the ideal moving pulley

The main cable has tension T on both sides of pulley C. Its two parallel portions pull uphill.

For the negligible-mass pulley,

$$T + T - F_{\text{link}} = 0 \quad \Rightarrow \quad \boxed{F_{\text{link}} = 2T}.$$

Thus, the pulley pulls the log **uphill with force** $2T$.

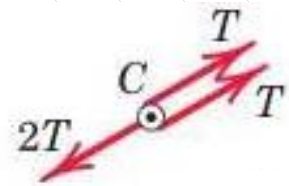
② Find the normal force and kinetic friction

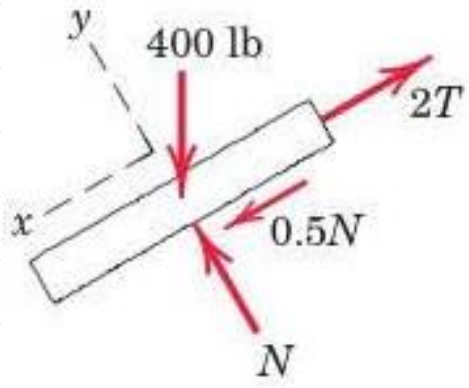
Perpendicular to the straight ramp, acceleration is zero:

$$N - 400\cos 30^\circ = 0 \quad \Rightarrow \quad \boxed{N = 346.4\,\text{lb}}.$$

Since the log slides uphill, friction acts downhill:

$$\boxed{F_k = \mu_k N = 0.5(346.4) = 173.2\,\text{lb}}.$$





③ Apply Newton's second law along the ramp

Using **downhill as positive**, consistent with s_C:

$$\underbrace{400\sin 30^\circ}_{\text{weight component}} + \underbrace{173.2}_{\text{friction}} - \underbrace{2T}_{\text{uphill pull}} = \frac{400}{32.2}a_C.$$

Substitute $a_C = -a_A/2$:

$$373.2 - 2T = -\frac{200}{32.2}a_A.$$

Rearrange:

$$\boxed{2T - 373.2 = \frac{200}{32.2}a_A}.$$

Block A: Acceleration and final velocity

① Apply Newton's second law to block A

Taking downward as positive,

$$\underbrace{250}_{\text{weight}} - \underbrace{T}_{\text{upward tension}} = \frac{250}{32.2} a_A.$$

Hence,

$$T = 250 - \frac{250}{32.2} a_A.$$

② Substitute into the log equation

$$2\left(250 - \frac{250}{32.2} a_A\right) - 373.2 = \frac{200}{32.2} a_A.$$

Collect terms:

$$500 - 373.2 = \frac{700}{32.2} a_A.$$

$$\boxed{a_A = 5.83\,\text{ft/s}^2 \quad \text{downward}}$$

Consequently,

$$\boxed{a_C = -2.92\,\text{ft/s}^2, \qquad T \approx 205\,\text{lb}}.$$

③ Find the velocity after the 20 ft drop

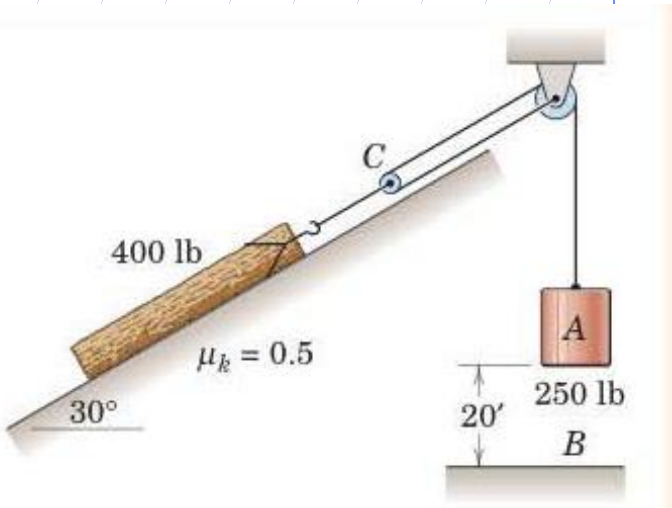
The forces—and therefore the accelerations—remain constant during the motion:

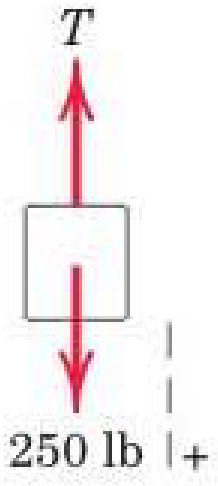
$$v_A^2 = v_{A0}^2 + 2a_A \Delta s_A.$$

Starting from rest,

$$v_A = \sqrt{0 + 2(5.83)(20)}$$

$$\boxed{v_A = 15.3\,\text{ft/s} \quad \text{downward}}.$$

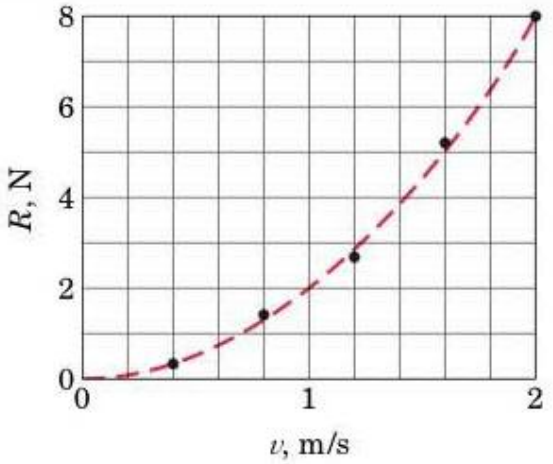


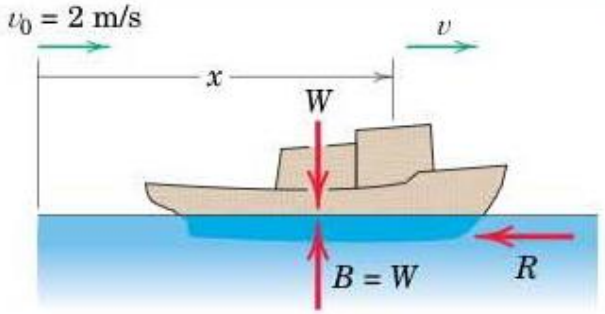


Check: $T < 250\,\text{lb}$, allowing the block to accelerate downward, while $2T > 373.2\,\text{lb}$, allowing the log to accelerate uphill.

Sample Problem 3/4

The design model for a new ship has a mass of 10 kg and is tested in an experimental towing tank to determine its resistance to motion through the water at various speeds. The test results are plotted on the accompanying graph, and the resistance R may be closely approximated by the dashed parabolic curve shown. If the model is released when it has a speed of 2 m/s, determine the time t required for it to reduce its speed to 1 m/s and the corresponding travel distance x.





Solution. We approximate the resistance-velocity relation by $R = kv^2$ and find k by substituting $R = 8$ N and $v = 2$ m/s into the equation, which gives $k = 8/2^2 = 2\ \text{N}\cdot\text{s}^2/\text{m}^2$. Thus, $R = 2v^2$.

The only horizontal force on the model is R, so that

① $[\Sigma F_x = ma_x] \qquad -R = ma_x \qquad \text{or} \qquad -2v^2 = 10\,\dfrac{dv}{dt}$

We separate the variables and integrate to obtain

$$\int_0^t dt = -5\int_2^v \frac{dv}{v^2} \qquad t = 5\left(\frac{1}{v} - \frac{1}{2}\right)\text{s}$$

Thus, when $v = v_0/2 = 1$ m/s, the time is $t = 5(\frac{1}{1} - \frac{1}{2}) = 2.5$ s. *Ans.*

The distance traveled during the 2.5 seconds is obtained by integrating $v = dx/dt$. Thus, $v = 10/(5 + 2t)$ so that

② $$\int_0^x dx = \int_0^{2.5} \frac{10}{5+2t}\,dt \qquad x = \frac{10}{2}\ln(5+2t)\Big|_0^{2.5} = 3.47\ \text{m} \qquad \textit{Ans.}$$

Helpful Hints

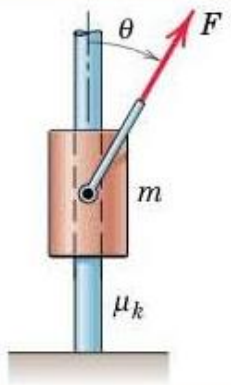
① Be careful to observe the minus sign for R.

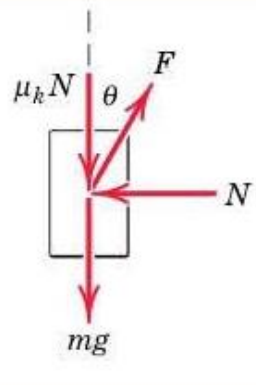
② *Suggestion:* Express the distance x after release in terms of the velocity v and see if you agree with the resulting relation $x = 5\ln(v_0/v)$.

(2D – Advanced)

Sample Problem 3/5

The collar of mass m slides up the vertical shaft under the action of a force F of constant magnitude but variable direction. If $\theta = kt$ where k is a constant and if the collar starts from rest with $\theta = 0$, determine the magnitude F of the force which will result in the collar coming to rest as θ reaches $\pi/2$. The coefficient of kinetic friction between the collar and shaft is μ_k.





Solution. After drawing the free-body diagram, we apply the equation of motion in the y-direction to get

① $[\Sigma F_y = ma_y]$ $$F\cos\theta - \mu_k N - mg = m\frac{dv}{dt}$$

where equilibrium in the horizontal direction requires $N = F\sin\theta$. Substituting $\theta = kt$ and integrating first between general limits give

$$\int_0^t (F\cos kt - \mu_k F\sin kt - mg)\,dt = m\int_0^v dv$$

which becomes

$$\frac{F}{k}[\sin kt + \mu_k(\cos kt - 1)] - mgt = mv$$

For $\theta = \pi/2$ the time becomes $t = \pi/2k$, and $v = 0$ so that

② $$\frac{F}{k}[1 + \mu_k(0 - 1)] - \frac{mg\pi}{2k} = 0 \qquad \text{and} \qquad F = \frac{mg\pi}{2(1 - \mu_k)} \qquad \textit{Ans.}$$

Helpful Hints

① If θ were expressed as a function of the vertical displacement y instead of the time t, the acceleration would become a function of the displacement and we would use $v\,dv = a\,dy$.

② We see that the results do not depend on k, the rate at which the force changes direction.

Kinetics – Curvilinear Motion

Rectangular coordinates

$$\Sigma F_x = ma_x$$
$$\Sigma F_y = ma_y$$

where $a_x = \ddot{x}$ and $a_y = \ddot{y}$

Normal and tangential coordinates

$$\Sigma F_n = ma_n$$
$$\Sigma F_t = ma_t$$

where $a_n = \rho\dot{\beta}^2 = v^2/\rho = v\dot{\beta}$, $a_t = \dot{v}$, and $v = \rho\dot{\beta}$

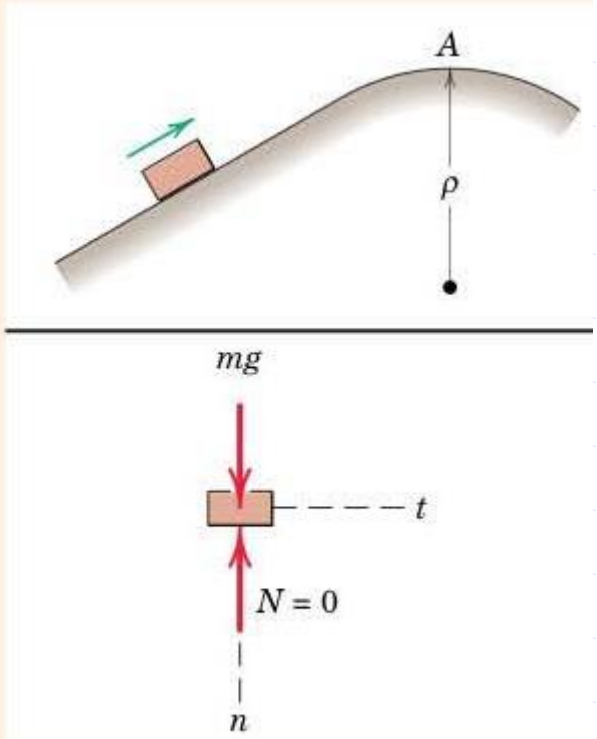
Polar coordinates

$$\Sigma F_r = ma_r$$
$$\Sigma F_\theta = ma_\theta$$

where $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Sample Problem 3/6

Determine the maximum speed v which the sliding block may have as it passes point A without losing contact with the surface.



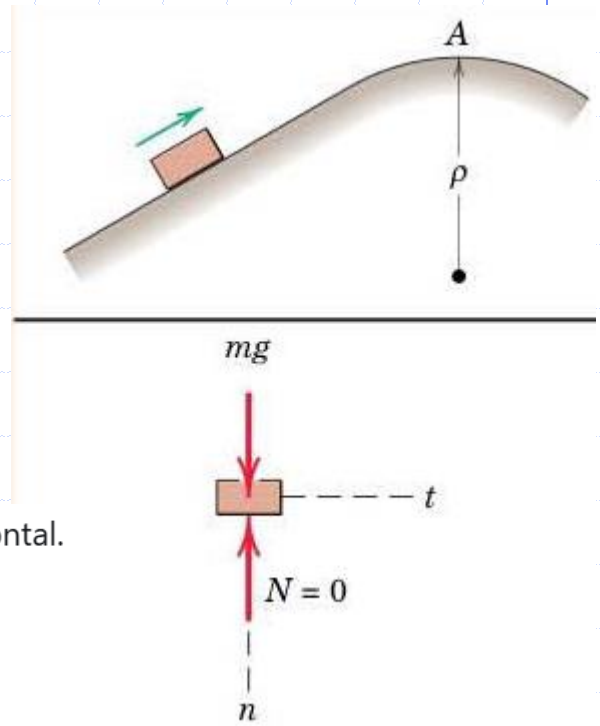
Solution. The condition for loss of contact is that the normal force N which the surface exerts on the block goes to zero. Summing forces in the normal direction gives

$$[\Sigma F_n = ma_n] \qquad mg = m\frac{v^2}{\rho} \qquad v = \sqrt{g\rho} \qquad \textit{Ans.}$$

If the speed at A were less than $\sqrt{g\rho}$, then an upward normal force exerted by the surface on the block would exist. In order for the block to have a speed at A which is greater than $\sqrt{g\rho}$, some type of constraint, such as a second curved surface above the block, would have to be introduced to provide additional downward force.

Given

- A block slides over the crest at A.
- The radius of curvature at A is ρ.
- Gravitational acceleration is g.
- The surface can push on the block but cannot pull it toward the surface.



Required

Find the maximum speed $v_{\max}$ at A without losing contact.

Strategy

① Draw the free-body diagram at A:

- Weight mg, downward.
- Contact force N, upward.

② Choose $+n$ **downward toward the center of curvature**; the tangent at A is horizontal.

③ Apply Newton's second law in the normal direction:

$$\boxed{\sum F_n = ma_n, \qquad a_n = \frac{v^2}{\rho}}.$$

④ At the limiting condition for maintaining contact,

$$\boxed{N = 0}.$$

Only the normal equation is needed to determine the limiting speed.

① Write the normal equation while contact is maintained

Taking downward as positive,

$$\underbrace{mg}_{\text{toward the center}} - \underbrace{N}_{\text{away from the center}} = m\frac{v^2}{\rho}.$$

Rearrange:

$$\boxed{N = m\left(g - \frac{v^2}{\rho}\right)}.$$

As speed increases, the required inward acceleration increases, and the contact force decreases.

② Apply the limiting condition $N = 0$

$$mg = m\frac{v_{\max}^2}{\rho}.$$

Cancel m and solve:

$$\boxed{v_{\max} = \sqrt{g\rho}}.$$

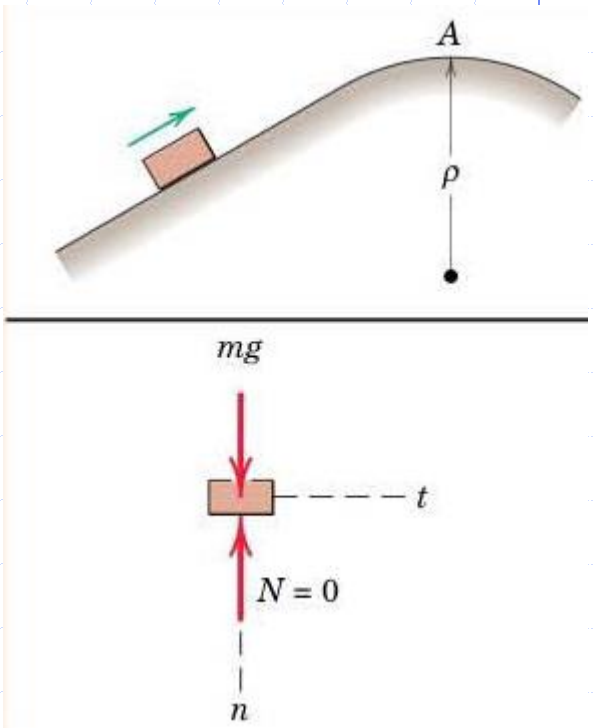
③ Interpret the result

Speed at A	Contact condition
$v < \sqrt{g\rho}$	$N > 0$: contact is maintained.
$v = \sqrt{g\rho}$	$N = 0$: limiting contact condition.
$v > \sqrt{g\rho}$	Following the surface would require $N < 0$, so the block loses contact.

Key takeaway: At the limiting speed, the contact force is zero, but the normal acceleration is **not**:

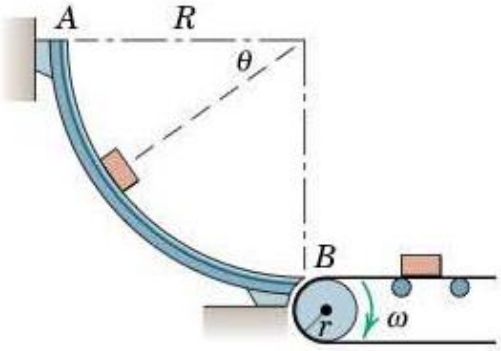
$$\boxed{a_n = \frac{v_{\max}^2}{\rho} = g}.$$

Gravity alone provides the required inward acceleration.

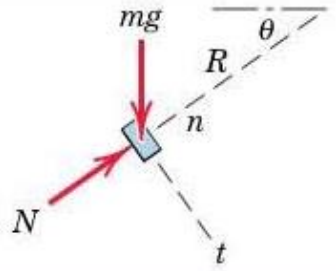


Sample Problem 3/7

Small objects are released from rest at A and slide down the smooth circular surface of radius R to a conveyor B. Determine the expression for the normal contact force N between the guide and each object in terms of θ and specify the correct angular velocity ω of the conveyor pulley of radius r to prevent any sliding on the belt as the objects transfer to the conveyor.



Solution. The free-body diagram of the object is shown together with the coordinate directions n and t. The normal force N depends on the n-component of the acceleration which, in turn, depends on the velocity. The velocity will be cumulative according to the tangential acceleration a_t. Hence, we will find a_t first for any general position.



$$[\Sigma F_t = ma_t] \qquad mg\cos\theta = ma_t \qquad a_t = g\cos\theta$$

① Now we can find the velocity by integrating

$$[v\,dv = a_t\,ds] \qquad \int_0^v v\,dv = \int_0^\theta g\cos\theta\,d(R\theta) \qquad v^2 = 2gR\sin\theta$$

We obtain the normal force by summing forces in the positive n-direction, which is the direction of the n-component of acceleration.

$$[\Sigma F_n = ma_n] \qquad N - mg\sin\theta = m\frac{v^2}{R} \qquad N = 3mg\sin\theta \qquad \textit{Ans.}$$

The conveyor pulley must turn at the rate $v = r\omega$ for $\theta = \pi/2$, so that

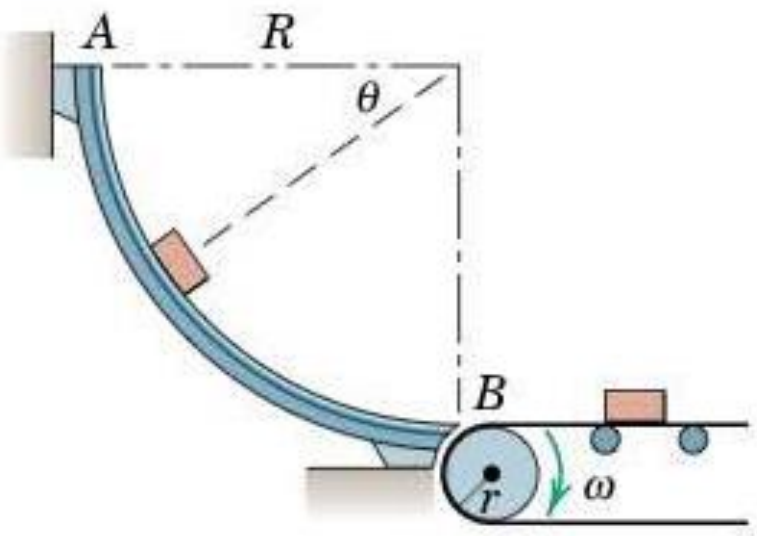
$$\omega = \sqrt{2gR}/r \qquad \textit{Ans.}$$

Helpful Hint

① It is essential here that we recognize the need to express the tangential acceleration as a function of position so that v may be found by integrating the kinematical relation $v\,dv = a_t\,ds$, in which all quantities are measured along the path.

Given

- Objects start from rest at A: $v_A = 0$.
- The circular guide is smooth: **no friction**.
- Guide radius: R.
- Conveyor-pulley radius: r.
- Let each object have mass m.
- The angular position increases from $\theta = 0$ at A to $\theta = \pi/2$ at B.



Required

① Normal contact force N as a function of θ.

② Conveyor-pulley angular velocity ω for transfer without sliding.

Strategy

Choose $+t$ down the guide and $+n$ toward its center of curvature.

1. Apply $\sum F_t = ma_t$, then integrate to find $v(\theta)$.
2. Apply $\sum F_n = mv^2/R$ to find $N(\theta)$.
3. Match the belt speed to the object's speed at B:

$$\boxed{r\omega = v_B}.$$

Free-body diagram: Only two forces act on the object—weight mg downward and contact force N toward the center.

Tangential equation: Find the speed

① Apply Newton's second law along the tangent

The contact force has no tangential component:

$$\sum F_t = ma_t: \qquad mg\cos\theta = ma_t.$$

$$\boxed{a_t = g\cos\theta}$$

This acceleration varies with position, so constant-acceleration formulas do not apply.

② Relate acceleration to speed and distance

Using the chain rule,

$$a_t = \frac{dv}{dt} = \frac{dv}{ds}\underbrace{\frac{ds}{dt}}_{v} \quad \Rightarrow \quad v\,dv = a_t\,ds.$$

Along the circular guide,

$$s = R\theta \quad \Rightarrow \quad ds = R\,d\theta \qquad (\theta \text{ in radians}).$$

Therefore,

$$v\,dv = gR\cos\theta\,d\theta.$$

③ Integrate from rest at A to a general position

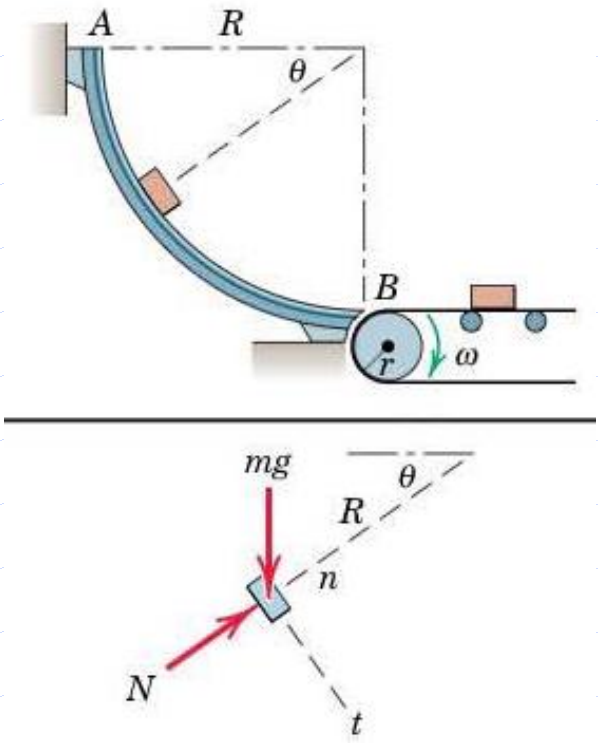
$$\int_0^v u\,du = gR\int_0^\theta \cos\phi\,d\phi$$

$$\frac{v^2}{2} = gR\sin\theta.$$

$$\boxed{v^2 = 2gR\sin\theta}$$

At B, $\theta = \pi/2$:

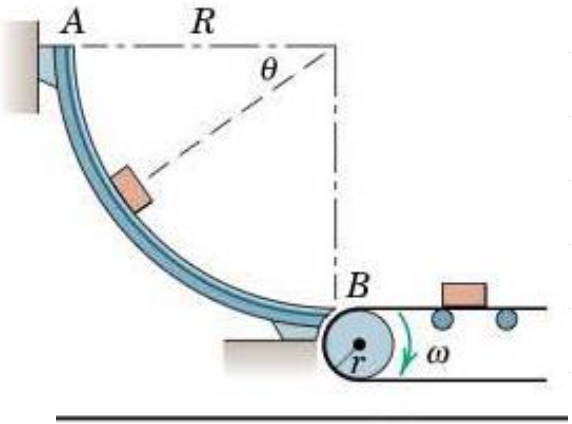
$$\boxed{v_B = \sqrt{2gR}}.$$



Normal force and conveyor angular velocity

① Apply Newton's second law toward the center

$$\sum F_n = m\frac{v^2}{R}: \qquad \underbrace{N}_{\text{toward the center}} - \underbrace{mg\sin\theta}_{\text{away from the center}} = m\frac{v^2}{R}.$$



Substitute $v^2 = 2gR\sin\theta$:

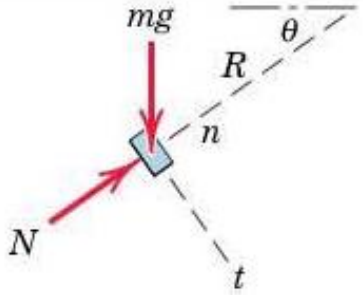
$$N - mg\sin\theta = m\frac{2gR\sin\theta}{R} = 2mg\sin\theta.$$

Thus,

$$\boxed{N(\theta) = 3mg\sin\theta}.$$

Check at the bottom of the guide:

$$\theta = \frac{\pi}{2} \quad \Rightarrow \quad \boxed{N_B = 3mg}.$$



This is the contact force **just before leaving the curved guide**. It exceeds the weight because an upward normal acceleration is required.

② Match the belt speed to the arrival speed

For transfer without sliding,

$$v_{\text{belt}} = v_B.$$

Assuming no slip between belt and pulley,

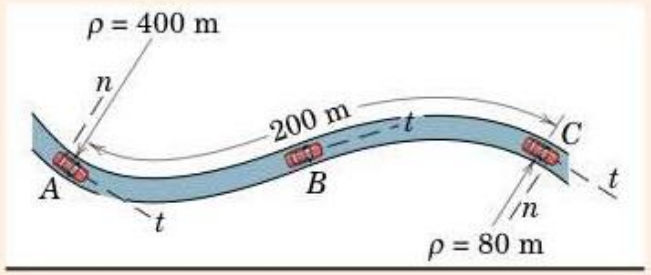
$$r\omega = \sqrt{2gR}.$$

$$\boxed{\omega = \frac{\sqrt{2gR}}{r}\ \text{rad/s}, \quad \text{clockwise}}$$

Clockwise rotation moves the upper belt to the right, matching the object's velocity at B.

Key distinction: R determines the object's curved path; r converts the required belt speed into pulley angular velocity.

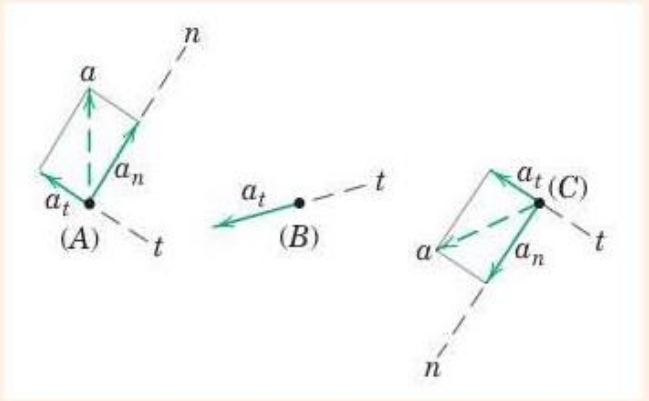
Sample Problem 3/8



A 1500-kg car enters a section of curved road in the horizontal plane and slows down at a uniform rate from a speed of 100 km/h at A to a speed of 50 km/h as it passes C. The radius of curvature ρ of the road at A is 400 m and at C is 80 m. Determine the total horizontal force exerted by the road on the tires at positions A, B, and C. Point B is the inflection point where the curvature changes direction.

Solution. The car will be treated as a particle so that the effect of all forces exerted by the road on the tires will be treated as a single force. Since the motion is described along the direction of the road, normal and tangential coordinates will be used to specify the acceleration of the car. We will then determine the forces from the accelerations.

The constant tangential acceleration is in the negative t-direction, and its magnitude is given by



① $[v_C^2 = v_A^2 + 2a_t\,\Delta s]$ $\qquad a_t = \left|\dfrac{(50/3.6)^2 - (100/3.6)^2}{2(200)}\right| = 1.447 \text{ m/s}^2$

The normal components of acceleration at A, B, and C are

② $[a_n = v^2/\rho]$

At A, $\quad a_n = \dfrac{(100/3.6)^2}{400} = 1.929 \text{ m/s}^2$

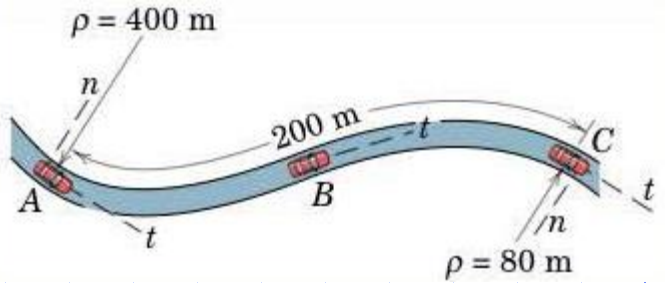
At B, $\quad a_n = 0$

At C, $\quad a_n = \dfrac{(50/3.6)^2}{80} = 2.41 \text{ m/s}^2$

Helpful Hints

① Recognize the numerical value of the conversion factor from km/h to m/s as 1000/3600 or 1/3.6.

② Note that a_n is always directed toward the center of curvature.



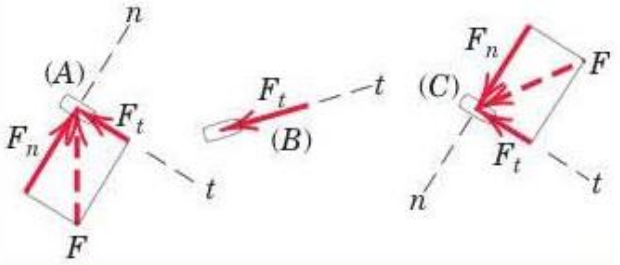
Application of Newton's second law in both the n- and t-directions to the free-body diagrams of the car gives

$[\Sigma F_t = ma_t]$ $\quad F_t = 1500(1.447) = 2170 \text{ N}$

③ $[\Sigma F_n = ma_n]$ At A, $\quad F_n = 1500(1.929) = 2890 \text{ N}$

At B, $\quad F_n = 0$

At C, $\quad F_n = 1500(2.41) = 3620 \text{ N}$



③ Note that the direction of F_n must agree with that of a_n.

Thus, the total horizontal force acting on the tires becomes

At A, $\quad F = \sqrt{F_n^2 + F_t^2} = \sqrt{(2890)^2 + (2170)^2} = 3620 \text{ N}$ *Ans.*

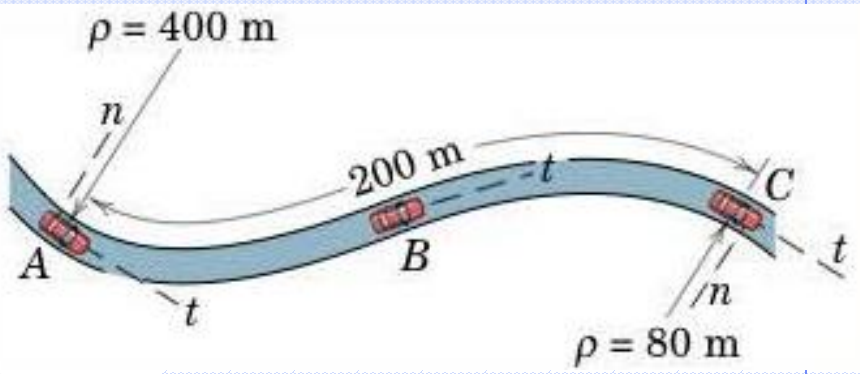
At B, $\quad F = F_t = 2170 \text{ N}$ *Ans.*

④ At C, $\quad F = \sqrt{F_n^2 + F_t^2} = \sqrt{(3620)^2 + (2170)^2} = 4220 \text{ N}$ *Ans.*

④ The angle made by **a** and **F** with the direction of the path can be computed if desired.

Given

- Car mass: $m = 1500\,\text{kg}$.
- Speed decreases uniformly from $100\,\text{km/h}$ at A to $50\,\text{km/h}$ at C.
- Distance along the road: $s_{AC} = 200\,\text{m}$.
- Radii of curvature:



$$\rho_A = 400\,\text{m}, \qquad \rho_C = 80\,\text{m}.$$

- B is an inflection point.
- The road lies in a horizontal plane.

Required

Determine the total **horizontal force exerted by the road on the tires** at A, B, and C.

Strategy

Choose $+t$ along the motion and $+n$ toward the local center of curvature.

① Find the constant tangential acceleration from

$$v_C^2 = v_A^2 + 2a_t s_{AC}.$$

② Calculate $a_n = v^2/\rho$ at each point.

③ Apply Newton's second law:

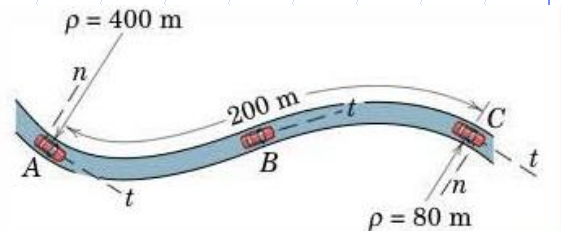
$$\boxed{F_t = ma_t, \qquad F_n = ma_n}.$$

④ Combine the perpendicular components:

$$\boxed{F = \sqrt{F_t^2 + F_n^2}}.$$

① Convert the speeds

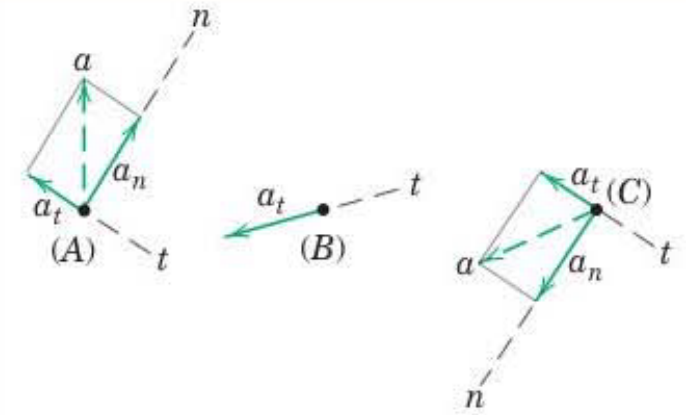
$$v_A = \frac{100}{3.6} = 27.778\,\text{m/s}, \qquad v_C = \frac{50}{3.6} = 13.889\,\text{m/s}.$$



② Determine the constant tangential acceleration

$$a_t = \frac{v_C^2 - v_A^2}{2s_{AC}} = \frac{(50/3.6)^2 - (100/3.6)^2}{2(200)}$$

$$\boxed{a_t = -1.447\,\text{m/s}^2}$$

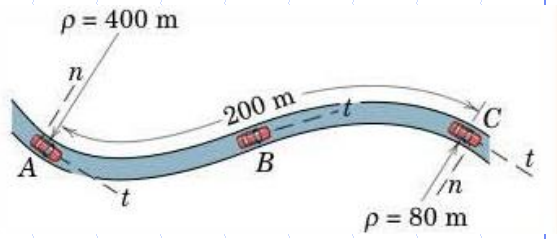


The negative sign means acceleration **opposite the direction of motion**.

③ Determine the normal acceleration at each point

Position	Normal acceleration $a_n = v^2/\rho$	Tangential acceleration a_t
A	$\frac{(100/3.6)^2}{400} = \boxed{1.929\,\text{m/s}^2}$	$-1.447\,\text{m/s}^2$
B	Zero curvature: $\boxed{a_{nB} = 0}$	$-1.447\,\text{m/s}^2$
C	$\frac{(50/3.6)^2}{80} = \boxed{2.411\,\text{m/s}^2}$	$-1.447\,\text{m/s}^2$

Observation: We do not need v_B, because the curvature at B is zero.



① Identify the horizontal forces

Treat the combined action of the road on all tires as one horizontal resultant:

$$\mathbf{F} = F_t\mathbf{e}_t + F_n\mathbf{e}_n.$$

The vertical support force balances the weight and is not part of the requested horizontal force.

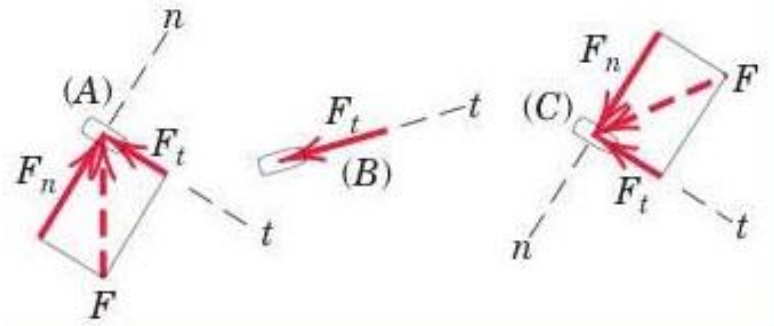
② Apply Newton's second law along the tangent

At all three positions,

$$\sum F_t = ma_t: \qquad F_t = 1500(-1.447).$$

$$\boxed{F_t = -2170\,\text{N}}$$

Thus, the tangential force has magnitude $2.17\,\text{kN}$ and acts backward along the road.



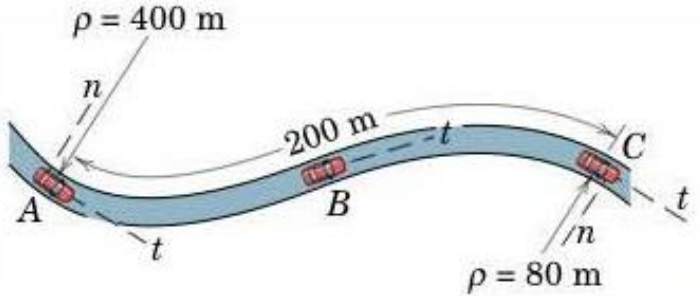
③ Apply Newton's second law toward the center of curvature

$$\sum F_n = ma_n.$$

Position	Normal force component
A	$F_{nA} = 1500(1.929) \approx \boxed{2890\,\text{N}}$
B	$F_{nB} = 1500(0) = \boxed{0}$
C	$F_{nC} = 1500(2.411) \approx \boxed{3620\,\text{N}}$

Important distinction: Here, F_n is the **horizontal turning-force component**, directed toward the center of curvature. It is different from the vertical support force exerted by the road.

Resultant horizontal force at A, B, and C



The tangential and normal directions are perpendicular:

$$\boxed{F = \sqrt{F_t^2 + F_n^2}}.$$

At A: Braking and turning

$$F_A = \sqrt{(-2170)^2 + 2890^2}$$

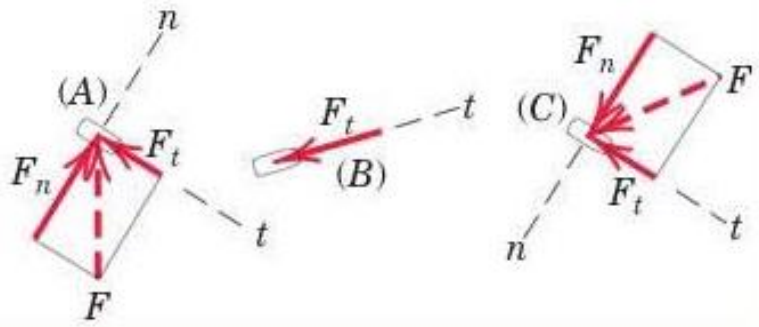
$$\boxed{F_A \approx 3620\,\text{N} = 3.62\,\text{kN}}$$

At B: Braking only

$$F_B = \sqrt{(-2170)^2 + 0^2}$$

$$\boxed{F_B = 2170\,\text{N} = 2.17\,\text{kN}}$$

At C: Braking and sharper turning



$$F_C = \sqrt{(-2170)^2 + 3620^2}$$

$$\boxed{F_C \approx 4220\,\text{N} = 4.22\,\text{kN}}$$

At A and C, the resultant points **backward and inward**; at B, it points directly backward along the tangent.

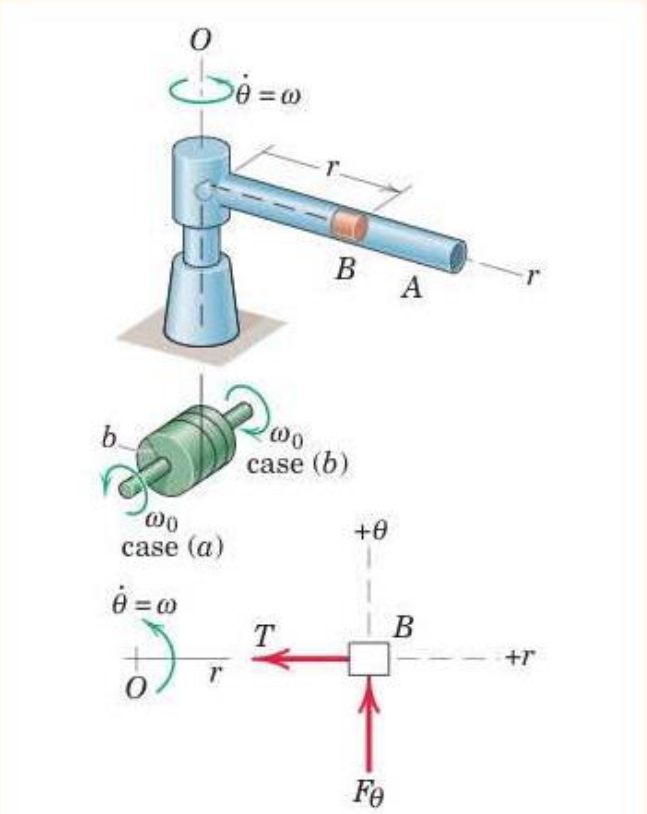
Teaching takeaway: Although the speed at C is half that at A, its radius of curvature is only one-fifth as large:

$$\frac{a_{nC}}{a_{nA}} = \left(\frac{1}{2}\right)^2 (5) = 1.25.$$

The tighter turn therefore requires a **larger normal force—and a larger total horizontal force—at** C.

Sample Problem 3/10

Tube A rotates about the vertical O-axis with a constant angular rate $\dot{\theta} = \omega$ and contains a small cylindrical plug B of mass m whose radial position is controlled by the cord which passes freely through the tube and shaft and is wound around the drum of radius b. Determine the tension T in the cord and the horizontal component F_θ of force exerted by the tube on the plug if the constant angular rate of rotation of the drum is ω_0 first in the direction for case (a) and second in the direction for case (b). Neglect friction.



Solution. With r a variable, we use the polar-coordinate form of the equations of motion, Eqs. 3/8. The free-body diagram of B is shown in the horizontal plane and discloses only T and F_θ. The equations of motion are

$$[\Sigma F_r = ma_r] \qquad -T = m(\ddot{r} - r\dot{\theta}^2)$$

$$[\Sigma F_\theta = ma_\theta] \qquad F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Case (a). With $\dot{r} = +b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \qquad F_\theta = 2mb\omega_0\omega \qquad \textit{Ans.}$$

① ***Case (b).*** With $\dot{r} = -b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \qquad F_\theta = -2mb\omega_0\omega \qquad \textit{Ans.}$$

Helpful Hint

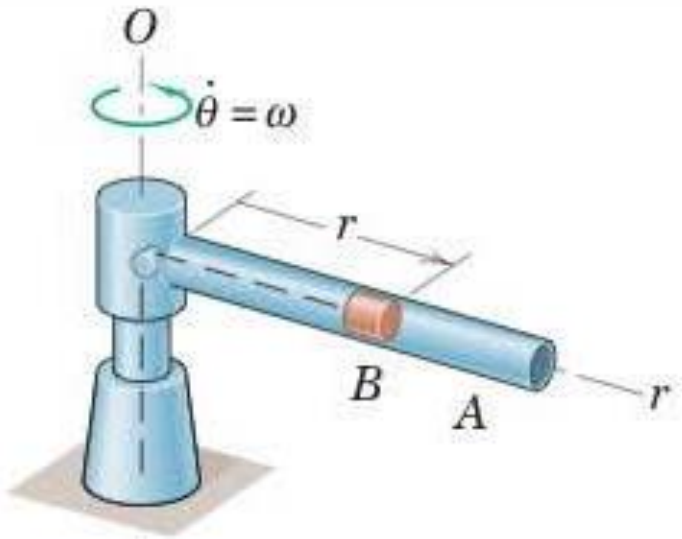
① The minus sign shows that F_θ is in the direction opposite to that shown on the free-body diagram.

Given

- Plug B has mass m and radial position r.
- The horizontal tube rotates at constant angular rate:

$$\dot{\theta} = \omega, \qquad \ddot{\theta} = 0.$$

- A drum of radius b controls the cord at constant angular speed ω_0.
- Neglect friction; assume the cord remains taut and does not slip on the drum.

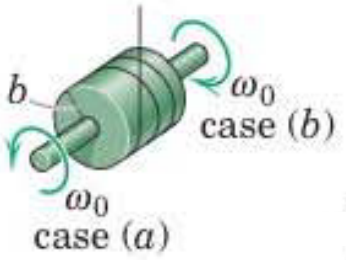


Required

For each drum-rotation direction, determine:

① Cord tension T.

② Horizontal force component F_θ exerted by the tube on the plug.



Strategy

Choose $+r$ outward along the tube and $+\theta$ in the direction of tube rotation.

1. Relate the plug's radial speed to the drum speed:

$$\boxed{\dot{r} = \pm b\omega_0}.$$

2. Draw the plug's free-body diagram in the horizontal plane.
3. Apply Newton's second law in polar coordinates:

$$\boxed{\sum F_r = m(\ddot{r} - r\dot{\theta}^2)}$$

$$\boxed{\sum F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})}.$$

Radial/Transverse Components

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

① Identify the forces

In the horizontal plane:

- Tension T acts inward, opposite $+r$.
- The frictionless tube exerts F_θ perpendicular to its axis.

Vertically, the tube's supporting force balances the plug's weight.

② Apply the radial equation

$$\sum F_r = ma_r: \qquad -T = m(\ddot{r} - r\dot{\theta}^2).$$

Since the drum speed and radius are constant,

$$\dot{r} = \pm b\omega_0 = \text{constant} \quad \Rightarrow \quad \ddot{r} = 0.$$

With $\dot{\theta} = \omega$,

$$-T = -mr\omega^2 \quad \Rightarrow \quad \boxed{T = mr\omega^2}.$$

③ Apply the transverse equation

$$\sum F_\theta = ma_\theta: \qquad F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}).$$

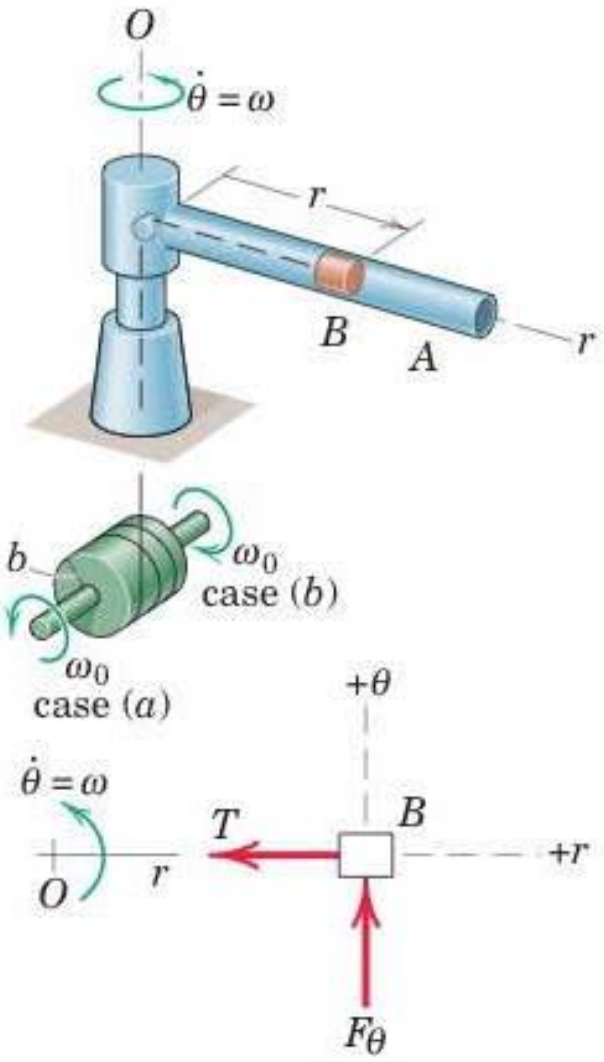
Because $\ddot{\theta} = 0$,

$$\boxed{F_\theta = 2m\dot{r}\,\omega}.$$

Key observation: Constant angular speed does **not** make the transverse acceleration zero:

$$\boxed{a_\theta = 2\dot{r}\,\omega}.$$

The plug is also moving radially while the tube rotates.



Evaluate cases (a) and (b)

Case (a): Cord pays out; the plug moves outward

$$\dot{r} = +b\omega_0, \qquad \ddot{r} = 0.$$

Therefore,

$$\boxed{T = mr\omega^2}$$

$$F_\theta = 2m(+b\omega_0)\omega \quad \Rightarrow \quad \boxed{F_\theta = +2mb\omega_0\omega}.$$

The tube pushes the plug in the **positive transverse direction**, as assumed in the free-body diagram.

Case (b): Cord winds in; the plug moves inward

$$\dot{r} = -b\omega_0, \qquad \ddot{r} = 0.$$

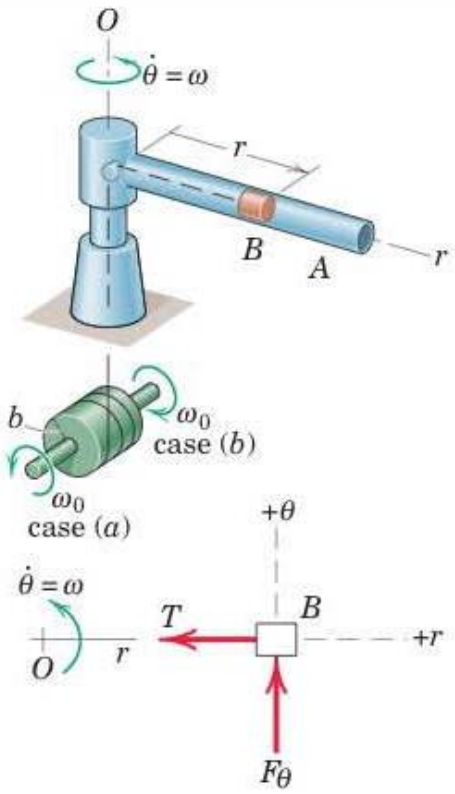
Therefore,

$$\boxed{T = mr\omega^2}$$

$$F_\theta = 2m(-b\omega_0)\omega \quad \Rightarrow \quad \boxed{F_\theta = -2mb\omega_0\omega}.$$

The negative sign means that the tube's horizontal force acts **opposite the direction assumed in the free-body diagram**.

Compare the cases: At the same radial position r, reversing the drum leaves the tension unchanged but reverses F_θ. As the plug moves, $T = mr\omega^2$ increases outward and decreases inward.



2. Work and Energy Principles



Energy and Momentum Methods

- Integration of $\bar{F} = m\bar{a}$ yields two new methods of analysis:

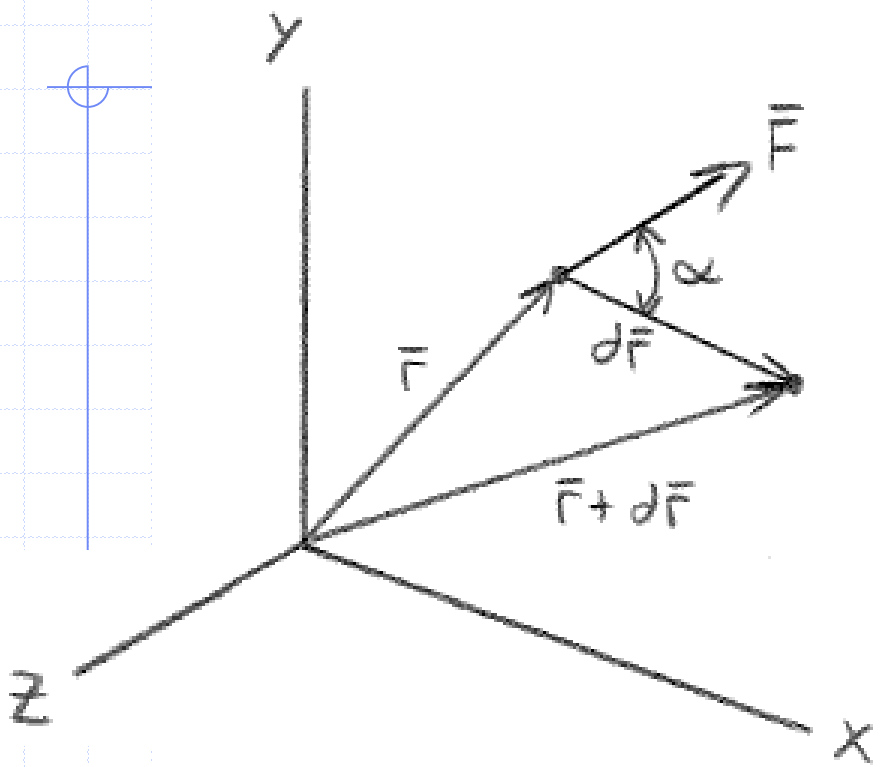
① Work - Energy ——— $\int (\,)\,ds$

② Impulse - Momentum ——— $\int (\,)\,dt$

Advantages:- Determination of $\bar{a}$ not required

- ① relates directly speed - disp.
- ② " " Vel. - time

Work of a Force



We define

$$dU = \bar{F} \cdot d\bar{r}$$

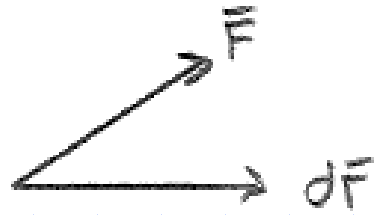
Differential Work of $\bar{F}$

(N-m = J, ft-lbf)

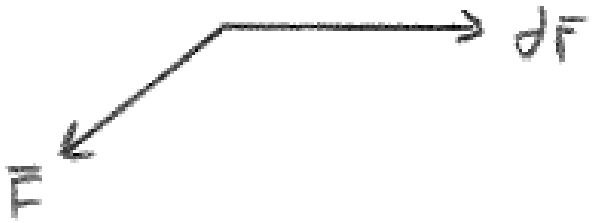
Letting $|\bar{F}| = F$, $|d\bar{r}| = ds$

$$dU = F\,ds\cos\alpha$$

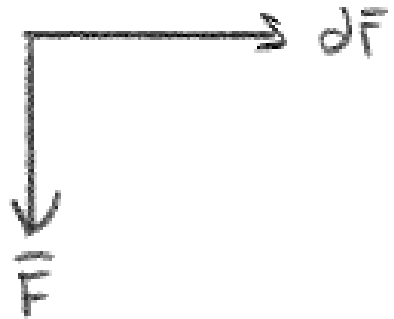
Ex



$dU > 0$



$dU < 0$



$dU = 0$

Comments:

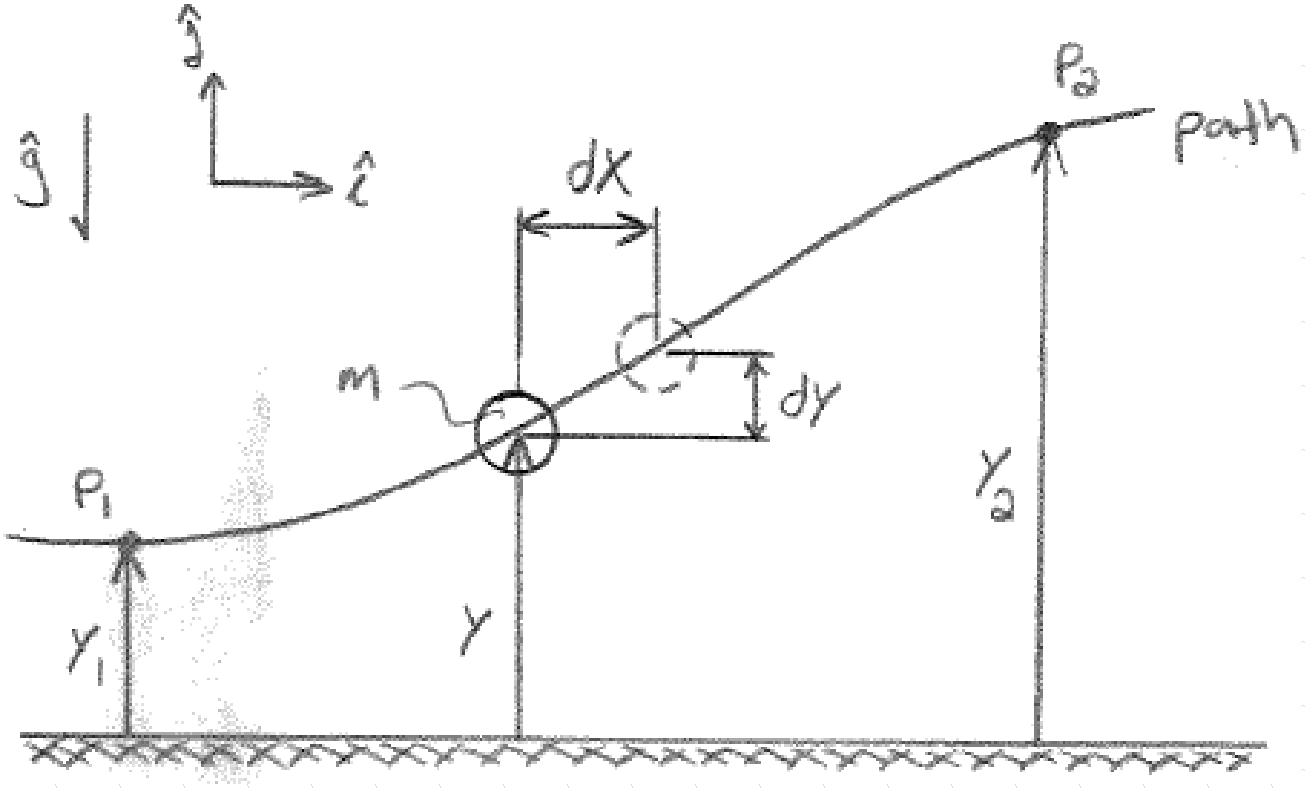
- dU gives a measure of how parallel forces and displacements are.
- Only that component of F tangent to path of motion does work.

Over a finite path from P_1 to P_2

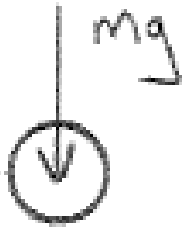
$$U_{1\to 2} = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{r}$$

Work

Work of Force due to Gravity: $P_1 \to P_2$



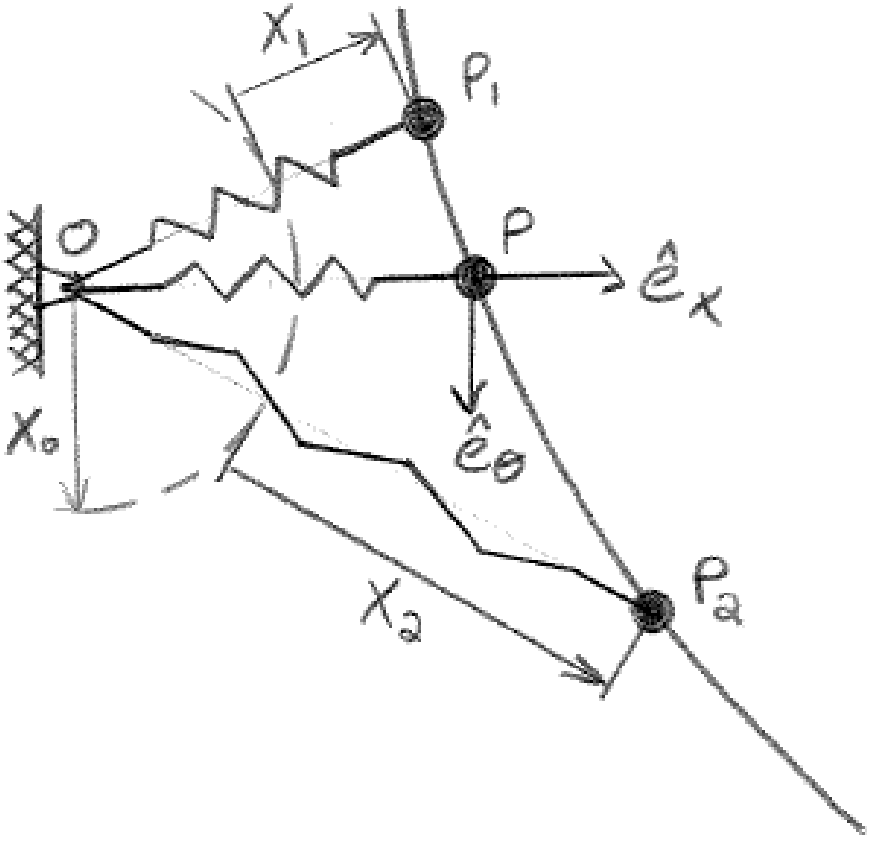
FBD:



$$U_{1\to 2} = -mg(Y_2 - Y_1) = -mg\,\Delta y$$

Work of a Spring Force: $P_1 \to P_2$

$$U_{1\to 2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$



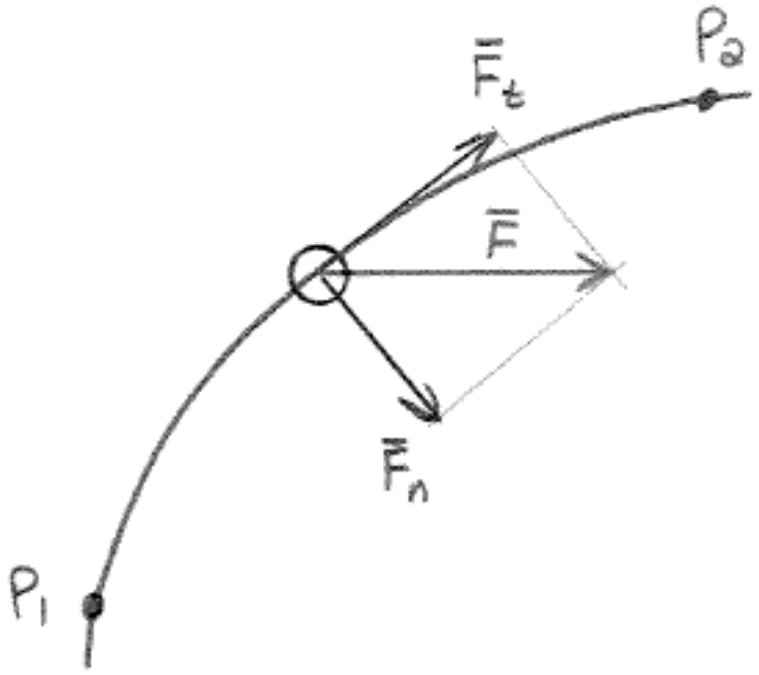
FBD:



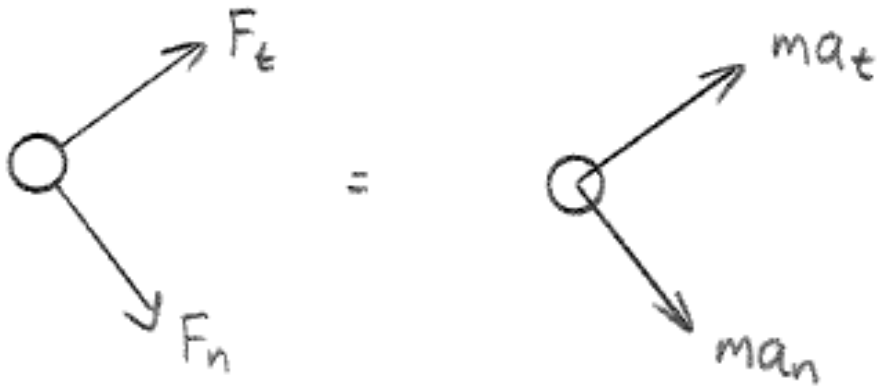
$$dU = -kx\,dx$$

x_0 = undeformed length

Kinetic Energy of a Particle. Principle of Work-Energy.



FBD:



Consider only tangential dynamics.

$$F_t = ma_t$$

$$F_t = ma_t$$

$$= m\frac{dv}{dt} = m\frac{dv}{ds}\cancel{\frac{ds}{dt}}^{v} = mv\frac{dv}{ds}$$

$$\Rightarrow \quad F_t\,ds = mv\,dv \qquad \text{—— (*)}$$

Integrate (*) from P_1 to P_2, where:

@ P_1: $s = s_1$, $v = v_1$

@ P_2: $s = s_2$, $v = v_2$

$$\int_{s_1}^{s_2} F_t\,ds = m\int_{v_1}^{v_2} v\,dv = m\left(\frac{v_2^2}{2} - \frac{v_1^2}{2}\right)$$

$$\Rightarrow \int_{s_1}^{s_2} F_t \, ds = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \quad \text{---} \quad (**)$$

We define $\boxed{T = \frac{1}{2} m v^2}$ Kinetic Energy

Since $\bar{v} \cdot \bar{v} = |\bar{v}|^2 = v^2$, can also write

$$\boxed{T = \frac{1}{2} m \bar{v} \cdot \bar{v}}$$

Finally, note $\int_{s_1}^{s_2} F_t \, ds = U_{1 \to 2}$

Thus from (**),

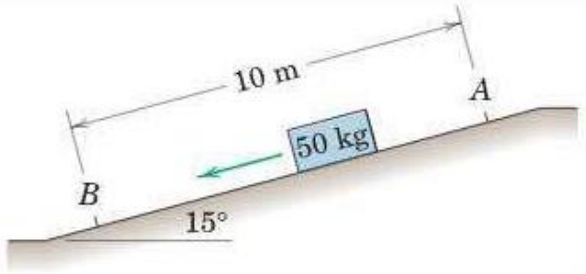
$$U_{1\to 2} = T_2 - T_1$$

$$\Rightarrow \boxed{T_1 + U_{1\to 2} = T_2}$$

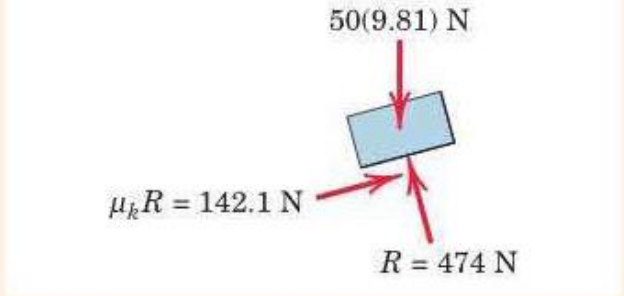
Principle of Work - Energy

Sample Problem 3/11

Calculate the velocity v of the 50-kg crate when it reaches the bottom of the chute at B if it is given an initial velocity of 4 m/s down the chute at A. The coefficient of kinetic friction is 0.30.



Solution. The free-body diagram of the crate is drawn and includes the normal force R and the kinetic friction force F calculated in the usual manner. The work done by the weight is positive, whereas that done by the friction force is negative. The total work done on the crate during the motion is



① $[U = Fs]$ $\quad U_{1\text{-}2} = 50(9.81)(10 \sin 15°) - 142.1(10) = -151.9$ J

The work-energy equation gives

$$[T_1 + U_{1\text{-}2} = T_2] \qquad \tfrac{1}{2}mv_1^2 + U_{1\text{-}2} = \tfrac{1}{2}mv_2^2$$

$$\tfrac{1}{2}(50)(4)^2 - 151.9 = \tfrac{1}{2}(50)v_2^2$$

$$v_2 = 3.15 \text{ m/s} \qquad \textit{Ans.}$$

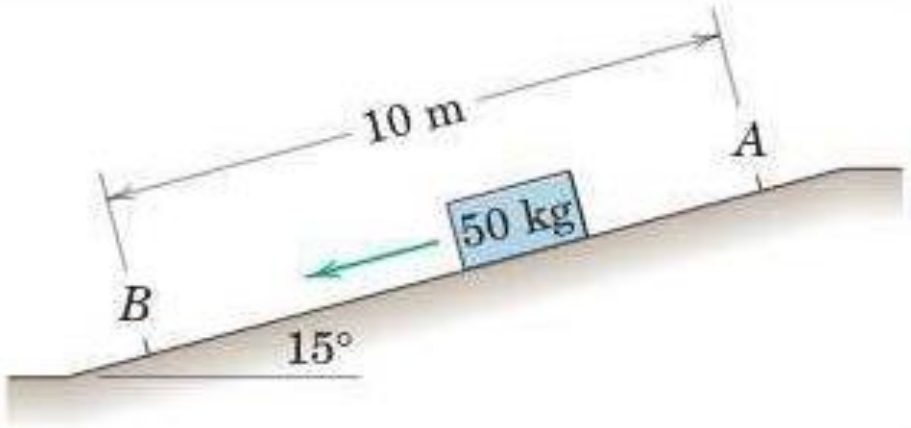
Since the net work done is negative, we obtain a decrease in the kinetic energy.

Helpful Hint

① The work due to the weight depends only on the *vertical* distance traveled.

Given

- Crate mass: $m = 50\,\text{kg}$
- Initial velocity: $v_A = 4\,\text{m/s}$, down the chute
- Distance traveled: $s = 10\,\text{m}$
- Inclination: $\theta = 15^\circ$
- Coefficient of kinetic friction: $\mu_k = 0.30$



Required

The velocity v_B when the crate reaches B.

Strategy

① Find the normal reaction and kinetic friction:

$$N = mg\cos\theta, \qquad F_k = \mu_k N$$

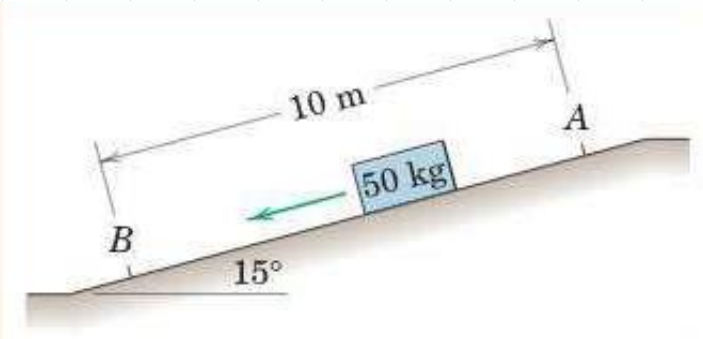
② Calculate the work of each force over $A \to B$:

$$U_W = mg\,s\sin\theta, \qquad U_f = -F_k s, \qquad U_N = 0$$

③ Apply the **principle of work and energy**:

$$\boxed{\frac{1}{2}mv_A^2 + U_{A\to B} = \frac{1}{2}mv_B^2}$$

① Determine the contact forces



There is no acceleration perpendicular to the straight chute:

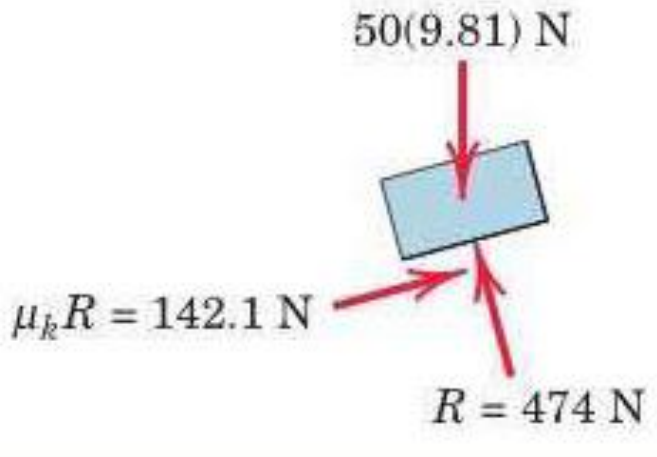
$$N = 50(9.81)\cos 15^\circ = 473.8\,\text{N}$$

$$F_k = 0.30(473.8) = 142.1\,\text{N}$$

② Calculate the work

$$U_W = \underbrace{mg\,s\sin 15^\circ}_{\text{weight assists the motion}} = 1269.5\,\text{J}$$

$$U_f = \underbrace{-\mu_k mg\cos 15^\circ\,s}_{\text{friction opposes the motion}} = -1421.4\,\text{J}$$



The normal reaction is perpendicular to the displacement, so $U_N = 0$.

$$\boxed{U_{A\to B} = 1269.5 - 1421.4 = -151.9\,\text{J}}$$

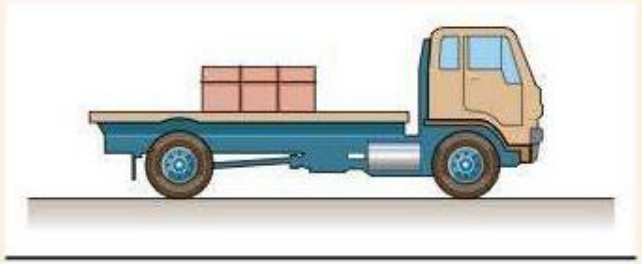
③ Apply PWE

$$\frac{1}{2}(50)(4)^2 - 151.9 = \frac{1}{2}(50)v_B^2$$

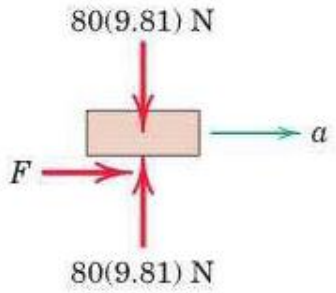
$$\boxed{v_B = 3.15\,\text{m/s} \quad \text{down the chute}}$$

Negative net work reduces kinetic energy: the crate slows down from 4.00 to $3.15\,\text{m/s}$, even while moving downhill.

Sample Problem 3/12



The flatbed truck, which carries an 80-kg crate, starts from rest and attains a speed of 72 km/h in a distance of 75 m on a level road with constant acceleration. Calculate the work done by the friction force acting on the crate during this interval if the static and kinetic coefficients of friction between the crate and the truck bed are (*a*) 0.30 and 0.28, respectively, or (*b*) 0.25 and 0.20, respectively.



Solution. If the crate does not slip on the bed, its acceleration will be that of the truck, which is

$$[v^2 = 2as] \qquad a = \frac{v^2}{2s} = \frac{(72/3.6)^2}{2(75)} = 2.67 \text{ m/s}^2$$

Case (a). This acceleration requires a friction force on the block of

$$[F = ma] \qquad F = 80(2.67) = 213 \text{ N}$$

which is less than the maximum possible value of $\mu_s N = 0.30(80)(9.81) = 235$ N. Therefore, the crate does not slip and the work done by the actual static friction force of 213 N is

① $[U = Fs] \qquad U_{1\text{-}2} = 213(75) = 16\,000 \text{ J} \quad \text{or} \quad 16 \text{ kJ} \qquad \textit{Ans.}$

Case (b). For $\mu_s = 0.25$, the maximum possible friction force is $0.25(80)(9.81) = 196.2$ N, which is slightly less than the value of 213 N required for no slipping. Therefore, we conclude that the crate slips, and the friction force is governed by the kinetic coefficient and is $F = 0.20(80)(9.81) = 157.0$ N. The acceleration becomes

$$[F = ma] \qquad a = F/m = 157.0/80 = 1.962 \text{ m/s}^2$$

The distances traveled by the crate and the truck are in proportion to their accelerations. Thus, the crate has a displacement of $(1.962/2.67)75 = 55.2$ m, and the work done by kinetic friction is

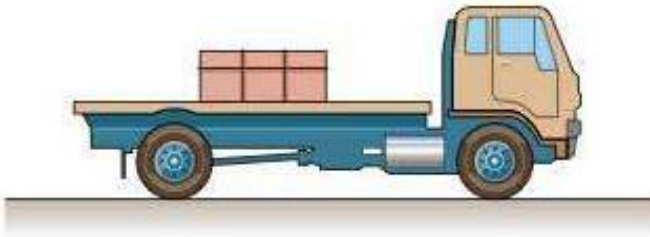
② $[U = Fs] \qquad U_{1\text{-}2} = 157.0(55.2) = 8660 \text{ J} \quad \text{or} \quad 8.66 \text{ kJ} \qquad \textit{Ans.}$

Helpful Hints

① We note that static friction forces do no work when the contacting surfaces are both at rest. When they are in motion, however, as in this problem, the static friction force acting on the crate does positive work and that acting on the truck bed does negative work.

② This problem shows that a kinetic friction force can do positive work when the surface which supports the object and generates the friction force is in motion. If the supporting surface is at rest, then the kinetic friction force acting on the moving part always does negative work.

Given



$$m_C = 80\,\text{kg}, \qquad v_{T0} = v_{C0} = 0$$

$$v_T = 72\,\text{km/h} = 20\,\text{m/s}, \qquad s_T = 75\,\text{m}$$

The truck accelerates uniformly on a level road.

Case	μ_s	μ_k
(a)	0.30	0.28
(b)	0.25	0.20

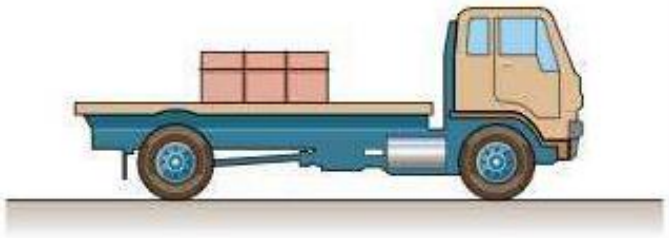
Required: Work done by friction **on the crate** during the truck's 75-m displacement.

Strategy

① Calculate the truck's acceleration.
② Assume no slipping and compare ma_T with $\mu_s N$.
③ Determine the actual friction force and the crate's displacement.
④ Calculate $U_f = F s_C$.

Use the ground reference frame. Assume the crate remains on the truck bed throughout the interval.

Truck kinematics and the no-slip requirement



① Calculate the truck's acceleration

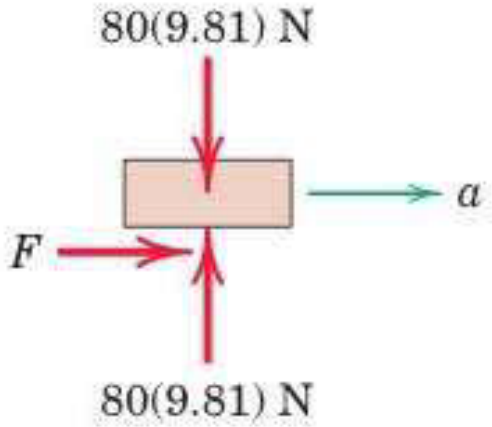
$$v_T^2 = v_{T0}^2 + 2a_T s_T$$

$$\boxed{a_T = \frac{20^2}{2(75)} = 2.667\,\text{m/s}^2}$$

② Determine the normal reaction on the crate

$$\sum F_y = 0: \qquad N - m_C g = 0$$

$$\boxed{N = 80(9.81) = 784.8\,\text{N}}$$



③ Find the friction required for no slipping

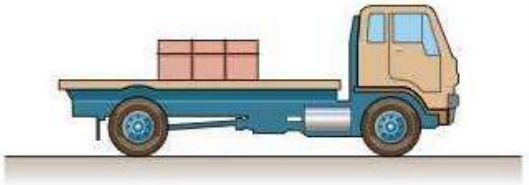
If the crate moves with the truck, $a_C = a_T$:

$$\boxed{F_{\text{required}} = m_C a_T = 80(2.667) = 213.3\,\text{N}}$$

Friction acts **forward**, preventing the crate from slipping backward relative to the bed.

Case (a): No slipping

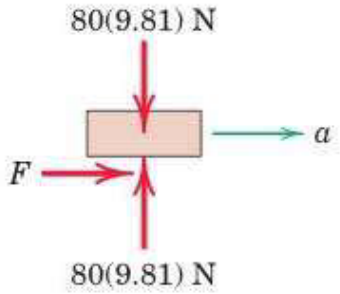
$$\mu_s = 0.30, \qquad \mu_k = 0.28$$



① Check the available static friction

$$F_{s,\max} = \mu_s N = 0.30(784.8) = 235.4\,\text{N}$$

$$\underbrace{213.3\,\text{N}}_{\text{required}} < \underbrace{235.4\,\text{N}}_{\text{available maximum}} \implies \boxed{\text{No slipping}}$$



② Calculate the work of actual static friction

$$F_s = 213.3\,\text{N}, \qquad s_C = s_T = 75\,\text{m}$$

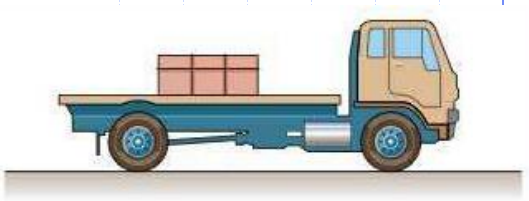
$$U_f = F_s s_C = \left[80\left(\frac{20^2}{2(75)}\right)\right](75)$$

$$\boxed{U_f = 16\,000\,\text{J} = 16.0\,\text{kJ}}$$

Static friction does positive work: both the force and the crate's displacement point forward.

Case (b): Slipping and crate acceleration

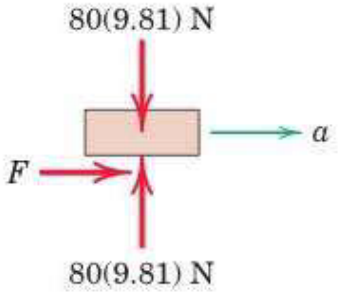
$$\mu_s = 0.25, \qquad \mu_k = 0.20$$



① Test the no-slip assumption

$$F_{s,\max} = 0.25(784.8) = 196.2\,\text{N}$$

$$\underbrace{213.3\,\text{N}}_{\text{required}} > \underbrace{196.2\,\text{N}}_{\text{available maximum}} \implies \boxed{\text{The crate slips}}$$



② Use kinetic friction

$$\boxed{F_k = \mu_k N = 0.20(784.8) = 156.96\,\text{N}}$$

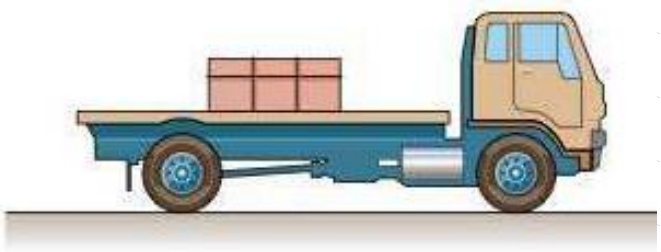
③ Apply Newton's second law to the crate

$$F_k = m_C a_C$$

$$\boxed{a_C = \frac{156.96}{80} = 1.962\,\text{m/s}^2 \quad \text{forward}}$$

Since $a_C < a_T$, the truck moves ahead of the crate. The crate slides backward **relative to the bed**.

Case (b): Crate displacement and friction work



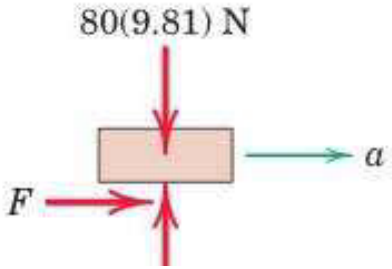
① Find the crate's ground displacement

Both start from rest and move for the same time with constant accelerations:

$$s_C = \frac{1}{2}a_C t^2, \qquad s_T = \frac{1}{2}a_T t^2$$

Dividing eliminates time:

$$\frac{s_C}{s_T} = \frac{a_C}{a_T}$$



$$\boxed{s_C = \frac{1.962}{20^2/[2(75)]}(75) = 55.18\,\text{m}}$$

② Calculate the work on the crate

$$U_f = \underbrace{F_k}_{\text{forward}} \underbrace{s_C}_{\text{forward ground displacement}} = 156.96(55.18)$$

$$\boxed{U_f = 8.66\,\text{kJ}}$$

Use the crate's displacement—not the truck's 75 m or the relative sliding distance. Kinetic friction does positive work on the crate because it accelerates the crate forward.

Instructor note: The implied backward displacement relative to the truck is $75 - 55.18 = 19.82\,\text{m}$. Thus, the textbook solution assumes a sufficiently long bed and ignores the possibility of the crate falling off.

Power – Efficiency

Accordingly, the power P developed by a force $\mathbf{F}$ which does an amount of work U is $P = dU/dt = \mathbf{F} \cdot d\mathbf{r}/dt$. Because $d\mathbf{r}/dt$ is the velocity $\mathbf{v}$ of the point of application of the force, we have

$$P = \mathbf{F} \cdot \mathbf{v} \qquad \textbf{(3/16)}$$

Power is clearly a scalar quantity, and in SI it has the units of $\text{N} \cdot \text{m/s} = \text{J/s}$. The special unit for power is the *watt* (W), which equals one joule per second (J/s). In U.S. customary units, the unit for mechanical power is the *horsepower* (hp). These units and their numerical equivalences are

$$1 \text{ W} = 1 \text{ J/s}$$

$$1 \text{ hp} = 550 \text{ ft-lb/sec} = 33{,}000 \text{ ft-lb/min}$$

$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

Power – Efficiency (cont.)

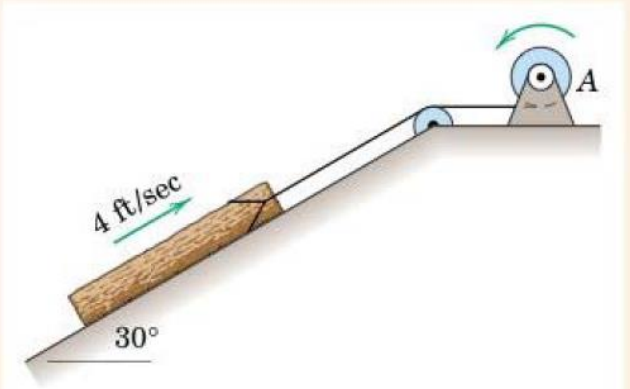
Efficiency

The ratio of the work done *by* a machine to the work done *on* the machine during the same time interval is called the *mechanical efficiency* e_m of the machine. This definition assumes that the machine operates uniformly so that there is no accumulation or depletion of energy within it. Efficiency is always less than unity since every device operates with some loss of energy and since energy cannot be created within the machine. In mechanical devices which involve moving parts, there will always be some loss of energy due to the negative work of kinetic friction forces. This work is converted to heat energy which, in turn, is dissipated to the surroundings. The mechanical efficiency at any instant of time may be expressed in terms of mechanical power P by

$$e_m = \frac{P_{\text{output}}}{P_{\text{input}}} \qquad \textbf{(3/17)}$$

Sample Problem 3/14

The power winch A hoists the 800-lb log up the 30° incline at a constant speed of 4 ft/sec. If the power output of the winch is 6 hp, compute the coefficient of kinetic friction μ_k between the log and the incline. If the power is suddenly increased to 8 hp, what is the corresponding instantaneous acceleration a of the log?



Solution. From the free-body diagram of the log, we get $N = 800 \cos 30° = 693$ lb, and the kinetic friction force becomes $693\mu_k$. For constant speed, the forces are in equilibrium so that

$$[\Sigma F_x = 0] \qquad T - 693\mu_k - 800 \sin 30° = 0 \qquad T = 693\mu_k + 400$$

The power output of the winch gives the tension in the cable

① $$[P = Tv] \qquad T = P/v = 6(550)/4 = 825 \text{ lb}$$

Substituting T gives

$$825 = 693\mu_k + 400 \qquad \mu_k = 0.613 \qquad \textit{Ans.}$$

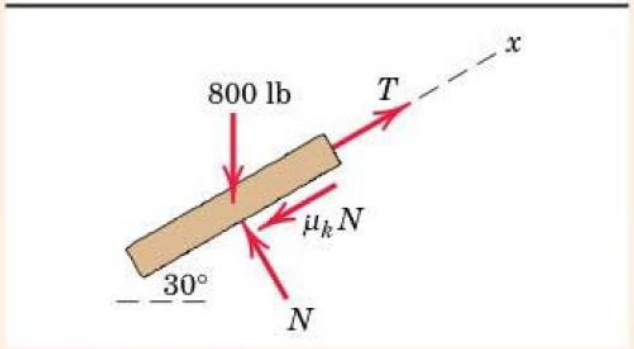
When the power is increased, the tension momentarily becomes

$$[P = Tv] \qquad T = P/v = 8(550)/4 = 1100 \text{ lb}$$

and the corresponding acceleration is given by

$$[\Sigma F_x = ma_x] \qquad 1100 - 693(0.613) - 800 \sin 30° = \frac{800}{32.2} a$$

② $$a = 11.07 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

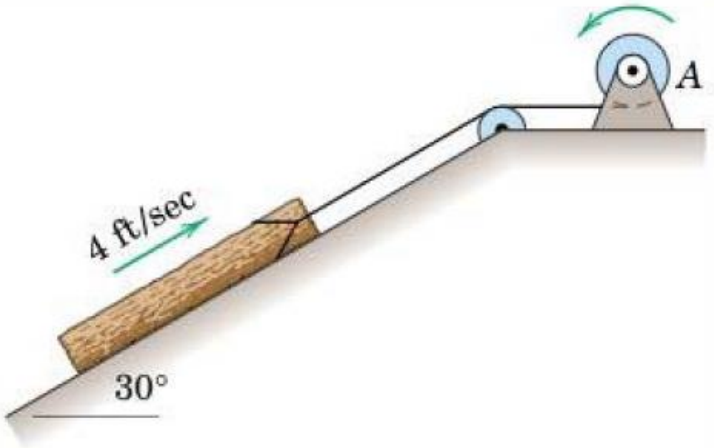


Helpful Hints

① Note the conversion from horsepower to ft-lb/sec.

② As the speed increases, the acceleration will drop until the speed stabilizes at a value higher than 4 ft/sec.

Given



- Log weight: $W = 800\ \text{lb}$
- Incline angle: $\theta = 30^\circ$
- Initial constant speed: $v = 4\ \text{ft/s}$
- Initial power: $P_1 = 6\ \text{hp}$
- Increased power: $P_2 = 8\ \text{hp}$

Required: μ_k and the acceleration immediately after the power increase.

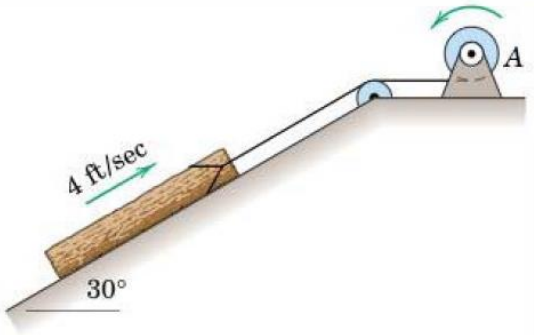
Strategy

1. Draw the FBD: tension uphill, friction downhill, weight vertically downward, and normal reaction perpendicular to the incline.
2. Use $P = Tv$ and the initial condition $a = 0$ to determine μ_k.
3. Calculate the new tension at the unchanged instantaneous speed, then apply $\sum F_x = ma$.

Include the unit reminders:

$$1\ \text{hp} = 550\ \text{ft} \cdot \text{lb/s}, \qquad m = \frac{W}{g}, \quad g = 32.2\ \text{ft/s}^2.$$

Determine the coefficient of kinetic friction



Taking $+x$ uphill:

$$N = W\cos 30^\circ = 800\cos 30^\circ = 692.82 \text{ lb}.$$

The initial cable tension is

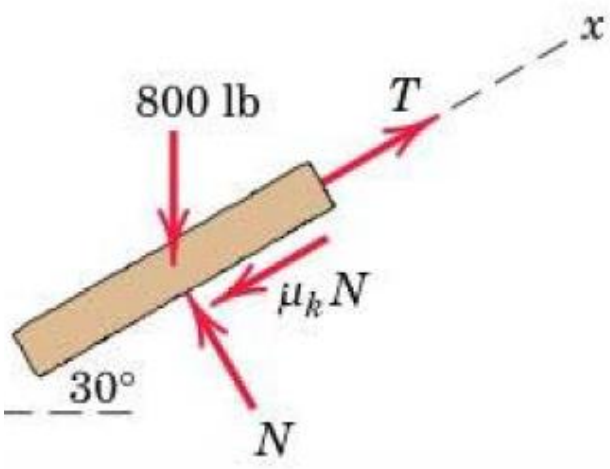
$$T_1 = \frac{P_1}{v} = \frac{6(550)}{4} = 825 \text{ lb}.$$

Since the speed is constant:

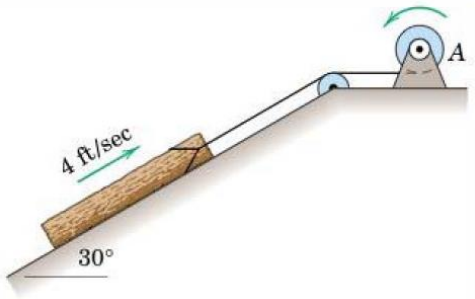
$$T_1 - \mu_k N - W\sin 30^\circ = 0.$$

Therefore,

$$\boxed{\mu_k = \frac{825 - 800\sin 30^\circ}{692.82} = 0.6134}$$



Determine the instantaneous acceleration

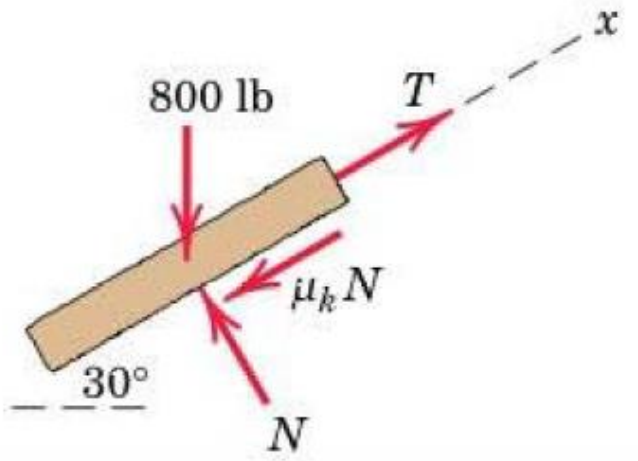


Immediately after the increase, $v = 4$ ft/s, so

$$T_2 = \frac{P_2}{v} = \frac{8(550)}{4} = 1100 \text{ lb}.$$

Using the unrounded friction force from slide 2:

$$F_k = \mu_k N = 425 \text{ lb}.$$



Apply Newton's second law:

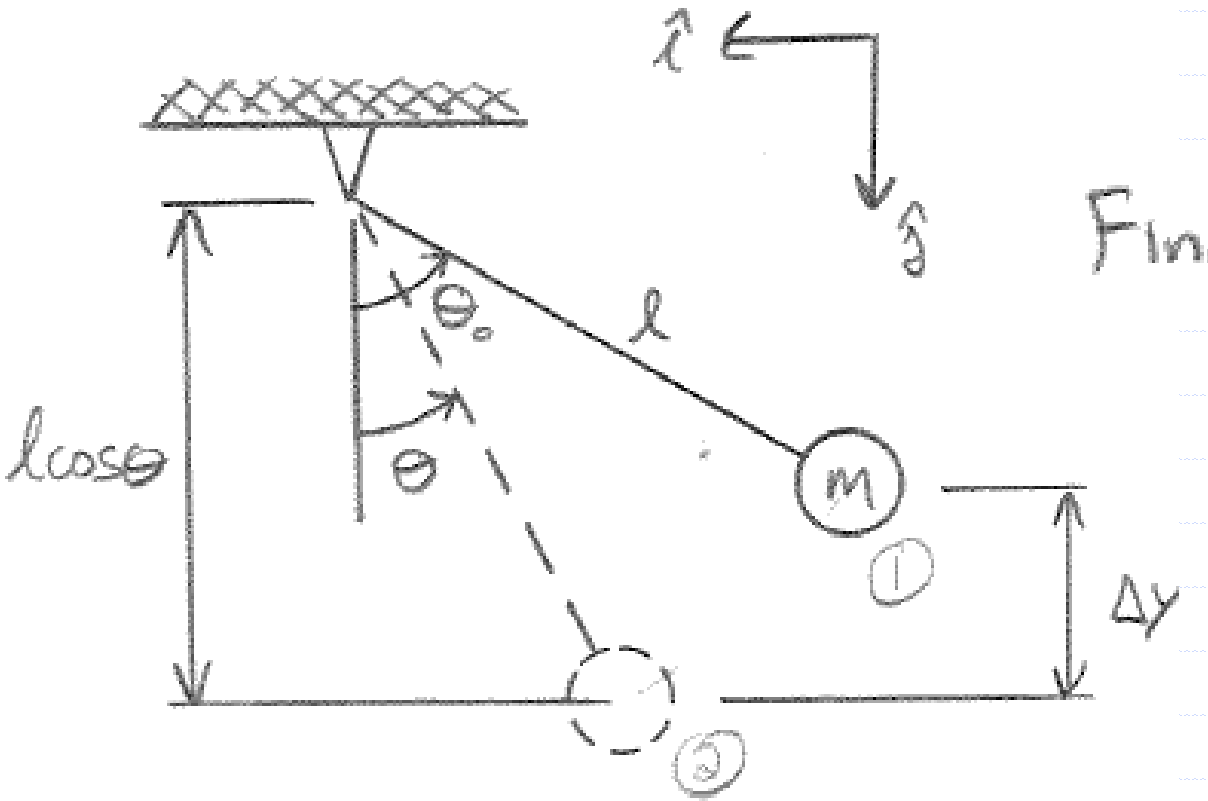
$$1100 - 425 - 800 \sin 30^\circ = \frac{800}{32.2} a,$$

$$\boxed{a = 11.07 \text{ ft/s}^2 \quad \text{up the incline}}.$$

Close with one short observation: **At constant power, $T = P/v$ decreases as the speed increases, so this acceleration is instantaneous—not constant throughout the subsequent motion.**

Example:

m is released @ (1) from $\theta = \theta_0$ with $v_0 = 0$.



Find $v = v(\theta)$

Invoke PWE: $T_1 + U_{1\to 2} = T_2$

$T_1 = \frac{1}{2} m v_1^2 = 0$

$v_0 = 0$

$T_2 = \frac{1}{2} m v_2^2 = \frac{1}{2} m v^2$

v

$U_{1\to 2} = U_{1\to 2}^{mg} + U_{1\to 2}^{T}$ ($U_{1\to 2}^{T} = 0$)

$U_{1\to 2}^{mg} = mg(y_2 - y_1) = mg\Delta y$



Where $\Delta y = l\cos\theta - l\cos\theta_0$

$\therefore U_{1\to 2} = mgl(\cos\theta - \cos\theta_0)$

PWE gives $0 + mgl(\cos\theta - \cos\theta_0) = \frac{1}{2}mv^2$

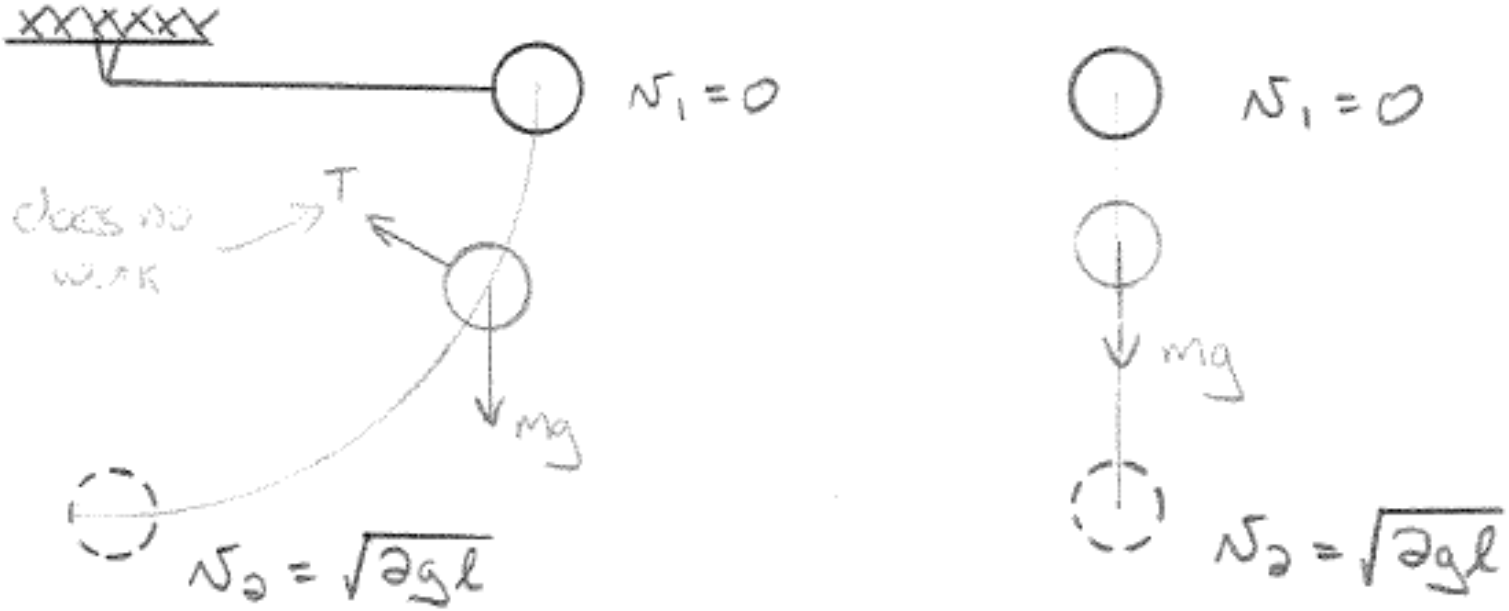
$\therefore v(\theta) = \sqrt{2gl(\cos\theta - \cos\theta_0)}$

Consider $\theta_0 = \frac{\pi}{2}$, $\theta = 0$

Then $v = \sqrt{2gl\,[\cos(0) - \cos(\pi/2)]} = \sqrt{2gl}$

→ Same as if m simply dropped!

Why?

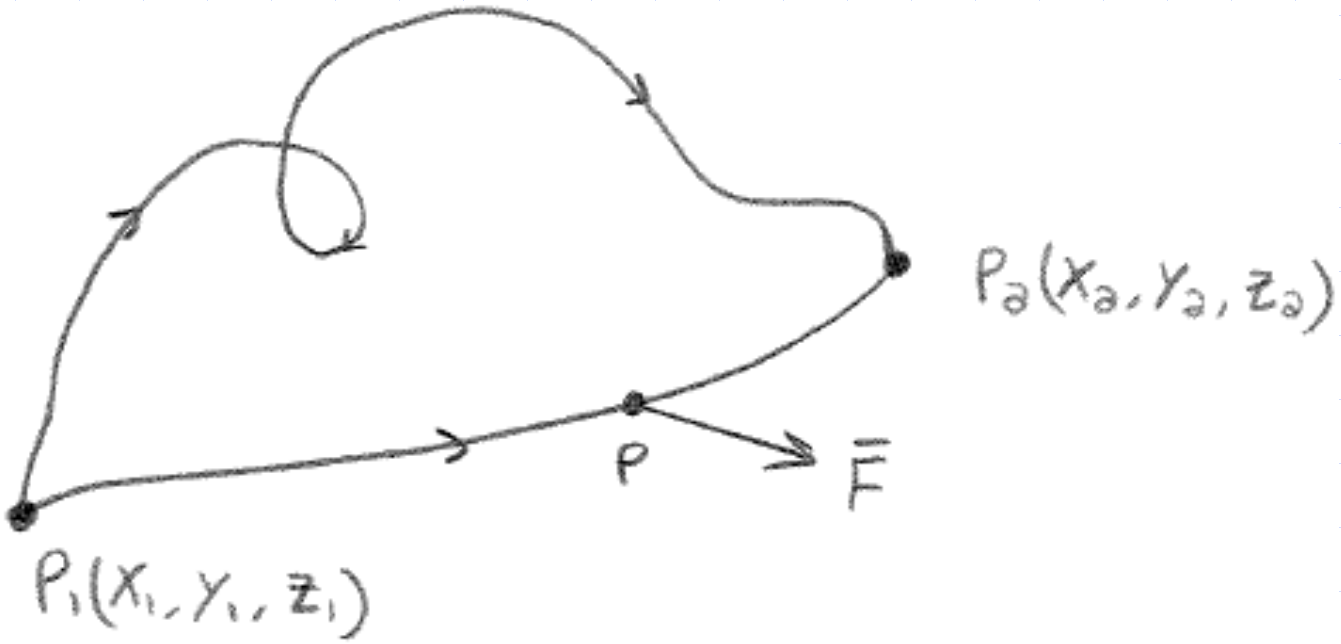


$$T_1 + U_{1\to 2} = T_2$$

Same both cases

Conservative Forces

A force $\bar{F}$ acting on a particle P is said to be <u>conservative</u> if its work $U_{1\to 2}$ is <u>independent</u> of the path followed by P in moving from P_1 to P_2.



FACT: If $\bar{F}$ is conservative, then

$$U_{1\to 2} = \int_1^2 \bar{F}\cdot d\bar{r} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)$$

@ 1 @ 2

V is called a potential function.

Potential Energy

Recall work of force due to weight: $(F = mg)$

$$U_{1\to 2}^{\text{weight}} = -mg(y_2 - y_1)$$

$$= mgy_1 - mgy_2$$

Path independent $\Rightarrow F = mg$ is conservative

$$\therefore \quad \boxed{V_{\text{weight}} = mgy}$$

Recall work of a spring Force: $(F = Kx)$

$$U_{1\to 2}^{\text{spring}} = \frac{1}{2}Kx_1^2 - \frac{1}{2}Kx_2^2$$

Path independent $\Rightarrow$ $F = Kx$ is conservative

$\therefore$ $$\boxed{V_{\text{spring}} = \frac{1}{2}Kx^2}$$

Nonconservative Forces

A force $\vec{F}$ is <u>nonconservative</u> if it is not conservative.

- are path dependent
- cannot be expressed as a change in potential energy

Examples:

- friction forces $f = \mu N$

Principle of Work - Energy Revisited

Recall $T_1 + U_{1\to 2} = T_2$

Separate work done by conservative and nonconservative forces:

Let $\bar{F} = \bar{F}^C + \bar{F}^{NC}$

Thus $T_1 + V_1 - V_2 + U_{1\to 2}^{NC} = T_2$

$$\Rightarrow \boxed{T_1 + V_1 + U_{1\to 2}^{NC} = T_2 + V_2}$$

$\therefore \; U_{1\to 2}^{NC} = 0 \Rightarrow$ Conservation of Energy

Datums for Potential Energy

Consider PWE:

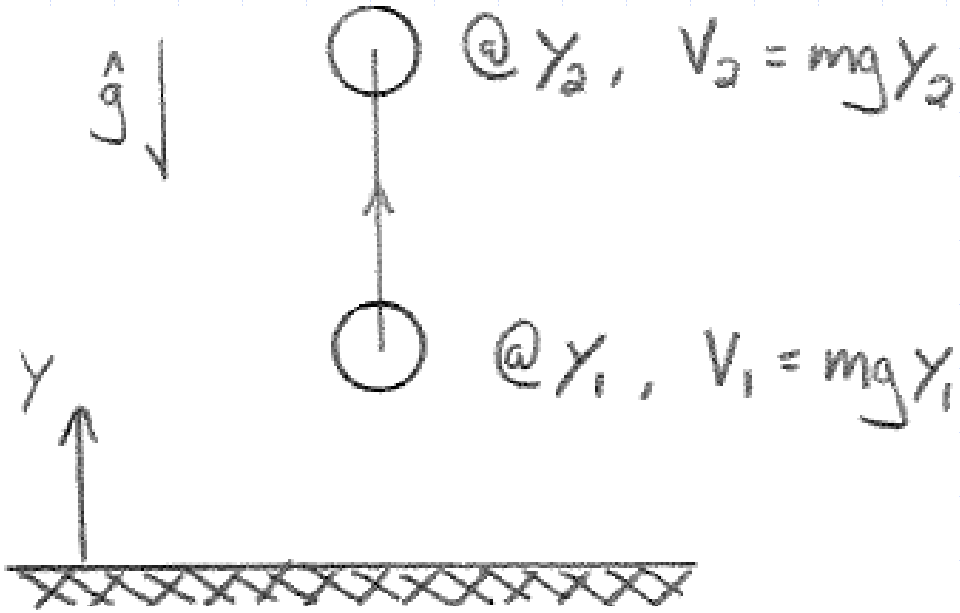
$$T_1 + V_1 + U_{1\to 2}^{NC} = T_2 + V_2$$

$$\Rightarrow U_{1\to 2}^{NC} = \underbrace{T_2 - T_1}_{\Delta T} + \underbrace{V_2 - V_1}_{\Delta V}$$

$\therefore$ only changes in energy are important.

FACT: If a conservative force is constant, can set $V = 0$ at an arbitrary datum.

ΔV due to Weight



$$\Delta V = V_2 - V_1 = mgy_2 - mgy_1 = \int_{y_1}^{y_2} mg\, dy$$

Only dist. from y_1 to y_2 matters!

EX | Set datum at y_1

@ y_1: $V_1 = 0$ (at datum)

@ y_2: $V_2 = mg \cdot$ (dist. between y_1 and y_2)

$= mg(y_2 - y_1)$

$\therefore \Delta V = V_2 - \cancel{V_1}^{0} = mg(y_2 - y_1)$ ✓

EX Set datum at y_2

@ y_2: $V_2 = 0$ (at datum)

@ y_1: $V_1 = -mg \cdot$ (dist. between y_1 and y_2)

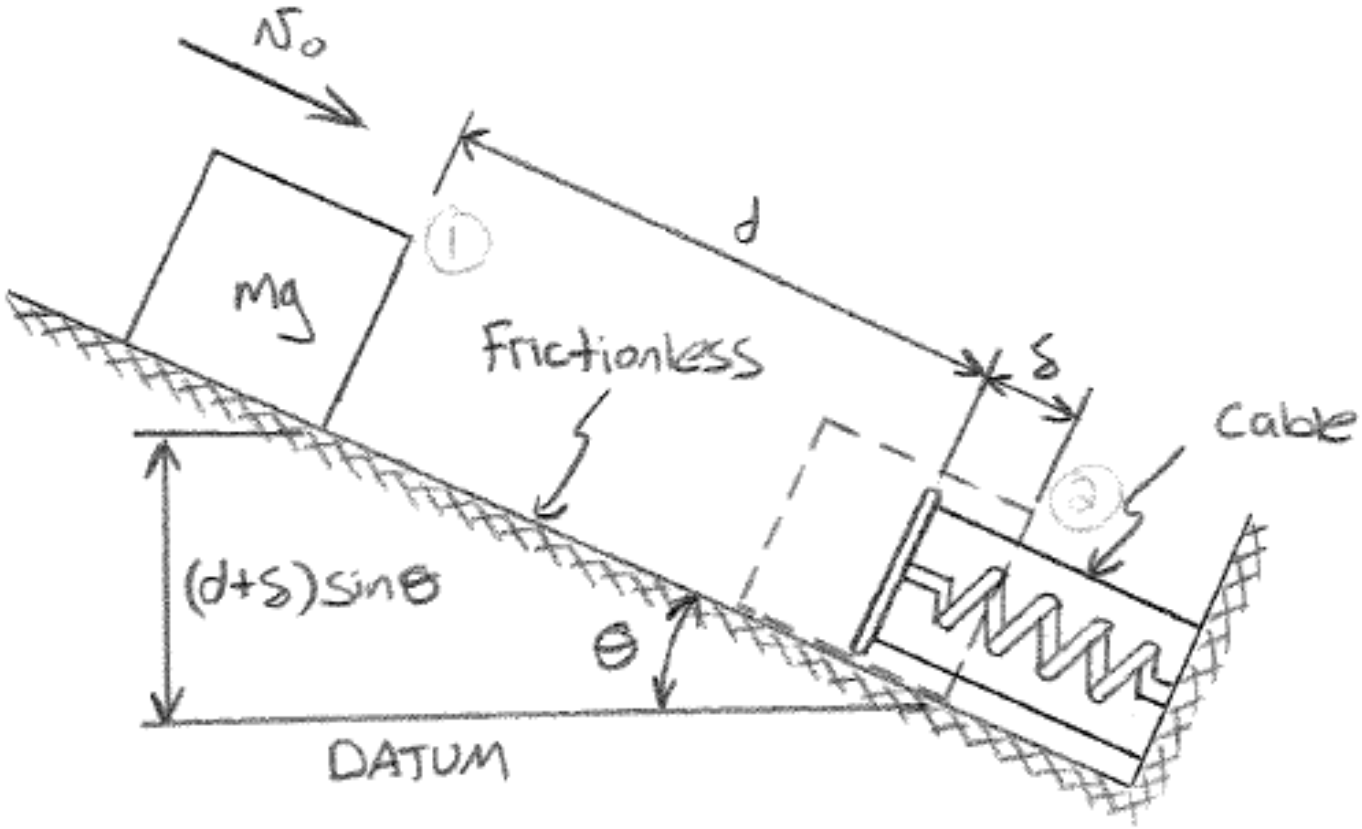
$= -mg(y_2 - y_1)$

$\therefore \quad \Delta V = \overset{0}{\cancel{V_2}} - V_1 = +mg(y_2 - y_1)$

Key: Above datum $\rightarrow$ V pos.

Below datum $\rightarrow$ V neg.

Example:



Neglect Friction

Spring is pre-deformed by $X_0 = 6\,in$

Find maximum additional deformation δ in the spring.

Data: $V_0 = 8\ ft/s$ (initial speed @ state ①)

$mg = 200\ lbf$

$\theta = 20°$

$K = 125\ lbs/in$ (spring constant)

$d = 25\ ft$

Invoke PWE with the block as the system.

$$T_1 + V_1 + U_{1\to 2}^{NC} = T_2 + V_2 \quad \text{——} \quad (*)$$

$$T_1 = \frac{1}{2} m v_0^2 \qquad (v_1 = v_0 \text{ @ } ①)$$

$$V_1 = mg(d+\delta)\sin\theta$$

$$U_{1\to 2}^{NC} = 0 \qquad (\text{all acting forces are conservative})$$

$$T_2 = 0 \qquad (v_2 = 0 \text{ @ max deformation})$$

$$V_2 = \underbrace{\frac{1}{2} k (x_0+\delta)^2}_{\text{"total deformed" energy}} - \underbrace{\frac{1}{2} k x_0^2}_{\text{"pre-deformed" energy}}$$

Plug into (*) ...

$$\frac{1}{2}mv_0^2 + mg(d+s)\sin\theta + 0 = 0 + \frac{1}{2}k(x_0+s)^2 - \frac{1}{2}kx_0^2$$

$\longrightarrow$ quadratic in s

Put in the form

$$(\quad)s^2 + (\quad)s + (\quad) = 0$$

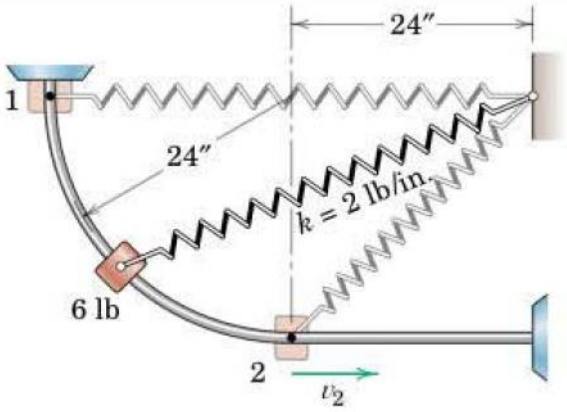
solve $s_1 = 1.204$ ft

$s_2 = -2.113$ ft

$\therefore \quad s = 1.204$ ft

Sample Problem 3/16

The 6-lb slider is released from rest at position 1 and slides with negligible friction in a vertical plane along the circular rod. The attached spring has a stiffness of 2 lb/in. and has an unstretched length of 24 in. Determine the velocity of the slider as it passes position 2.



Solution. The work done by the weight and the spring force on the slider will be treated using potential-energy methods. The reaction of the rod on the slider is normal to the motion and does no work. Hence, $U'_{1\text{-}2} = 0$. We define the datum to be at the level of position 1, so that the gravitational potential energies are

① $$V_1 = 0$$

$$V_2 = -mgh = -6\left(\frac{24}{12}\right) = -12 \text{ ft-lb}$$

The initial and final elastic (spring) potential energies are

$$V_1 = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(2)(12)\left(\frac{24}{12}\right)^2 = 48 \text{ ft-lb}$$

$$V_2 = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(2)(12)\left(\frac{24\sqrt{2}}{12} - \frac{24}{12}\right)^2 = 8.24 \text{ ft-lb}$$

Substitution into the alternative work-energy equation yields

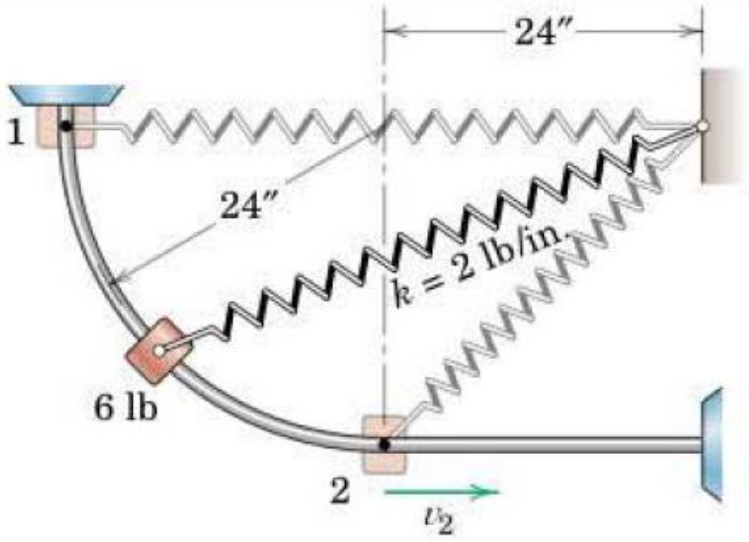
$$[T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2] \qquad 0 + 48 + 0 = \frac{1}{2}\left(\frac{6}{32.2}\right)v_2^2 - 12 + 8.24$$

$$v_2 = 23.6 \text{ ft/sec} \qquad \textit{Ans.}$$

Helpful Hint

① Note that if we evaluated the work done by the spring force acting on the slider by means of the integral $\int \mathbf{F} \cdot d\mathbf{r}$, it would necessitate a lengthy computation to account for the change in the magnitude of the force, along with the change in the angle between the force and the tangent to the path. Note further that v_2 depends only on the end conditions of the motion and does not require knowledge of the shape of the path.

Given



- Slider weight: $W = 6\ \text{lb}$
- Released from rest: $v_1 = 0$
- Circular-rod radius: $R = 24\ \text{in} = 2\ \text{ft}$
- Spring stiffness: $k = 2\ \text{lb/in} = 24\ \text{lb/ft}$
- Unstretched spring length: $L_0 = 24\ \text{in} = 2\ \text{ft}$
- Negligible friction

Required: velocity at position 2.

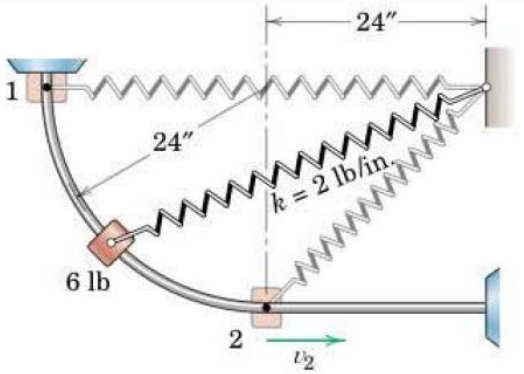
Strategy

1. Choose the gravitational datum at position 1.
2. Calculate gravitational and spring potential energies at both positions.
3. Apply the energy equation to determine v_2.

The rod's normal reaction is perpendicular to the motion and does no work. With negligible friction,

$$U_{1\to 2}^{\text{nc}} = 0.$$

Gravitational potential energy



Taking position 1 as the datum:

$$y_1 = 0, \qquad y_2 = -2 \text{ ft}.$$

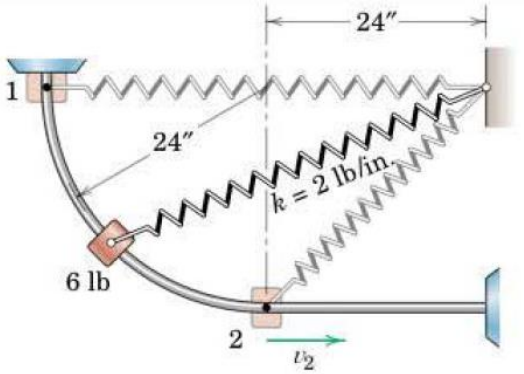
Using $V_g = Wy$:

$$\boxed{V_{g1} = 6(0) = 0}$$

$$\boxed{V_{g2} = 6(-2) = -12 \text{ ft} \cdot \text{lb}}.$$

The negative value means position 2 lies below the chosen datum.

Spring potential energy



Use

$$V_s = \frac{1}{2}kx^2, \qquad x = L - L_0,$$

where x is the **spring extension**, not its total length.

At position 1:

$$L_1 = 24 + 24 = 48 \text{ in} = 4 \text{ ft},$$

$$x_1 = 4 - 2 = 2 \text{ ft},$$

$$\boxed{V_{s1} = \frac{1}{2}(24)(2)^2 = 48 \text{ ft} \cdot \text{lb}}.$$

At position 2, the spring forms the hypotenuse of a right triangle:

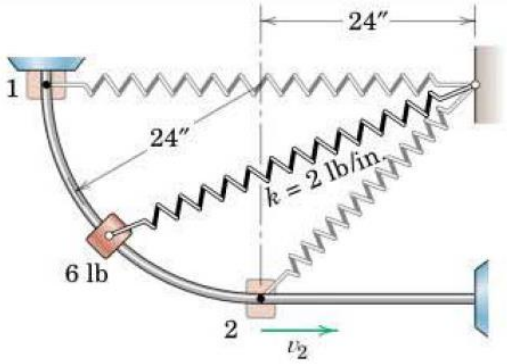
$$L_2 = \sqrt{2^2 + 2^2} = 2\sqrt{2} \text{ ft},$$

$$x_2 = 2\sqrt{2} - 2 = 0.8284 \text{ ft},$$

$$\boxed{V_{s2} = \frac{1}{2}(24)(2\sqrt{2} - 2)^2 = 8.24 \text{ ft} \cdot \text{lb}}.$$

The spring shortens but remains stretched at position 2.

Apply the energy equation



Retain every term explicitly:

$$\boxed{T_1 + V_{g1} + V_{s1} + U^{\text{nc}}_{1\to 2} = T_2 + V_{g2} + V_{s2}}$$

Since the slider starts from rest:

$$\underbrace{0}_{T_1} + \underbrace{0}_{V_{g1}} + \underbrace{48}_{V_{s1}} + \underbrace{0}_{U^{\text{nc}}_{1\to 2}} = \underbrace{\frac{1}{2}\left(\frac{6}{32.2}\right)v_2^2}_{T_2} \underbrace{-12}_{V_{g2}} + \underbrace{8.24}_{V_{s2}}.$$

Thus,

$$\frac{1}{2}\left(\frac{6}{32.2}\right)v_2^2 = 48 + 12 - 8.24 = 51.76 \text{ ft} \cdot \text{lb},$$

$$\boxed{v_2 = 23.6 \text{ ft/s} \quad \text{to the right}}.$$

Both gravity and the spring release potential energy, increasing the slider's kinetic energy. Their work is already represented by the potential-energy terms and must not be added again to $U^{\text{nc}}_{1\to 2}$.

3. Impulse and Momentum Methods

Principle of Impulse - Momentum

We have already seen,

$\bar{F} = m\bar{a}$, $\int (\ ) ds \longrightarrow$ Principle of Work - Energy

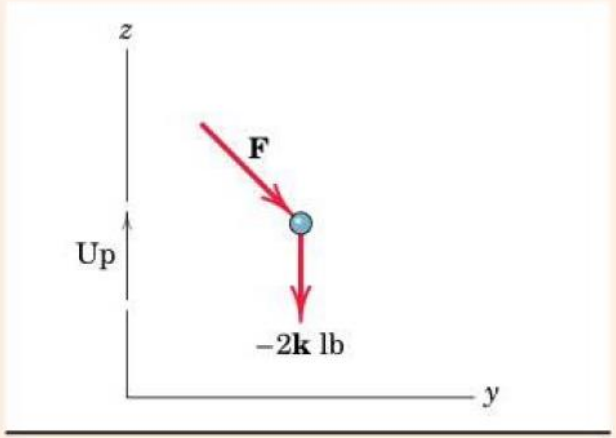
Now, $\int (\ ) dt \longrightarrow$ Principle of Impulse - Momentum

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

$m\bar{v}$ ← linear momentum

Sample Problem 3/20

A 2-lb particle moves in the vertical y-z plane (z up, y horizontal) under the action of its weight and a force $\mathbf{F}$ which varies with time. The linear momentum of the particle in pound-seconds is given by the expression $\mathbf{G} = \frac{3}{2}(t^2 + 3)\mathbf{j} - \frac{2}{3}(t^3 - 4)\mathbf{k}$, where t is the time in seconds. Determine $\mathbf{F}$ and its magnitude for the instant when $t = 2$ sec.



Solution. The weight expressed as a vector is $-2\mathbf{k}$ lb. Thus, the force-momentum equation becomes

① $[\Sigma\mathbf{F} = \dot{\mathbf{G}}]$ $$\mathbf{F} - 2\mathbf{k} = \frac{d}{dt}[\tfrac{3}{2}(t^2 + 3)\mathbf{j} - \tfrac{2}{3}(t^3 - 4)\mathbf{k}]$$
$$= 3t\mathbf{j} - 2t^2\mathbf{k}$$

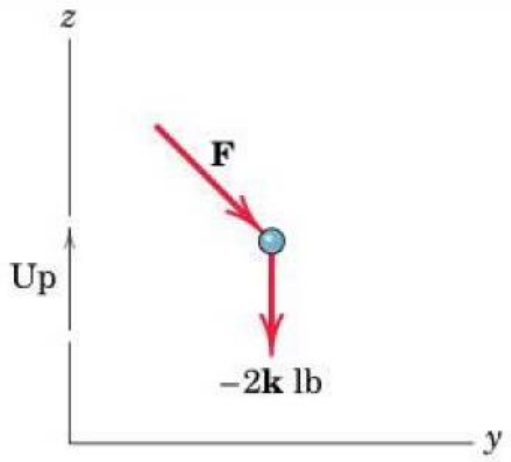
For $t = 2$ sec, $\mathbf{F} = 2\mathbf{k} + 3(2)\mathbf{j} - 2(2^2)\mathbf{k} = 6\mathbf{j} - 6\mathbf{k}$ lb *Ans.*

Thus, $F = \sqrt{6^2 + 6^2} = 6\sqrt{2}$ lb *Ans.*

Helpful Hint

① Don't forget that $\Sigma\mathbf{F}$ includes *all* external forces acting on the particle, including the weight.

Given



- Particle weight: $W = 2$ lb.
- Motion in the vertical y–z plane, with $+z$ upward.
- External forces: applied force $\mathbf{F}(t)$ and weight $\mathbf{W} = -2\mathbf{k}$ lb.
- Linear momentum:

$$\mathbf{G}(t) = \frac{3}{2}(t^2 + 3)\mathbf{j} - \frac{2}{3}(t^3 - 4)\mathbf{k} \qquad [\text{lb} \cdot \text{s}],$$

  where t is in seconds.

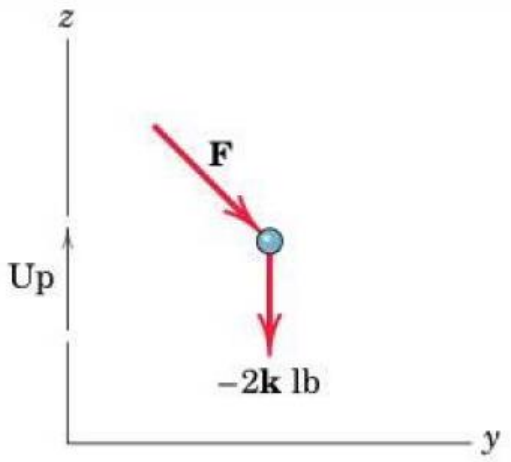
Required: $\mathbf{F}$ and its magnitude at $t = 2$ s.

Strategy

1. Differentiate momentum using $\sum \mathbf{F} = d\mathbf{G}/dt$.
2. Include the weight and isolate $\mathbf{F}(t)$.
3. Substitute $t = 2$ s and calculate the magnitude.

Place the supplied force diagram alongside these points.

Force from the rate of change of momentum



Differentiate the momentum:

$$\sum \mathbf{F} = \frac{d\mathbf{G}}{dt} = \boxed{3t\,\mathbf{j} - 2t^2\,\mathbf{k} \quad [\text{lb}]}.$$

The resultant includes **both** the applied force and weight:

$$\mathbf{F} - 2\mathbf{k} = 3t\,\mathbf{j} - 2t^2\,\mathbf{k}.$$

Therefore,

$$\boxed{\mathbf{F}(t) = 3t\,\mathbf{j} + (2 - 2t^2)\mathbf{k} \quad [\text{lb}]}.$$

At $t = 2$ s:

$$\mathbf{F}(2) = 3(2)\mathbf{j} + \left[2 - 2(2)^2\right]\mathbf{k} = \boxed{6\mathbf{j} - 6\mathbf{k}\ \text{lb}}.$$

Its magnitude is

$$\boxed{|\mathbf{F}| = \sqrt{6^2 + (-6)^2} = 6\sqrt{2} = 8.49\ \text{lb}}.$$

Teaching takeaway: At this instant, the resultant vertical force is 8 lb downward: 6 lb from the applied force plus 2 lb from the weight.

$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

$$\Rightarrow \quad \bar{F}\,dt = d(m\bar{v})$$

$$\Rightarrow \quad \int_{t_1}^{t_2} \bar{F}\,dt = \int_{m\bar{v}_1}^{m\bar{v}_2} d(m\bar{v})$$

$$= m\bar{v}_2 - m\bar{v}_1$$

Define

$$\boxed{Imp_{1\to 2} = \int_{t_1}^{t_2} \bar{F}\,dt}$$

Linear Impulse

Then $$Imp_{1\to 2} = m\bar{v}_2 - m\bar{v}_1$$

$$m\bar{v}_1 + Imp_{1\to 2} = m\bar{v}_2$$

Principle of Impulse - Momentum

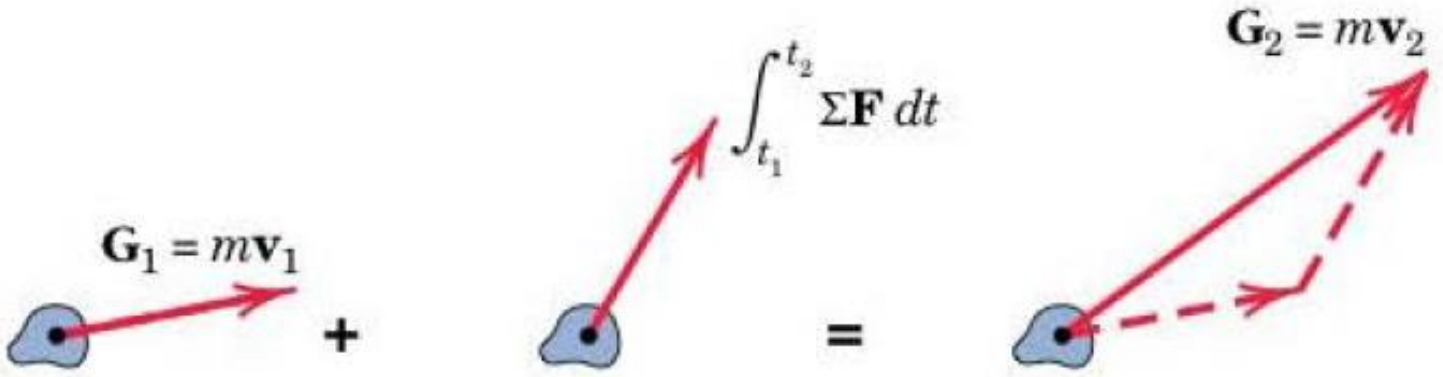
$Imp_{1\to 2} = 0$ (In some direction) $\Rightarrow$ Conservation of Momentum (In that direction)

Average Impulsive Force

$$\bar{F}_{AVE} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \bar{F}\, dt = \frac{1}{\Delta t}\, Imp_{1\to 2}$$

$$\Rightarrow \quad \boxed{Imp_{1\to 2} = \bar{F}_{AVE}\,\Delta t}$$

Impulse – Momentum Diagram

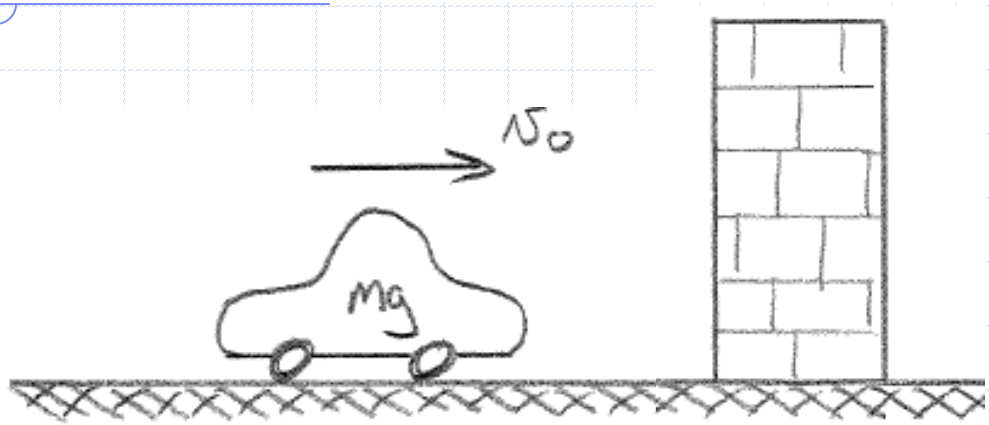


initial momentum $m\mathbf{v}_1$

external linear impulses

final linear momentum $m\mathbf{v}_2$

Example:

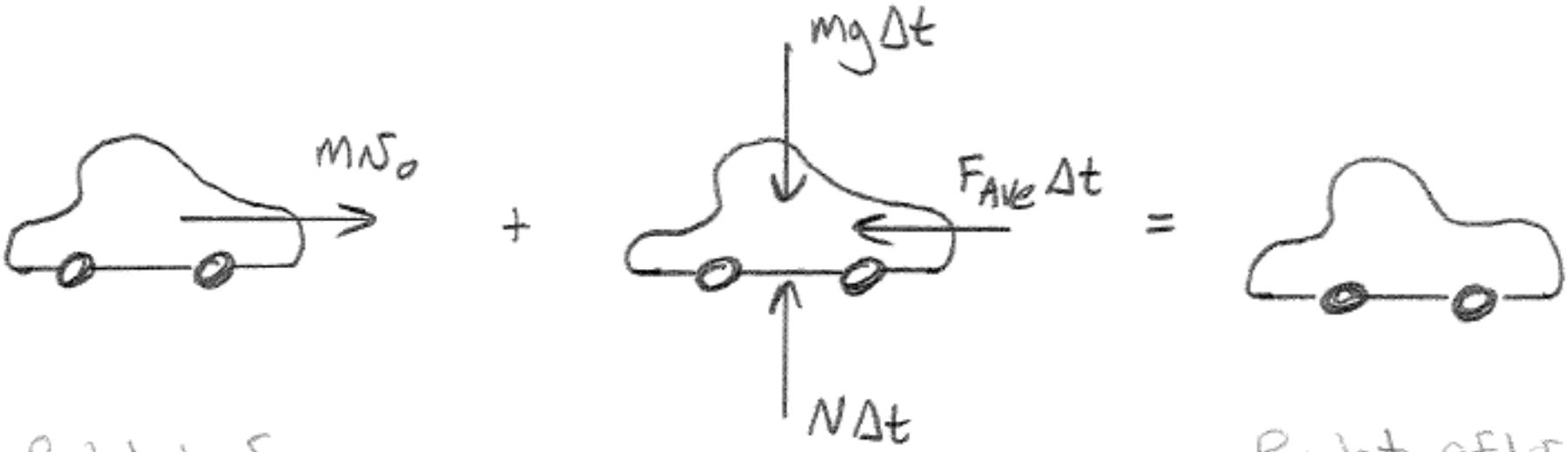


Data: $mg = 3000 \text{ lbs}$

$v_0 = 2.5 \text{ mph} = 3.667 \text{ ft/s}$

Car hits wall, comes to rest in $\Delta t = 75 \text{ ms}$.

Find F_{AVE} between wall and car bumper during collision.

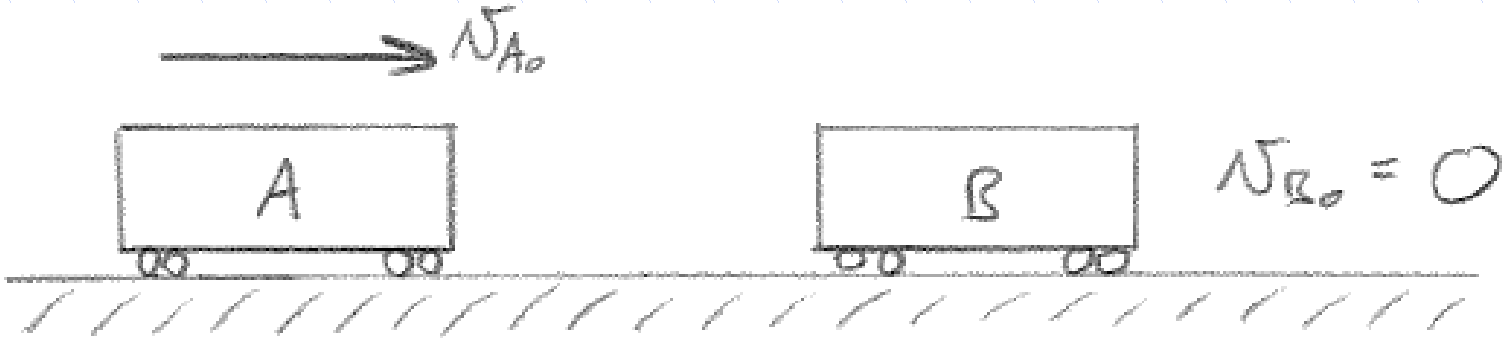


$$\xrightarrow{+} \Sigma \text{Mom:} \quad mv_o - F_{Ave}\Delta t = 0$$

$$\Rightarrow \quad F_{Ave} = \frac{mv_o}{\Delta t} = \frac{mgv_o}{g\Delta t}$$

$$F_{Ave} = \frac{(3000\,\text{lbf})(3.667\,\text{ft/s})}{(32.174\,\text{ft/s}^2)(0.075\,\text{s})} = 4558.55\,\text{lbf}$$

Example:



A hits B and they become coupled.

Data: $m_A = 45000\ kg$

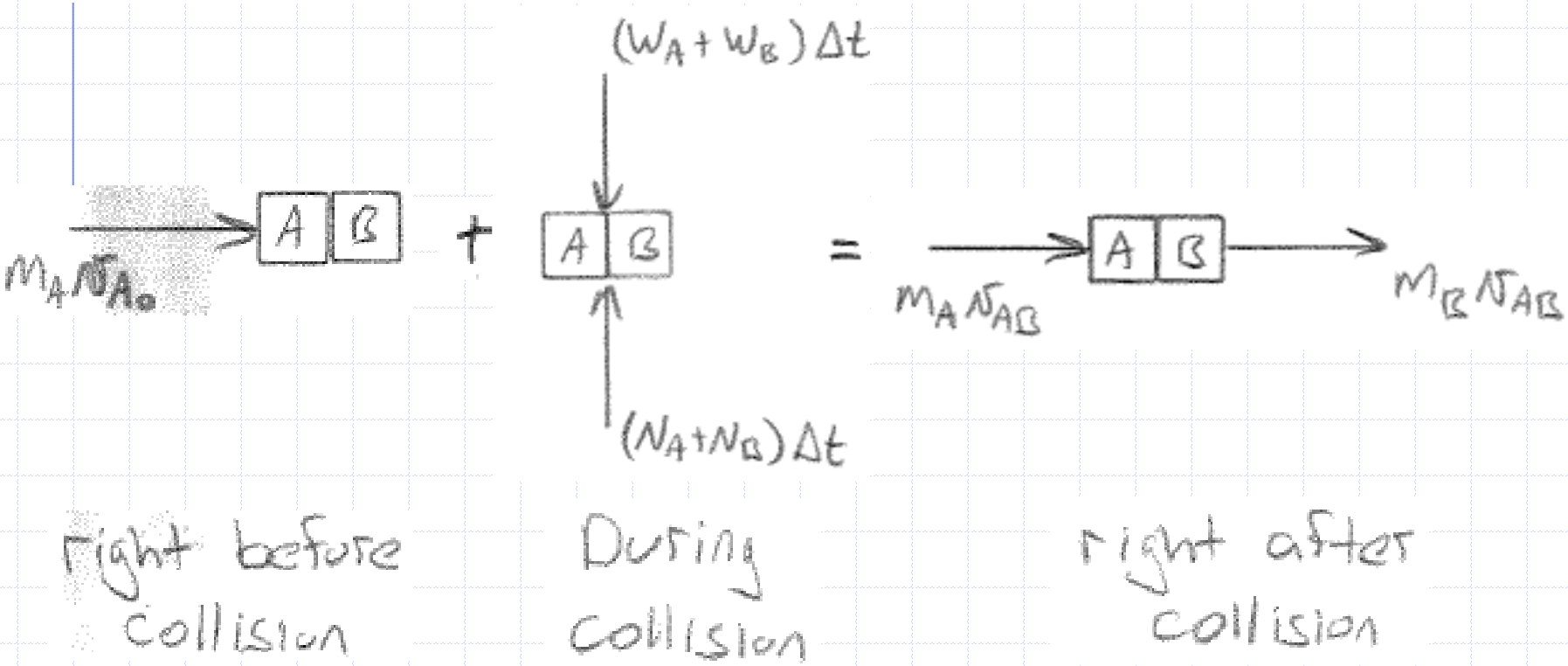
$m_B = 35000\ kg$

$v_{A_0} = 3\ km/hr = 0.8333\ m/s$

Determine:

(A) Final velocity of the coupled cars

(B) Magnitude of Ave. impulsive force acting on each car if coupling takes 0.5 s.



$$\rightarrow \oplus \sum \text{Mom}: \quad m_A v_{A_0} + 0 = (m_A + m_B) v_{AB}$$

$$\Rightarrow \quad v_{AB} = \frac{m_A}{m_A + m_B} v_{A_0}$$

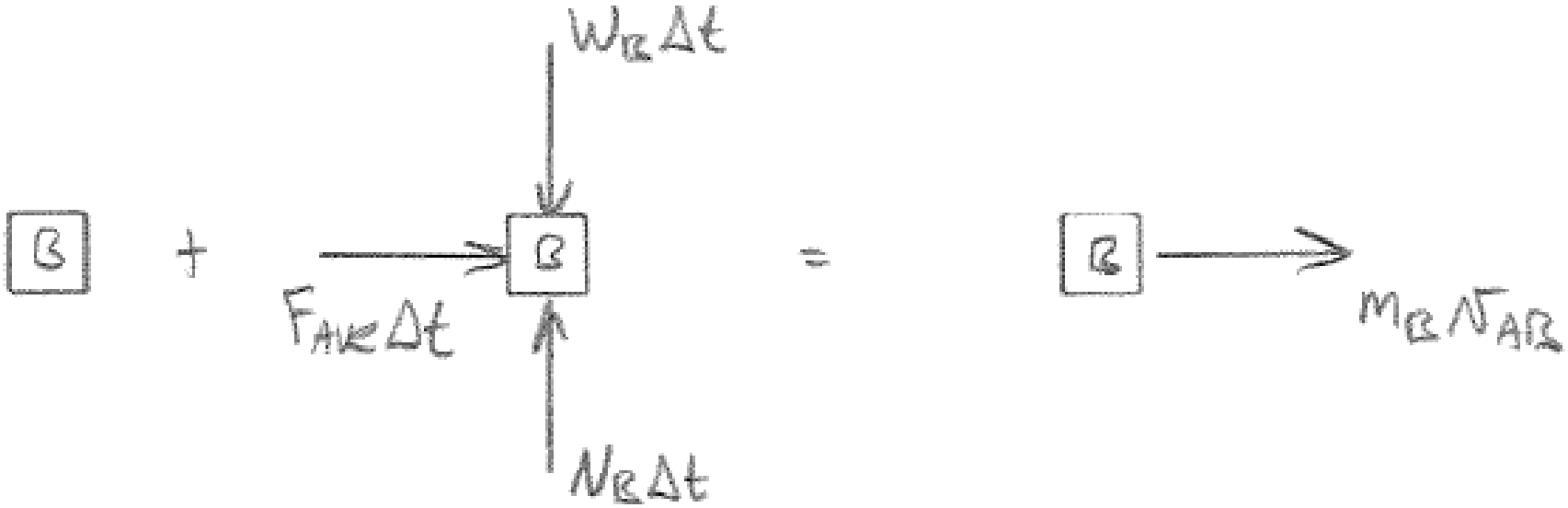
$$v_{AB} = \frac{(45000\ \text{kg})(0.8333\ \text{m/s})}{(45000\ \text{kg}) + (35000\ \text{kg})}$$

$$= 0.5357\ \text{m/s}$$

$$= 1.93\ \text{km/hr}$$

$$\therefore \quad \overline{v}_{AB} = 1.93\ \text{km/hr} \rightarrow$$

PART B: Invoke PIM with B as system.

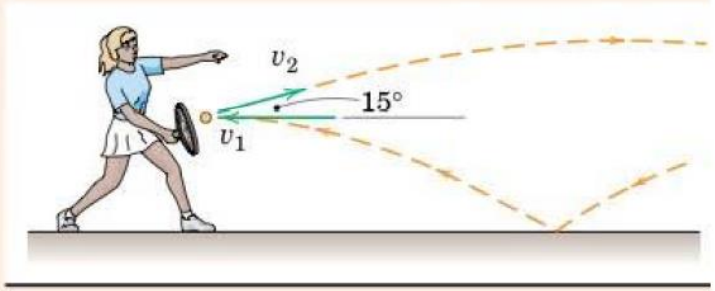


$\rightarrow \oplus \Sigma M_{OM}$: $\quad 0 + F_{AVE} \Delta t = m_B \Delta v_{AB}$

$$\Rightarrow \quad F_{AVE} = \frac{m_B \Delta v_{AB}}{\Delta t}$$

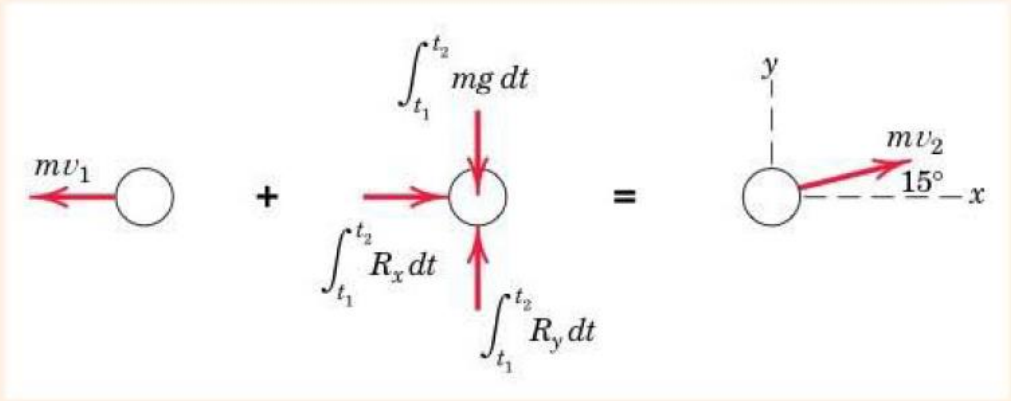
$$F_{AVE} = \frac{(25000\,Kg)(0.5357\,m/s)}{(0.3\,s)} \left(\frac{1\,KN}{1000\,N}\right) = 44.643\,KN$$

Sample Problem 3/19



A tennis player strikes the tennis ball with her racket when the ball is at the uppermost point of its trajectory as shown. The horizontal velocity of the ball just before impact with the racket is $v_1 = 50$ ft/sec and just after impact its velocity is $v_2 = 70$ ft/sec directed at the 15° angle as shown. If the 4-oz ball is in contact with the racket for 0.02 sec, determine the magnitude of the average force $\mathbf{R}$ exerted by the racket on the ball. Also determine the angle β made by $\mathbf{R}$ with the horizontal.

Solution. We construct the impulse-momentum diagrams for the ball as follows:



①

Helpful Hints

① Recall that for the impulse-momentum diagrams, initial linear momentum goes in the first diagram, all external linear impulses go in the second diagram, and final linear momentum goes in the third diagram.

② $[m(v_x)_1 + \int_{t_1}^{t_2} \Sigma F_x\, dt = m(v_x)_2]$ $\quad -\frac{4/16}{32.2}(50) + R_x(0.02) = \frac{4/16}{32.2}(70\cos 15^\circ)$

$[m(v_y)_1 + \int_{t_1}^{t_2} \Sigma F_y\, dt = m(v_y)_2]$

$$\frac{4/16}{32.2}(0) + R_y(0.02) - (4/16)(0.02) = \frac{4/16}{32.2}(70\sin 15^\circ)$$

We can now solve for the impact forces as

$$R_x = 45.7 \text{ lb}$$

$$R_y = 7.28 \text{ lb}$$

We note that the impact force $R_y = 7.28$ lb is considerably larger than the 0.25-lb weight of the ball. Thus, the weight mg, a nonimpulsive force, could have been neglected as small in comparison with R_y. Had we neglected the weight, the computed value of R_y would have been 7.03 lb.

We now determine the magnitude and direction of $\mathbf{R}$ as

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{45.7^2 + 7.28^2} = 46.2 \text{ lb} \qquad \textit{Ans.}$$

$$\beta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{7.28}{45.7} = 9.06^\circ \qquad \textit{Ans.}$$

② For the linear impulse $\int_{t_1}^{t_2} R_x\, dt$, the average impact force R_x is a constant, so that it can be brought outside the integral sign, resulting in $R_x \int_{t_1}^{t_2} dt = R_x(t_2 - t_1) = R_x \Delta t$. The linear impulse in the y-direction has been similarly treated.

Given

- Ball weight:

$$W = 4\ \text{oz} = \frac{4}{16} = 0.25\ \text{lb}, \qquad m = \frac{W}{g} = \frac{0.25}{32.2}\ \text{slugs}.$$

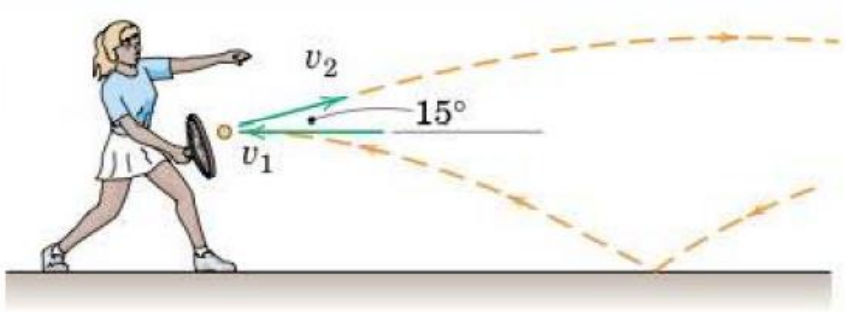
- Before impact: $v_1 = 50\ \text{ft/s}$, horizontally **to the left**.
- After impact: $v_2 = 70\ \text{ft/s}$, 15° **above the positive horizontal**.
- Contact duration: $\Delta t = 0.020\ \text{s}$.

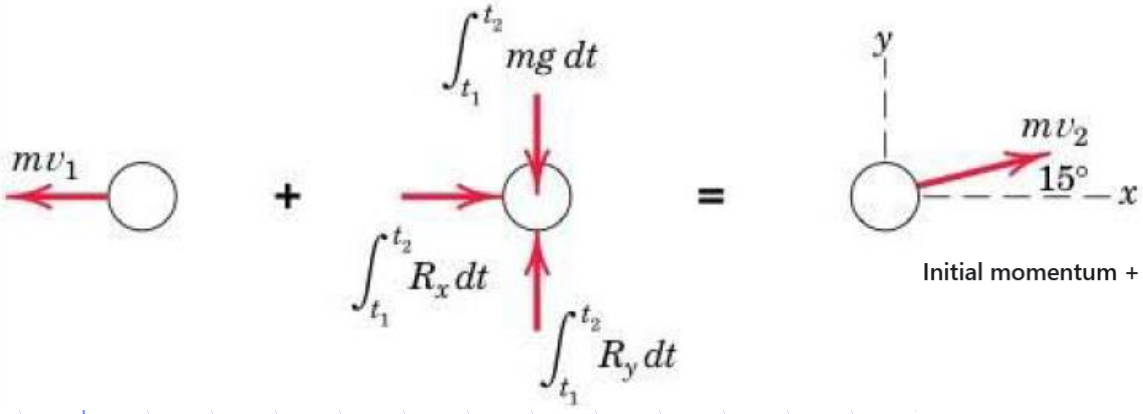
Required: magnitude R and direction β of the **average force exerted by the racket on the ball**.

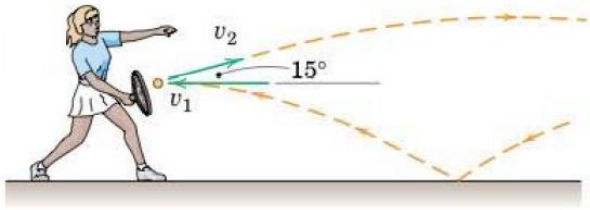
Strategy

1. Draw the impulse–momentum diagram.
2. Apply linear impulse–momentum in x and y.
3. Determine R_x, R_y, then calculate R and β.

Take $+x$ rightward and $+y$ upward.







Initial momentum + external impulses = final momentum

$$m\mathbf{v}_1 + \int_{t_1}^{t_2} \sum \mathbf{F}\,dt = m\mathbf{v}_2.$$

The racket's average force components satisfy

$$\int_{t_1}^{t_2} R_x(t)\,dt = \overline{R}_x \Delta t, \qquad \int_{t_1}^{t_2} R_y(t)\,dt = \overline{R}_y \Delta t.$$

For simplicity, denote these averages by R_x and R_y.

Horizontal direction

$$m(v_x)_1 + R_x \Delta t = m(v_x)_2$$

$$\boxed{-m(50) + R_x(0.020) = m(70\cos 15^\circ)}.$$

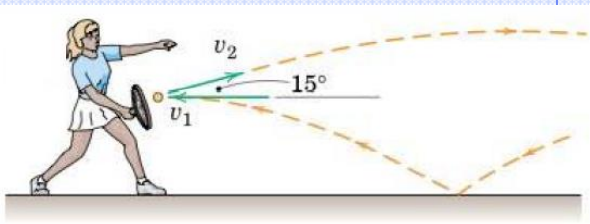
Vertical direction

$$m(v_y)_1 + R_y \Delta t - W \Delta t = m(v_y)_2$$

$$\boxed{m(0) + R_y(0.020) - 0.25(0.020) = m(70\sin 15^\circ)}.$$

Sign reminder: the incoming ball moves left, and its vertical velocity is zero at the highest point.

Average racket force and direction



From the horizontal equation:

$$R_x = \frac{m(70\cos 15^\circ + 50)}{0.020}$$

$$\boxed{R_x = 45.7 \text{ lb}}.$$

From the vertical equation:

$$R_y = \frac{m(70\sin 15^\circ)}{0.020} + 0.25$$

$$\boxed{R_y = 7.28 \text{ lb}}.$$

Therefore,

$$\boxed{R = \sqrt{R_x^2 + R_y^2} = 46.2 \text{ lb}}$$

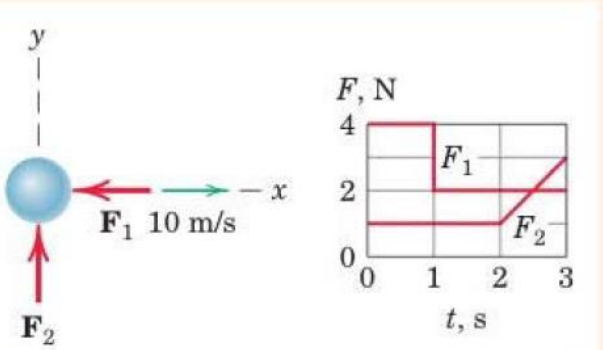
and

$$\boxed{\beta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = 9.06^\circ \text{ above the positive } x\text{-axis}}.$$

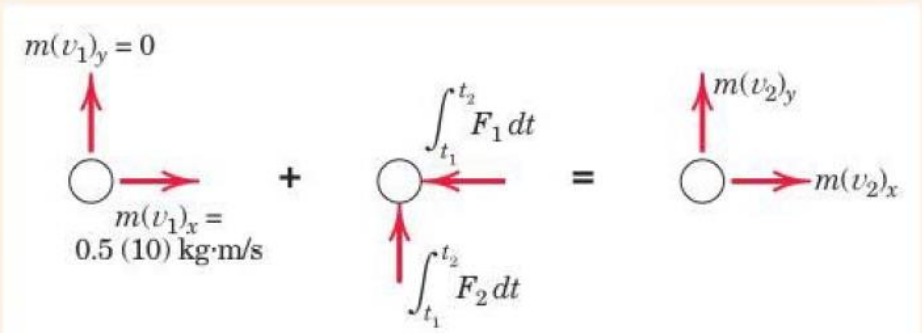
Closing observation: the force angle β differs from the outgoing velocity angle 15°, because impulse depends on the **change in momentum**, not just the final momentum.

Sample Problem 3/21

A particle with a mass of 0.5 kg has a velocity of 10 m/s in the x-direction at time $t = 0$. Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the particle, and their magnitudes change with time according to the graphical schedule shown. Determine the velocity $\mathbf{v}_2$ of the particle at the end of the 3-s interval. The motion occurs in the horizontal x-y plane.



Solution. First, we construct the impulse-momentum diagrams as shown.



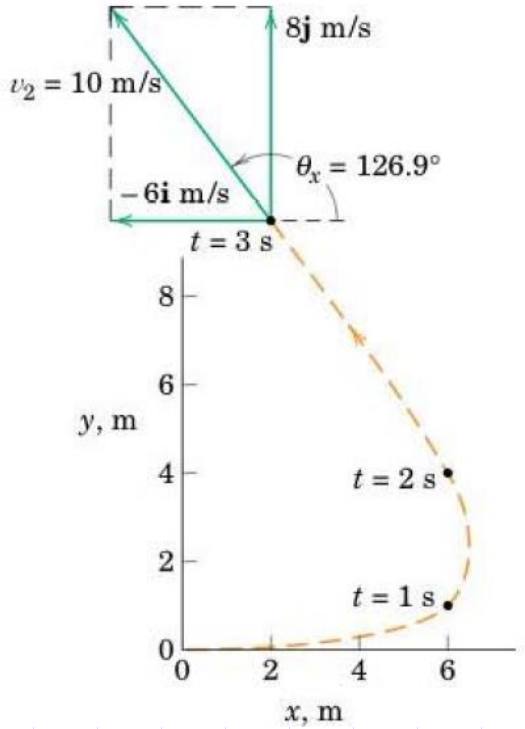
Then the impulse-momentum equations follow as

① $[m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x \, dt = m(v_2)_x]$ $\quad 0.5(10) - [4(1) + 2(3 - 1)] = 0.5(v_2)_x$

$(v_2)_x = -6$ m/s

$[m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y \, dt = m(v_2)_y]$ $\quad 0.5(0) + [1(2) + 2(3 - 2)] = 0.5(v_2)_y$

$(v_2)_y = 8$ m/s



Thus,

$$\mathbf{v}_2 = -6\mathbf{i} + 8\mathbf{j} \text{ m/s} \qquad \text{and} \qquad v_2 = \sqrt{6^2 + 8^2} = 10 \text{ m/s}$$

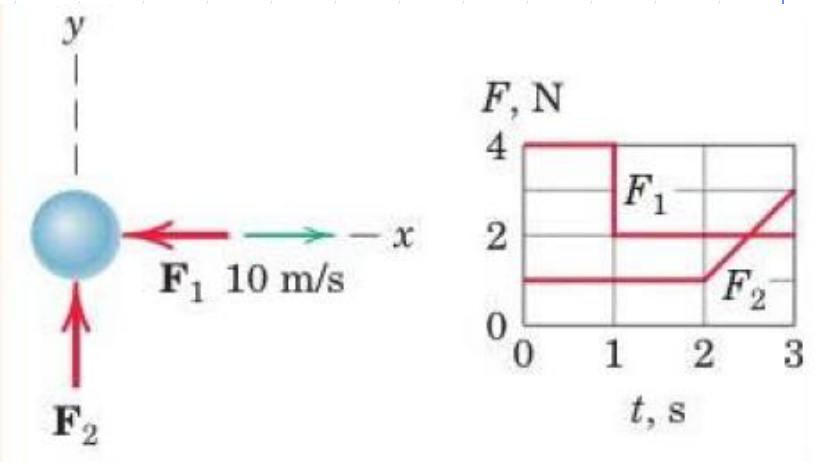
$$\theta_x = \tan^{-1}\frac{8}{-6} = 126.9^\circ \qquad \textit{Ans.}$$

Although not called for, the path of the particle for the first 3 seconds is plotted in the figure. The velocity at $t = 3$ s is shown together with its components.

Helpful Hint

① The impulse in each direction is the corresponding area under the force-time graph. Note that F_1 is in the negative x-direction, so its impulse is negative.

Given



- Mass: $m = 0.5\ \text{kg}$.
- Initial velocity: $\mathbf{v}_1 = 10\mathbf{i}\ \text{m/s}$.
- Time interval: $0 \le t \le 3\ \text{s}$.
- F_1 acts in the **negative x-direction**.
- F_2 acts in the **positive y-direction**.
- Force magnitudes vary according to the supplied graph.
- Motion occurs in the horizontal x–y plane.

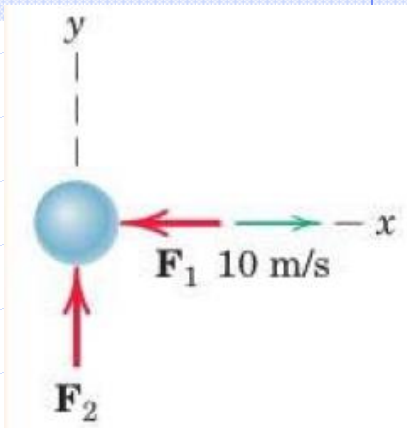
Required: velocity $\mathbf{v}_2$, its magnitude, and its direction at $t = 3\ \text{s}$.

Strategy

1. Calculate each impulse from the area under its force–time curve.
2. Apply impulse–momentum separately in x and y.
3. Combine the final velocity components.

Place the small particle-and-force diagram beside these points.

Impulses from the force–time graph

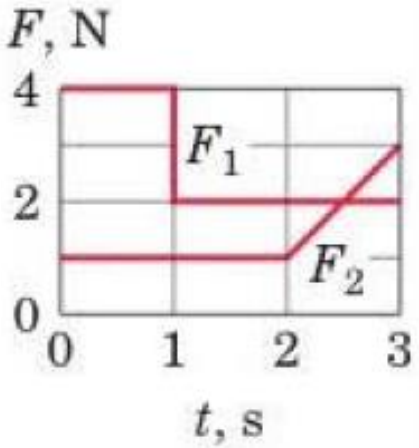


For F_1, the area comprises two rectangles:

$$J_1 = \int_0^3 F_1\,dt = 4(1) + 2(3-1) = \boxed{8\ \text{N}\cdot\text{s}}.$$

Since F_1 acts leftward:

$$\boxed{\mathbf{J}_1 = -8\mathbf{i}\ \text{N}\cdot\text{s}}.$$



For F_2, the area comprises a rectangle and a trapezoid:

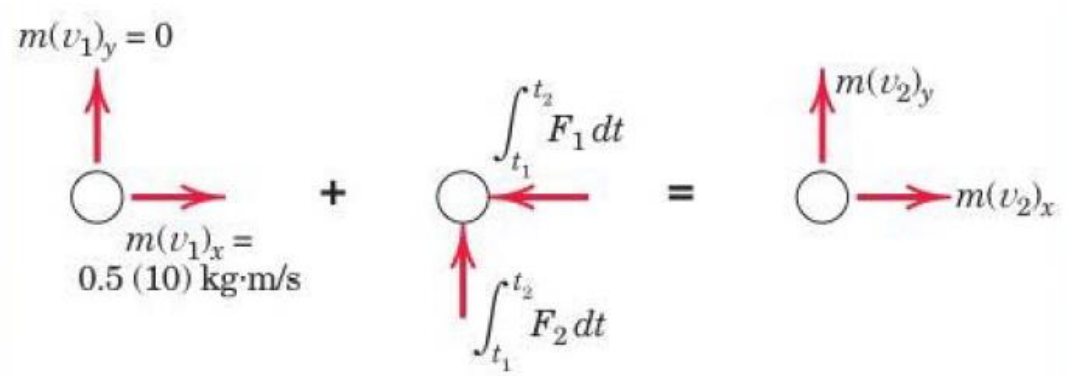
$$J_2 = \int_0^3 F_2\,dt = 1(2) + \frac{1+3}{2}(3-2) = \boxed{4\ \text{N}\cdot\text{s}}.$$

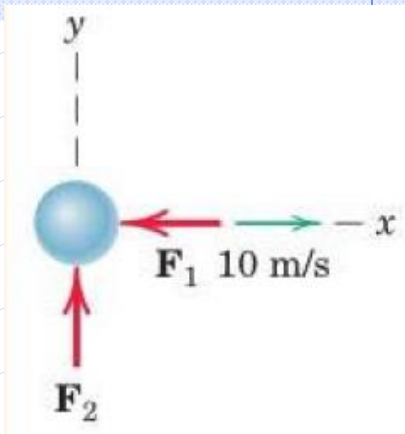
Since F_2 acts upward in the diagram:

$$\boxed{\mathbf{J}_2 = +4\mathbf{j}\ \text{N}\cdot\text{s}}.$$

Key point: the graph gives force **magnitudes**; assign impulse signs from the force directions.

Impulse–momentum diagram and equations





$$\boxed{m\mathbf{v}_1 + \int_0^3 \sum \mathbf{F}\,dt = m\mathbf{v}_2}$$

In the x-direction

$$m(v_1)_x - J_1 = m(v_2)_x$$

$$\boxed{0.5(10) - 8 = 0.5(v_2)_x}.$$

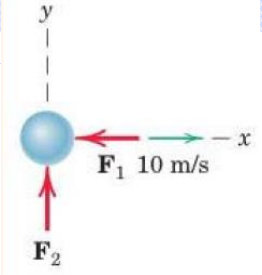
In the y-direction

$$m(v_1)_y + J_2 = m(v_2)_y$$

$$\boxed{0.5(0) + 4 = 0.5(v_2)_y}.$$

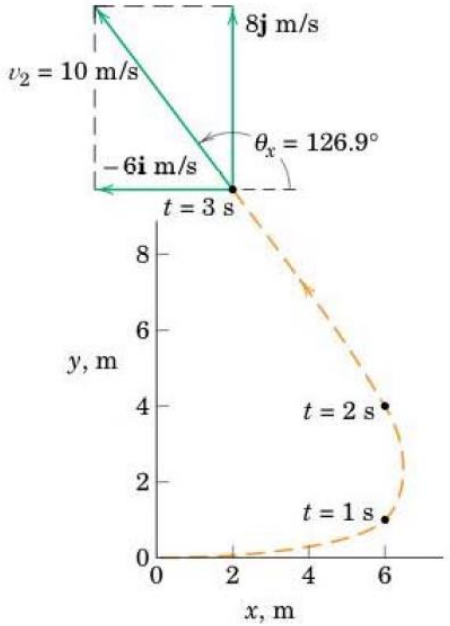
The final component arrows in the supplied diagram represent **assumed positive directions**. A negative answer means the actual component points oppositely.

Final velocity and direction



Solving the component equations:

$$\boxed{(v_2)_x = -6 \text{ m/s}}, \qquad \boxed{(v_2)_y = 8 \text{ m/s}}.$$



Therefore,

$$\boxed{\mathbf{v}_2 = -6\mathbf{i} + 8\mathbf{j} \text{ m/s}},$$

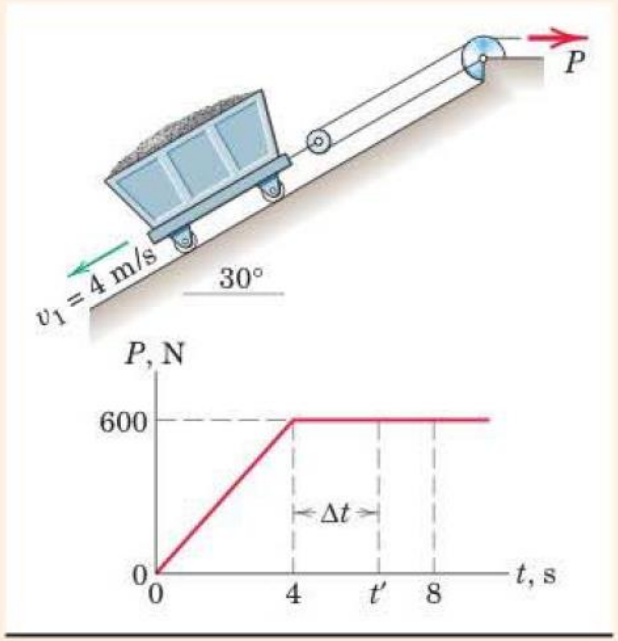
$$\boxed{v_2 = \sqrt{(-6)^2 + 8^2} = 10 \text{ m/s}}.$$

Because $v_x < 0$ and $v_y > 0$, the velocity lies in **quadrant II**:

$$\theta = 180^\circ - \tan^{-1}\left(\frac{8}{6}\right) = \boxed{126.9^\circ \text{ counterclockwise from } +x}.$$

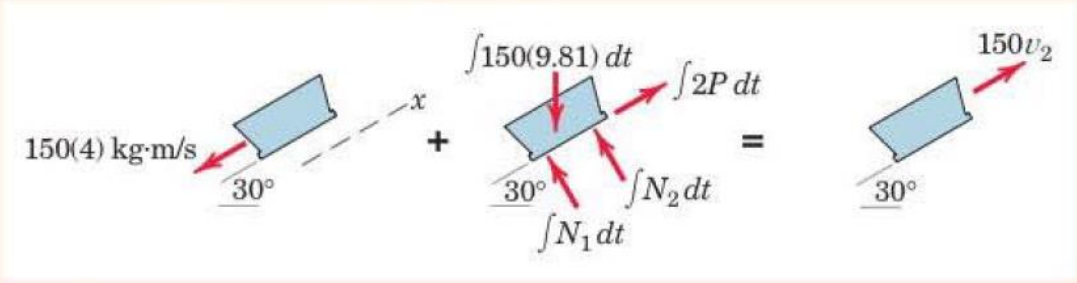
Closing observation: the final speed equals the initial speed, but the velocity has changed because its direction has changed.

Sample Problem 3/22



The loaded 150-kg skip is rolling down the incline at 4 m/s when a force P is applied to the cable as shown at time $t = 0$. The force P is increased uniformly with the time until it reaches 600 N at $t = 4$ s, after which time it remains constant at this value. Calculate (a) the time t' at which the skip reverses its direction and (b) the velocity v of the skip at $t = 8$ s. Treat the skip as a particle.

Solution. The stated variation of P with the time is plotted, and the impulse-momentum diagrams of the skip are drawn.



Part (a). The skip reverses direction when its velocity becomes zero. We will assume that this condition occurs at $t = 4 + \Delta t$ s. The impulse-momentum equation applied consistently in the positive x-direction gives

$$m(v_1)_x + \int \Sigma F_x \, dt = m(v_2)_x$$

① $$150(-4) + \tfrac{1}{2}(4)(2)(600) + 2(600)\Delta t - 150(9.81)\sin 30°(4 + \Delta t) = 150(0)$$

$$\Delta t = 2.46 \text{ s} \qquad t' = 4 + 2.46 = 6.46 \text{ s} \qquad \textit{Ans.}$$

Part (b). Applying the momentum equation to the entire 8-s interval gives

$$m(v_1)_x + \int \Sigma F_x \, dt = m(v_2)_x$$

$$150(-4) + \tfrac{1}{2}(4)(2)(600) + 4(2)(600) - 150(9.81)\sin 30°(8) = 150(v_2)_x$$

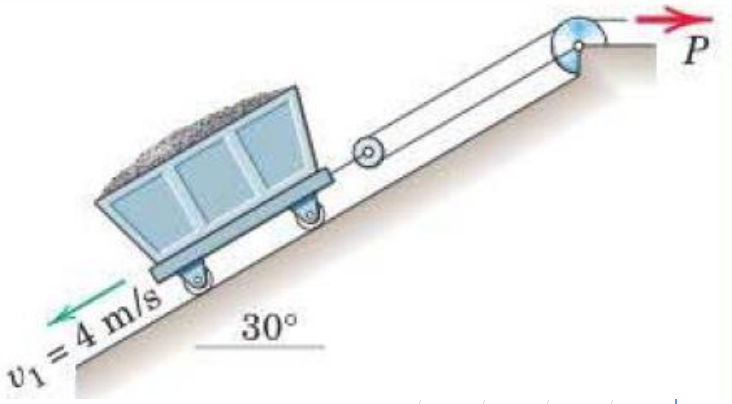
$$(v_2)_x = 4.76 \text{ m/s} \qquad \textit{Ans.}$$

The same result is obtained by analyzing the interval from t' to 8 s.

Helpful Hint

① The impulse-momentum diagram keeps us from making the error of using the impulse of P rather than $2P$ or of forgetting the impulse of the component of the weight. The first term in the linear impulse is the triangular area of the P-t relation for the first 4 s, doubled for the force of $2P$.

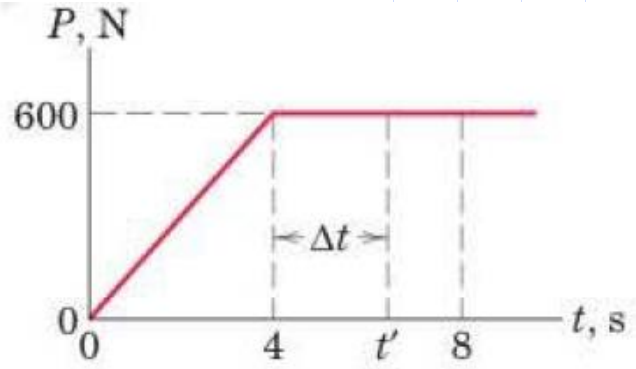
Given



- Skip mass: $m = 150\ \text{kg}$
- Incline angle: 30°
- Initial velocity: $4\ \text{m/s}$ **down the incline**
- Applied cable force:

$$P(t) = \begin{cases} 150t\ \text{N}, & 0 \le t \le 4\ \text{s}, \\ 600\ \text{N}, & 4 \le t \le 8\ \text{s}. \end{cases}$$

- Treat the skip as a particle; neglect rolling resistance and use ideal pulleys and cable.

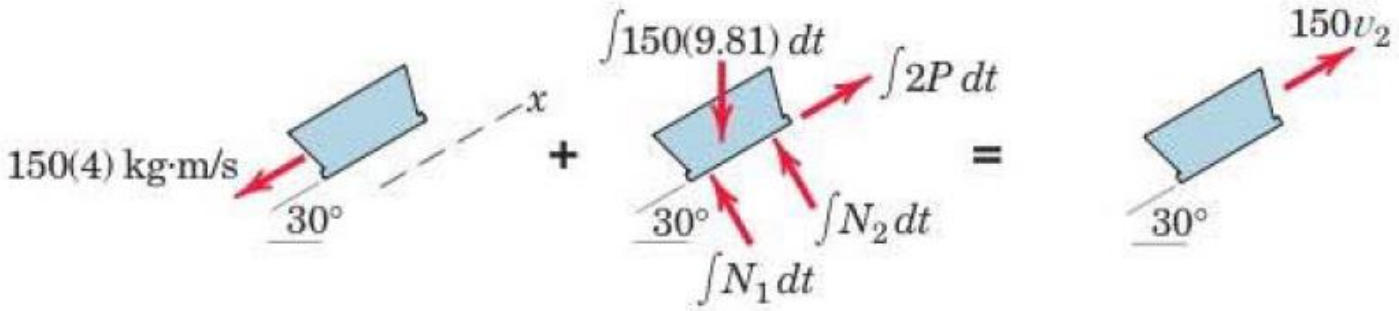


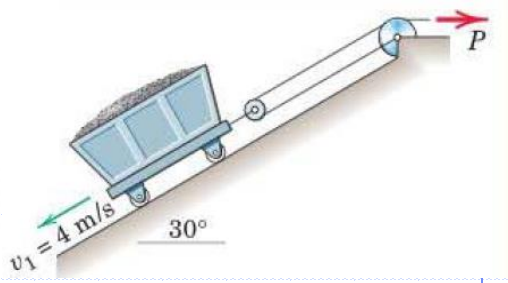
Required

1. Time t' when the skip reverses direction.
2. Velocity at $t = 8\ \text{s}$.

Strategy

- Take $+x$ **up the incline**, so $v_{1x} = -4\ \text{m/s}$.
- Obtain cable impulse from **twice the area under the P–t graph**.
- Apply impulse–momentum along the incline.
- At reversal, $v_x = 0$.

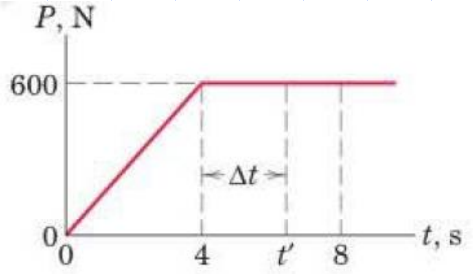




Forces on the skip

- **Cable pull:** two parallel cable segments each exert P, giving $2P$ uphill.
- **Weight:** mg vertically downward; its downhill component is

$$mg\sin 30^\circ = 150(9.81)\sin 30^\circ = 735.75\ \text{N}.$$



- **Normal reactions:** N_1, N_2, perpendicular to the incline. They contribute no x-impulse.

Therefore,

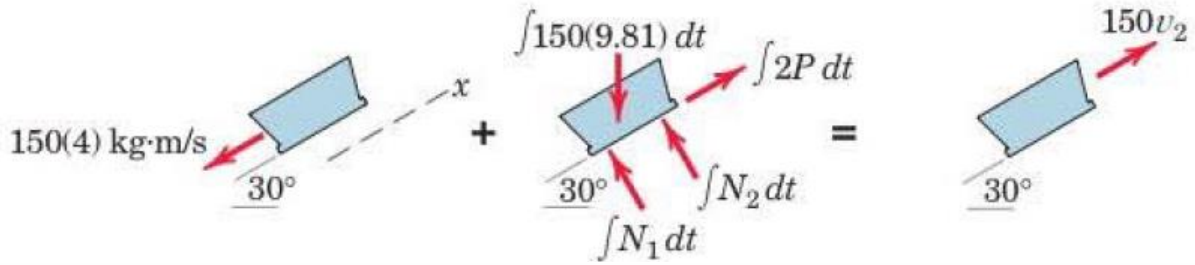
$$\sum F_x = 2P - mg\sin 30^\circ.$$

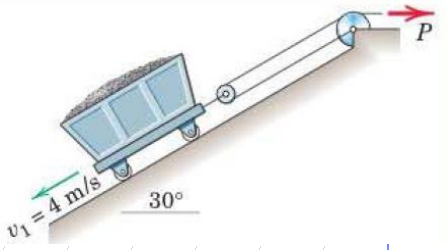
The impulse–momentum equation from $t = 0$ to any time t is

$$mv_{1x} + \int_0^t \sum F_x\, dt = mv_x(t),$$

$$\boxed{-600 + 2\int_0^t P\, dt - 735.75t = 150v_x(t)}.$$

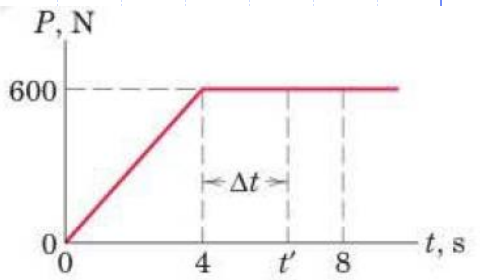
Diagram reminder: initial momentum points downhill; cable impulse points uphill; the weight component contributes a downhill impulse.





Part (a): Time of reversal

First check the velocity at the end of the force ramp:



$$-600 + 2\left[\frac{1}{2}(4)(600)\right] - 735.75(4) = 150v_x(4),$$

$$\boxed{v_x(4) = -7.62 \text{ m/s}}.$$

The skip is still moving downhill. Since the net force increases steadily during the ramp, changing from downhill to uphill, the skip has not reversed during this interval.

Let

$$t' = 4 + \Delta t.$$

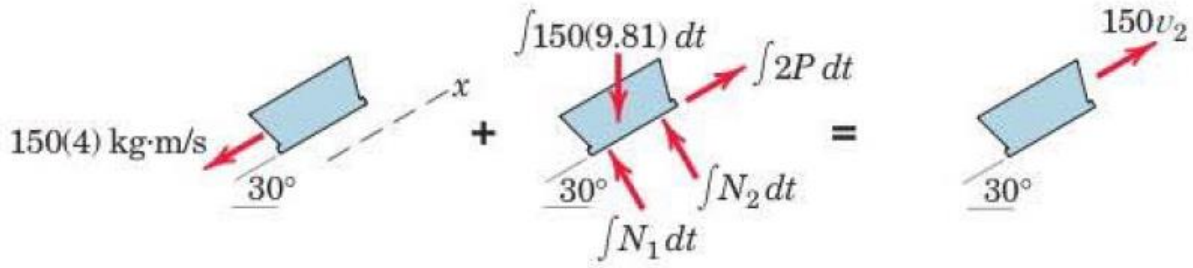
At reversal, $v_x(t') = 0$. Applying impulse–momentum over the entire interval:

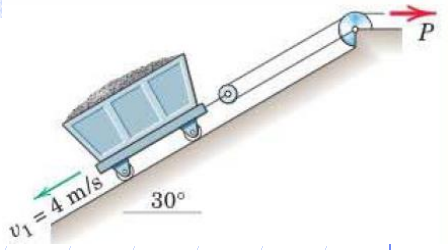
$$-600 + \underbrace{2\left[\frac{1}{2}(4)(600)\right]}_{\text{cable impulse, 0–4 s}} + \underbrace{2(600)\Delta t}_{\text{cable impulse after 4 s}} - \underbrace{735.75(4+\Delta t)}_{\text{weight-component impulse}} = 0.$$

Simplifying:

$$-1143 + 464.25\Delta t = 0,$$

$$\boxed{\Delta t = 2.46 \text{ s}}, \qquad \boxed{t' = 6.46 \text{ s}}.$$

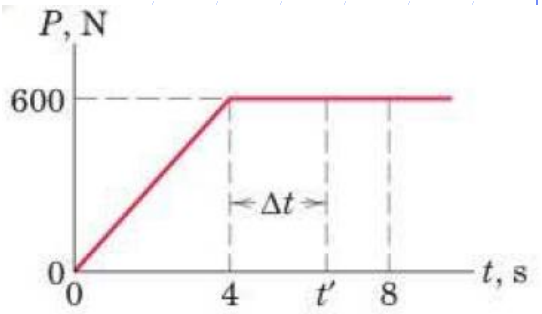




Part (b): Velocity at $t = 8$ s

Apply impulse–momentum over the full eight seconds:

$$mv_{1x} + \int_0^8 (2P - mg\sin 30^\circ)\, dt = mv_{2x}.$$



The cable impulse is twice the triangular-plus-rectangular area:

$$2\int_0^8 P\, dt = 2\left[\frac{1}{2}(4)(600) + (8-4)(600)\right] = 7200\ \text{N}\cdot\text{s}.$$

Hence,

$$150(-4) + 7200 - 150(9.81)\sin 30^\circ(8) = 150v_{2x},$$

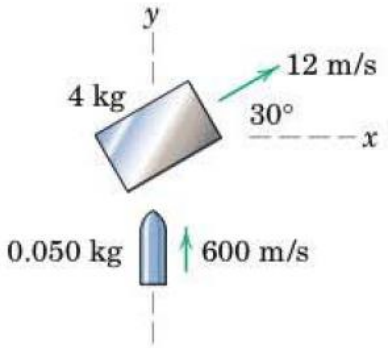
$$-600 + 7200 - 5886 = 150v_{2x},$$

$$\boxed{v_{2x} = +4.76\ \text{m/s}}.$$

The positive sign indicates **motion up the incline**.

Sample Problem 3/23

The 50-g bullet traveling at 600 m/s strikes the 4-kg block centrally and is embedded within it. If the block slides on a smooth horizontal plane with a velocity of 12 m/s in the direction shown prior to impact, determine the velocity $\mathbf{v}_2$ of the block and embedded bullet immediately after impact.



Solution. Since the force of impact is internal to the system composed of the block and bullet and since there are no other external forces acting on the system in the plane of motion, it follows that the linear momentum of the system is conserved. Thus,

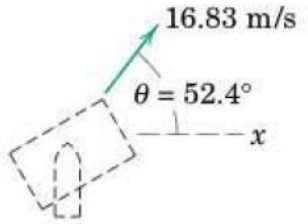
① $[\mathbf{G}_1 = \mathbf{G}_2]$ $\quad 0.050(600\mathbf{j}) + 4(12)(\cos 30^\circ\mathbf{i} + \sin 30^\circ\mathbf{j}) = (4 + 0.050)\mathbf{v}_2$

$$\mathbf{v}_2 = 10.26\mathbf{i} + 13.33\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

The final velocity and its direction are given by

$[v = \sqrt{v_x^2 + v_y^2}]$ $\quad v_2 = \sqrt{(10.26)^2 + (13.33)^2} = 16.83$ m/s $\qquad$ *Ans.*

$[\tan\theta = v_y/v_x]$ $\quad \tan\theta = \dfrac{13.33}{10.26} = 1.299 \quad \theta = 52.4^\circ \qquad$ *Ans.*



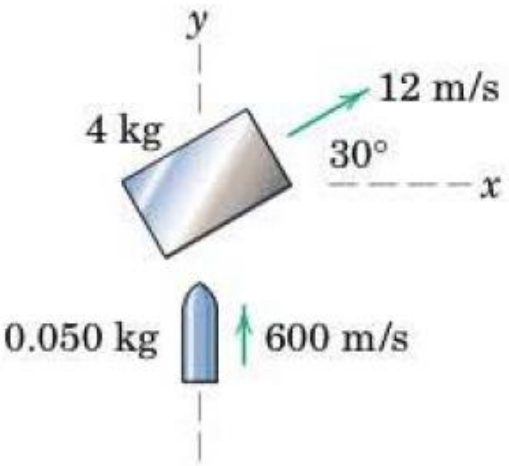
Helpful Hint

① Working with the vector form of the principle of conservation of linear momentum is clearly equivalent to working with the component form.

Given

- Bullet mass: $m_b = 50\ \text{g} = 0.050\ \text{kg}$
- Bullet velocity:

$$\mathbf{v}_{b1} = 600\mathbf{j}\ \text{m/s}$$



- Block mass: $m_B = 4\ \text{kg}$
- Initial block velocity:

$$\mathbf{v}_{B1} = 12\cos 30^\circ\,\mathbf{i} + 12\sin 30^\circ\,\mathbf{j}\ \text{m/s}$$

- Smooth horizontal plane.
- The bullet embeds in the block, so both have the same final velocity.

Required: common velocity $\mathbf{v}_2$, its magnitude, and its direction immediately after impact.

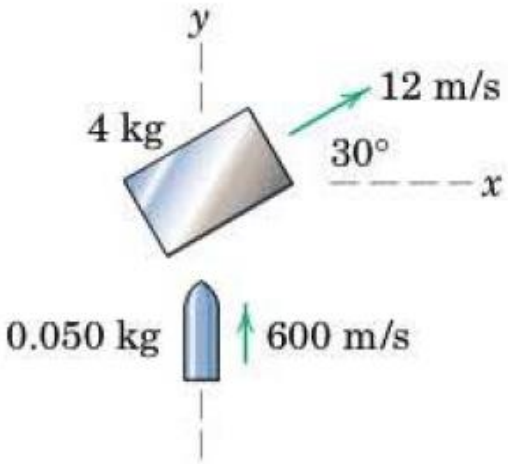
Strategy

1. Select the **bullet and block together** as the system.
2. Apply conservation of linear momentum in the horizontal x–y plane.
3. Determine the final velocity components, magnitude, and direction.

The bullet–block contact impulses are internal and cancel; weight and normal reactions act perpendicular to the plane, so there is no external impulse in x or y.

Thus,

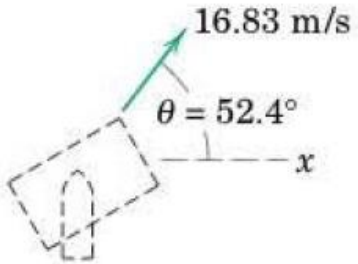
$$\mathbf{G}_1 + \underbrace{\mathbf{J}_{\text{ext},xy}}_{0} = \mathbf{G}_2,$$



$$\boxed{m_b\mathbf{v}_{b1} + m_B\mathbf{v}_{B1} = (m_b + m_B)\mathbf{v}_2}.$$

Substitute:

$$0.050(600\mathbf{j}) + 4(12)\,(\cos 30^\circ\,\mathbf{i} + \sin 30^\circ\,\mathbf{j}) = 4.050\mathbf{v}_2.$$

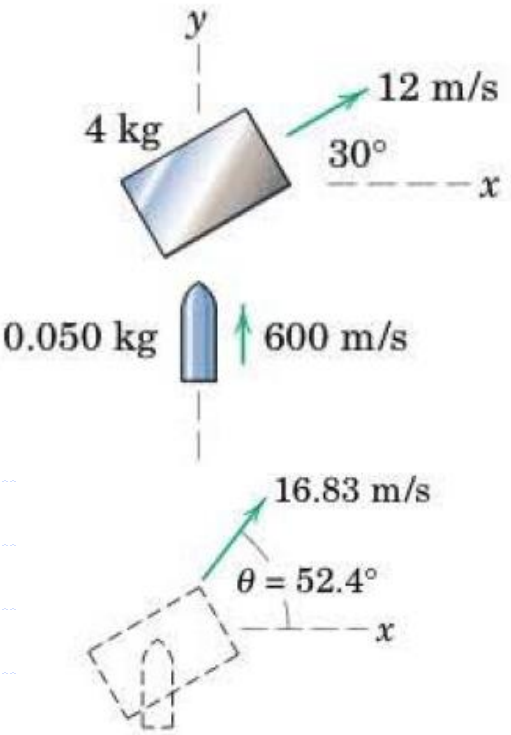


Collect components:

$$\boxed{41.569\mathbf{i} + 54.000\mathbf{j} = 4.050\mathbf{v}_2}.$$

Physical distinction: momentum is conserved, but kinetic energy is not conserved because the bullet embeds in the block.

Final velocity, magnitude, and direction



Divide by the combined mass:

$$\mathbf{v}_2 = \frac{41.569}{4.050}\mathbf{i} + \frac{54.000}{4.050}\mathbf{j},$$

$$\boxed{\mathbf{v}_2 = 10.26\mathbf{i} + 13.33\mathbf{j}\ \text{m/s}}.$$

The speed is

$$v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \boxed{16.83\ \text{m/s}}.$$

Both components are positive, placing the velocity in quadrant I:

$$\theta = \tan^{-1}\left(\frac{v_{2y}}{v_{2x}}\right) = \boxed{52.4^\circ\ \text{counterclockwise from}\ +x}.$$

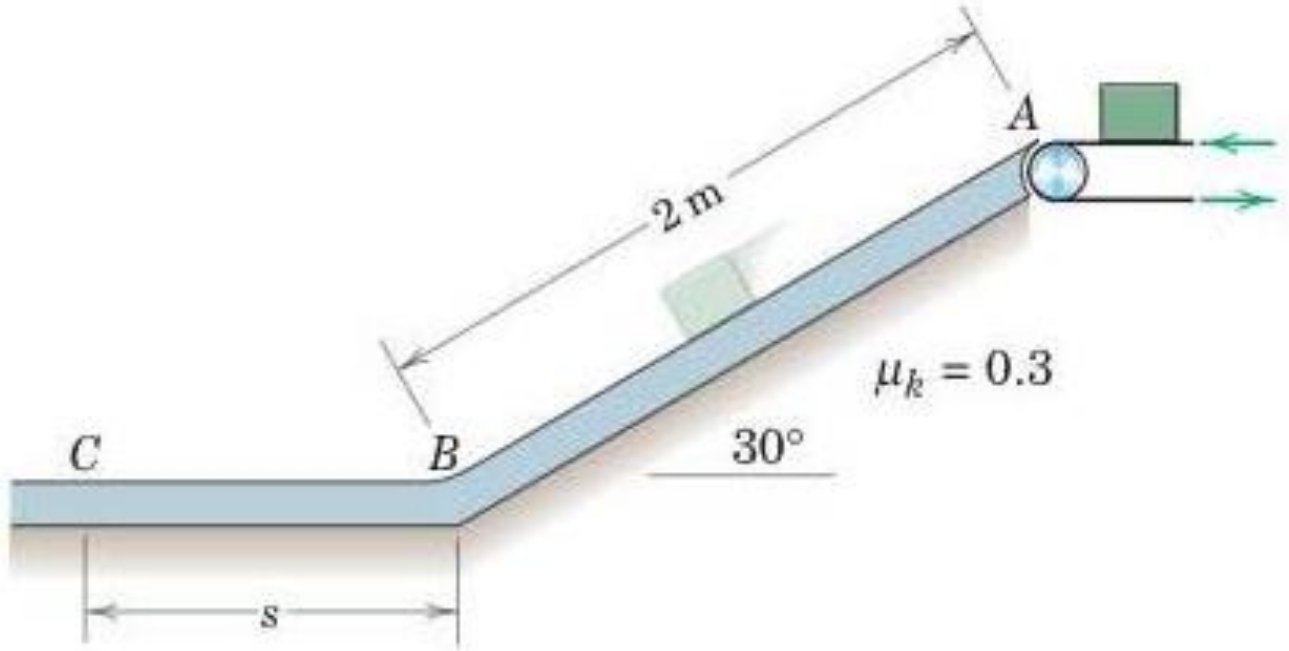
Use the final figure or a component triangle beside these results.

Closing observation: the bullet adds momentum in $+y$, turning the combined motion from the block's initial 30° direction toward $+y$.

Additional Problems



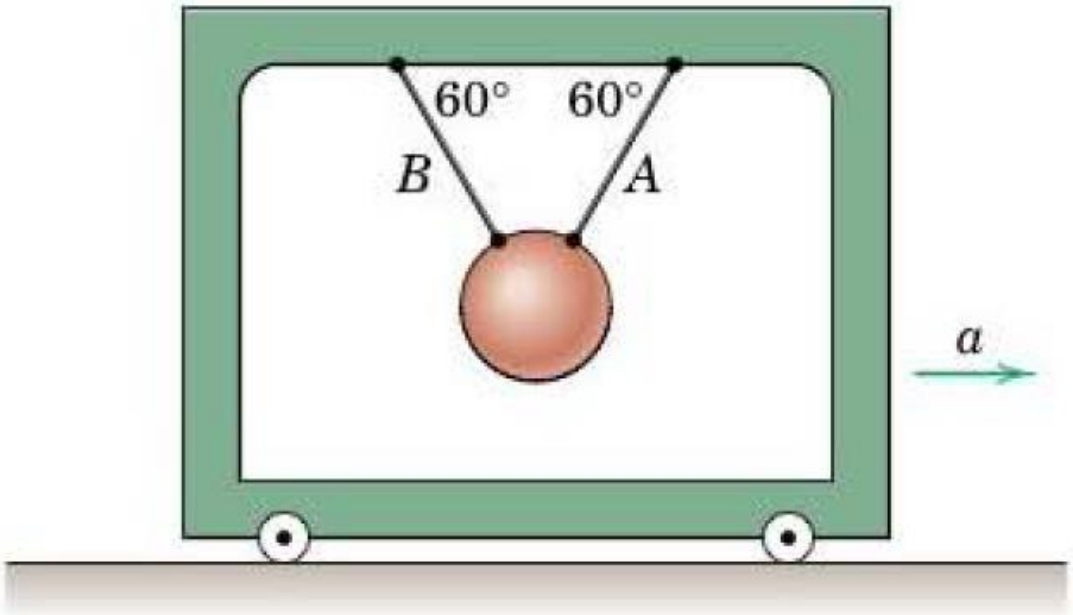
3/16 A small package is deposited by the conveyor belt onto the 30° ramp at A with a velocity of 0.8 m/s. Calculate the distance s on the level surface BC at which the package comes to rest. The coefficient of kinetic friction for the package and supporting surface from A to C is 0.3.



Problem 3/16

3/17 The steel ball is suspended from the accelerating frame by the two cords A and B. Determine the acceleration a of the frame which will cause the tension in A to be twice that in B.

$$Ans.\ a = \frac{g}{3\sqrt{3}}$$



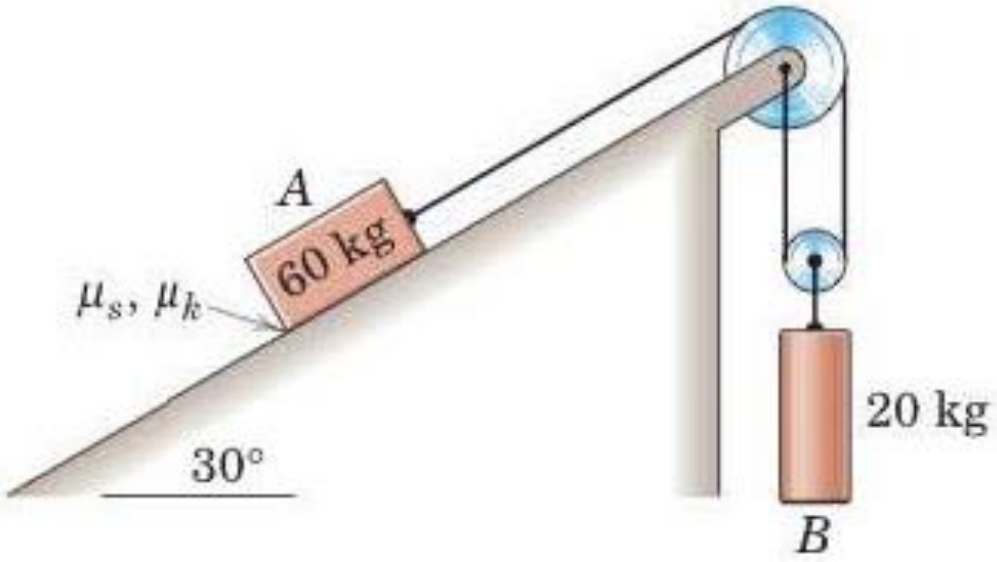
Problem 3/17

3/27 The system is released from rest with the cable taut. For the friction coefficients $\mu_s = 0.25$ and $\mu_k = 0.20$, calculate the acceleration of each body and the tension T in the cable. Neglect the small mass and friction of the pulleys.

Ans. $a_A = 1.450 \text{ m/s}^2$ down incline

$a_B = 0.725 \text{ m/s}^2$ up

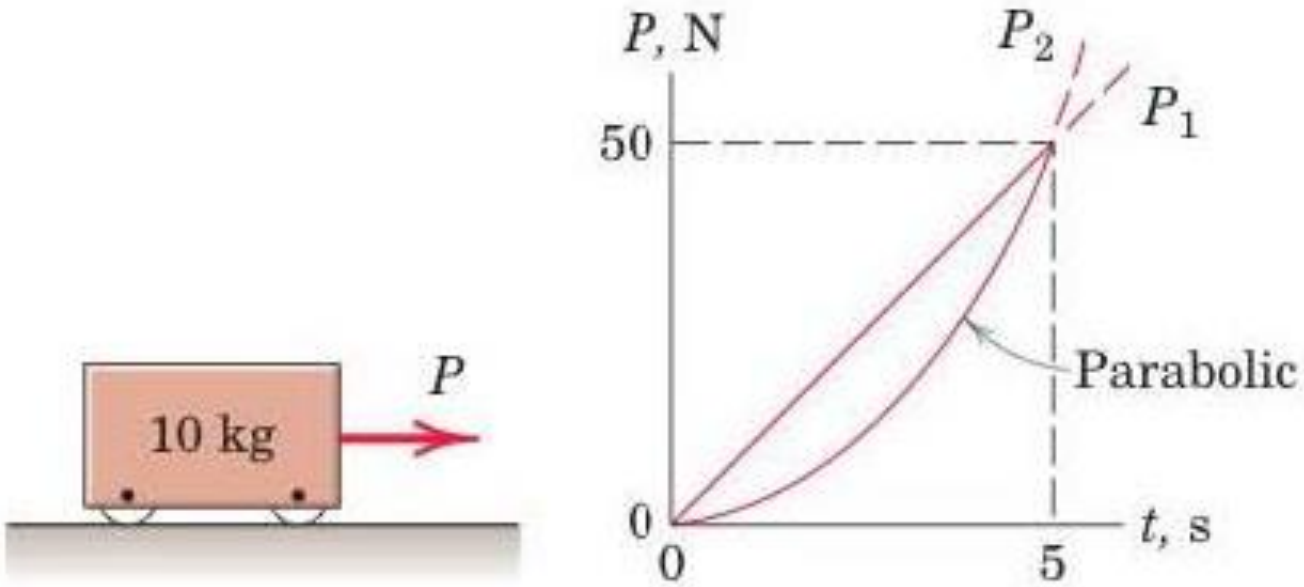
$T = 105.4 \text{ N}$



Problem 3/27

3/31 A force P is applied to the initially stationary cart. Determine the velocity and displacement at time $t = 5$ s for each of the force histories P_1 and P_2. Neglect friction.

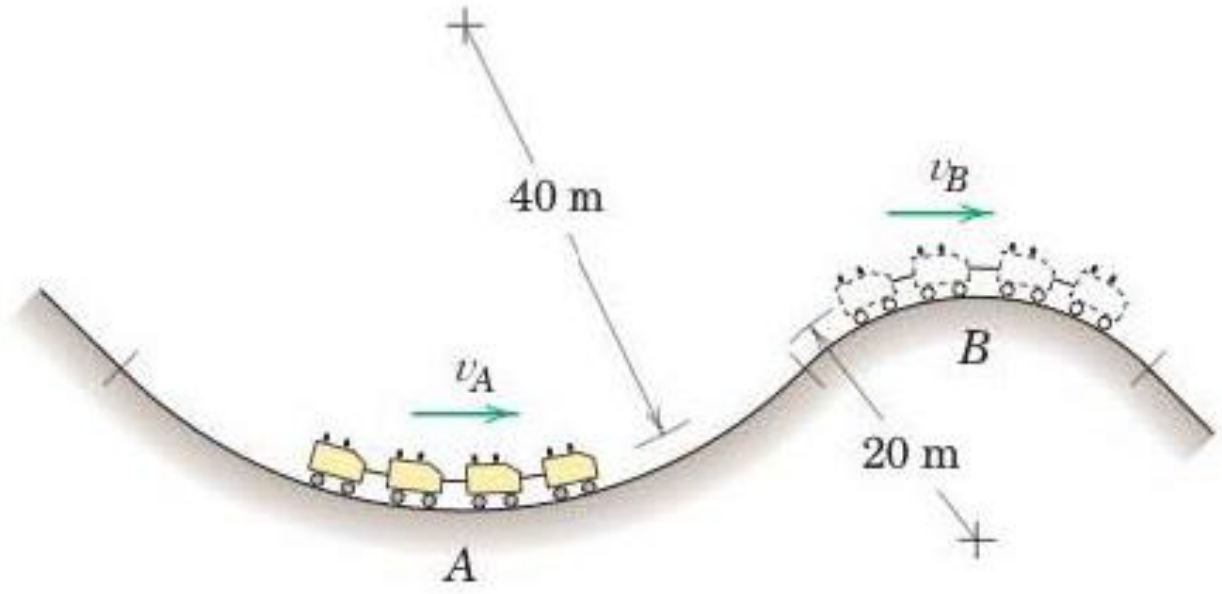
Ans. For P_1: $v = 12.5$ m/s, $s = 20.8$ m
For P_2: $v = 8.33$ m/s, $s = 10.42$ m



Problem 3/31

3/71 The cars of an amusement park ride have a speed $v_A = 22$ m/s at A and a speed $v_B = 12$ m/s at B. If a 75-kg rider sits on a spring scale (which registers the normal force exerted on it), determine the scale readings as the car passes points A and B. Assume that the person's arms and legs do not support appreciable force.

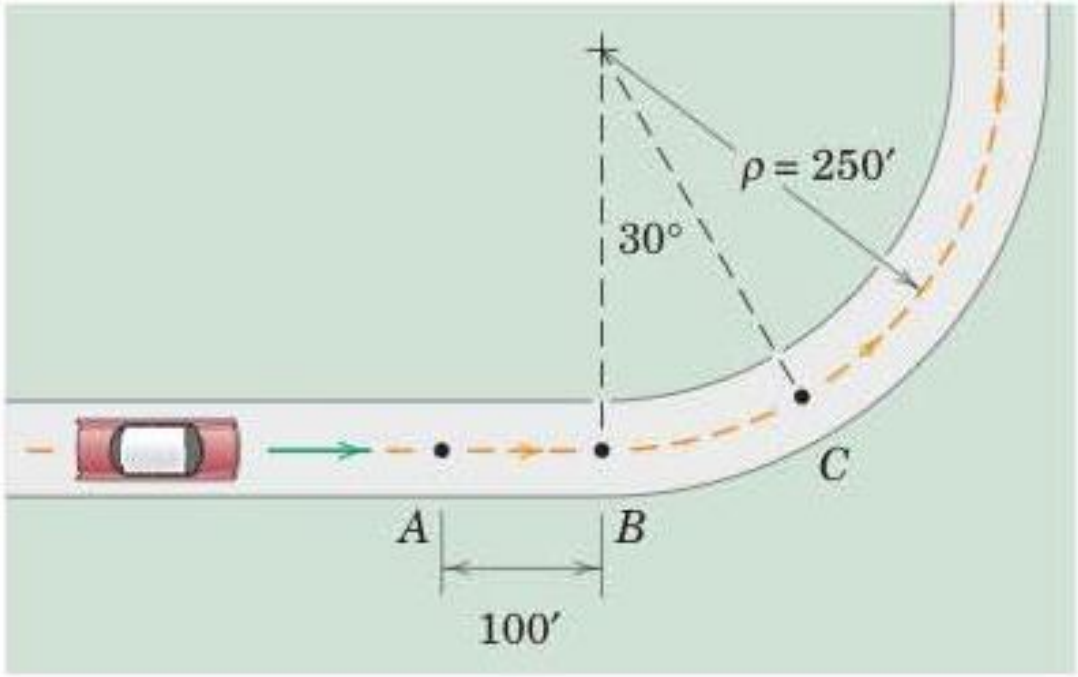
Ans. $N_A = 1643$ N, $N_B = 195.8$ N



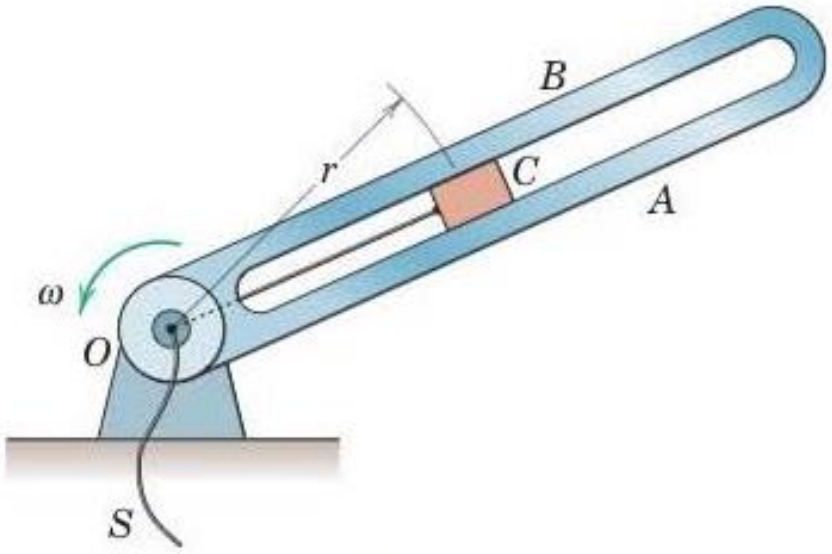
Problem 3/71

3/91 The 3000-lb car is traveling at 60 mi/hr on the straight portion of the road, and then its speed is reduced uniformly from A to C, at which point it comes to rest. Compute the magnitude F of the total friction force exerted by the road on the car (a) just before it passes point B, (b) just after it passes point B, and (c) just before it stops at point C.

Ans. (a) $F = 1562$ lb, (b) $F = 2260$ lb
(c) $F = 1562$ lb



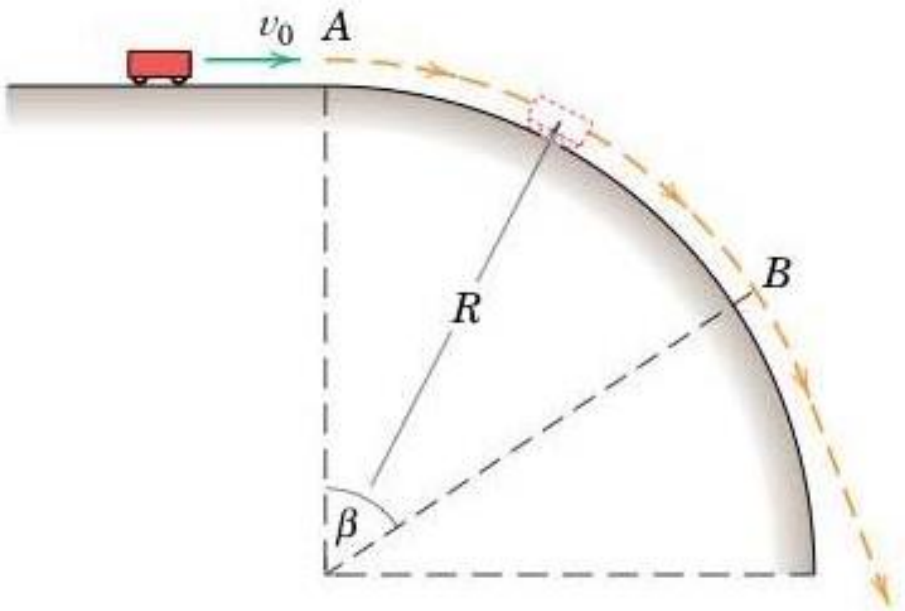
3/86 The slotted arm revolves in the horizontal plane about the fixed vertical axis through point O. The 3-lb slider C is drawn toward O at the constant rate of 2 in./sec by pulling the cord S. At the instant for which $r = 9$ in., the arm has a counterclockwise angular velocity $\omega = 6$ rad/sec and is slowing down at the rate of 2 rad/sec^2. For this instant, determine the tension T in the cord and the magnitude N of the force exerted on the slider by the sides of the smooth radial slot. Indicate which side, A or B, of the slot contacts the slider.



Problem 3/86

3/93 A small vehicle enters the top A of the circular path with a horizontal velocity v_0 and gathers speed as it moves down the path. Determine an expression for the angle β which locates the point where the vehicle leaves the path and becomes a projectile. Evaluate your expression for $v_0 = 0$. Neglect friction and treat the vehicle as a particle.

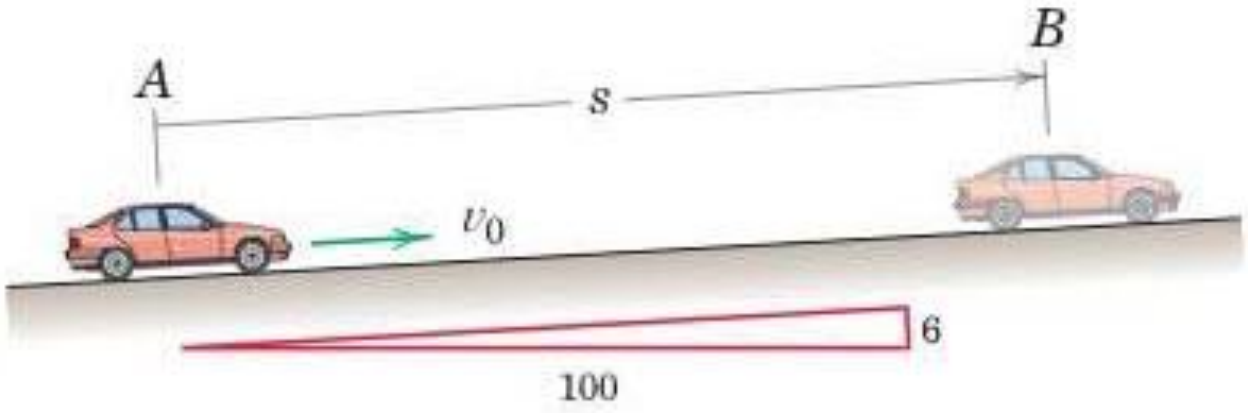
$$Ans.\ \beta = \cos^{-1}\left(\frac{2}{3} + \frac{{v_0}^2}{3gR}\right),\ \beta = 48.2°$$



Problem 3/93

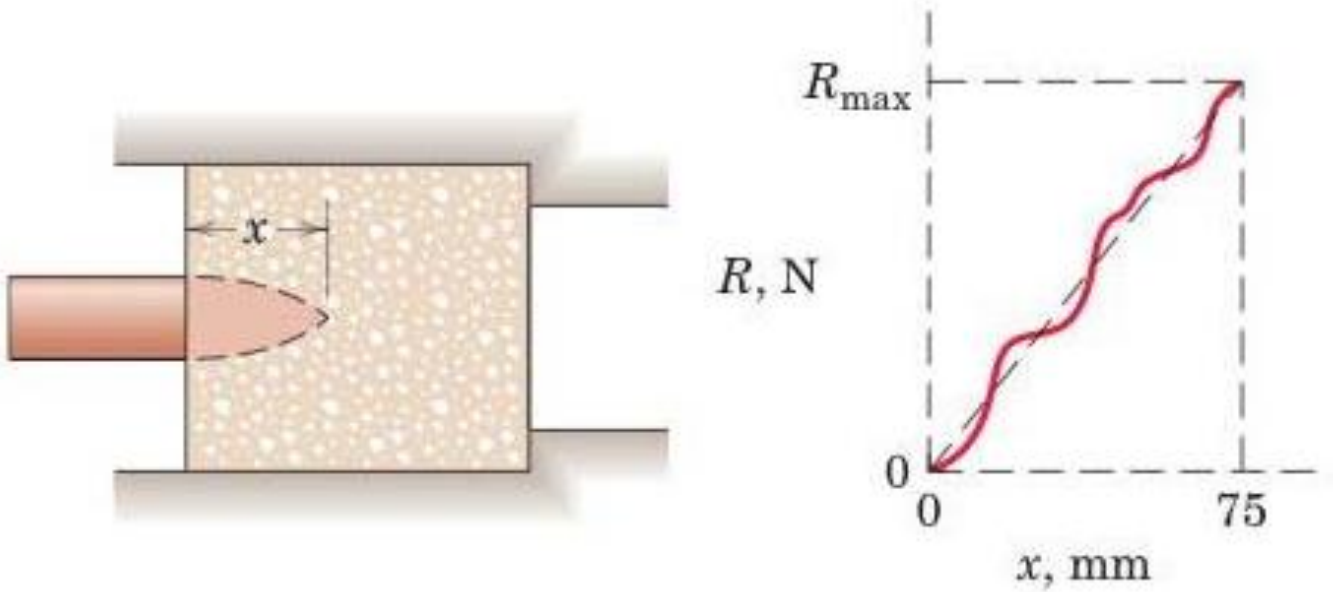
3/109 The car is moving with a speed $v_0 = 65$ mi/hr up the 6-percent grade, and the driver applies the brakes at point A, causing all wheels to skid. The coefficient of kinetic friction for the rain-slicked road is $\mu_k = 0.6$. Determine the stopping distance s_{AB}. Repeat your calculations for the case when the car is moving downhill from B to A.

Ans. $s_{AB} = 214$ ft, $s_{BA} = 262$ ft



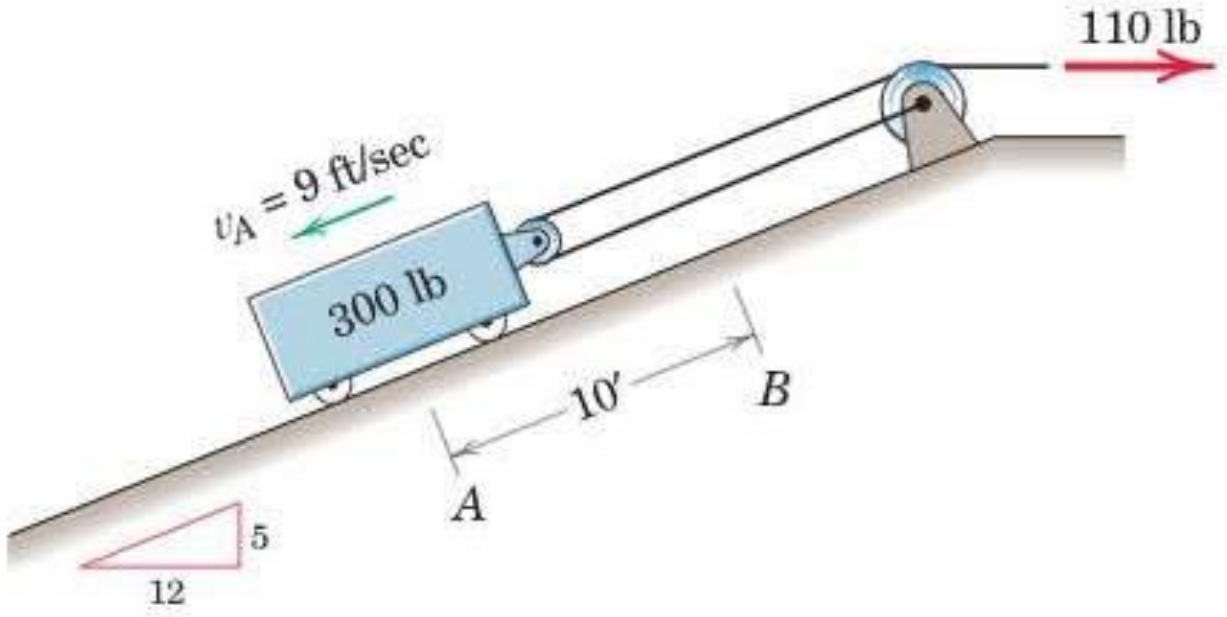
Problem 3/109

3/122 The resistance R to penetration x of a 0.25-kg projectile fired with a velocity of 600 m/s into a certain block of fibrous material is shown in the graph. Represent this resistance by the dashed line and compute the velocity v of the projectile for the instant when $x = 25$ mm if the projectile is brought to rest after a total penetration of 75 mm.



Problem 3/122

3/130 The 300-lb carriage has an initial velocity of 9 ft/sec down the incline at A, when a constant force of 110 lb is applied to the hoisting cable as shown. Calculate the velocity of the carriage when it reaches B. Show that in the absence of friction this velocity is independent of whether the initial velocity of the carriage at A was up or down the incline.



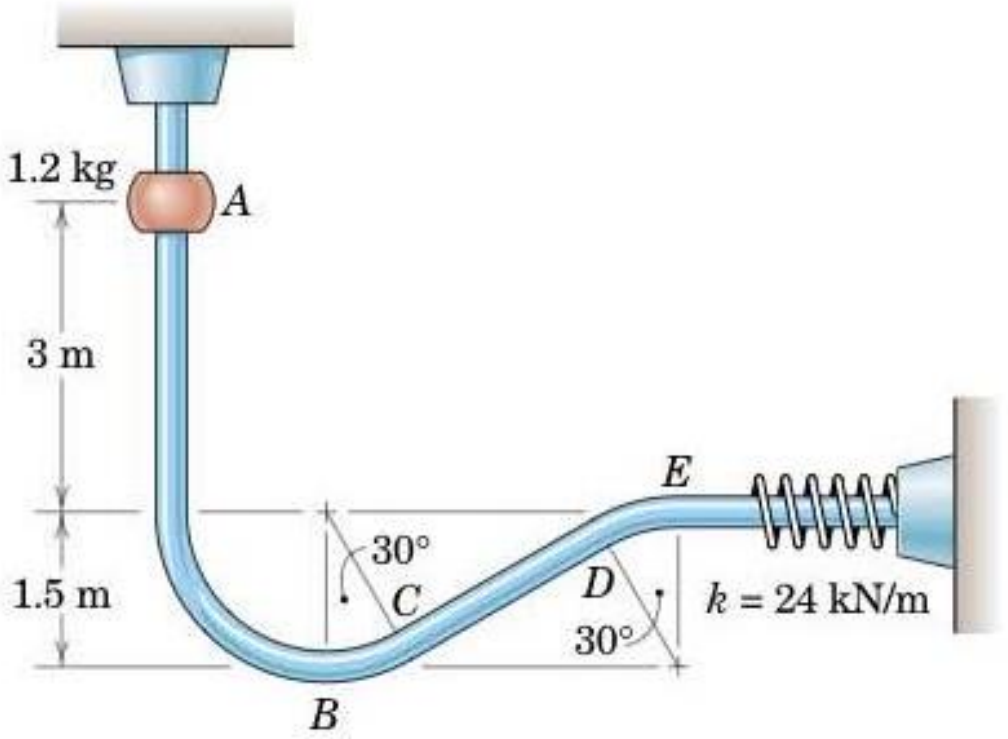
Problem 3/130

3/121 A car with a mass of 1500 kg starts from rest at the bottom of a 10-percent grade and acquires a speed of 50 km/h in a distance of 100 m with constant acceleration up the grade. What is the power P delivered to the drive wheels by the engine when the car reaches this speed?

Ans. $P = 40.4$ kW

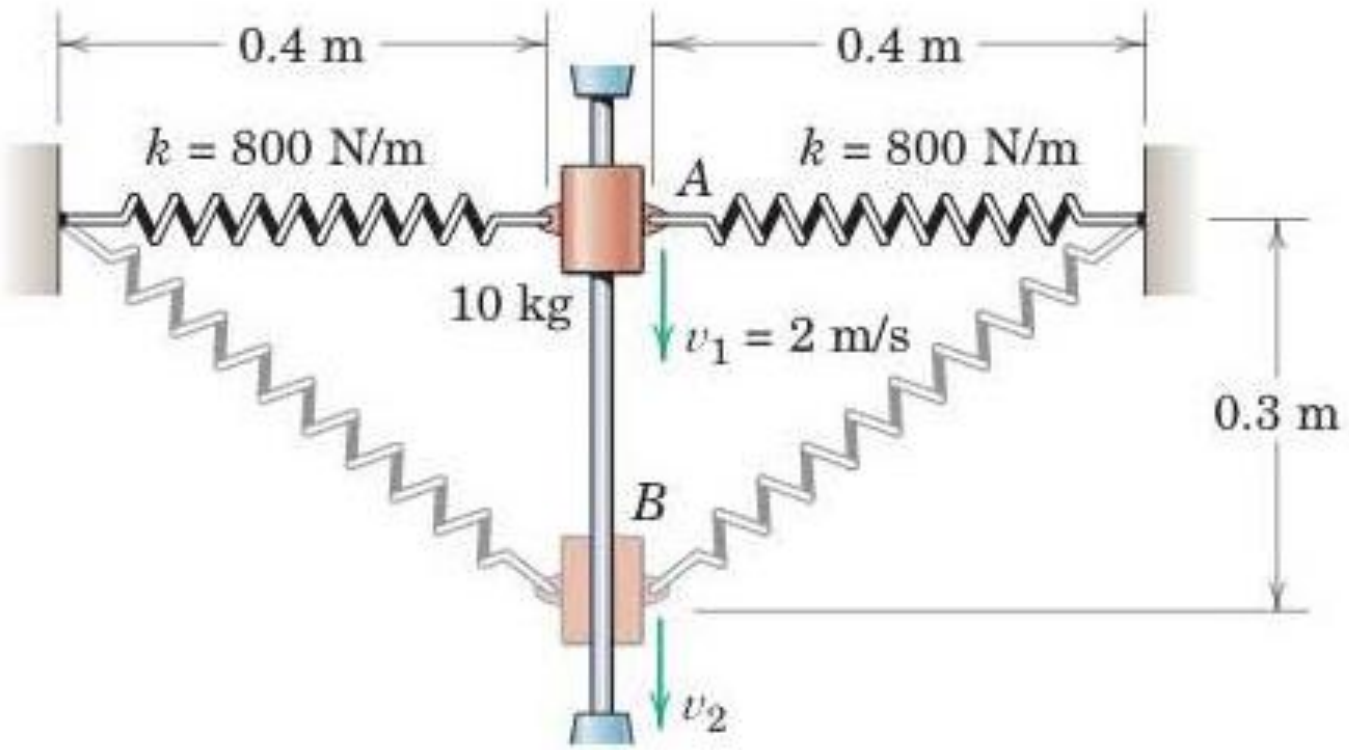
3/149 The 1.2-kg slider is released from rest in position A and slides without friction along the vertical-plane guide shown. Determine (a) the speed v_B of the slider as it passes position B and (b) the maximum deflection δ of the spring.

Ans. $(a)\ v_B = 9.40$ m/s, $(b)\ \delta = 54.2$ mm



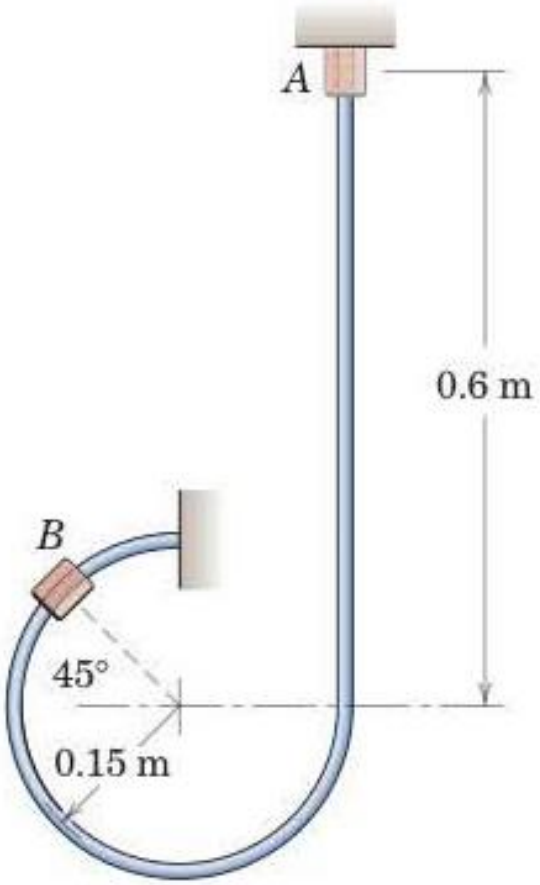
Problem 3/149

3/156 The 10-kg collar slides on the smooth vertical rod and has a velocity $v_1 = 2$ m/s in position A where each spring is stretched 0.1 m. Calculate the velocity v_2 of the collar as it passes point B.



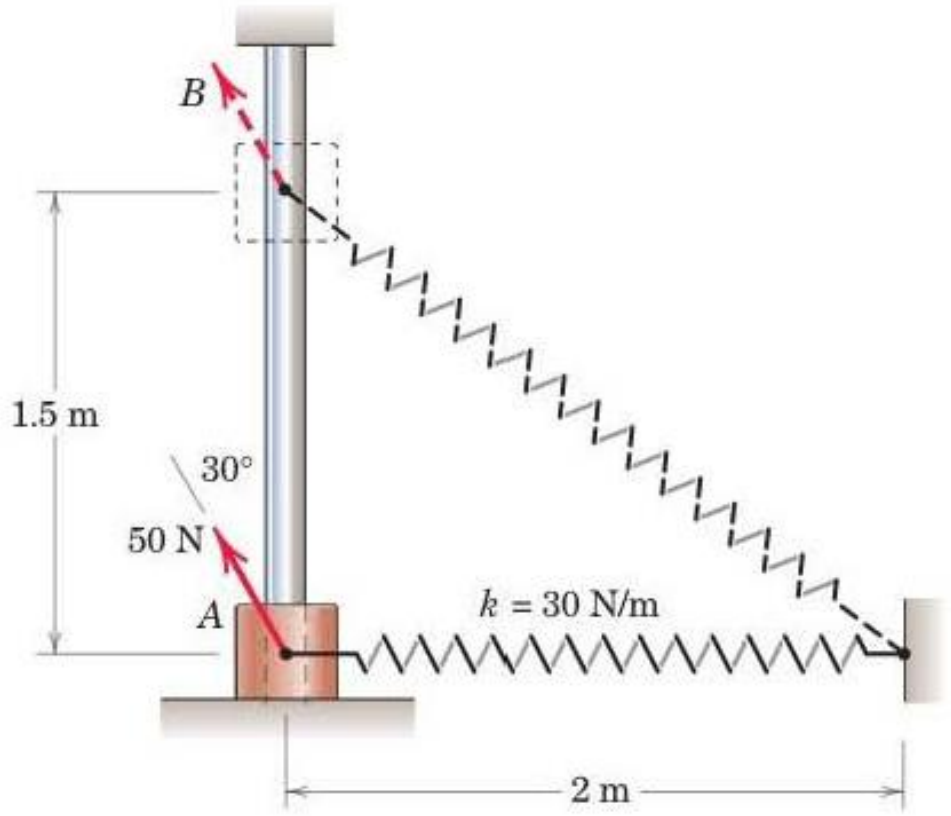
Problem 3/156

3/152 A bead with a mass of 0.25 kg is released from rest at A and slides down and around the fixed smooth wire. Determine the force N between the wire and the bead as it passes point B.



Problem 3/152

3/166 The collar has a mass of 2 kg and is attached to the light spring, which has a stiffness of 30 N/m and an unstretched length of 1.5 m. The collar is released from rest at A and slides up the smooth rod under the action of the constant 50-N force. Calculate the velocity v of the collar as it passes position B.



Problem 3/166

3/207 Car B is initially stationary and is struck by car A moving with initial speed $v_1 = 20$ mi/hr. The cars become entangled and move together with speed v' after the collision. If the time duration of the collision is 0.1 sec, determine (a) the common final speed v', (b) the average acceleration of each car during the collision, and (c) the magnitude R of the average force exerted by each car on the other car during the impact. All brakes are released duiring the collision.

Ans. (a) $v' = 13.33$ mi/hr

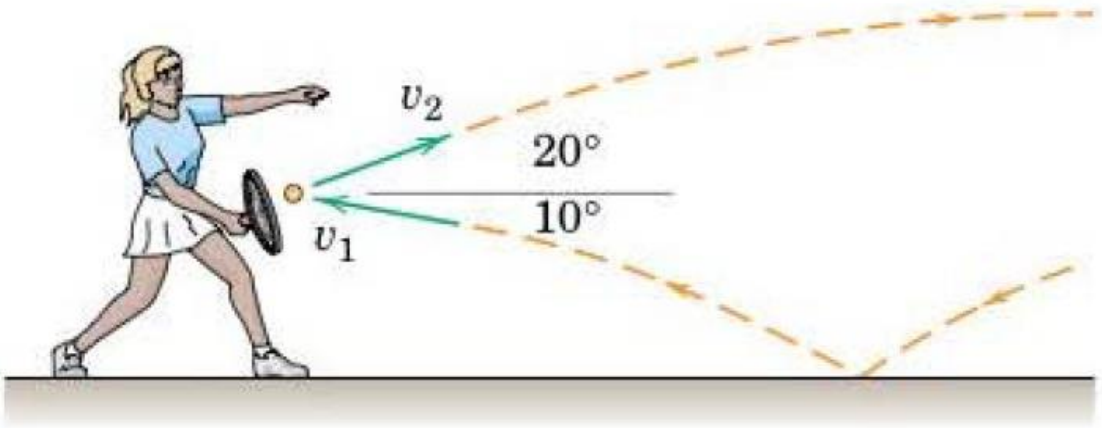
(b) $a_A = -97.8$ ft/sec^2, $a_B = 195.6$ ft/sec^2

(c) $R = 12{,}150$ lb



Problem 3/207

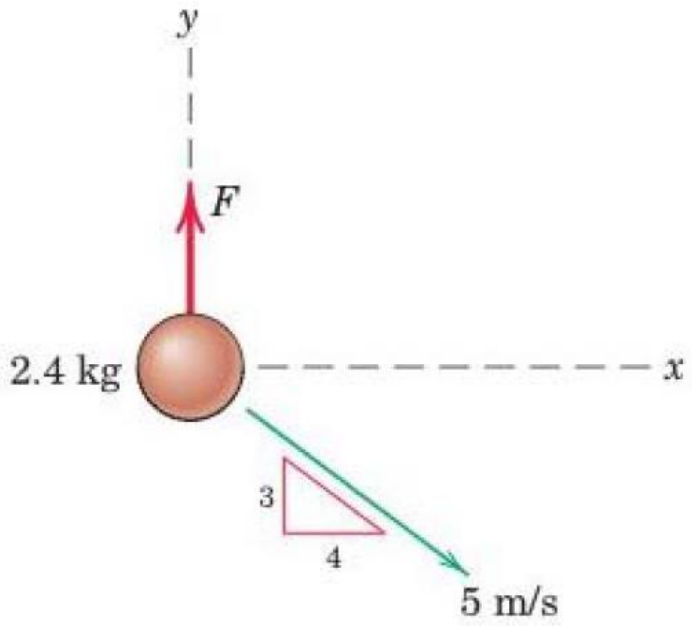
3/222 A tennis player strikes the tennis ball with her racket while the ball is still rising. The ball speed before impact with the racket is $v_1 = 15$ m/s and after impact its speed is $v_2 = 22$ m/s, with directions as shown in the figure. If the 60-g ball is in contact with the racket for 0.05 s, determine the magnitude of the average force $\mathbf{R}$ exerted by the racket on the ball. Find the angle β made by $\mathbf{R}$ with the horizontal. Comment on the treatment of the ball weight during impact.



Problem 3/222

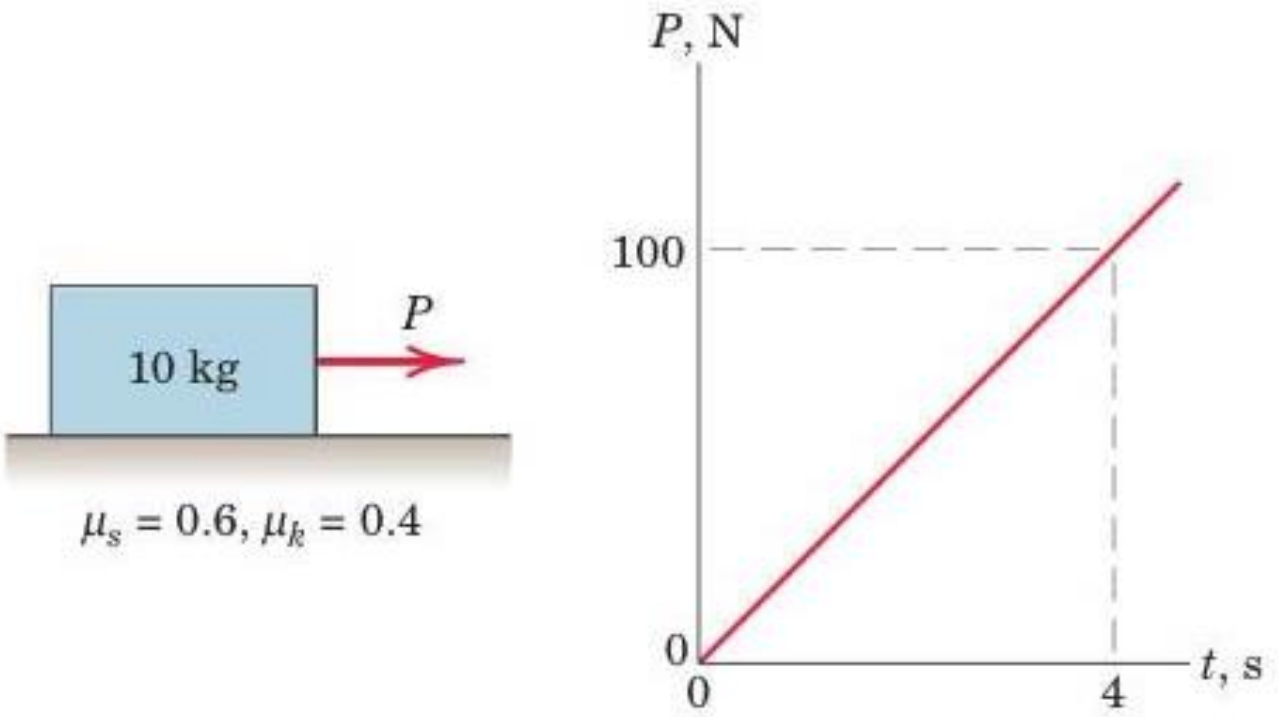
3/205 The 2.4-kg particle moves in the horizontal x-y plane and has the velocity shown at time $t = 0$. If the force $F = 2 + 3t^2/4$ newtons, where t is time in seconds, is applied to the particle in the y-direction beginning at time $t = 0$, determine the velocity v of the particle 4 seconds after F is applied and specify the corresponding angle θ measured counterclockwise from the x-axis to the direction of the velocity.

Ans. $v = 8.06$ m/s, $\theta = 60.3°$



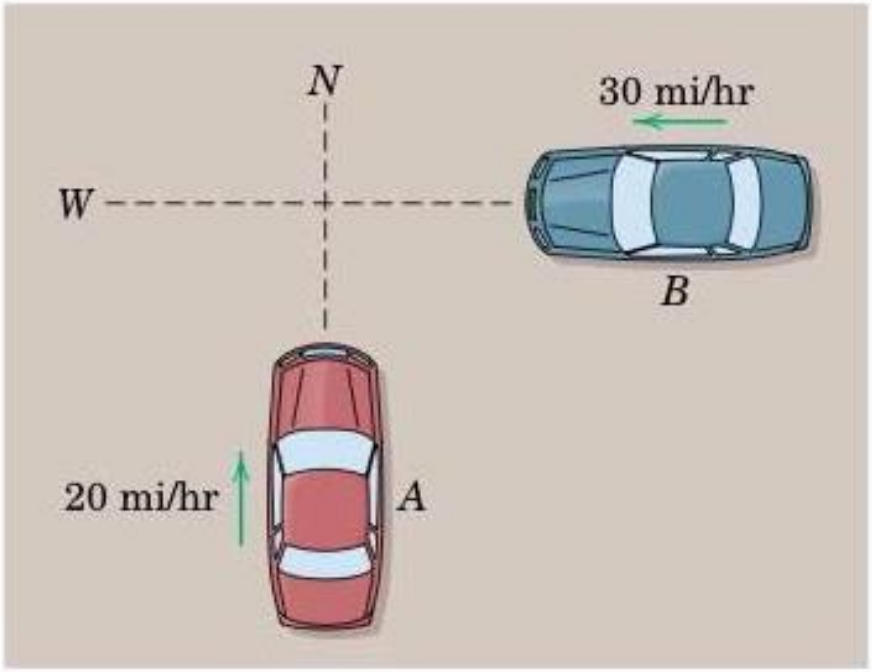
Problem 3/205

3/214 The force P, which is applied to the 10-kg block initially at rest, varies linearly with the time as indicated. If the coefficients of static and kinetic friction between the block and the horizontal surface are 0.6 and 0.4, respectively, determine the velocity of the block when $t = 4$ s.



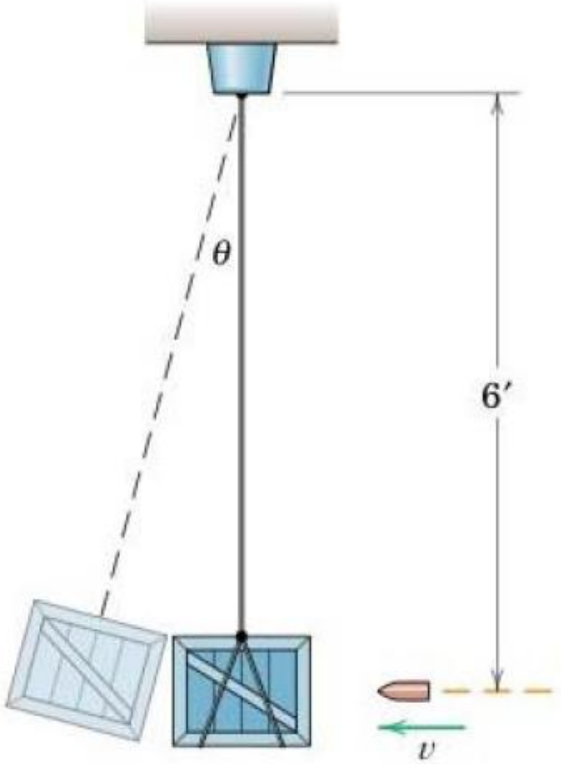
Problem 3/214

3/208 Car B weighing 3200 lb and traveling west at 30 mi/hr collides with car A weighing 3400 lb and traveling north at 20 mi/hr as shown. If the two cars become entangled and move together as a unit after the crash, compute the magnitude v of their common velocity immediately after the impact and the angle θ made by the velocity vector with the north direction.



Problem 3/208

3/218 The ballistic pendulum is a simple device to measure projectile velocity v by observing the maximum angle θ to which the box of sand with embedded projectile swings. Calculate the angle θ if the 2-oz projectile is fired horizontally into the suspended 50-lb box of sand with a velocity $v = 2000$ ft/sec. Also find the percentage of energy lost during impact.



Problem 3/218