Chapter 3 Equilibrium (2D)

STATICS, AGE-1330

Ahmed M El-Sherbeeny, PhD

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Introduction

When a body is in equilibrium, the resultant of *all* forces acting on it is zero. Thus, the resultant force **R** and the resultant couple **M** are both zero, and we have the equilibrium equations

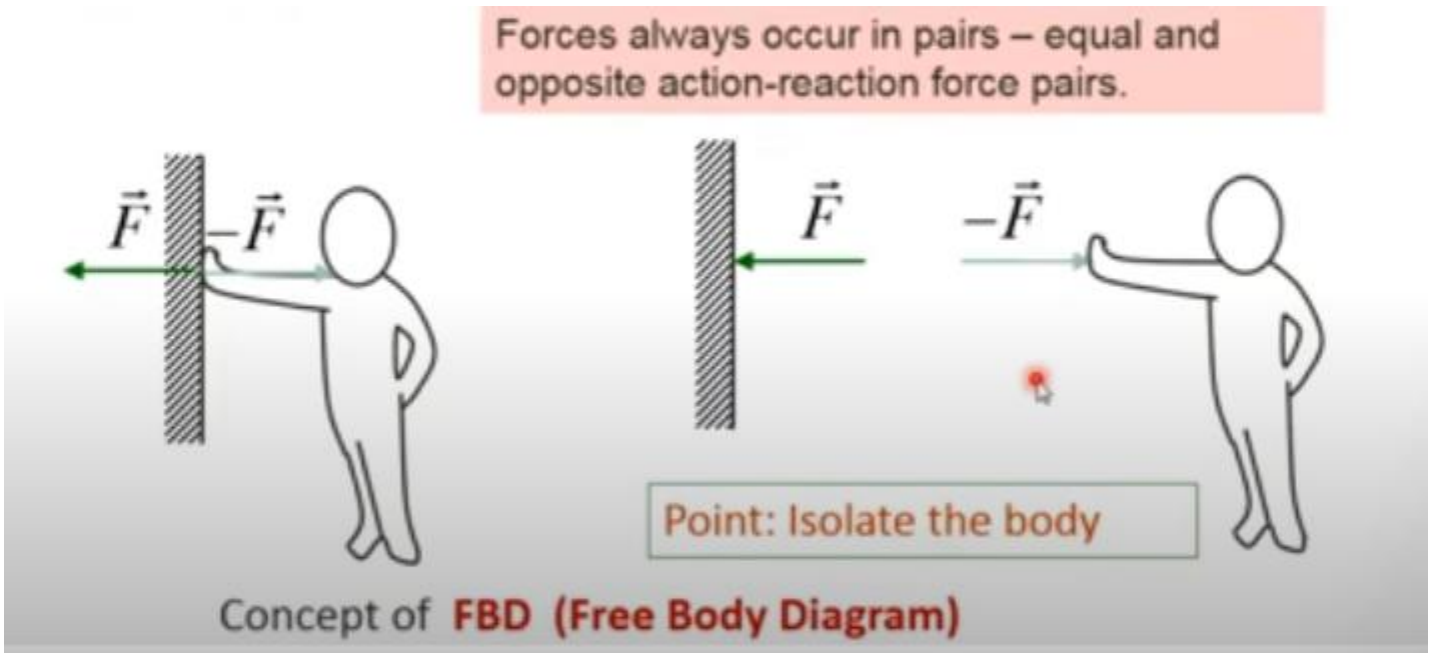
$$\mathbf{R} = \Sigma\mathbf{F} = \mathbf{0} \qquad \mathbf{M} = \Sigma\mathbf{M} = \mathbf{0} \tag{3/1}$$

EQUILIBRIUM IN TWO DIMENSIONS

System Isolation and the Free-Body Diagram

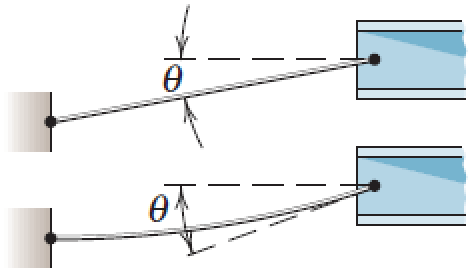
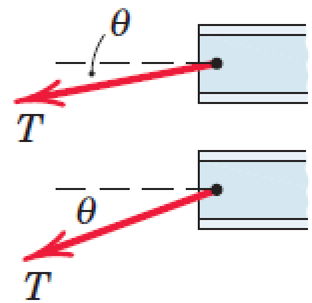
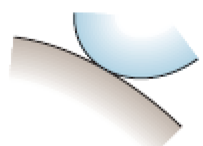
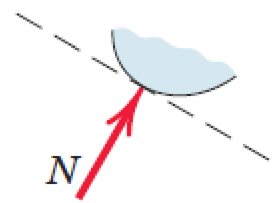
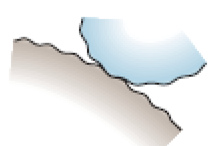
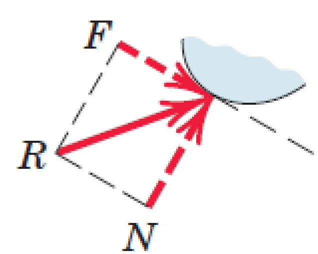
A *mechanical system is defined as a body or group of bodies which* can be conceptually isolated from all other bodies. A system may be a single body or a combination of connected bodies. The bodies may be rigid or nonrigid.

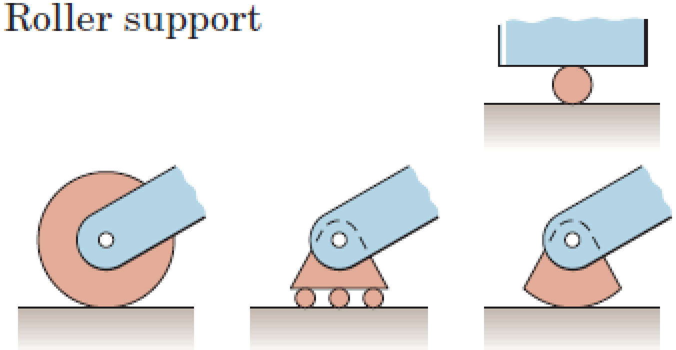
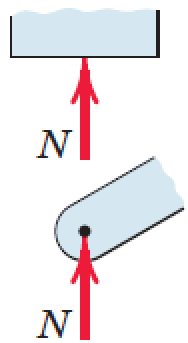
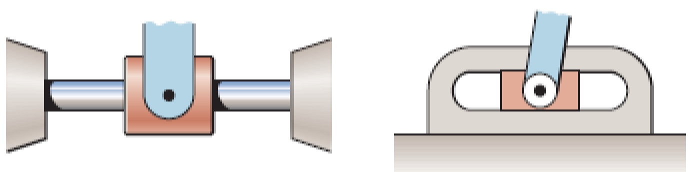
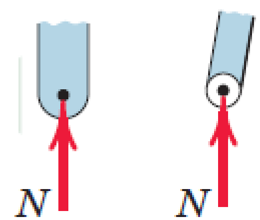
Once we decide which body or combination of bodies to analyze, we then treat this body or combination as a single body *isolated from all* surrounding bodies. This isolation is accomplished by means of the ***free-body diagram,***



the free-body diagram is the most important single step in the solution of problems in mechanics.

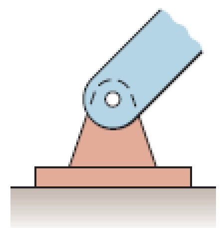
Modeling the Action of Forces

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component *F* (frictional force) as well as a normal component *N* of the resultant contact force *R*.

4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

<table>
<tr><th colspan="2">MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (cont.)</th></tr>
<tr><th>Type of Contact and Force Origin</th><th>Action on Body to Be Isolated</th></tr>
<tr><td>

6. Pin connection



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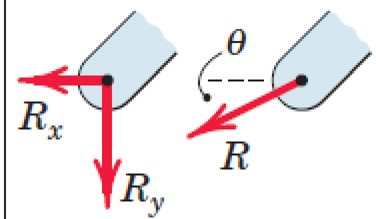
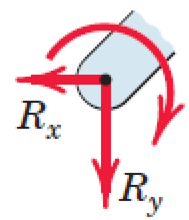
Pin free to turn



Pin not free to turn



A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.

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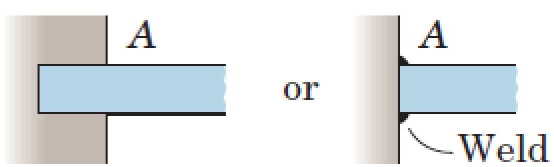
<tr><td>

7. Built-in or fixed support



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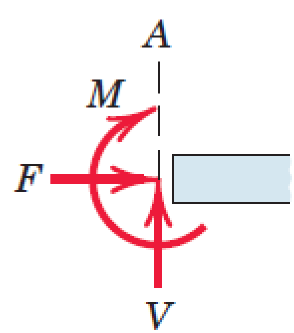

A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.

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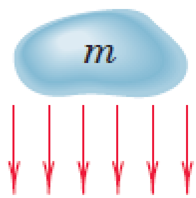
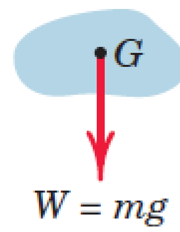
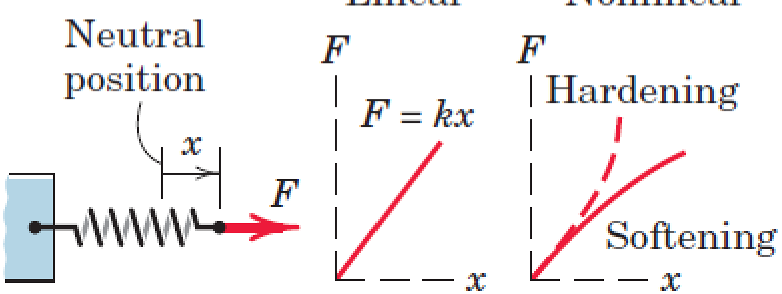
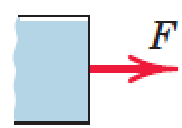
</table>

8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

The representations are *not free-body diagrams, but are* merely elements used to construct free-body diagrams. Study these nine conditions and identify them in the problem work so that you can draw the correct free-body diagrams.

Construction of FBD

- Decide which system to isolate. The system chosen should usually involve one or more of the desired unknowns.
- Next isolate the chosen system by drawing a diagram which represents its complete external boundary.
- Identify all forces which acts on the isolated system as applied by the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system.
- Show the choice of coordinate axes directly on the diagram.

KEY CONCEPTS

Construction of Free-Body Diagrams

The full procedure for drawing a free-body diagram which isolates a body or system consists of the following steps.

Step 1. Decide which system to isolate. The system chosen should usually involve one or more of the desired unknown quantities.

Step 2. Next isolate the chosen system by drawing a diagram which represents its *complete external boundary*. This boundary defines the isolation of the system from *all* other attracting or contacting bodies, which are considered removed. This step is often the most crucial of all. Make certain that you have *completely isolated* the system before proceeding with the next step.

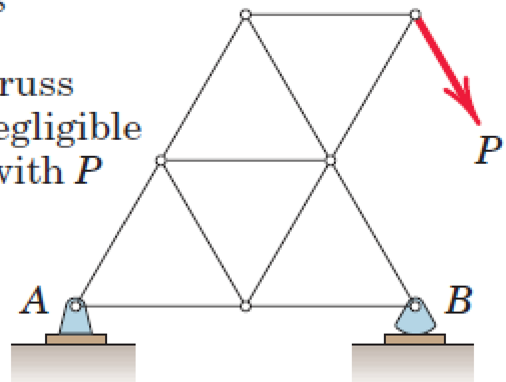
Step 3. Identify all forces which act *on* the isolated system as applied *by* the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system. Make a systematic traverse of the entire boundary to identify all contact forces. Include body forces such as weights, where appreciable. Represent all known forces by vector arrows, each with its proper magnitude, direction, and sense indicated. Each unknown force should be represented by a vector arrow with the unknown magnitude or direction indicated by symbol. If the sense of the vector is also unknown, you must arbitrarily assign a sense. The subsequent calculations with the equilibrium equations will yield a positive quantity if the correct sense was assumed and a negative quantity if the incorrect sense was assumed. It is necessary to be *consistent* with the assigned characteristics of unknown forces throughout all of the calculations. If you are consistent, the solution of the equilibrium equations will reveal the correct senses.

Step 4. Show the choice of coordinate axes directly on the diagram. Pertinent dimensions may also be represented for convenience. Note, however, that the free-body diagram serves the purpose of focusing attention on the action of the external forces, and therefore the diagram should not be cluttered with excessive extraneous information. Clearly distinguish force arrows from arrows representing quantities other than forces. For this purpose a colored pencil may be used.

Examples of Free-Body Diagrams

<table>
<tr><th colspan="2">SAMPLE FREE-BODY DIAGRAMS</th></tr>
<tr><th>Mechanical System</th><th>Free-Body Diagram of Isolated Body</th></tr>
<tr><td>
1. Plane truss

Weight of truss assumed negligible compared with P



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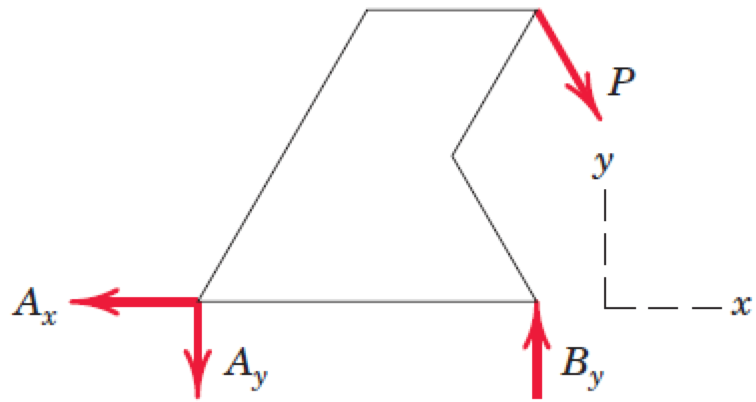

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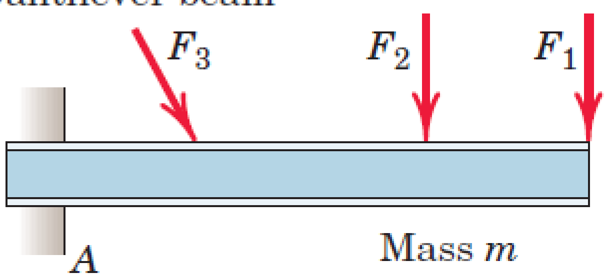
<tr><td>
2. Cantilever beam



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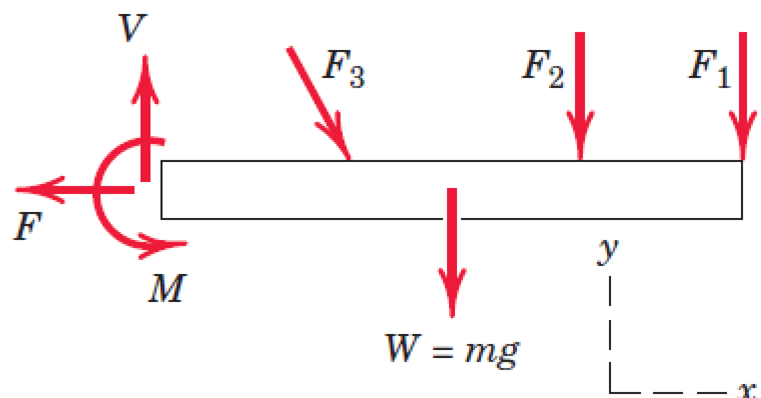

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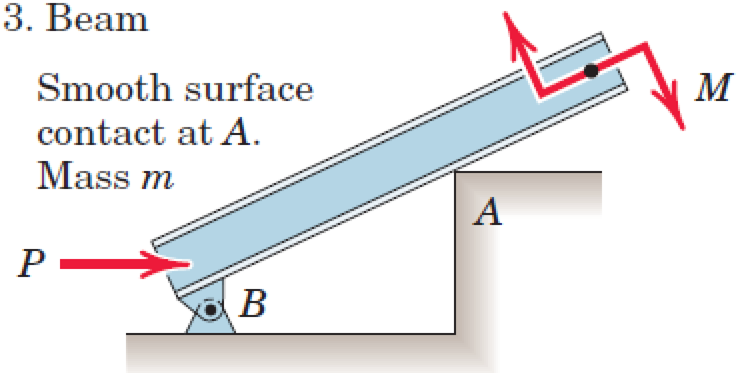
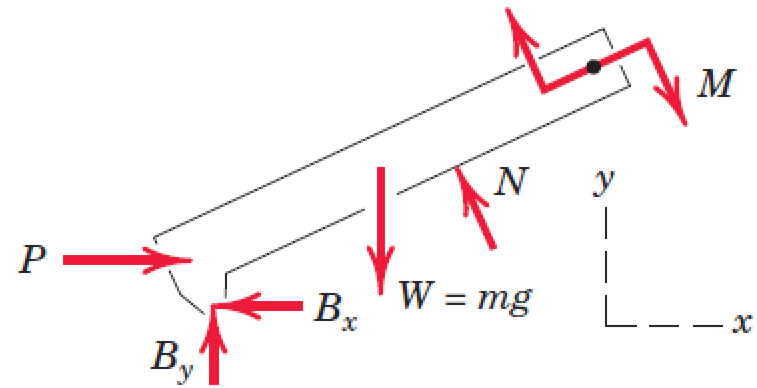
3. Beam

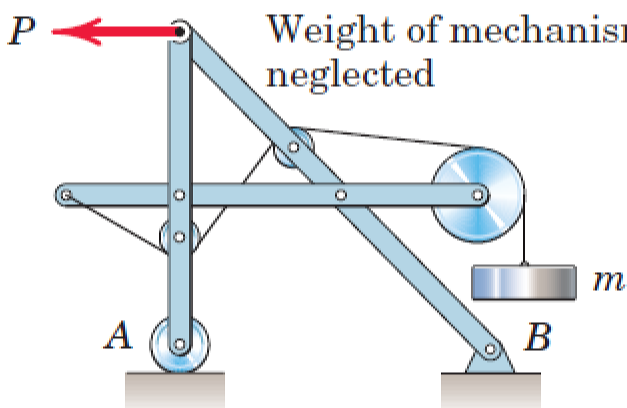
Smooth surface contact at A.
Mass m

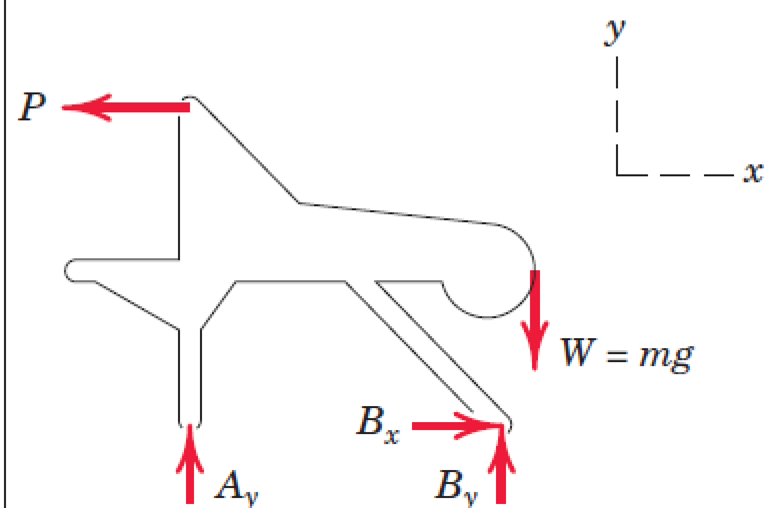




4. Rigid system of interconnected bodies analyzed as a single unit

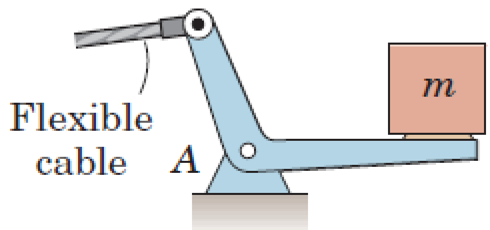
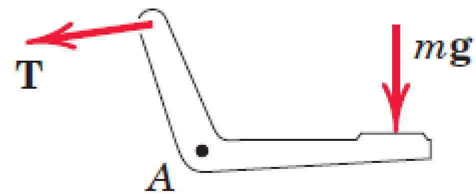
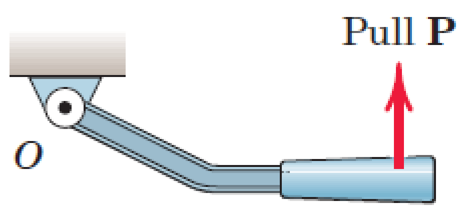
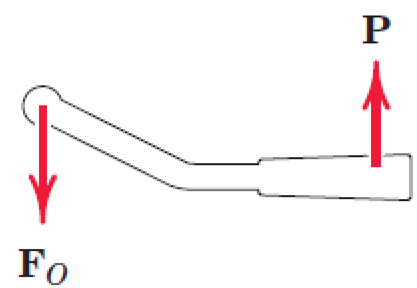
Weight of mechanism neglected

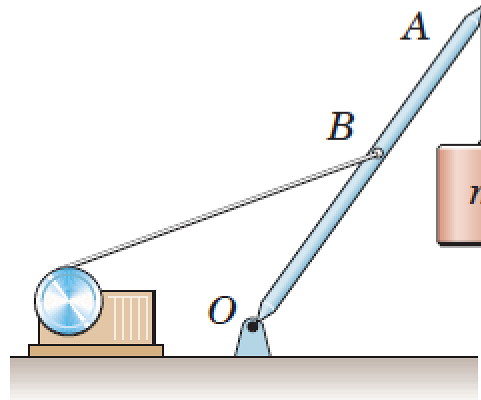
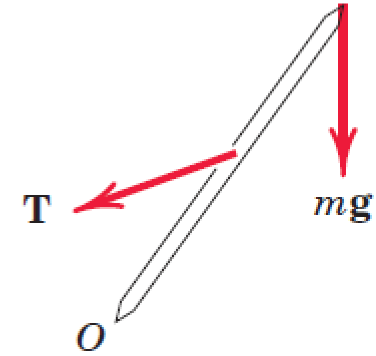
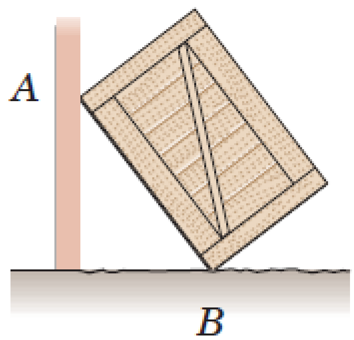
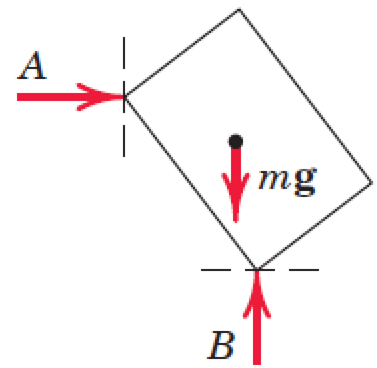
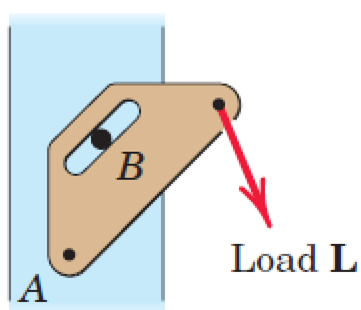
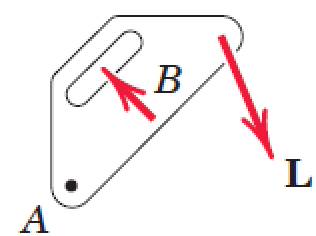




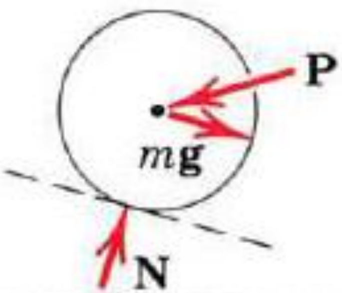
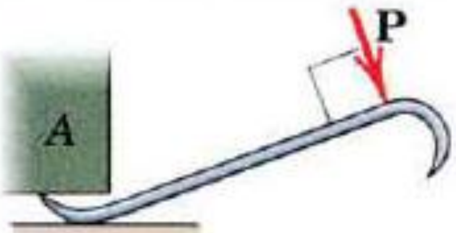
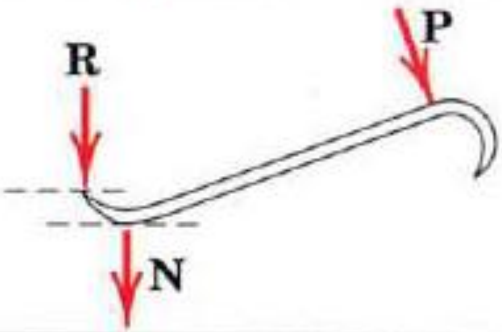
FREE-BODY DIAGRAM EXERCISES

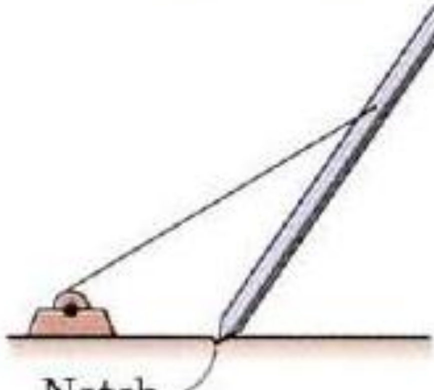
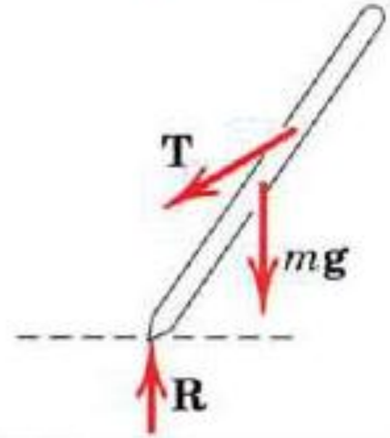
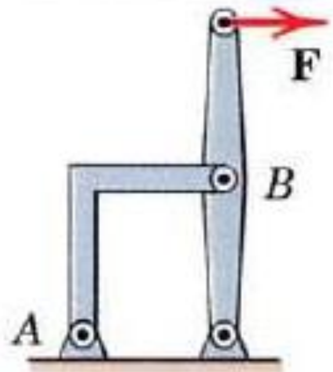
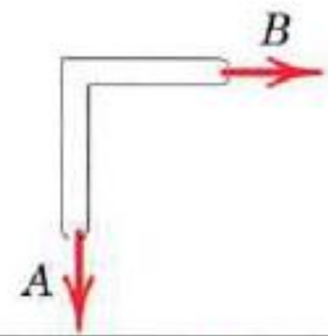
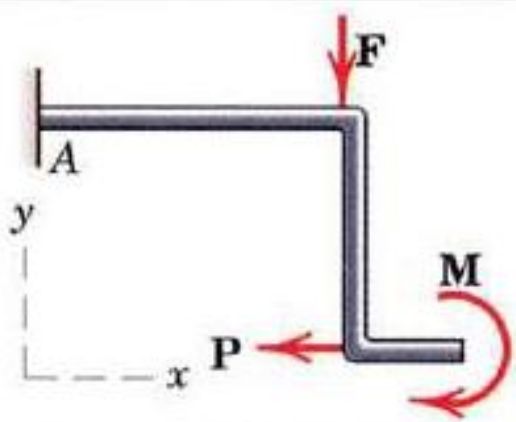
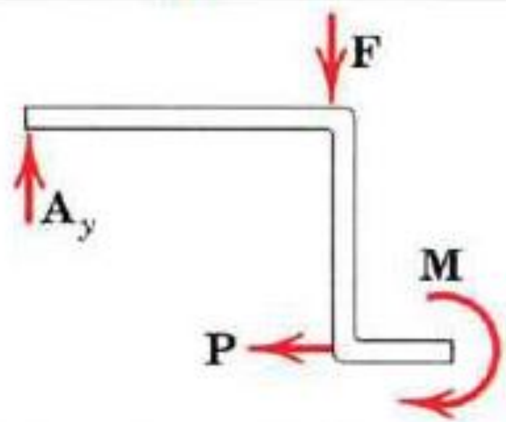
In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and an *incomplete* free-body diagram (FBD) of the isolated body is shown on the right. Add whatever forces are necessary in each case to form a complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

	Body	Incomplete FBD
1. Bell crank supporting mass m with pin support at A.		
2. Control lever applying torque to shaft at O.		

3. Boom *OA*, of negligible mass compared with mass *m*. Boom hinged at *O* and supported by hoisting cable at *B*.		
4. Uniform crate of mass *m* leaning against smooth vertical wall and supported on a rough horizontal surface.		
5. Loaded bracket supported by pin connection at *A* and fixed pin in smooth slot at *B*.		

3/B In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and either a *wrong* or an *incomplete* free-body diagram (FBD) is shown on the right. Make whatever changes or additions are necessary in each case to form a correct and complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

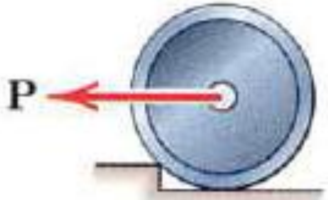
	Body	Wrong or Incomplete *FBD*
1. Lawn roller of mass m being pushed up incline θ.		
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		

3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.		
4. Supporting angle bracket for frame; Pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

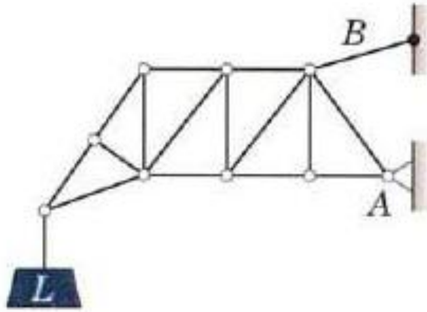
3/C Draw a complete and correct free-body diagram of each of the bodies designated in the statements. The weights of the bodies are significant only if the mass is stated. All forces, known and unknown, should be labeled. (*Note*: The sense of some reaction components cannot always be determined without numerical calculation.)

1. Uniform horizontal bar of mass m suspended by vertical cable at A and supported by rough inclined surface at B.

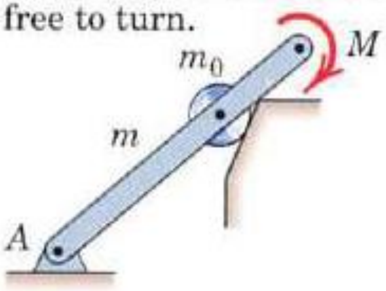
2. Wheel of mass m on verge of being rolled over curb by pull **P**.



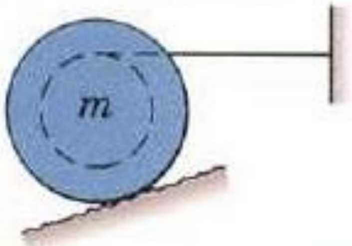
3. Loaded truss supported by pin joint at A and by cable at B.



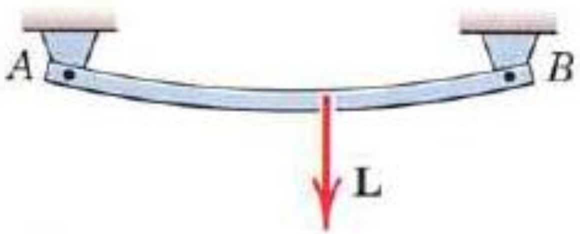
4. Uniform bar of mass m and roller of mass m_0 taken together. Subjected to couple M and supported as shown. Roller is free to turn.



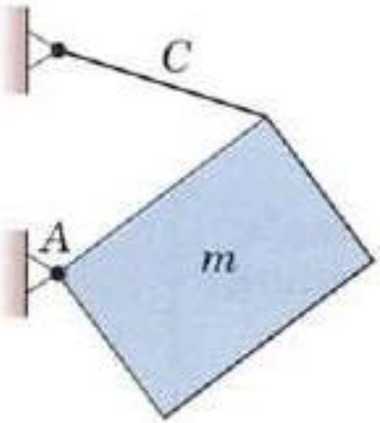
5. Uniform grooved wheel of mass m supported by a rough surface and by action of horizontal cable.



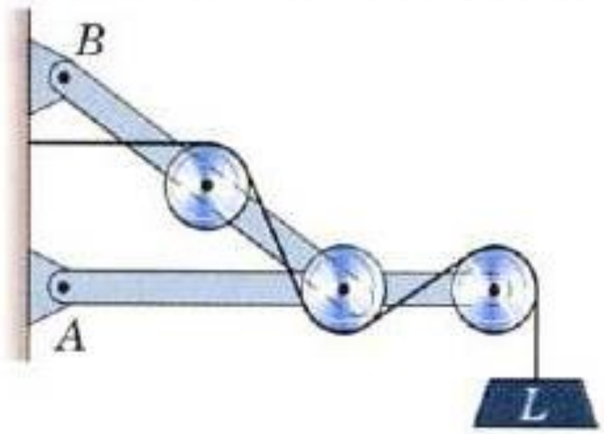
6. Bar, initially horizontal but deflected under load **L**. Pinned to rigid support at each end.



7. Uniform heavy plate of mass m supported in vertical plane by cable C and hinge A.



8. Entire frame, pulleys, and contacting cable to be isolated as a single unit.



Equilibrium Conditions

We defined equilibrium as the condition in which the resultant of all forces and moments acting on a body is zero. Stated in another way, a body is in equilibrium if all forces and moments applied to it are in balance. These requirements are contained in the vector equations of equilibrium, Eqs. 3/1, which in two dimensions may be written in scalar form as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0 \qquad \mathbf{(3/2)}$$

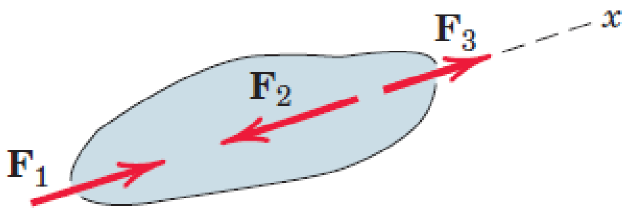
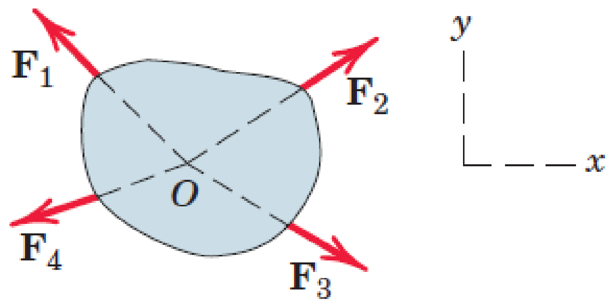
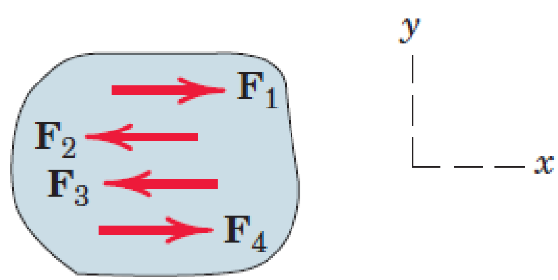
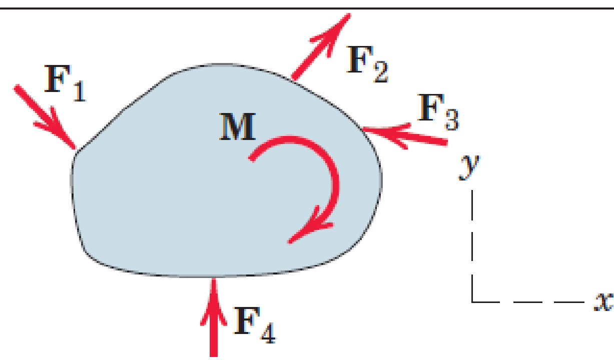
Categories of Equilibrium

Category 1, equilibrium of collinear forces, clearly requires only the one force equation in the direction of the forces (x-direction), since all other equations are automatically satisfied.

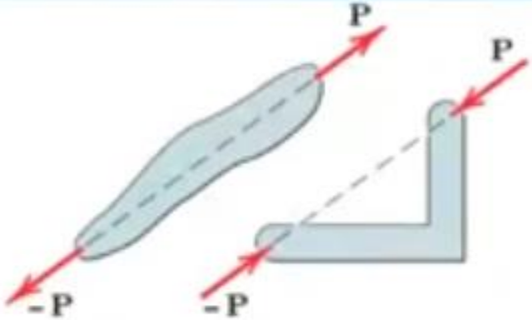
Category 2, equilibrium of forces which lie in a plane (x-y plane) and are concurrent at a point O, requires the two force equations only, since the moment sum about O, that is, about a z-axis through O, is necessarily zero. Included in this category is the case of the equilibrium of a particle.

Category 3, equilibrium of parallel forces in a plane, requires the one force equation in the direction of the forces (x-direction) and one moment equation about an axis (z-axis) normal to the plane of the forces.

Category 4, equilibrium of a general system of forces in a plane (x-y), requires the two force equations in the plane and one moment equation about an axis (z-axis) normal to the plane.

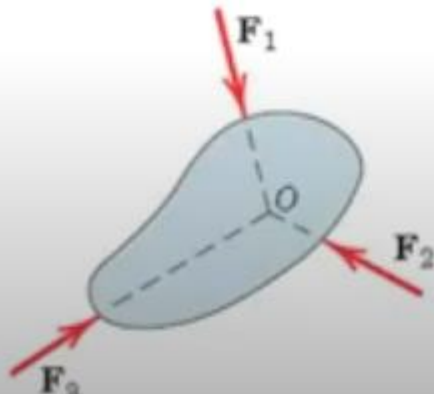
CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

Two- and Three-Force Members

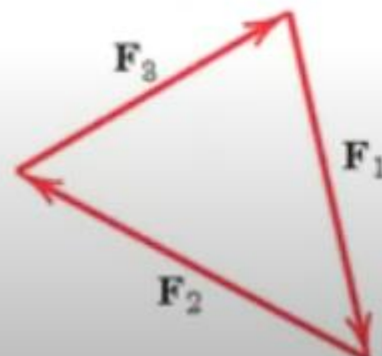


- A *two-force member* to be in equilibrium, the forces must be *equal, opposite,* and *collinear.*

- A three-force member equilibrium requires the lines of action of the three forces to be ***concurrent.***



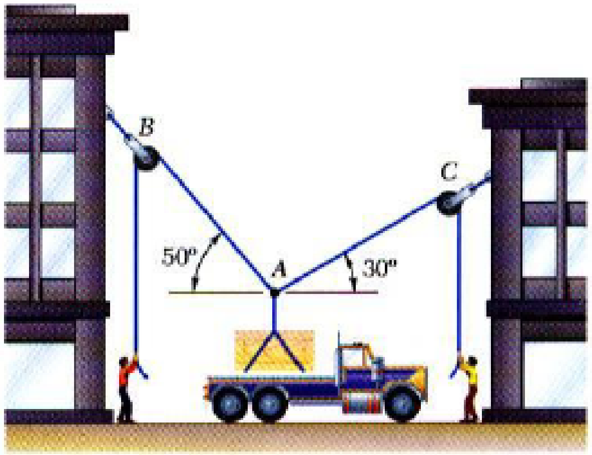
(a) Three-force member



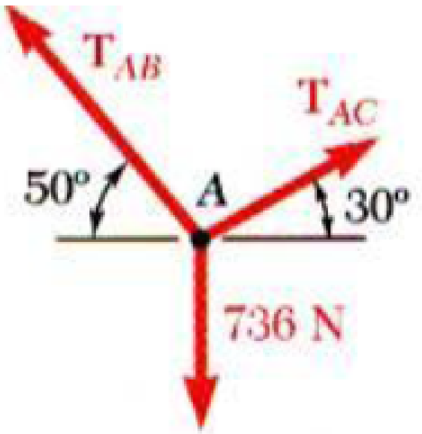
(b) Closed polygon satisfies $\Sigma F = 0$

Rigid Body Equilibrium

Free-Body Diagrams



Space Diagram: A sketch showing the physical conditions of the problem.



Free-Body Diagram: A sketch showing only the forces on the selected particle.

Rigid Body Equilibrium

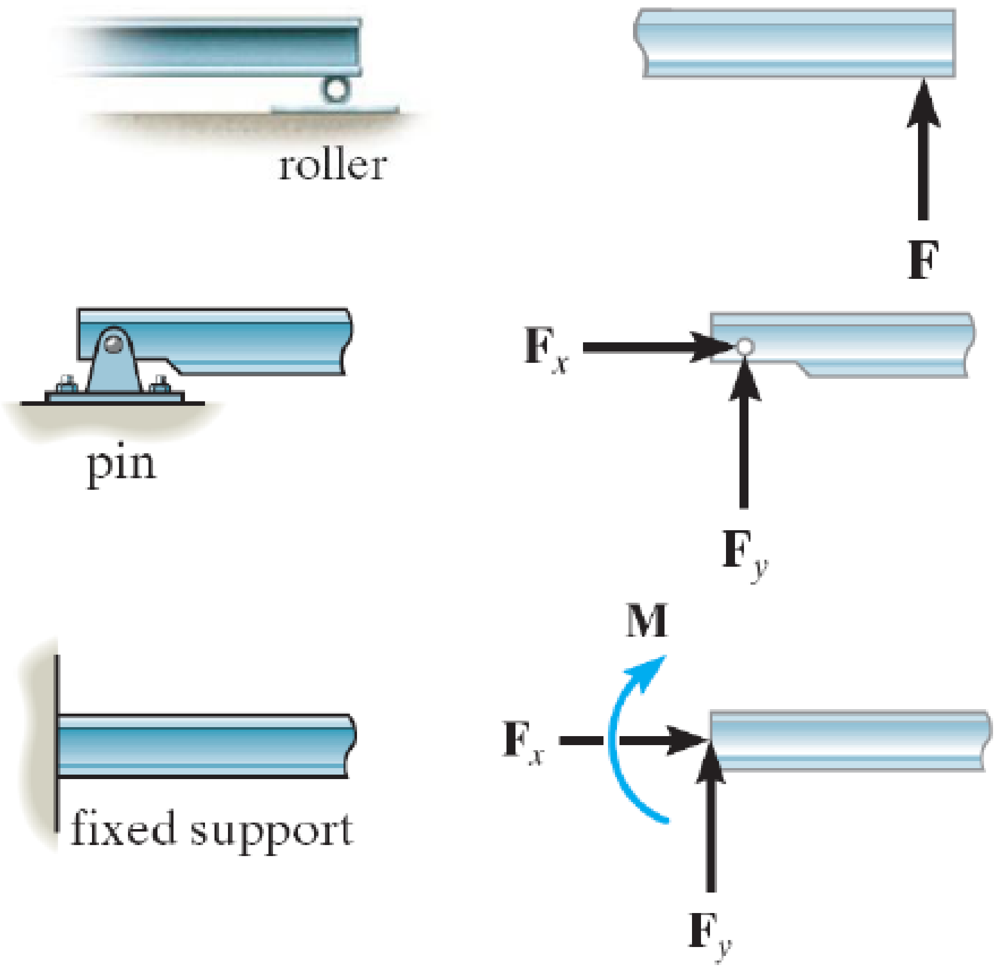
Support Reactions

Prevention of

Translation or

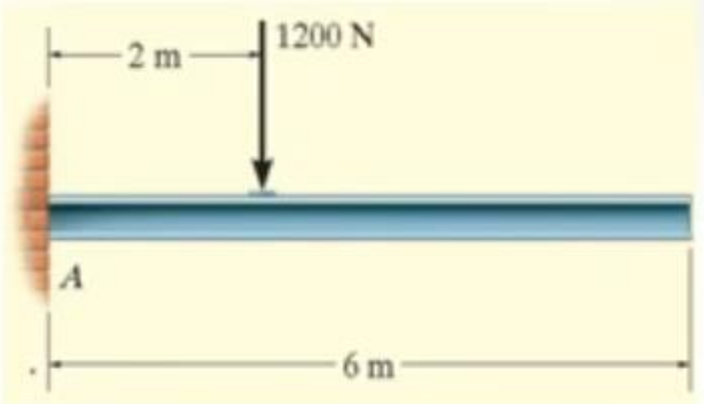
Rotation of a body

Restraints



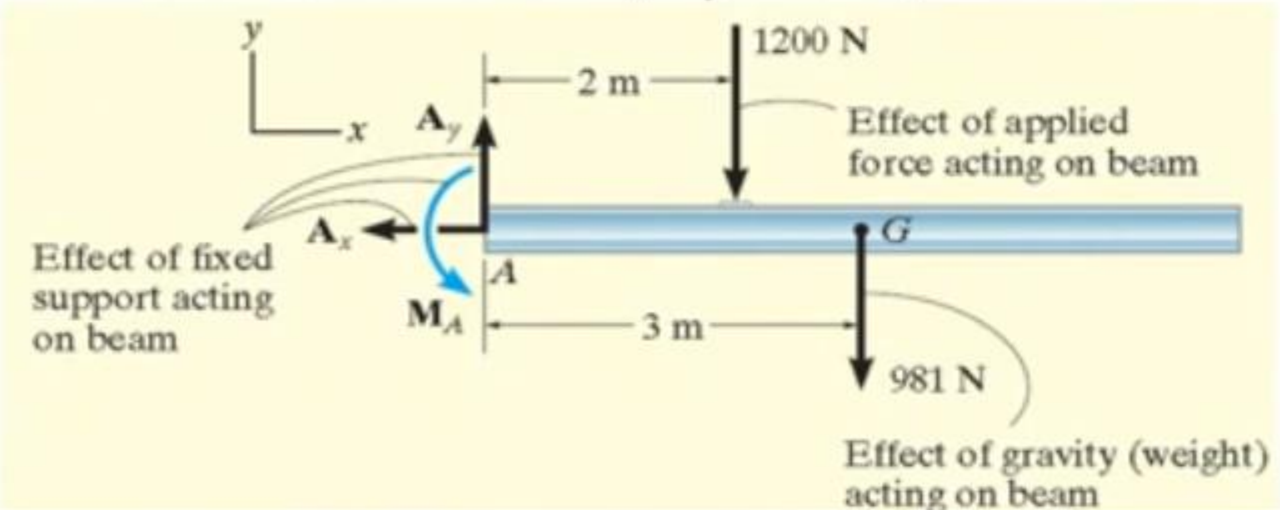
1- SAMPLE PROBLEM

Draw the free-body diagram of the uniform beam shown in Figure. The beam has a mass of 100 kg.



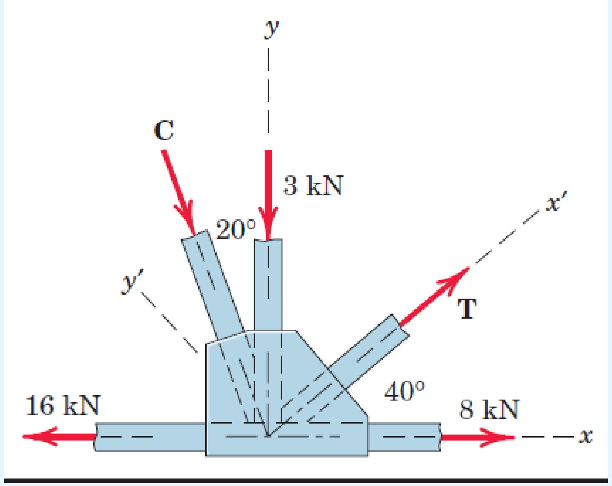
Solution.

The support at *A* is fixed, the wall exerts three reactions *on the beam* , denoted as $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{M}_A$



SAMPLE PROBLEM 3/1

Determine the magnitudes of the forces **C** and **T**, which, along with the other three forces shown, act on the bridge-truss joint.



Solution. The given sketch constitutes the free-body diagram of the isolated section of the joint in question and shows the five forces which are in equilibrium. ①

Solution I (scalar algebra). For the x-y axes as shown we have

$[\Sigma F_x = 0]$ $$8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$0.766T + 0.342C = 8 \qquad (a)$$

$[\Sigma F_y = 0]$ $$T\sin 40° - C\cos 20° - 3 = 0$$

$$0.643T - 0.940C = 3 \qquad (b)$$

Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN}$$ *Ans.*

Solution II (scalar algebra). To avoid a simultaneous solution, we may use axes x'-y' with the first summation in the y'-direction to eliminate reference to T. Thus, ②

$[\Sigma F_{y'} = 0]$ $$-C\cos 20° - 3\cos 40° - 8\sin 40° + 16\sin 40° = 0$$

$$C = 3.03 \text{ kN}$$ *Ans.*

$[\Sigma F_{x'} = 0]$ $$T + 8\cos 40° - 16\cos 40° - 3\sin 40° - 3.03\sin 20° = 0$$

$$T = 9.09 \text{ kN}$$ *Ans.*

Helpful Hints

① Since this is a problem of concurrent forces, no moment equation is necessary.

② The selection of reference axes to facilitate computation is always an important consideration. Alternatively in this example we could take a set of axes along and normal to the direction of **C** and employ a force summation normal to **C** to eliminate it.

Solution III (vector algebra). With unit vectors **i** and **j** in the x- and y-directions, the zero summation of forces for equilibrium yields the vector equation

$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad 8\mathbf{i} + (T\cos 40^\circ)\mathbf{i} + (T\sin 40^\circ)\mathbf{j} - 3\mathbf{j} + (C\sin 20^\circ)\mathbf{i} - (C\cos 20^\circ)\mathbf{j} - 16\mathbf{i} = \mathbf{0}$$

Equating the coefficients of the **i**- and **j**-terms to zero gives

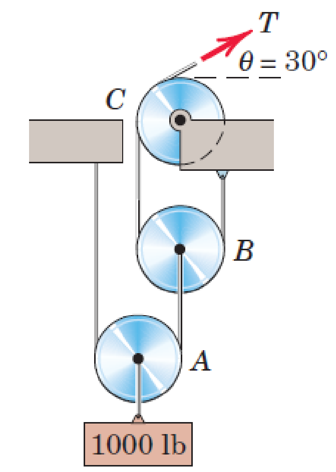
$$8 + T\cos 40^\circ + C\sin 20^\circ - 16 = 0$$

$$T\sin 40^\circ - 3 - C\cos 20^\circ = 0$$

which are the same, of course, as Eqs. (a) and (b), which we solved above.

SAMPLE PROBLEM 3/2

Calculate the tension T in the cable which supports the 1000-lb load with the pulley arrangement shown. Each pulley is free to rotate about its bearing, and the weights of all parts are small compared with the load. Find the magnitude of the total force on the bearing of pulley C.



Solution. The free-body diagram of each pulley is drawn in its relative position to the others. We begin with pulley A, which includes the only known force. With the unspecified pulley radius designated by r, the equilibrium of moments about its center O and the equilibrium of forces in the vertical direction require

① $[\Sigma M_O = 0]$ $\quad T_1 r - T_2 r = 0 \quad T_1 = T_2$

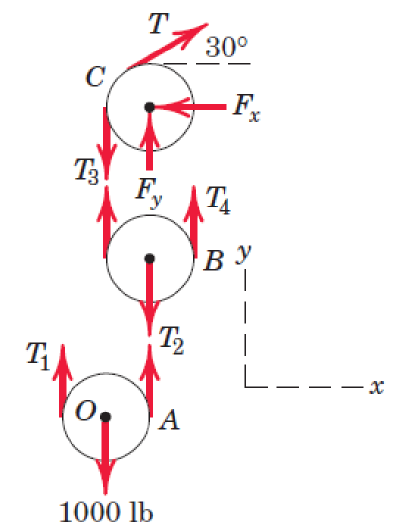
$[\Sigma F_y = 0]$ $\quad T_1 + T_2 - 1000 = 0 \quad 2T_1 = 1000 \quad T_1 = T_2 = 500 \text{ lb}$

From the example of pulley A we may write the equilibrium of forces on pulley B by inspection as

$$T_3 = T_4 = T_2/2 = 250 \text{ lb}$$

For pulley C the angle $\theta = 30°$ in no way affects the moment of T about the center of the pulley, so that moment equilibrium requires

$$T = T_3 \quad \text{or} \quad T = 250 \text{ lb}$$ *Ans.*



Equilibrium of the pulley in the x- and y-directions requires

$[\Sigma F_x = 0]$ $\quad 250 \cos 30° - F_x = 0 \quad F_x = 217 \text{ lb}$

$[\Sigma F_y = 0]$ $\quad F_y + 250 \sin 30° - 250 = 0 \quad F_y = 125 \text{ lb}$

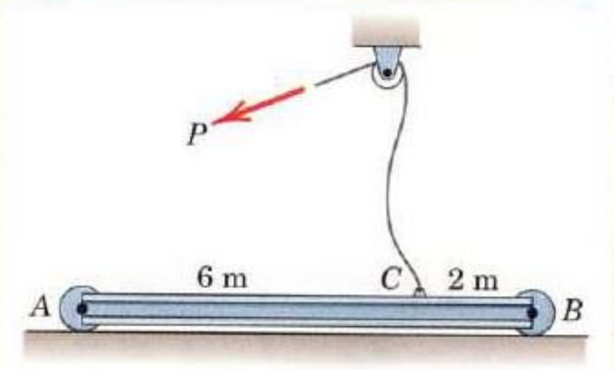
$[F = \sqrt{F_x^2 + F_y^2}]$ $\quad F = \sqrt{(217)^2 + (125)^2} = 250 \text{ lb}$ *Ans.*

Helpful Hint

① Clearly the radius r does not influence the results. Once we have analyzed a simple pulley, the results should be perfectly clear by inspection.

Sample Problem 3/3

The uniform 100-kg I-beam is supported initially by its end rollers on the horizontal surface at *A* and *B*. By means of the cable at *C* it is desired to elevate end *B* to a position 3 m above end *A*. Determine the required tension *P*, the reaction at *A*, and the angle θ made by the beam with the horizontal in the elevated position.



Solution. In constructing the free-body diagram, we note that the reaction on the roller at *A* and the weight are vertical forces. Consequently, in the absence of other horizontal forces, *P* must also be vertical. From Sample Problem 3/2 we see immediately that the tension *P* in the cable equals the tension *P* applied to the beam at *C*.

Moment equilibrium about *A* eliminates force *R* and gives

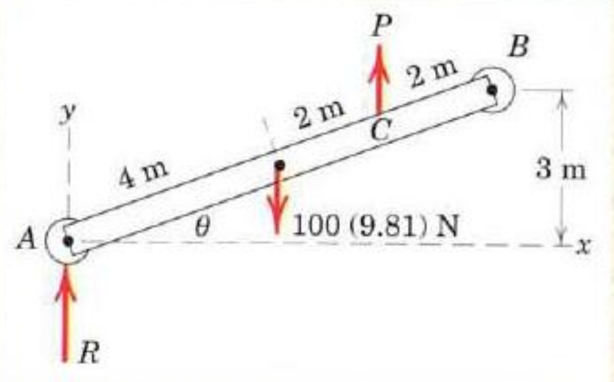
① $[\Sigma M_A = 0]$ $\quad P(6 \cos \theta) - 981(4 \cos \theta) = 0 \quad P = 654 \text{ N}$ *Ans.*

Equilibrium of vertical forces requires

$[\Sigma F_y = 0]$ $\quad 654 + R - 981 = 0 \quad R = 327 \text{ N}$ *Ans.*

The angle θ depends only on the specified geometry and is

$\sin \theta = 3/8 \quad \theta = 22.0°$ *Ans.*

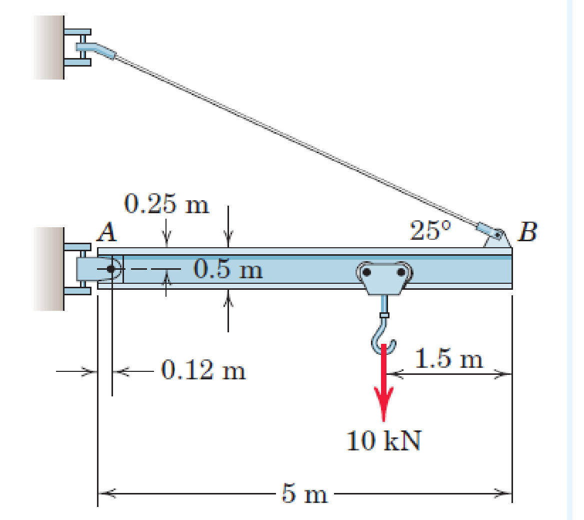


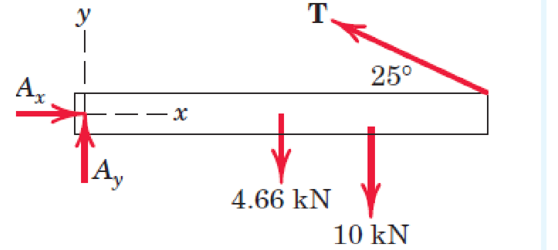
Helpful Hint

① Clearly the equilibrium of this parallel force system is independent of θ.

SAMPLE PROBLEM 3/4

Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.





Free-body diagram

Algebraic solution. The system is symmetrical about the vertical x-y plane through the center of the beam, so the problem may be analyzed as the equilibrium of a coplanar force system. The free-body diagram of the beam is shown in the figure with the pin reaction at A represented in terms of its two rectangular components. The weight of the beam is $95(10^{-3})(5)9.81 = 4.66$ kN and acts through its center. Note that there are three unknowns A_x, A_y, and T, which may be found from the three equations of equilibrium. We begin with a moment equation about A, which eliminates two of the three unknowns from the equa-
1 tion. In applying the moment equation about A, it is simpler to consider the moments of the x- and y-components of $\mathbf{T}$ than it is to compute the perpendicular distance from $\mathbf{T}$ to A. Hence, with the counterclockwise sense as positive we write

2 $[\Sigma M_A = 0]$ $$(T\cos 25°)0.25 + (T\sin 25°)(5 - 0.12) - 10(5 - 1.5 - 0.12) - 4.66(2.5 - 0.12) = 0$$

from which $T = 19.61$ kN *Ans.*

Equating the sums of forces in the x- and y-directions to zero gives

$[\Sigma F_x = 0]$ $A_x - 19.61\cos 25° = 0$ $A_x = 17.77$ kN

$[\Sigma F_y = 0]$ $A_y + 19.61\sin 25° - 4.66 - 10 = 0$ $A_y = 6.37$ kN

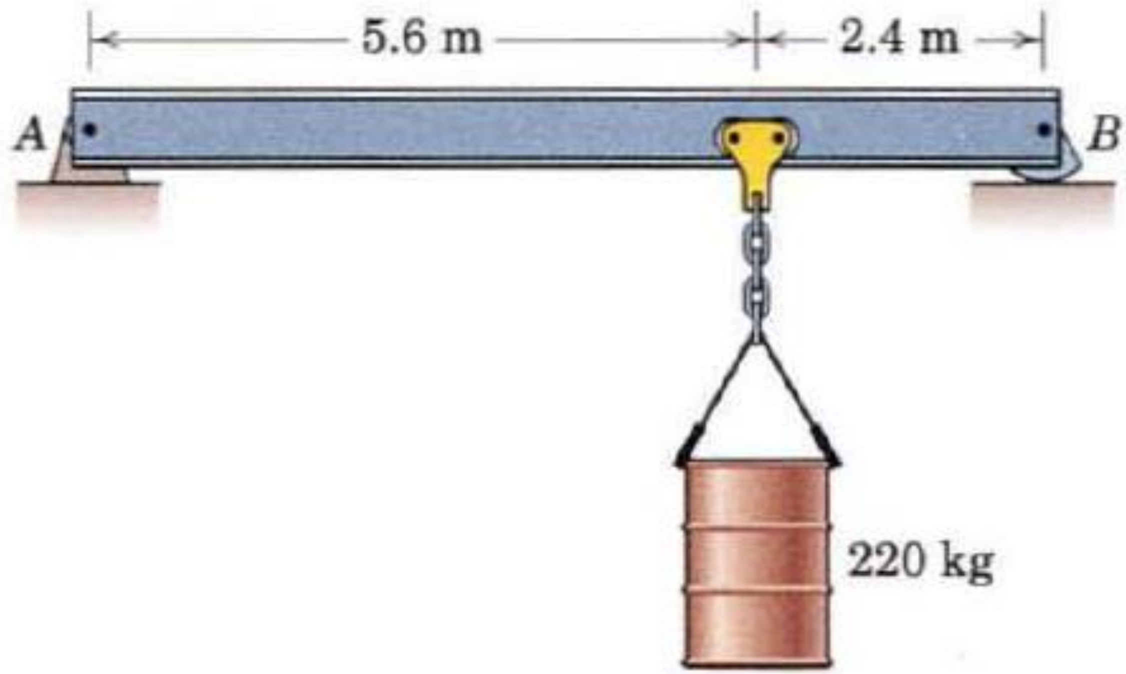
3 $[A = \sqrt{A_x^2 + A_y^2}]$ $A = \sqrt{(17.77)^2 + (6.37)^2} = 18.88$ kN *Ans.*

Helpful Hints

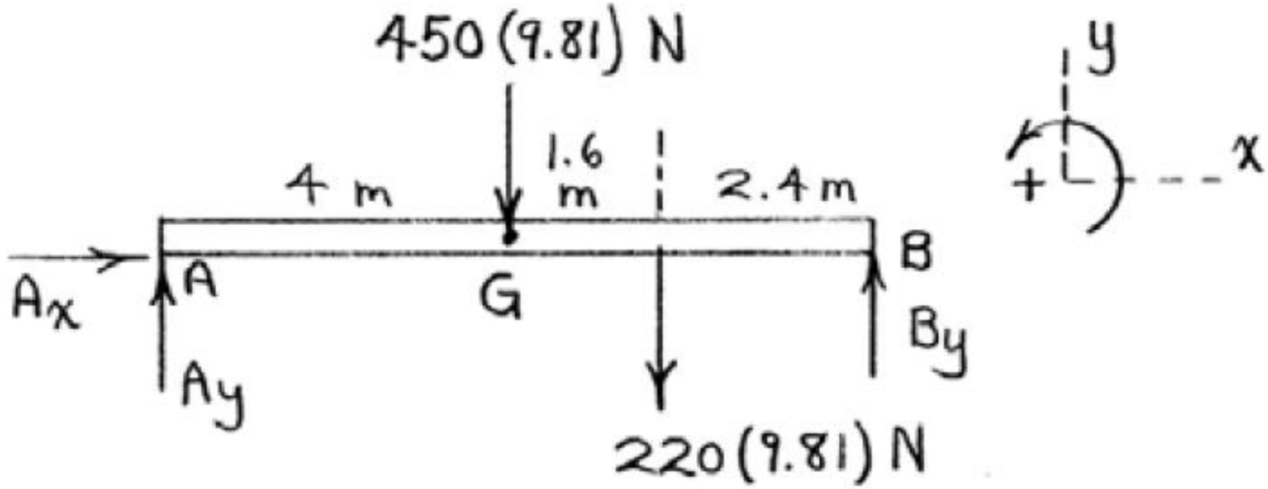
1 The justification for this step is Varignon's theorem, explained in Art. 2/4. Be prepared to take full advantage of this principle frequently.

2 The calculation of moments in two-dimensional problems is generally handled more simply by scalar alge-

3/4 The 450-kg uniform I-beam supports the load shown. Determine the reactions at the supports.



3/4



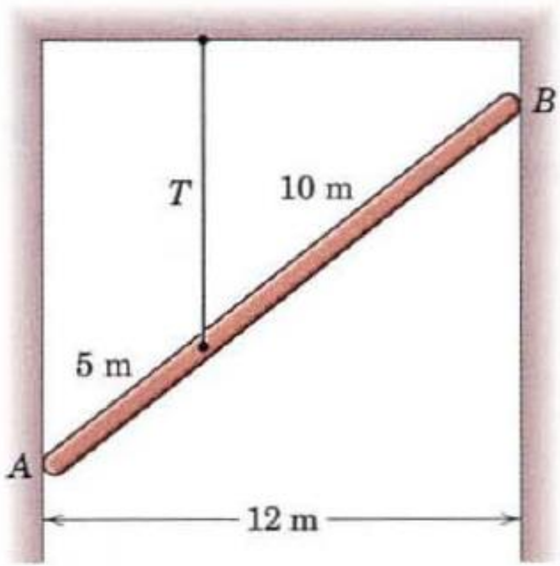
From $\Sigma F_x = 0$, $A_x = 0$

$$\Sigma M_A = 0: -450(9.81)4 - 220(9.81)(5.6) + B_y(8) = 0, \quad \underline{B_y = 3720 \text{ N}}$$

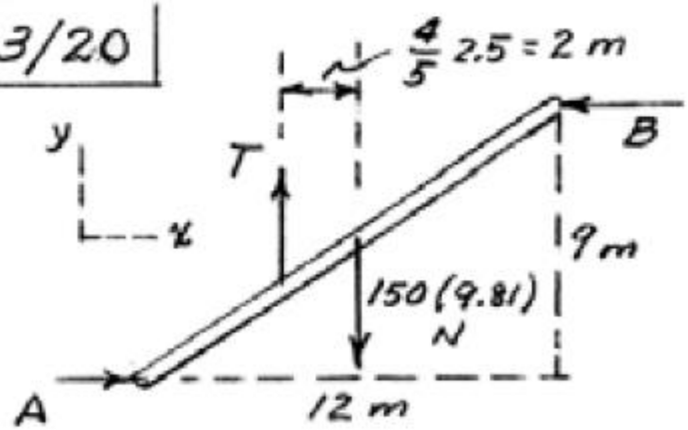
$$\Sigma F_y = 0: A_y - 450(9.81) - 220(9.81) + 3720 = 0$$

$$\underline{A_y = 2850 \text{ N}}$$

3/20 The uniform 15-m pole has a mass of 150 kg and is supported by its smooth ends against the vertical walls and by the tension T in the vertical cable. Compute the reactions at A and B.



3/20



$\Sigma F_y = 0;\quad T - 150(9.81) = 0$

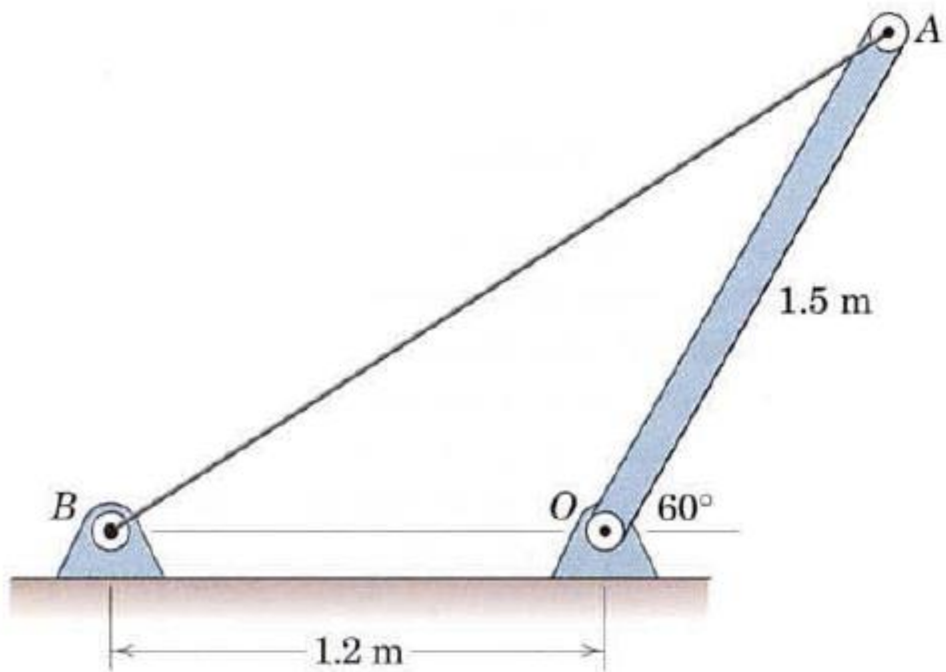
$T = 1472 \text{ N}$

$\Sigma F_x = 0;\quad A = B$

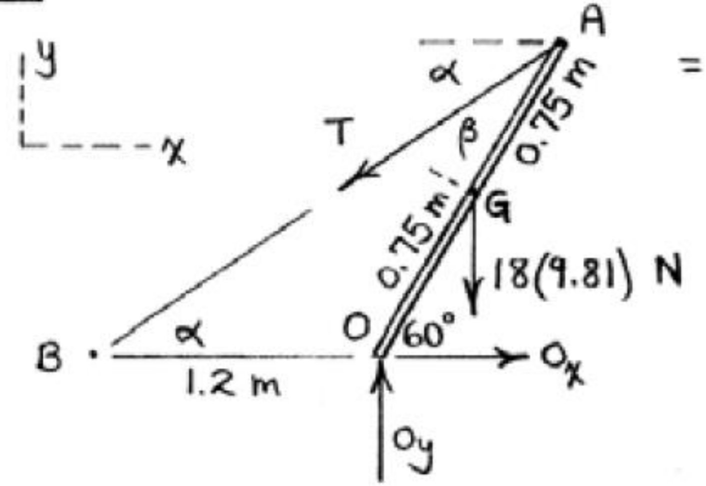
$\Sigma M = 0;\quad 1472(2) - A(9) = 0$

$\underline{A = B = 327 \text{ N}}$

3/32 The uniform 18-kg bar OA is held in the position shown by the smooth pin at O and the cable AB. Determine the tension T in the cable and the magnitude and direction of the external pin reaction at O.



3/32



$$\alpha = \tan^{-1}\left[\frac{1.5\sin 60°}{1.5\cos 60° + 1.2}\right] = 33.7°$$

$$\beta = 90° - \alpha - 30° = 26.3°$$

$$\curvearrowleft + \Sigma M_O = 0: \ T\sin 33.7°(1.2) - 18(9.81)(0.75)\cos 60° = 0$$

$$\underline{T = 99.5 \text{ N}}$$

$$\Sigma F_x = 0: \ -99.5\cos 33.7° + O_x = 0$$

$$O_x = 82.8 \text{ N}$$

$$\Sigma F_y = 0: \ -99.5\sin 33.7° - 18(9.81) + O_y = 0$$

$$O_y = 232 \text{ N}$$

So $O = 246$ N @ 70.3° CCW from $+x$-axis