

# Chapter 2 – Kinematics of Particles – p2

- ◆ Curvilinear Motion
- ◆ Projectile Motion ( $x-y$ )
- ◆ Relative & Dependent Curvilinear Motion
- ◆ Normal & Tangential Components ( $t-n$ )
- ◆ Radial & Transverse Components ( $r-\theta$ )
- ◆ Additional Problems

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# 1. Curvilinear Motion of Particles

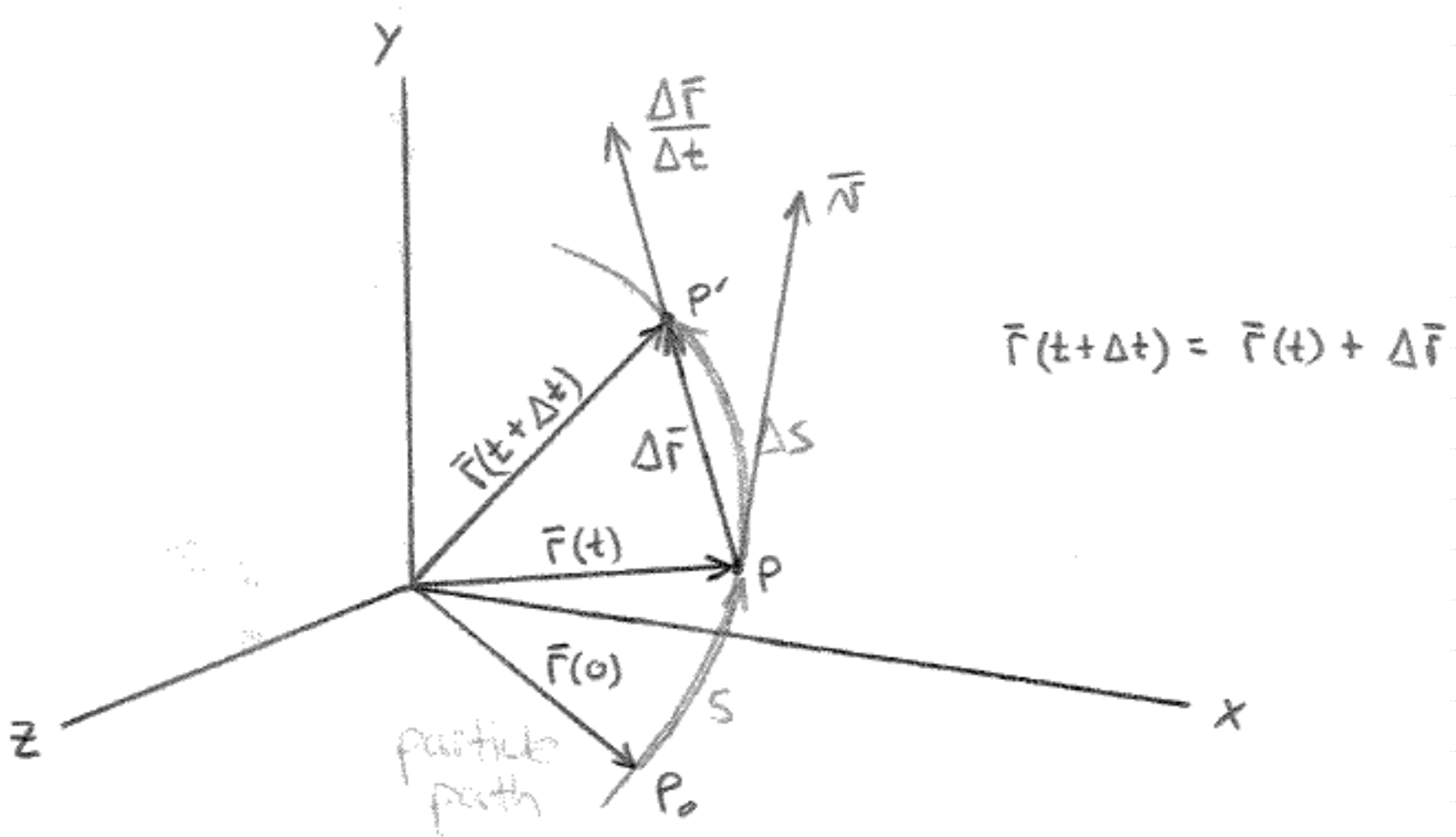


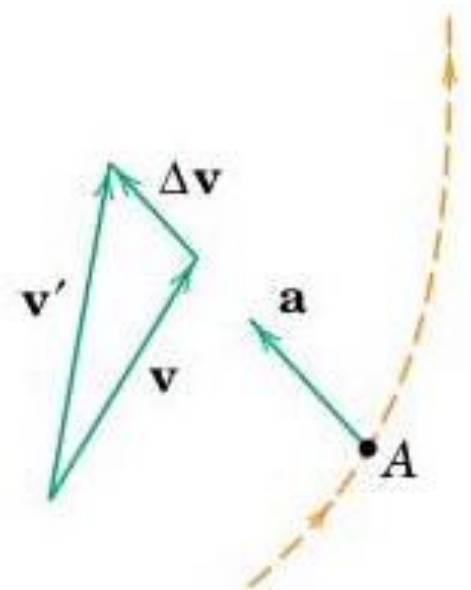
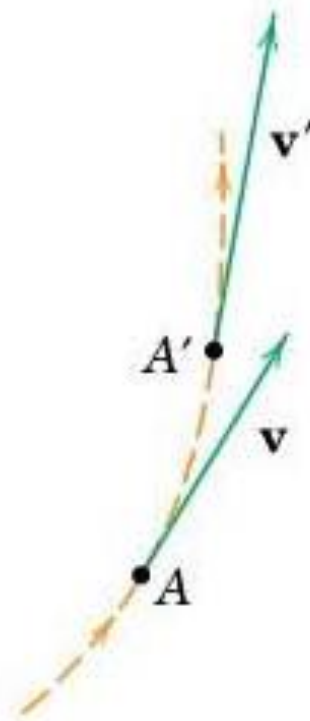
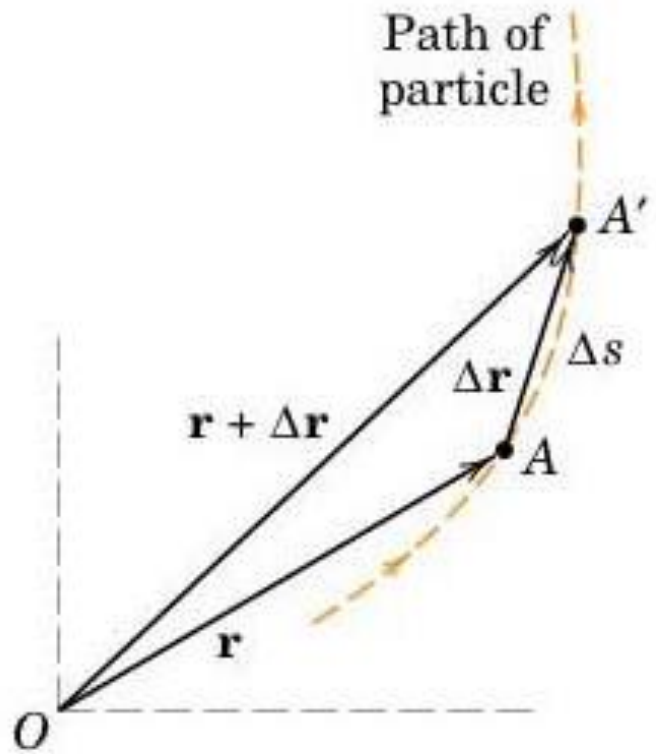
# Curvilinear Motion of Particles

→ Particle is constrained to move along a curved path.

Velocity  $\vec{v}$

→ Time-evolution of  $\vec{r}$



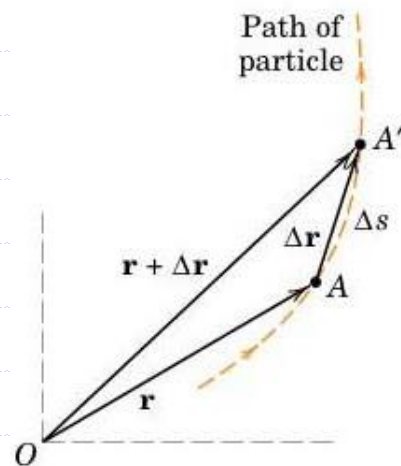


we define,

$$\text{Ave. Velocity} = \frac{\Delta \vec{r}}{\Delta t}$$

Letting  $\Delta t \rightarrow 0$ ,

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}$$



$\therefore$

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt}}$$

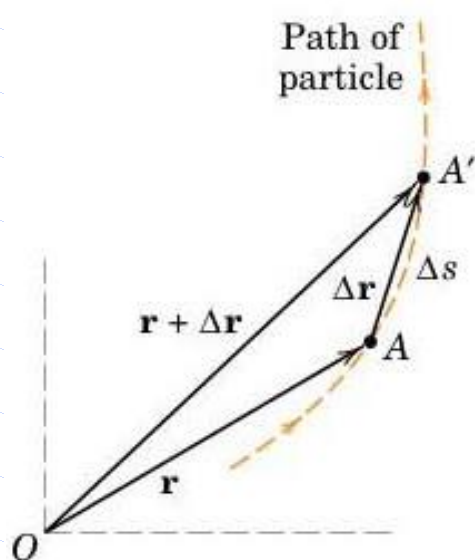
Instantaneous  
Velocity

## Comments:

- ①  $\Delta \vec{r}$  specifies dir. of Ave. vel.
- ② As  $\Delta t \rightarrow 0$ ,  $\vec{v}$  becomes tangent to particle path.

The particle speed is

Path of  
particle



$$\begin{aligned} v &= |\vec{v}| = \left| \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} \right| = \lim_{\Delta t \rightarrow 0} \left| \frac{\Delta \vec{r}}{\Delta t} \right| \\ &= \lim_{\Delta t \rightarrow 0} \frac{|\Delta \vec{r}|}{\Delta t} = \frac{ds}{dt} \end{aligned}$$

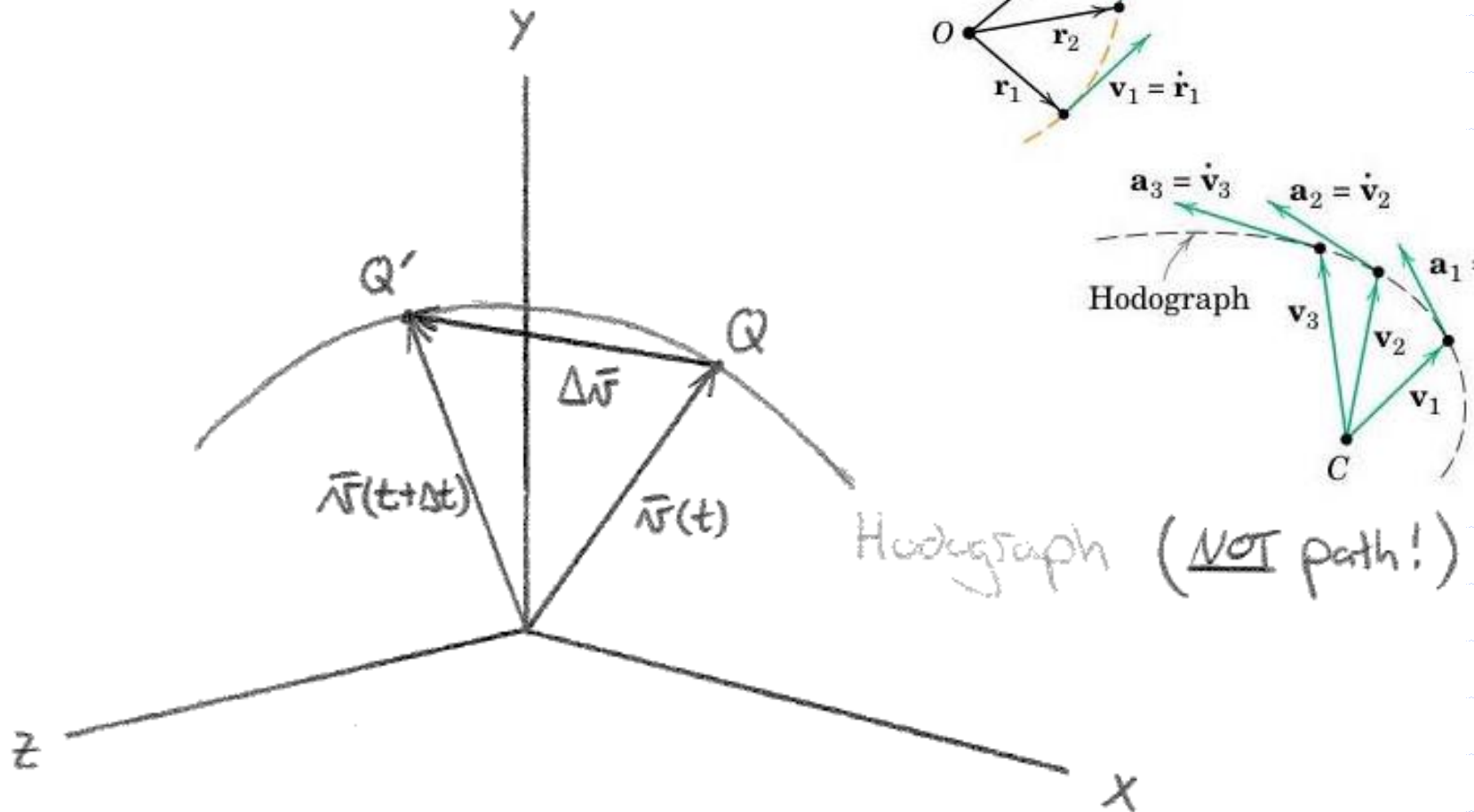
$\therefore$

$$\boxed{v = \frac{ds}{dt}}$$

Speed

# Acceleration $\bar{a}$

↳ Time-evolution of  $\bar{v}$



we define,

$$\text{Ave. Acceleration} = \frac{\Delta \bar{v}}{\Delta t}$$

Letting  $\Delta t \rightarrow 0$ ,

$$\bar{a} = \lim_{\Delta t \rightarrow 0} \frac{\bar{v}(t + \Delta t) - \bar{v}(t)}{\Delta t} = \frac{d\bar{v}}{dt}$$

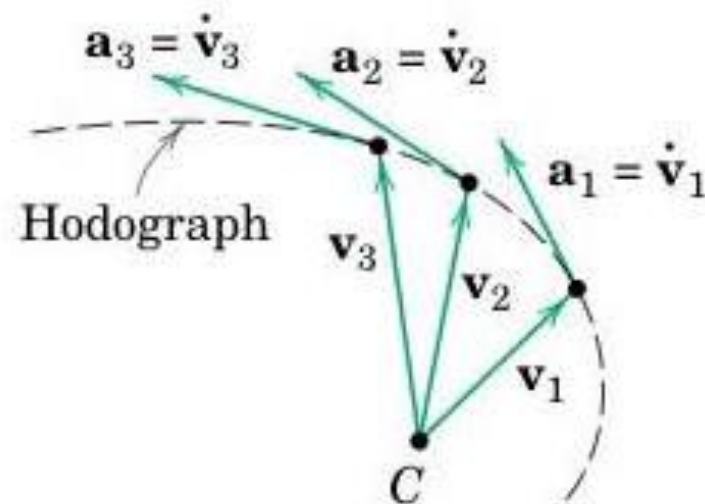
$\therefore$

$$\boxed{\bar{a} = \frac{d\bar{v}}{dt}}$$

Instantaneous  
Acceleration

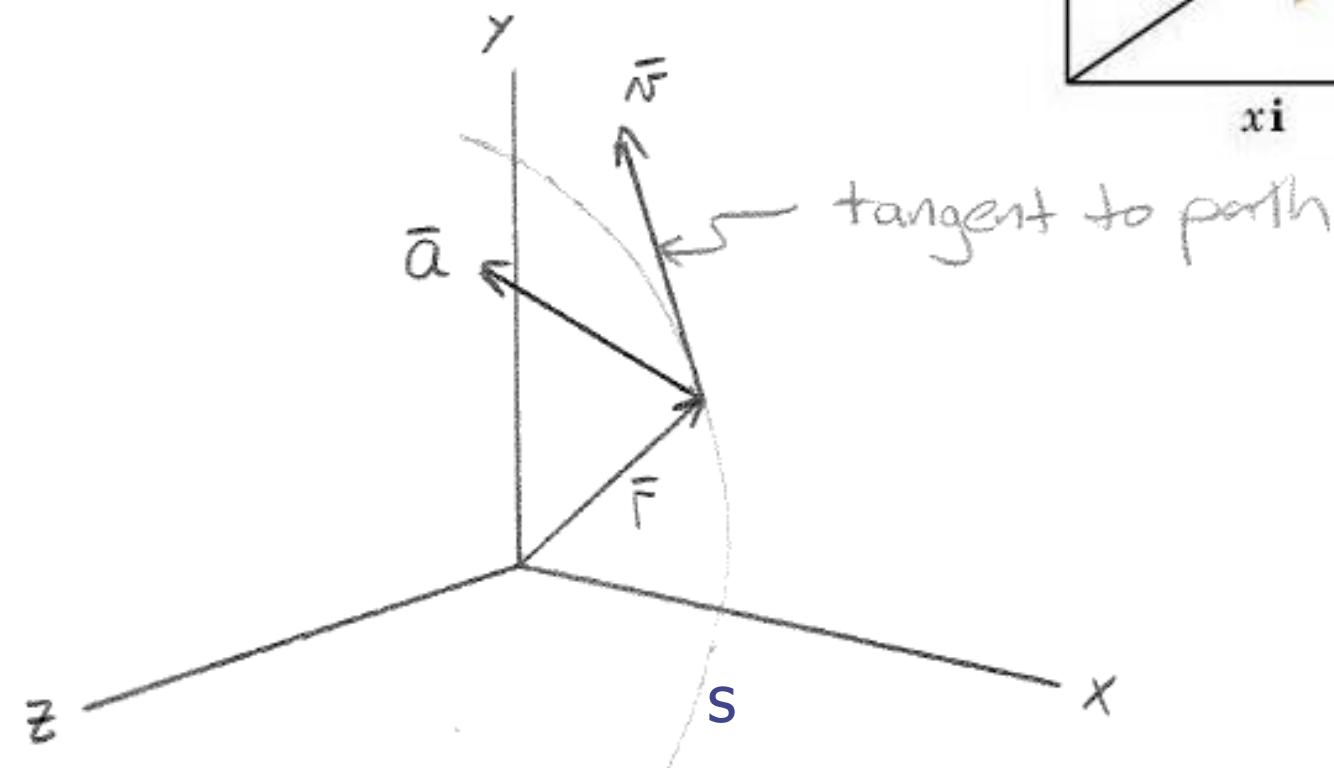
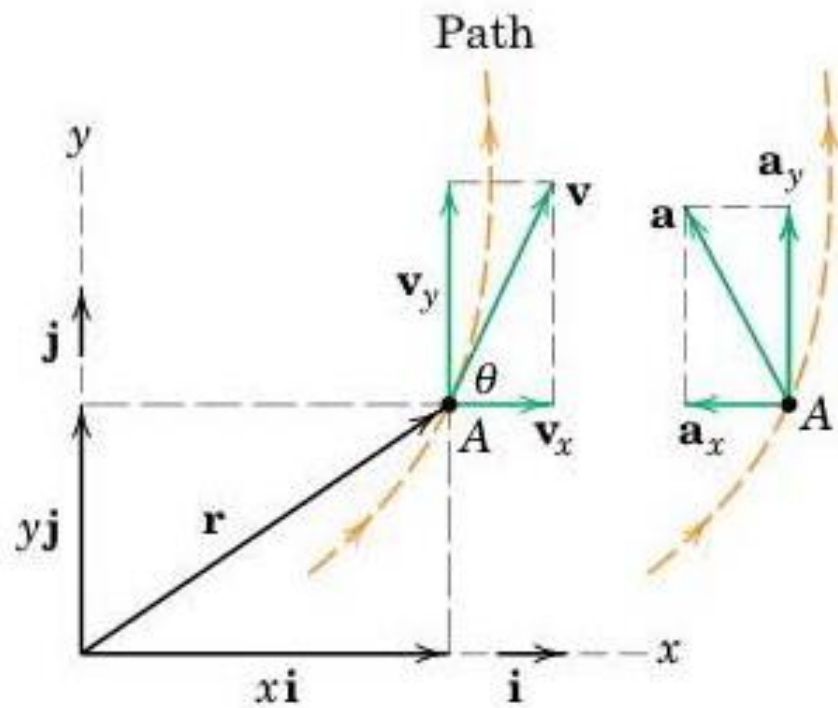
## Comments:

- ①  $\Delta \bar{a}$  specifies dir. of Ave. Acc.
- ③ As  $\Delta t \rightarrow 0$ ,  $\bar{a}$  becomes tangent to hodograph of motion, but NOT to particle path.





In summary:



## Sample Problem 2/5

The curvilinear motion of a particle is defined by  $v_x = 50 - 16t$  and  $y = 100 - 4t^2$ , where  $v_x$  is in meters per second,  $y$  is in meters, and  $t$  is in seconds. It is also known that  $x = 0$  when  $t = 0$ . Plot the path of the particle and determine its velocity and acceleration when the position  $y = 0$  is reached.

**Solution.** The  $x$ -coordinate is obtained by integrating the expression for  $v_x$ , and the  $x$ -component of the acceleration is obtained by differentiating  $v_x$ . Thus,

$$\left[ \int dx = \int v_x dt \right] \quad \int_0^x dx = \int_0^t (50 - 16t) dt \quad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \quad a_x = \frac{d}{dt} (50 - 16t) \quad a_x = -16 \text{ m/s}^2$$

The  $y$ -components of velocity and acceleration are

$$[v_y = \dot{y}] \quad v_y = \frac{d}{dt} (100 - 4t^2) \quad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \quad a_y = \frac{d}{dt} (-8t) \quad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of  $x$  and  $y$  for various values of  $t$  and plot  $x$  against  $y$  to obtain the path as shown.

When  $y = 0$ ,  $0 = 100 - 4t^2$ , so  $t = 5$  s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

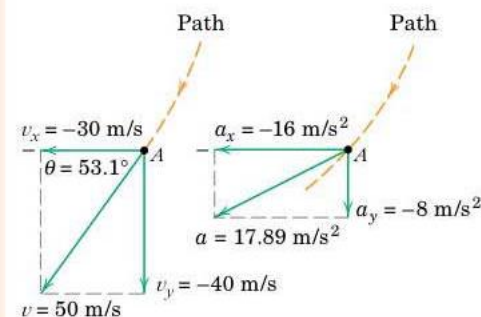
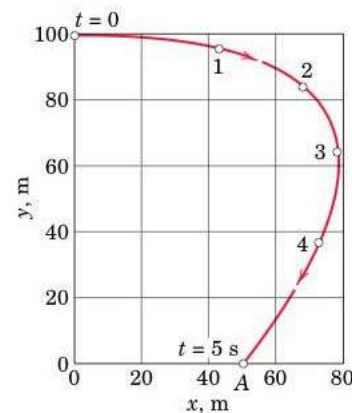
The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where  $y = 0$ . Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s}$$

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2$$

Ans.

Ans.



### Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

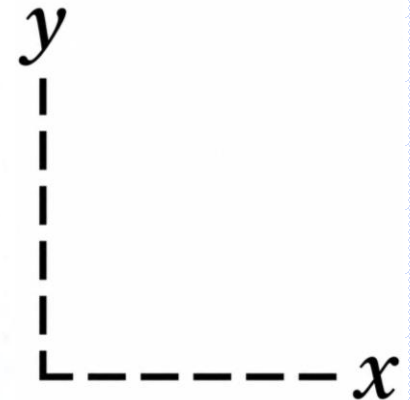
S/P : 2/5

Given : curv. motion  
 $v_x = 50 - 16t$  (m/s)  
 $y = 100 - 4t^2$  (m)  
 $x(t=0) = 0$

(more detailed solution)

Req. : A) particle path (y vs x i.t.o. t)  
 B)  $\bar{v}$  (y=0) = ?  
 $\bar{a}$  (y=0) = ?

Sol<sup>n</sup> : Plan :  $\left\{ \begin{array}{l} x \xrightarrow{\textcircled{1}} \int v_x dt \\ v_x \xrightarrow{\textcircled{2}} \downarrow \dot{v}_x \\ a_x \end{array} \right\} \left\{ \begin{array}{l} y \xrightarrow{\textcircled{3}} \downarrow \dot{y} \\ v_y \xrightarrow{\textcircled{4}} \downarrow \dot{v}_y \\ a_y \end{array} \right\}$



⑤ plot (y vs. x) for different values of  $t \rightarrow A)$

⑥ find @  $y=0 \left\{ \begin{array}{l} v_x \\ v_y \\ \bar{v} \end{array} \right\} \left\{ \begin{array}{l} a_x \\ a_y \\ \bar{a} \end{array} \right\} \rightarrow B)$

$$\textcircled{1} \quad v_x = \frac{dx}{dt} \Rightarrow \int_{x=x_0=0}^{x=x} dx = \int_{t=0}^{t=t} v_x dt = \int_0^t (50-16t) dt$$

$$x - \cancel{x_0} = 50t - \frac{16}{2}t^2 \Rightarrow \underline{x = 50t - 8t^2 \text{ (m)}}$$

$$\textcircled{2} \quad a_x = \dot{v}_x = \frac{dv_x}{dt} = \frac{d}{dt}(50-16t) = \underline{-16 \text{ (m/s}^2\text{)}}$$

$$\textcircled{3} \quad v_y = \frac{dy}{dt} = \dot{y} = \frac{d}{dt}(100-4t^2) = 0 - 8t = \underline{-8t \text{ (m/s)}}$$

$$\textcircled{4} \quad a_y = \dot{v}_y = \frac{dv_y}{dt} = \frac{d}{dt}(-8t) = \underline{-8 \text{ (m/s}^2\text{)}}$$

Cont. S/P : 2/5

$$x = 50t - 8t^2 \dots$$

so far :

$$v_x = 50 - 16t \dots$$

$$a_x = -16 \dots$$

$$y = 100 - 4t^2 \leftarrow$$

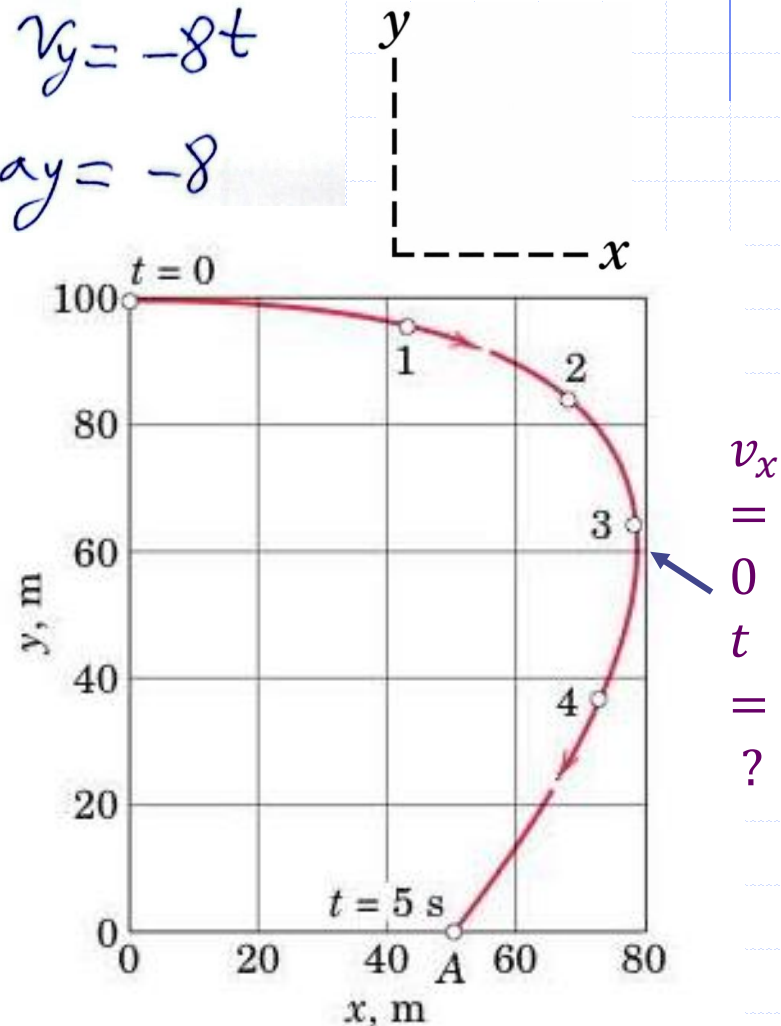
(When  
 $y = 0$ ,  
 $t = ?$ )

$$v_y = -8t$$

$$a_y = -8$$

⑤ plot :

| $t(s)$ | $x(m)$           | $y(m)$          |
|--------|------------------|-----------------|
| 0      | 0                | $100 - 0 = 100$ |
| 1      | $50 - 8 = 42$    | $100 - 4 = 96$  |
| 2      | $100 - 32 = 68$  | $100 - 16 = 84$ |
| 3      | $150 - 72 = 78$  | $100 - 36 = 64$ |
| 4      | $200 - 128 = 72$ | $100 - 64 = 36$ |
| 5      | $250 - 200 = 50$ | $100 - 100 = 0$ |



$$\textcircled{6} \text{ } ay=0 \Rightarrow t=5 \text{ s} :$$

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$\Rightarrow v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

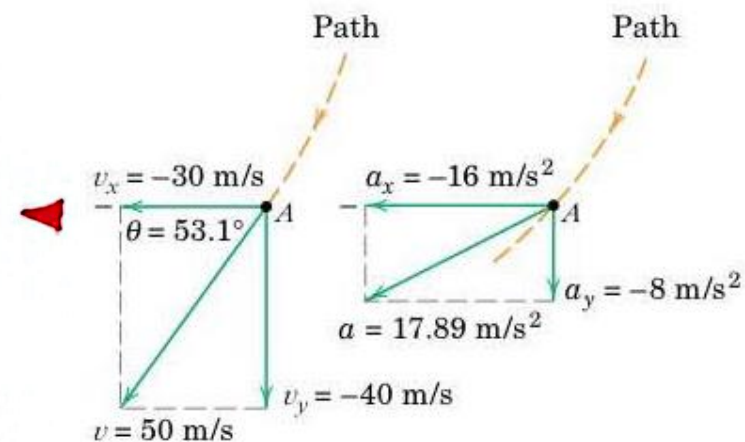
$$\boxed{\vec{v}_{y=0} = -30 \hat{i} - 40 \hat{j} \text{ (m/s)}}$$

$$a_x = -16 \text{ m/s}^2$$

$$a_y = -8 \text{ m/s}^2$$

$$\Rightarrow a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

$$\boxed{\vec{a}_{y=0} = -16 \hat{i} - 8 \hat{j} \text{ (m/s}^2\text{)}}$$



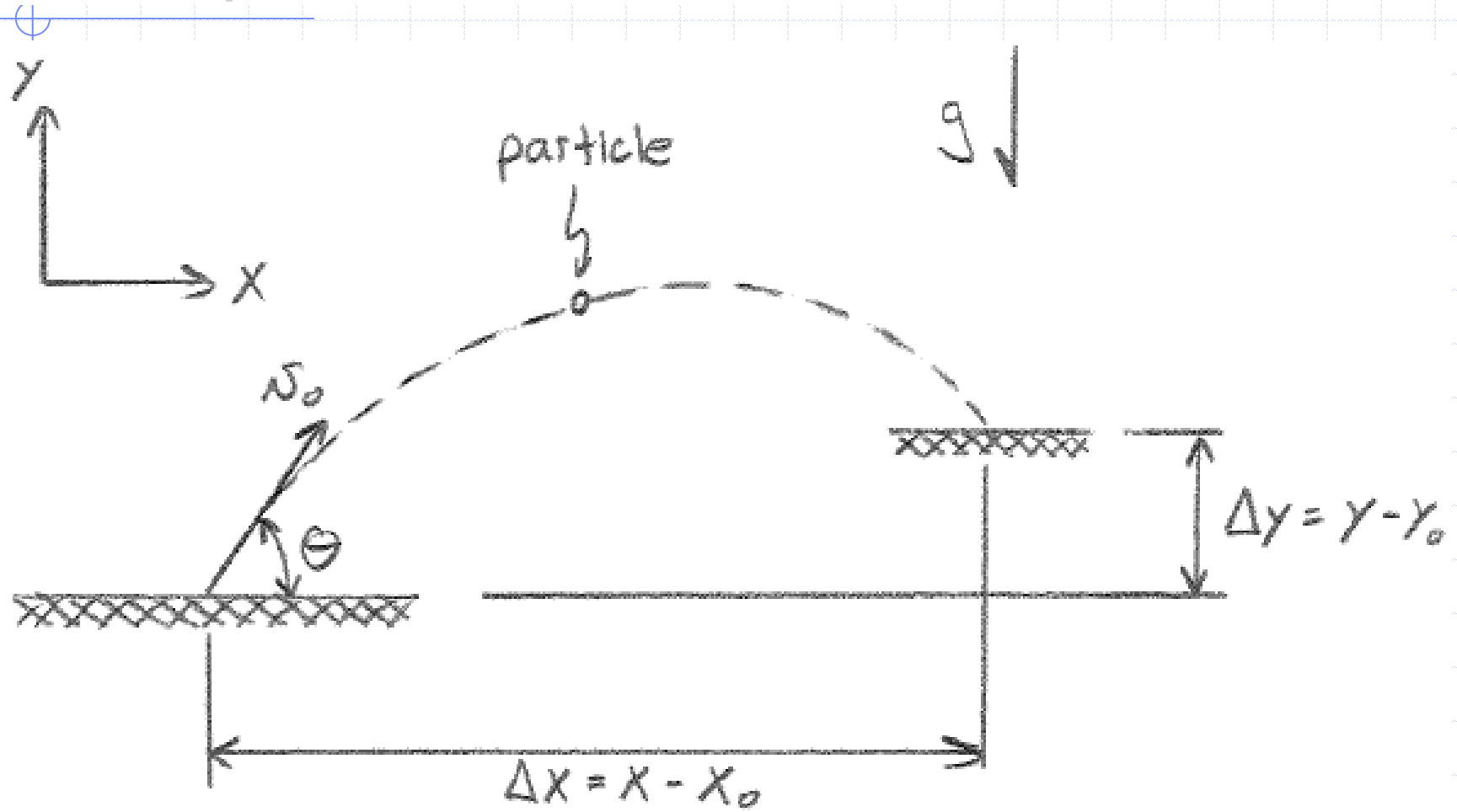
### Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

## 2. Projectile Motion ( $x$ - $y$ )



# Projectile Motion

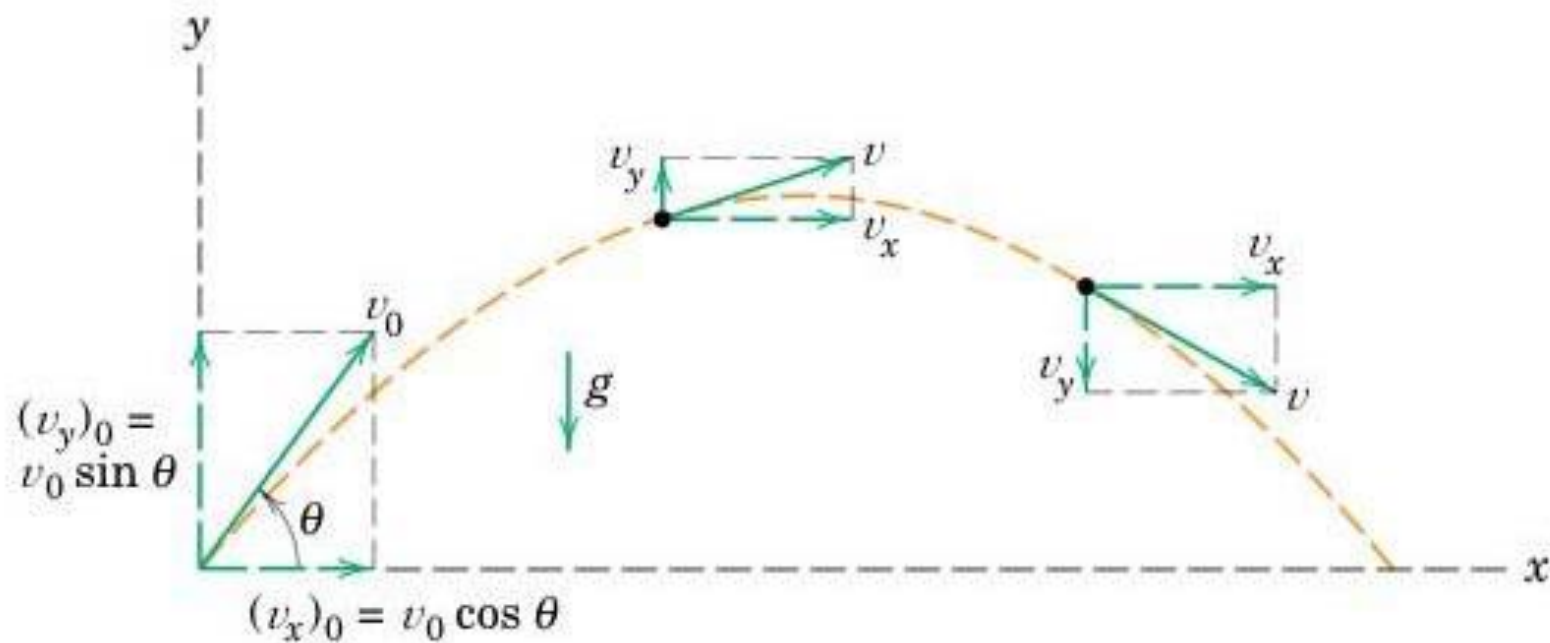


$a_z = 0 \Leftrightarrow$  motion remains in  $x-y$  plane

$a_y = -g \Leftrightarrow$  vertical motion is uniformly accelerated

$a_x = 0 \Leftrightarrow$  horizontal motion is uniform

Key: Analyze  $\uparrow$  and  $\rightarrow$  motion separately.



$$a_x = 0$$

$$a_y = -g$$

$$v_x = (v_x)_0$$

$$v_y = (v_y)_0 - gt$$

$$x = x_0 + (v_x)_0 t$$

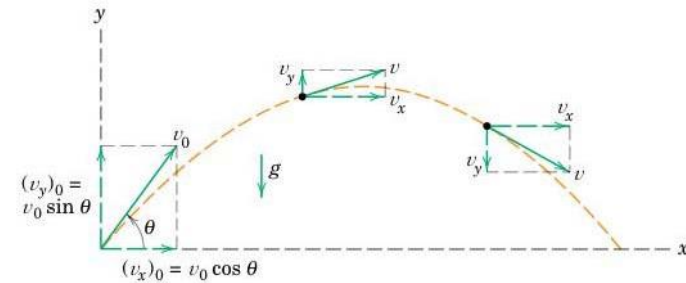
$$y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2$$

$$v_y^2 = (v_y)_0^2 - 2g(y - y_0)$$

# Example 1 (See prev. diagram)

Find an expression that relates directly

$\underbrace{\Delta x, \Delta y}_{\text{geometry}}, \underbrace{v_0, \theta}_{\text{initial velocity}}$



X-dynamics:  $a_x = 0, v_{x0} = v_0 \cos \theta$

Invoke  $x = x_0 + v_{x0} t$

$$\Rightarrow \Delta x = v_0 \cos \theta t$$

$$\Rightarrow t = \frac{\Delta x}{v_0 \cos \theta} \quad \text{--- (*)}$$

y-dynamics:  $a_y = -g$ ,  $v_{y0} = v_0 \sin \theta$

Invoke  $y - y_0 = v_{y0}t + \frac{1}{2}a_y t^2$

$$\Rightarrow \Delta y = v_0 \sin \theta t - \frac{1}{2} g t^2 \quad \text{--- (**)}$$

$$(*) \rightarrow (**): \quad \Delta y = v_0 \sin \theta \cdot \frac{\Delta x}{v_0 \cos \theta} - \frac{g}{2} \cdot \frac{(\Delta x)^2}{v_0^2 \cos^2 \theta}$$

$$\therefore \Delta y = \Delta x \tan \theta - \frac{g(\Delta x)^2}{2v_0^2 \cos^2 \theta} \quad \text{--- (***)}$$

(\*\*\*) in a different form... substitute:

$$\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$$

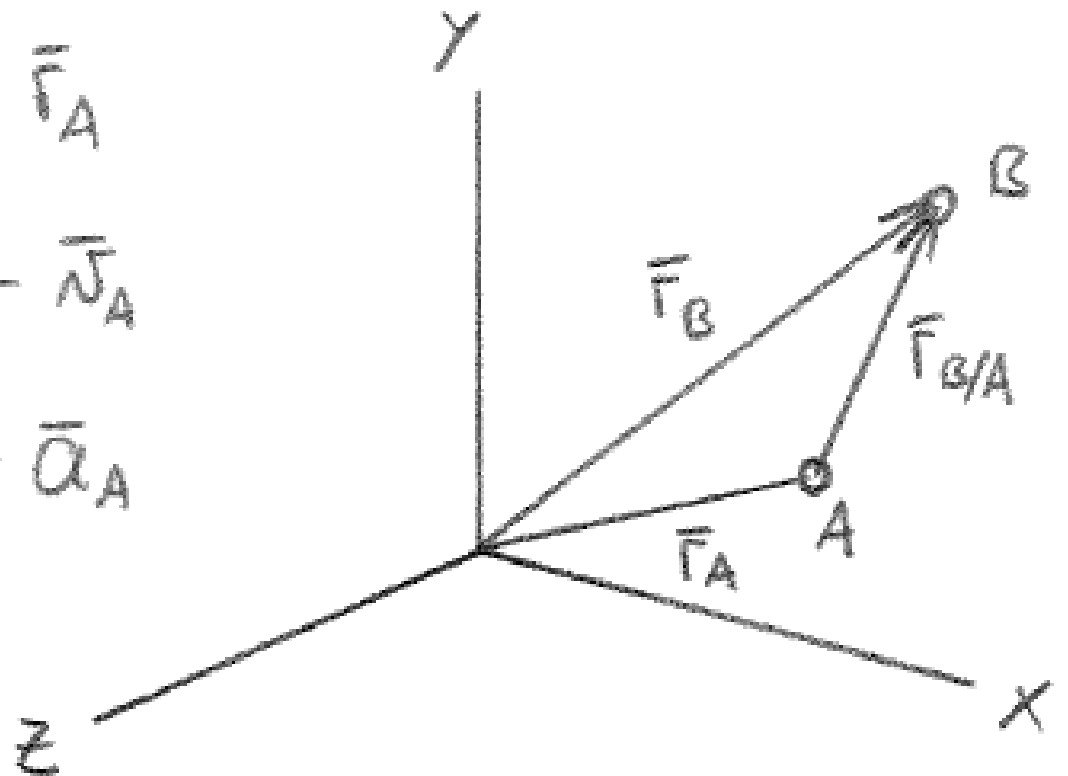
$$(\tan \theta)^2 - \frac{\partial v_0^2}{g \Delta x} (\tan \theta) + \left[ 1 + \frac{\partial v_0^2 \Delta x}{g (\Delta x)^2} \right] = 0$$

# 3. Relative & Dependent Curvilinear Motion

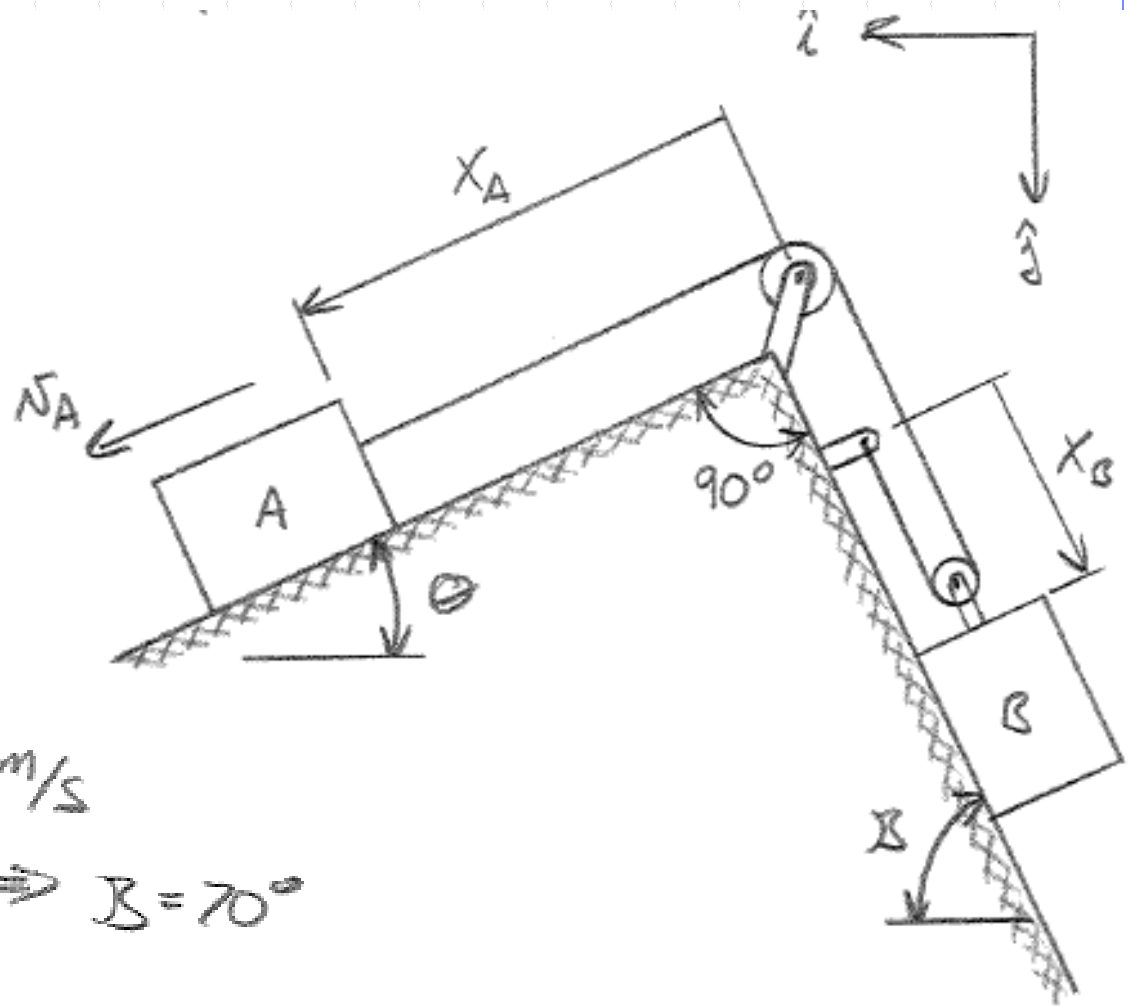


# Relative Curvilinear Motion

$$\begin{aligned}\bar{\mathbf{r}}_{B/A} &= \bar{\mathbf{r}}_B - \bar{\mathbf{r}}_A \\ \Rightarrow \bar{\mathbf{v}}_{B/A} &= \bar{\mathbf{v}}_B - \bar{\mathbf{v}}_A \\ \Rightarrow \bar{\mathbf{a}}_{B/A} &= \bar{\mathbf{a}}_B - \bar{\mathbf{a}}_A\end{aligned}$$



# Example 2



Data:  $\dot{x}_A = 80 \text{ mm/s}$

$$\theta = 20^\circ \Rightarrow \beta = 70^\circ$$

Find: (A)  $\vec{v}_B$   
 (B)  $\vec{v}_{A/B}$  ) Vectors!

PART A: Find  $\bar{v}_B$

Kinematic constraint:  $2x_B + x_A = \text{const.}$

$$\Rightarrow 2v_B + v_A = 0$$

$$\Rightarrow v_B = -\frac{1}{2} v_A$$

Thus

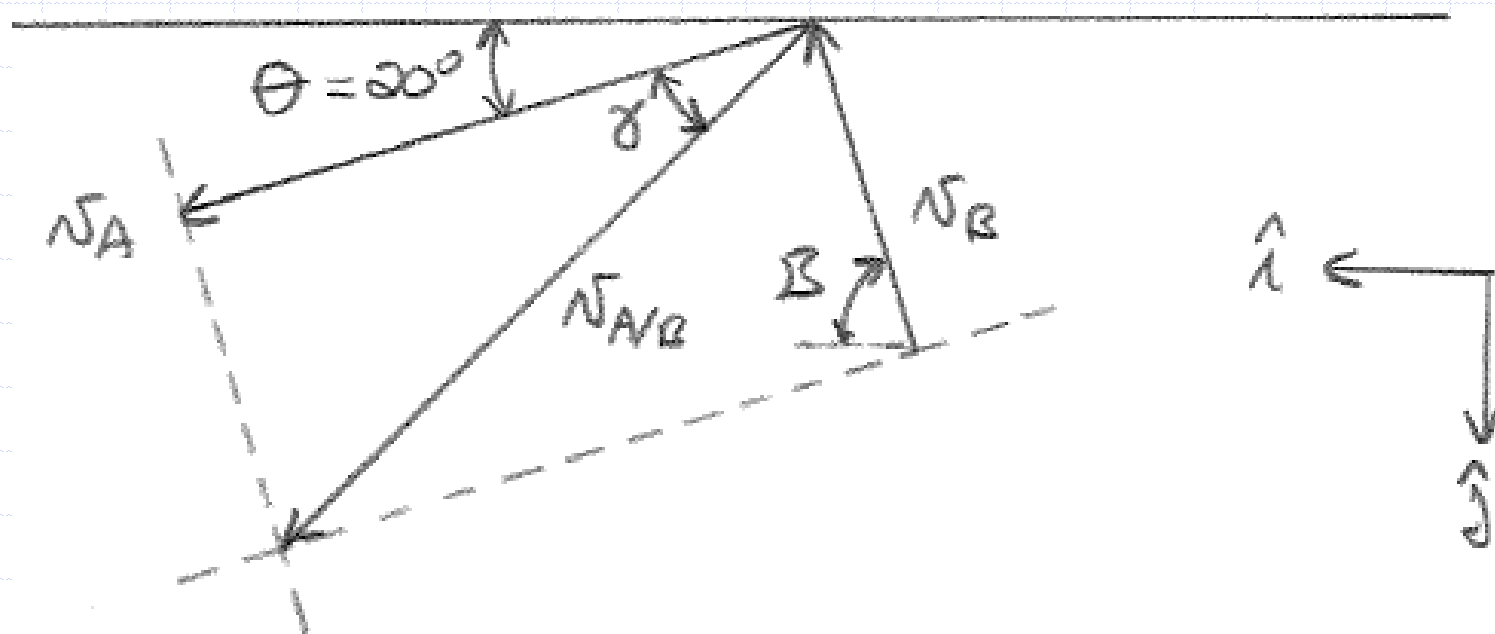
$$v_B = -\frac{1}{2} (80 \text{ mm/s}) = -40 \text{ mm/s}$$

$$\therefore \bar{v}_B = 40 \text{ mm/s} \nearrow 70^\circ$$

PART B:

find  $\vec{N}_{A/B} = \vec{N}_A - \vec{N}_B$

The geometry:



Two ways to solve:

- ① Geometrically - Here, easy! ... but won't be in general.
- ② Vectorially (rectangular components)

Method ②

Resolve

$$\vec{N}_A = N_A \cos \theta \hat{i} + N_A \sin \theta \hat{j}$$

$$\vec{N}_B = N_B \cos \beta \hat{i} - N_B \sin \beta \hat{j}$$

$$\begin{aligned}
 \text{Thus } \vec{V}_{A/B} &= \vec{V}_A - \vec{V}_B \\
 &= (V_A \cos \theta - V_B \cos \phi) \hat{i} \\
 &\quad + (V_A \sin \theta + V_B \sin \phi) \hat{j} \\
 &= (61.5 \text{ mm/s}) \hat{i} + (64.9 \text{ mm/s}) \hat{j}
 \end{aligned}$$

$$|\vec{V}_{A/B}| = \sqrt{(61.5 \text{ mm/s})^2 + (64.9 \text{ mm/s})^2} = 89.4 \text{ mm/s}$$

$$\gamma = \tan^{-1} \left( \frac{64.9 \text{ mm/s}}{61.5 \text{ mm/s}} \right) = 46.6^\circ$$

$$\therefore \vec{V}_{A/B} = 89.4 \text{ mm/s} \nearrow \gamma = 66.6^\circ$$

Error\*

## Sample Problem 2/15

In the pulley configuration shown, cylinder *A* has a downward velocity of 0.3 m/s. Determine the velocity of *B*. Solve in two ways.

**Solution (I).** The centers of the pulleys at *A* and *B* are located by the coordinates  $y_A$  and  $y_B$  measured from fixed positions. The total constant length of cable in the pulley system is

$$L = 3y_B + 2y_A + \text{constants}$$

where the constants account for the fixed lengths of cable in contact with the circumferences of the pulleys and the constant vertical separation between the two upper left-hand pulleys. Differentiation with time gives

①

$$0 = 3\dot{y}_B + 2\dot{y}_A$$

Substitution of  $v_A = \dot{y}_A = 0.3$  m/s and  $v_B = \dot{y}_B$  gives

②

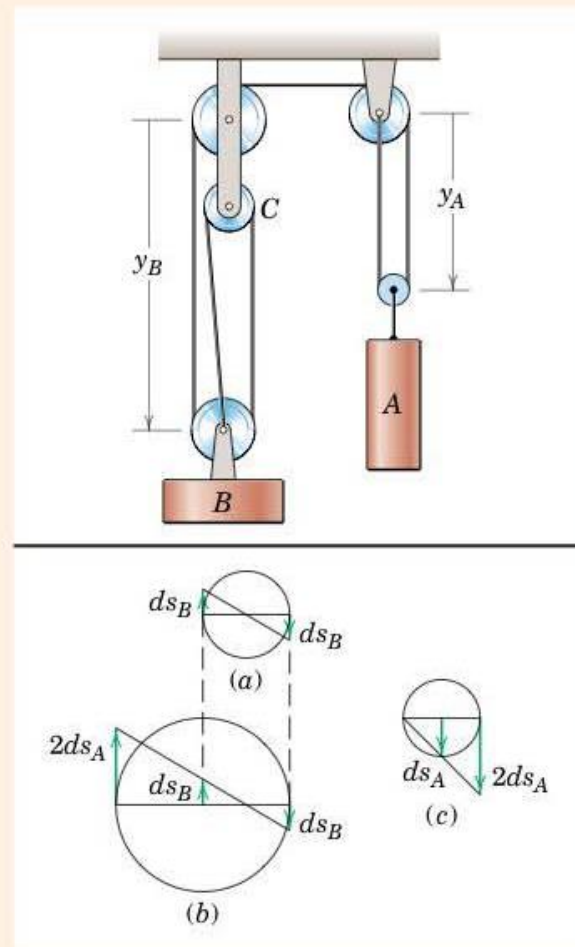
$$0 = 3(v_B) + 2(0.3) \quad \text{or} \quad v_B = -0.2 \text{ m/s} \quad \text{Ans.}$$

**Solution (II).** An enlarged diagram of the pulleys at *A*, *B*, and *C* is shown. During a differential movement  $ds_A$  of the center of pulley *A*, the left end of its horizontal diameter has no motion since it is attached to the fixed part of the cable. Therefore, the right-hand end has a movement of  $2ds_A$  as shown. This movement is transmitted to the left-hand end of the horizontal diameter of the pulley at *B*. Further, from pulley *C* with its fixed center, we see that the displacements on each side are equal and opposite. Thus, for pulley *B*, the right-hand end of the diameter has a downward displacement equal to the upward displacement  $ds_B$  of its center. By inspection of the geometry, we conclude that

$$2ds_A = 3ds_B \quad \text{or} \quad ds_B = \frac{2}{3}ds_A$$

Dividing by  $dt$  gives

$$|v_B| = \frac{2}{3}v_A = \frac{2}{3}(0.3) = 0.2 \text{ m/s (upward)} \quad \text{Ans.}$$



### Helpful Hints

- ① We neglect the small angularity of the cables between *B* and *C*.
- ② The negative sign indicates that the velocity of *B* is upward.

This is a **dependent-motion problem**: the cable connects the motions of  $A$  and  $B$

### Given

- Cylinder  $A$  moves downward at  $0.3 \text{ m/s}$ .
- The cable has constant total length.
- The upper pulleys, including  $C$ , have fixed centers.

### Required

Determine the **velocity of cylinder  $B$** : magnitude and direction.

### Coordinates

Measure  $y_A$  and  $y_B$  **vertically downward from fixed reference levels**, as shown in the diagram.

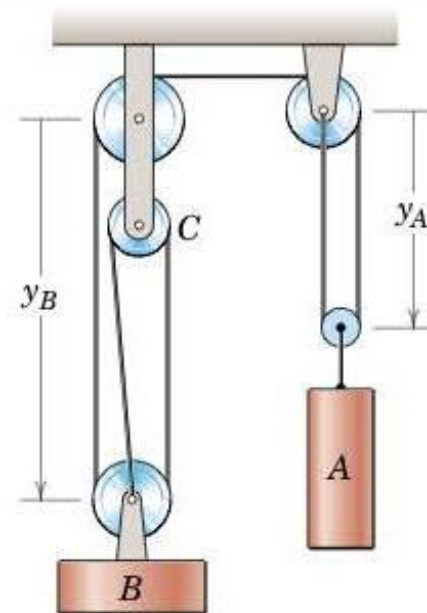
Therefore,

Downward is positive for both coordinates.

Each cylinder moves with its attached pulley, so their vertical velocities are equal.

### Solution plan

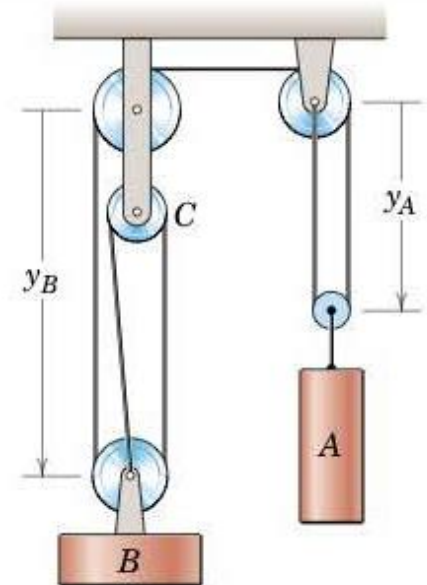
1. Identify the cable portions whose lengths change.
2. Express the constant cable length in terms of  $y_A$  and  $y_B$ .
3. Differentiate the length equation with respect to time.
4. Substitute the known velocity and interpret the result.



## ① Identify the changing cable lengths

Following the cable in the diagram, we find:

| Cable portion                                                          | Variable contribution to its length |
|------------------------------------------------------------------------|-------------------------------------|
| Upper-left fixed pulley to the moving pulley at $B$                    | $y_B$                               |
| Moving pulley at $B$ to fixed pulley $C$                               | $y_B$                               |
| Fixed pulley $C$ to the cable attachment on the moving assembly at $B$ | $y_B$                               |
| Upper-right fixed pulley to the moving pulley at $A$                   | $y_A$                               |
| Moving pulley at $A$ to the fixed cable end above it                   | $y_A$                               |



## ② Write the total cable length

Let  $K$  include all lengths that remain constant: fixed portions, pulley-contact lengths, and reference offsets.

$$L = \underbrace{y_B + y_B + y_B}_{\text{three changing portions at } B} + \underbrace{y_A + y_A}_{\text{two changing portions at } A} + K$$

Thus,

$$L = 3y_B + 2y_A + K$$

$$L = 3y_B + 2y_A + K$$

③ Differentiate with respect to time

Because the cable length is constant,

$$\frac{dL}{dt} = 0.$$

Also,  $K$  is constant, so

$$\frac{dK}{dt} = 0.$$

Therefore,

$$\begin{aligned}\frac{dL}{dt} &= 3\frac{dy_B}{dt} + 2\frac{dy_A}{dt} + \frac{dK}{dt} \\ 0 &= 3\dot{y}_B + 2\dot{y}_A.\end{aligned}$$

Using

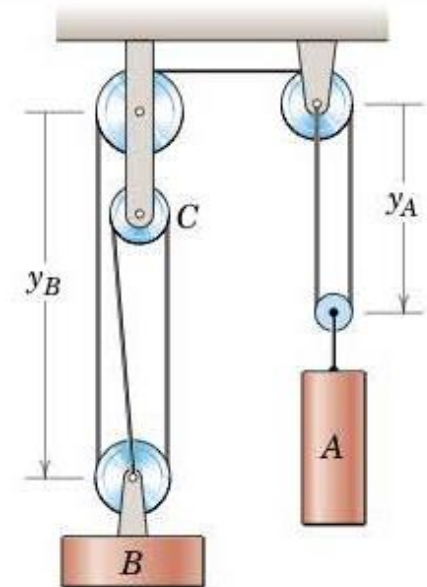
$$v_B = \dot{y}_B, \quad v_A = \dot{y}_A,$$

we obtain

$$\boxed{3v_B + 2v_A = 0}.$$

Rearranging,

$$3v_B = -2v_A \quad \Rightarrow \quad \boxed{v_B = -\frac{2}{3}v_A}.$$



#### ④ Substitute the known velocity

Cylinder *A* moves downward, our positive direction:

$$v_A = +0.3 \text{ m/s.}$$

Hence,

$$3v_B + 2(0.3) = 0$$

$$3v_B = -0.6$$

$$v_B = -0.2 \text{ m/s.}$$

The negative sign means that *B* moves **upward**.

Cylinder *B* has a velocity of 0.20 m/s upward.

#### Physical check

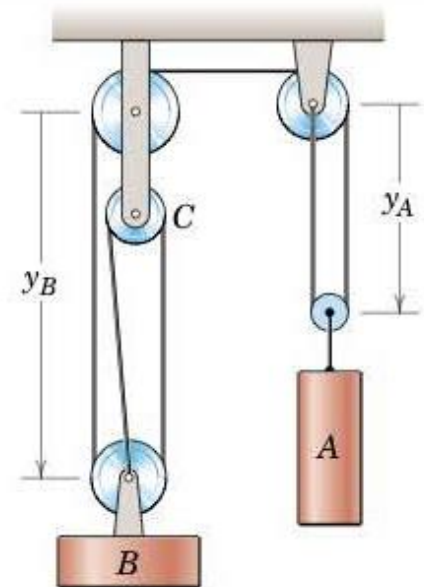
- The two portions at *A* lengthen at a combined rate of

$$2(0.3) = 0.6 \text{ m/s.}$$

- The three portions at *B* shorten at a combined rate of

$$3(0.2) = 0.6 \text{ m/s.}$$

The length gained equals the length lost, so the total cable length remains constant.



### Sample Problem 2/16

The tractor *A* is used to hoist the bale *B* with the pulley arrangement shown. If *A* has a forward velocity  $v_A$ , determine an expression for the upward velocity  $v_B$  of the bale in terms of  $x$ .

**Solution.** We designate the position of the tractor by the coordinate  $x$  and the position of the bale by the coordinate  $y$ , both measured from a fixed reference. The total constant length of the cable is

$$L = 2(h - y) + l = 2(h - y) + \sqrt{h^2 + x^2}$$

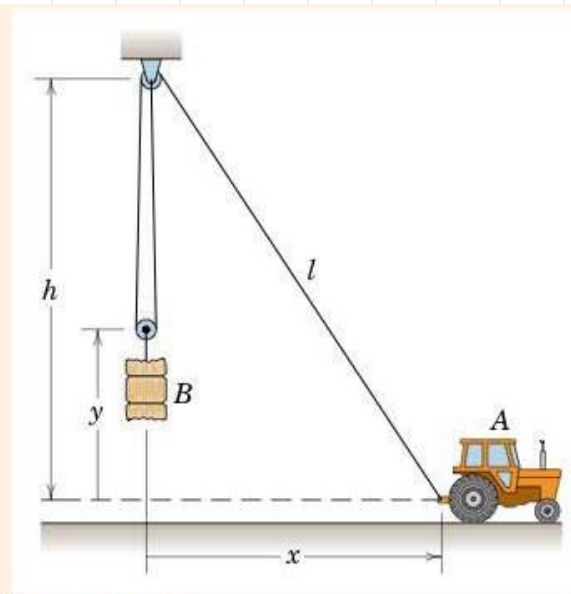
- ① Differentiation with time yields

$$0 = -2\dot{y} + \frac{x\dot{x}}{\sqrt{h^2 + x^2}}$$

Substituting  $v_A = \dot{x}$  and  $v_B = \dot{y}$  gives

$$v_B = \frac{1}{2} \frac{xv_A}{\sqrt{h^2 + x^2}}$$

Ans.



#### Helpful Hint

- ① Differentiation of the relation for a right triangle occurs frequently in mechanics.

**Given:** Tractor  $A$  moves forward with velocity  $v_A$ . The cable length is constant, and  $h$  is fixed.

**Required:** An expression for the upward velocity  $v_B$  in terms of  $x$ .

**Coordinates:**

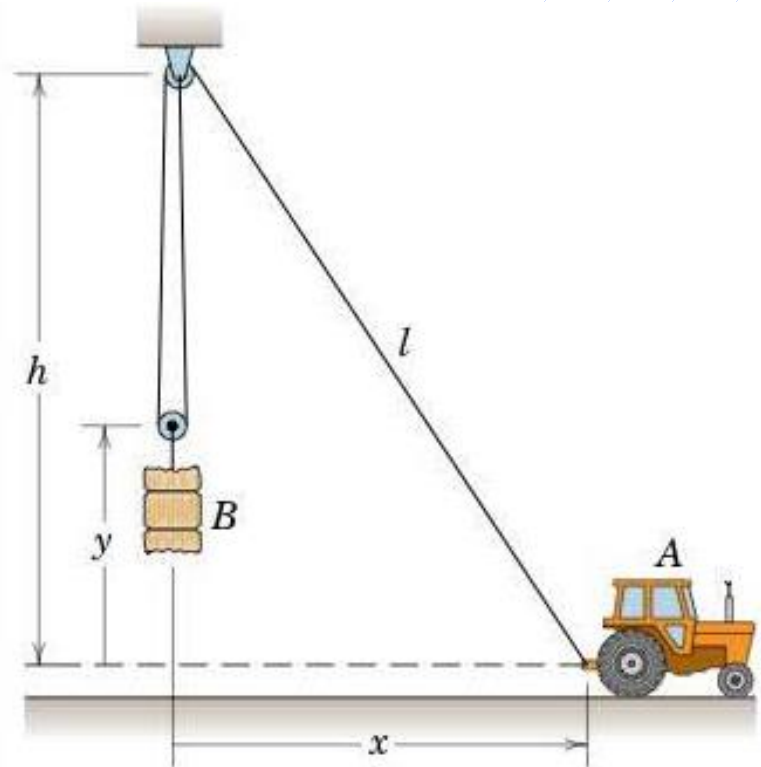
- $x$ : horizontal position of the tractor, positive to the right.
- $y$ : height of the moving pulley attached to the bale, positive upward.

Thus,

$$\dot{x} = v_A, \quad \dot{y} = v_B.$$

**Plan:**

- ① Express the cable length using the geometry.
- ② Differentiate with respect to time.
- ③ Substitute the velocities and interpret the result.



① Identify the changing cable portions

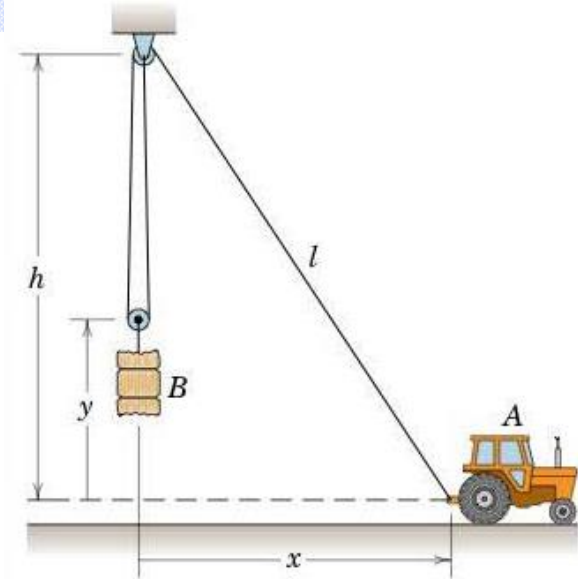
The moving pulley is supported by **two vertical cable portions**, each of length  $h - y$ .

The inclined portion has length  $l$ . From the right triangle,

$$l^2 = h^2 + x^2 \quad \Rightarrow \quad l = \sqrt{h^2 + x^2}.$$

Therefore, the total cable length is

$$L = \underbrace{2(h - y)}_{\text{two vertical portions}} + \underbrace{\sqrt{h^2 + x^2}}_{\text{inclined portion}} + K,$$



where  $K$  accounts for constant cable lengths around the pulleys, using the idealized geometry shown.

$$L = 2(h - y) + \sqrt{h^2 + x^2} + K$$

**Physical connection:** As the tractor moves right, the inclined portion lengthens. The two vertical portions must shorten, raising the bale.

## ② Differentiate the cable-length equation

Since  $L$ ,  $h$ , and  $K$  are constant,

$$0 = -2\dot{y} + \frac{d}{dt}(h^2 + x^2)^{1/2}.$$

Show the chain rule explicitly:

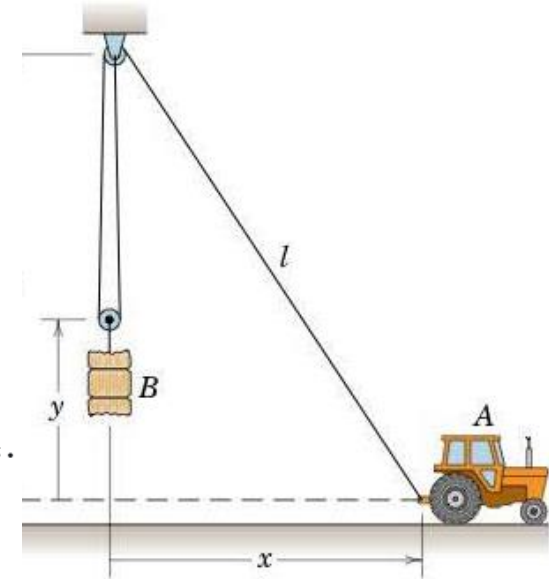
$$\frac{d}{dt}(h^2 + x^2)^{1/2} = \frac{1}{2}(h^2 + x^2)^{-1/2}(2x\dot{x}) = \frac{x\dot{x}}{\sqrt{h^2 + x^2}}.$$

Hence,

$$\boxed{\frac{d}{dt}[u(t)]^n = n[u(t)]^{n-1} \frac{du}{dt}}$$

$$0 = -2\dot{y} + \frac{x\dot{x}}{\sqrt{h^2 + x^2}}.$$

$$\boxed{\frac{d(x^2)}{dt} = \underbrace{\frac{d(x^2)}{dx}}_{2x} \underbrace{\frac{dx}{dt}}_{\text{rate of change of } x} = 2x \frac{dx}{dt}}$$



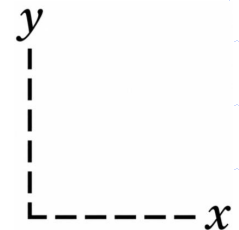
## ③ Substitute $\dot{x} = v_A$ and $\dot{y} = v_B$

$$2v_B = \frac{xv_A}{\sqrt{h^2 + x^2}}$$

$$\boxed{v_B = \frac{xv_A}{2\sqrt{h^2 + x^2}} \quad \text{upward}}$$

Check: For  $h > 0$  and  $x > 0$ ,

$$\frac{x}{\sqrt{h^2 + x^2}} < 1 \quad \Rightarrow \quad v_B < \frac{v_A}{2}.$$



The inclined cable lengthens more slowly than the tractor travels; its added length is supplied by the shortening of **two** vertical portions.

## Sample Problem 2/13

Passengers in the jet transport *A* flying east at a speed of 800 km/h observe a second jet plane *B* that passes under the transport in horizontal flight. Although the nose of *B* is pointed in the  $45^\circ$  northeast direction, plane *B* appears to the passengers in *A* to be moving away from the transport at the  $60^\circ$  angle as shown. Determine the true velocity of *B*.

**Solution.** The moving reference axes  $x$ - $y$  are attached to *A*, from which the relative observations are made. We write, therefore,

$$\textcircled{1} \quad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Next we identify the knowns and unknowns. The velocity  $\mathbf{v}_A$  is given in both magnitude and direction. The  $60^\circ$  direction of  $\mathbf{v}_{B/A}$ , the velocity which *B* appears to have to the moving observers in *A*, is known, and the true velocity of *B* is in the  $45^\circ$  direction in which it is heading. The two remaining unknowns are the magnitudes of  $\mathbf{v}_B$  and  $\mathbf{v}_{B/A}$ . We may solve the vector equation in any one of three ways.

**(I) Graphical.** We start the vector sum at some point *P* by drawing  $\mathbf{v}_A$  to a convenient scale and then construct a line through the tip of  $\mathbf{v}_A$  with the known direction of  $\mathbf{v}_{B/A}$ . The known direction of  $\mathbf{v}_B$  is then drawn through *P*, and the intersection *C* yields the unique solution enabling us to complete the vector triangle and scale off the unknown magnitudes, which are found to be

$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

**(II) Trigonometric.** A sketch of the vector triangle is made to reveal the trigonometry, which gives

$$\textcircled{4} \quad \frac{v_B}{\sin 60^\circ} = \frac{v_A}{\sin 75^\circ} \quad v_B = 800 \frac{\sin 60^\circ}{\sin 75^\circ} = 717 \text{ km/h} \quad \text{Ans.}$$

**(III) Vector Algebra.** Using unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ , we express the velocities in vector form as

$$\mathbf{v}_A = 800\mathbf{i} \text{ km/h} \quad \mathbf{v}_B = (v_B \cos 45^\circ)\mathbf{i} + (v_B \sin 45^\circ)\mathbf{j}$$

$$\mathbf{v}_{B/A} = (v_{B/A} \cos 60^\circ)(-\mathbf{i}) + (v_{B/A} \sin 60^\circ)\mathbf{j}$$

Substituting these relations into the relative-velocity equation and solving separately for the  $\mathbf{i}$  and  $\mathbf{j}$  terms give

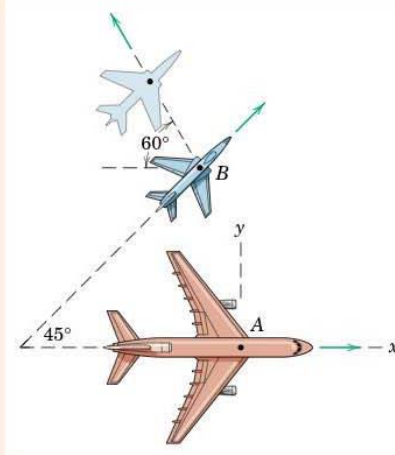
$$\text{(i-terms)} \quad v_B \cos 45^\circ = 800 - v_{B/A} \cos 60^\circ$$

$$\text{(j-terms)} \quad v_B \sin 45^\circ = v_{B/A} \sin 60^\circ$$

$\textcircled{5}$  Solving simultaneously yields the unknown velocity magnitudes

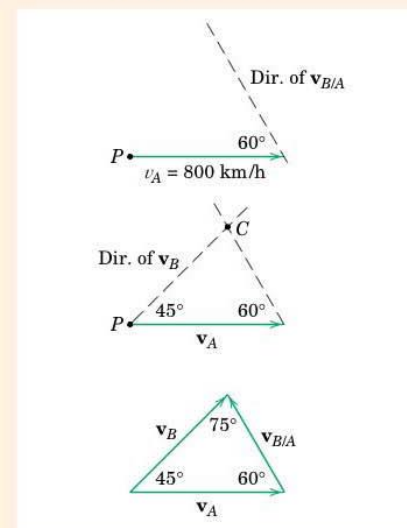
$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

It is worth noting the solution of this problem from the viewpoint of an observer in *B*. With reference axes attached to *B*, we would write  $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ . The apparent velocity of *A* as observed by *B* is then  $\mathbf{v}_{A/B}$ , which is the negative of  $\mathbf{v}_{B/A}$ .



### Helpful Hints

- $\textcircled{1}$  We treat each airplane as a particle.
- $\textcircled{2}$  We assume no side slip due to cross wind.
- $\textcircled{3}$  Students should become familiar with all three solutions.



- $\textcircled{4}$  We must be prepared to recognize the appropriate trigonometric relation, which here is the law of sines.
- $\textcircled{5}$  We can see that the graphical or trigonometric solution is shorter than the vector algebra solution in this particular problem.

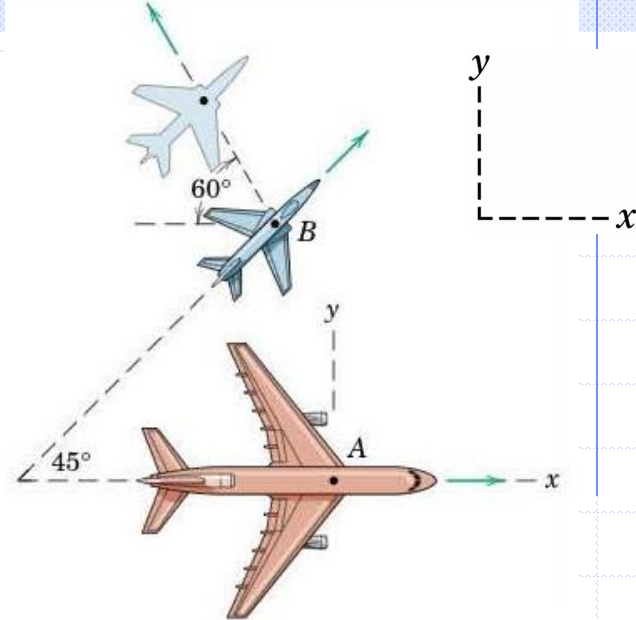
Given

- Plane  $A$  flies east at 800 km/h.
- Plane  $B$  heads  $45^\circ$  north of east.
- Observers in  $A$  see  $B$  moving  $60^\circ$  north of west.
- Assume no sideslip, so the true velocity of  $B$  follows its heading.

**Required:** The true velocity of  $B$ —its speed and direction relative to the ground.

Choose  $+x$  east and  $+y$  north.

**Strategy:** Relative velocity



$$\underbrace{\mathbf{v}_B}_{\text{B relative to ground}} = \underbrace{\mathbf{v}_A}_{\text{A relative to ground}} + \underbrace{\mathbf{v}_{B/A}}_{\text{B as observed from A}}$$

| Velocity           | Magnitude | Direction                |
|--------------------|-----------|--------------------------|
| $\mathbf{v}_A$     | 800 km/h  | East                     |
| $\mathbf{v}_B$     | Unknown   | $45^\circ$ north of east |
| $\mathbf{v}_{B/A}$ | Unknown   | $60^\circ$ north of west |

## Method I: Graphical solution

### ① Construct the velocity triangle

Use a convenient scale, for example,

$$1 \text{ cm} \equiv 100 \text{ km/h.}$$

1. From point  $P$ , draw  $\overrightarrow{PQ}$  8 cm east to represent  $\mathbf{v}_A$ .
2. From  $Q$ , draw a ray  $60^\circ$  north of west, along  $\mathbf{v}_{B/A}$ .
3. From  $P$ , draw a ray  $45^\circ$  north of east, along  $\mathbf{v}_B$ .
4. Label their intersection  $C$ .

The arrows satisfy

$$\underbrace{\overrightarrow{PQ}}_{\mathbf{v}_A} + \underbrace{\overrightarrow{QC}}_{\mathbf{v}_{B/A}} = \underbrace{\overrightarrow{PC}}_{\mathbf{v}_B}.$$

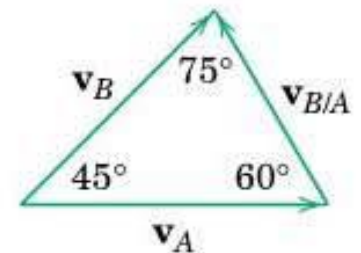
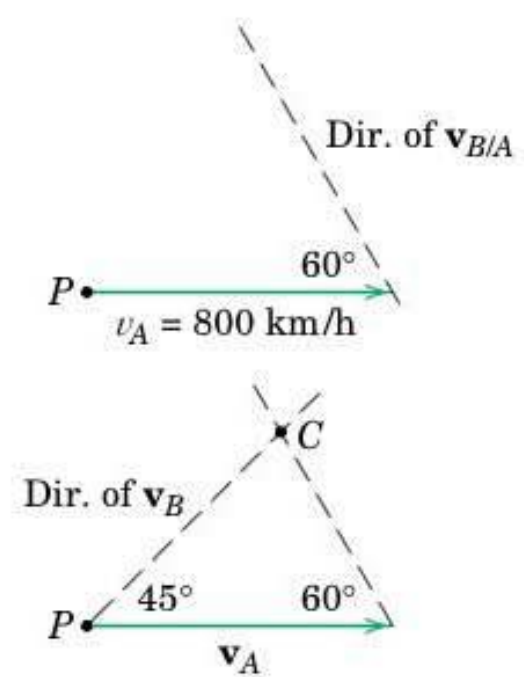
### ② Measure and convert using the scale

$$PC \approx 7.17 \text{ cm} \Rightarrow \boxed{v_B \approx 717 \text{ km/h}}$$

$$QC \approx 5.86 \text{ cm} \Rightarrow v_{B/A} \approx 586 \text{ km/h.}$$

**Answer:**  $B$  flies approximately 717 km/h,  $45^\circ$  north of east.

*Graphical accuracy depends on the drawing and measurement.*



## Method II: Trigonometric solution

### ① Identify the internal angles of the velocity triangle

The base angles are  $45^\circ$  and  $60^\circ$ . Therefore,

$$\angle C = 180^\circ - 45^\circ - 60^\circ = \boxed{75^\circ}.$$

### ② Apply the sine rule

Match each side with its **opposite angle**:

$$\frac{\underbrace{v_B}_{\text{opposite } 60^\circ}}{\sin 60^\circ} = \frac{\underbrace{800}_{\text{opposite } 75^\circ}}{\sin 75^\circ} = \frac{\underbrace{v_{B/A}}_{\text{opposite } 45^\circ}}{\sin 45^\circ}$$

Hence,

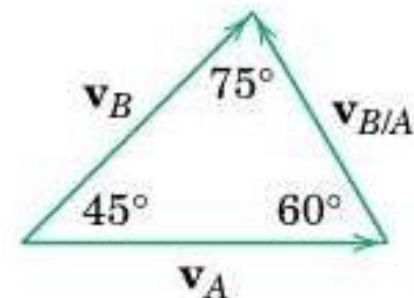
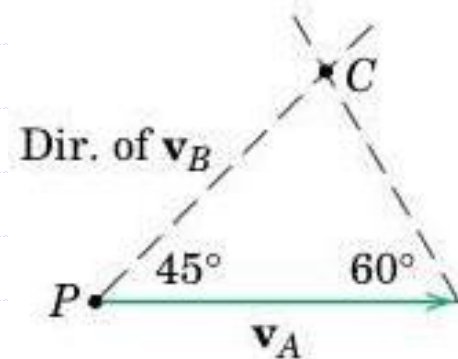
$$v_B = \frac{800 \sin 60^\circ}{\sin 75^\circ} = \boxed{717 \text{ km/h}}.$$

Also,

$$v_{B/A} = \frac{800 \sin 45^\circ}{\sin 75^\circ} = \boxed{586 \text{ km/h}}.$$

True velocity of  $B$ :

$$\boxed{717 \text{ km/h, } 45^\circ \text{ north of east}}$$



### Method III: Vector algebra

① Express the velocities in components

$$\mathbf{v}_A = 800 \mathbf{i}$$

$$\mathbf{v}_B = v_B \cos 45^\circ \mathbf{i} + v_B \sin 45^\circ \mathbf{j}$$

$$\mathbf{v}_{B/A} = \underbrace{-v_{B/A} \cos 60^\circ \mathbf{i}}_{\text{west: negative } x} + \underbrace{v_{B/A} \sin 60^\circ \mathbf{j}}_{\text{north: positive } y}.$$

② Substitute into  $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

Equate corresponding components:

$x$ -components — Equation (1):

$$v_B \cos 45^\circ = 800 - v_{B/A} \cos 60^\circ$$

$y$ -components — Equation (2):

$$v_B \sin 45^\circ = v_{B/A} \sin 60^\circ$$

③ Eliminate  $v_{B/A}$

From (2),

$$v_{B/A} = \frac{v_B \sin 45^\circ}{\sin 60^\circ}.$$

Substitute into (1) and collect the  $v_B$  terms:

$$v_B \left( \cos 45^\circ + \frac{\sin 45^\circ \cos 60^\circ}{\sin 60^\circ} \right) = 800.$$

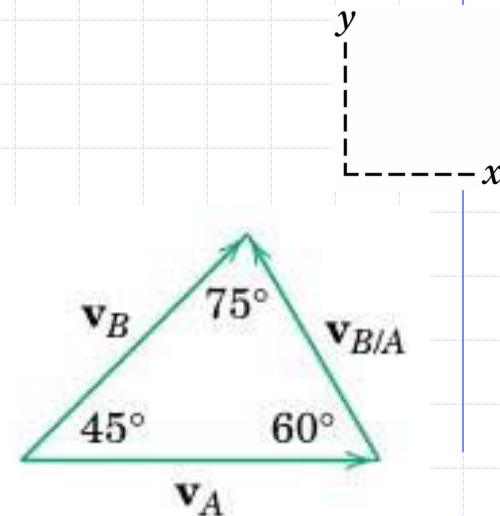
Thus,

$$\boxed{v_B = 717 \text{ km/h}}, \quad v_{B/A} = \frac{717 \sin 45^\circ}{\sin 60^\circ} \approx 586 \text{ km/h}.$$

In vector form,

$$\boxed{\mathbf{v}_B \approx 507 \mathbf{i} + 507 \mathbf{j} \text{ km/h}}.$$

Check: Equal positive east and north components give the required  $45^\circ$  northeast direction.



### Sample Problem 2/14

Car A is accelerating in the direction of its motion at the rate of  $3 \text{ ft/sec}^2$ . Car B is rounding a curve of 440-ft radius at a constant speed of 30 mi/hr. Determine the velocity and acceleration which car B appears to have to an observer in car A if car A has reached a speed of 45 mi/hr for the positions represented.

**Solution.** We choose nonrotating reference axes attached to car A since the motion of B with respect to A is desired.

**Velocity.** The relative-velocity equation is

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

and the velocities of A and B for the position considered have the magnitudes

$$v_A = 45 \frac{5280}{60^2} = 45 \frac{44}{30} = 66 \text{ ft/sec} \quad v_B = 30 \frac{44}{30} = 44 \text{ ft/sec}$$

The triangle of velocity vectors is drawn in the sequence required by the equation, and application of the law of cosines and the law of sines gives

$$\textcircled{1} \quad v_{B/A} = 58.2 \text{ ft/sec} \quad \theta = 40.9^\circ \quad \text{Ans.}$$

**Acceleration.** The relative-acceleration equation is

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

The acceleration of A is given, and the acceleration of B is normal to the curve in the  $n$ -direction and has the magnitude

$$[a_n = v^2/\rho] \quad a_B = (44)^2/440 = 4.4 \text{ ft/sec}^2$$

The triangle of acceleration vectors is drawn in the sequence required by the equation as illustrated. Solving for the  $x$ - and  $y$ -components of  $\mathbf{a}_{B/A}$  gives us

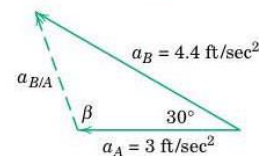
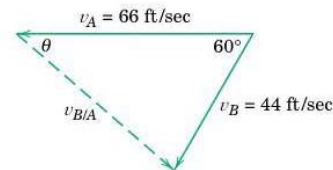
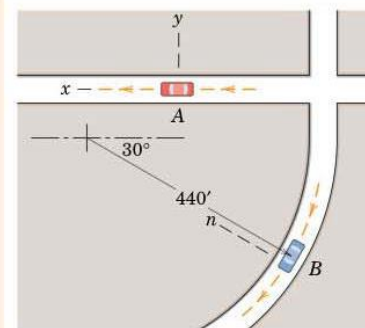
$$(a_{B/A})_x = 4.4 \cos 30^\circ - 3 = 0.810 \text{ ft/sec}^2$$

$$(a_{B/A})_y = 4.4 \sin 30^\circ = 2.2 \text{ ft/sec}^2$$

$$\text{from which } a_{B/A} = \sqrt{(0.810)^2 + (2.2)^2} = 2.34 \text{ ft/sec}^2 \quad \text{Ans.}$$

The direction of  $\mathbf{a}_{B/A}$  may be specified by the angle  $\beta$  which, by the law of sines, becomes

$$\textcircled{2} \quad \frac{4.4}{\sin \beta} = \frac{2.34}{\sin 30^\circ} \quad \beta = \sin^{-1} \left( \frac{2.34}{2.34} 0.5 \right) = 110.2^\circ \quad \text{Ans.}$$



### Helpful Hints

- ① Alternatively, we could use either a graphical or a vector algebraic solution.
- ② Be careful to choose between the two values  $69.8^\circ$  and  $180 - 69.8 = 110.2^\circ$ .

**Suggestion:** To gain familiarity with the manipulation of vector equations, it is suggested that the student rewrite the relative-motion equations in the form  $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$  and  $\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$  and redraw the vector polygons to conform with these alternative relations.

**Caution:** So far we are only prepared to handle motion relative to *nonrotating* axes. If we had attached the reference axes rigidly to car B, they would rotate with the car, and we would find that the velocity and acceleration terms relative to the rotating axes are *not* the negative of those measured from the nonrotating axes moving with A. Rotating axes are treated in Art. 5/7.

### Given

- Car  $A$  moves left at 45 mi/h, speeding up at  $3 \text{ ft/s}^2$ .
- Car  $B$  follows a circular curve of radius  $\rho = 440 \text{ ft}$  at a constant speed of 30 mi/h.
- The positions and directions are shown in the diagram.

### Required

Find the velocity and acceleration of  $B$  as observed from  $A$ :

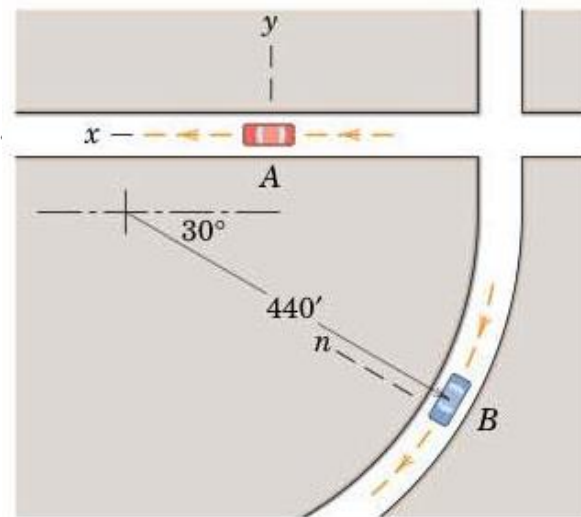
$$\mathbf{v}_{B/A}, \quad \mathbf{a}_{B/A}.$$

### Strategy

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$$

$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$$

- ① Convert speeds and resolve the velocities into components.
- ② Subtract corresponding components to find relative velocity.
- ③ After introducing normal and tangential acceleration, determine  $\mathbf{a}_B$  and find relative acceleration.



① Convert the speeds

$$v_A = 45 \left( \frac{5280}{3600} \right) = 66 \text{ ft/s}, \quad v_B = 30 \left( \frac{5280}{3600} \right) = 44 \text{ ft/s}.$$

② Express the velocity vectors

Car  $B$ 's velocity is tangent to the curve, directed  $60^\circ$  below the positive  $x$ -axis.

$$\mathbf{v}_A = 66 \mathbf{i}$$

$$\mathbf{v}_B = \underbrace{44 \cos 60^\circ}_{\text{left}} \mathbf{i} - \underbrace{44 \sin 60^\circ}_{\text{down}} \mathbf{j} = 22 \mathbf{i} - 38.1 \mathbf{j}.$$

③ Subtract and determine magnitude and direction

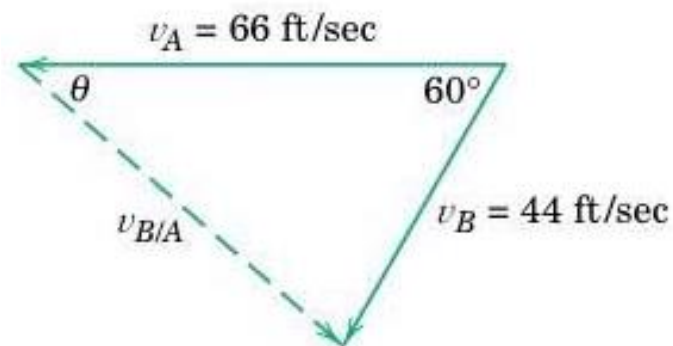
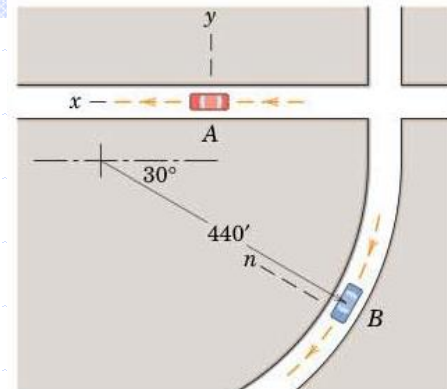
$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A = (22 - 66) \mathbf{i} - 38.1 \mathbf{j}$$

$$\mathbf{v}_{B/A} = -44 \mathbf{i} - 38.1 \mathbf{j} \text{ ft/s}$$

$$v_{B/A} = \sqrt{44^2 + 38.1^2} = 58.2 \text{ ft/s}$$

$$\theta = \tan^{-1} \left( \frac{38.1}{44} \right) = 40.9^\circ.$$

Direction:  $40.9^\circ$  below the horizontal toward the right, since both components are negative in our chosen axes.



② Apply the law of cosines to find  $v_{B/A}$

$$\underbrace{v_{B/A}^2}_{\text{side opposite } 60^\circ} = v_A^2 + v_B^2 - 2v_A v_B \cos 60^\circ.$$

Substitute:

$$v_{B/A} = \sqrt{66^2 + 44^2 - 2(66)(44) \cos 60^\circ} = \sqrt{3388}.$$

$$v_{B/A} = 58.2 \text{ ft/s}$$

③ Apply the law of sines to find  $\theta$

The side opposite  $\theta$  has length  $v_B = 44 \text{ ft/s}$ :

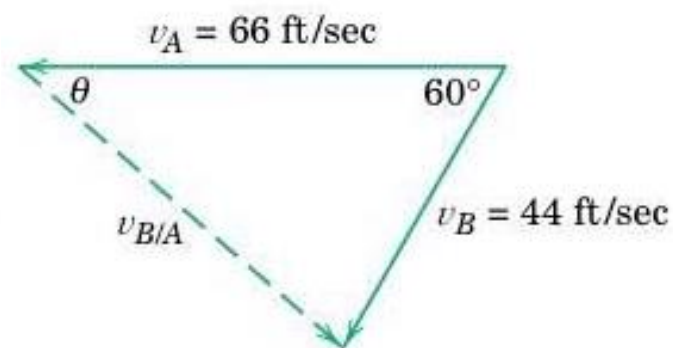
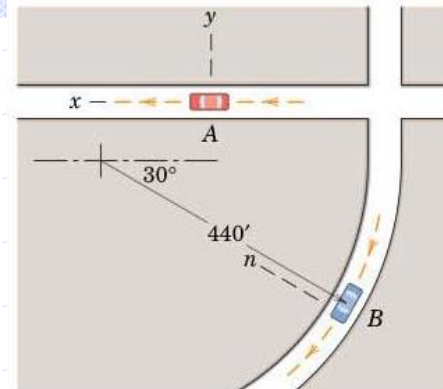
$$\frac{\underbrace{44}_{\text{opposite } \theta}}{\sin \theta} = \frac{\underbrace{58.2}_{\text{opposite } 60^\circ}}{\sin 60^\circ}.$$

Therefore,

$$\theta = \sin^{-1} \left( \frac{44 \sin 60^\circ}{58.2} \right) = 40.9^\circ.$$

Since  $44 < 58.2$ , their opposite angles satisfy  $\theta < 60^\circ$ , confirming the acute solution.

$$\mathbf{v}_{B/A} : 58.2 \text{ ft/s}, \quad 40.9^\circ \text{ below the rightward horizontal}$$



① Determine the acceleration of each car

Car  $A$  speeds up toward the left:

$$\mathbf{a}_A = 3 \mathbf{i} \text{ ft/s}^2.$$

For car  $B$ ,

$$\underbrace{a_{Bt} = \frac{dv_B}{dt}}_{\text{constant speed}} = 0, \quad \underbrace{a_{Bn} = \frac{v_B^2}{\rho}}_{\text{changing direction}} = \frac{44^2}{440} = 4.4 \text{ ft/s}^2.$$

The normal acceleration points toward the center of curvature,  $30^\circ$  above the positive  $x$ -axis:

$$\mathbf{a}_B = 4.4 \cos 30^\circ \mathbf{i} + 4.4 \sin 30^\circ \mathbf{j}.$$

② Subtract the acceleration vectors

$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$$

$$\mathbf{a}_{B/A} = (4.4 \cos 30^\circ - 3) \mathbf{i} + (4.4 \sin 30^\circ) \mathbf{j}$$

$$\mathbf{a}_{B/A} = 0.811 \mathbf{i} + 2.20 \mathbf{j} \text{ ft/s}^2$$

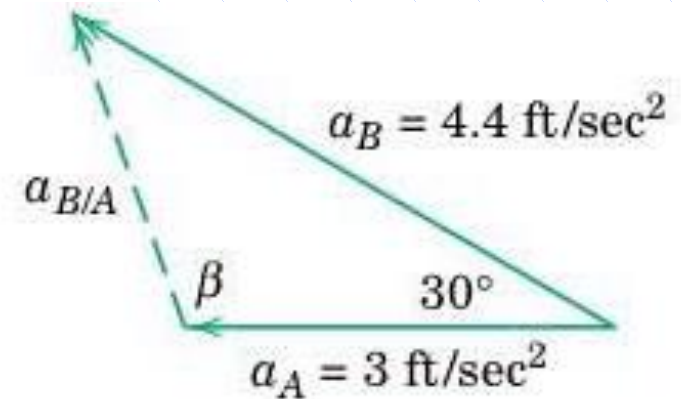
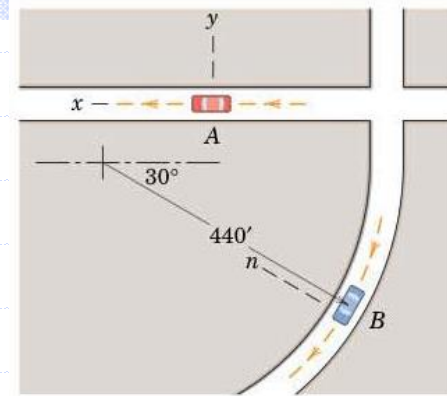
③ Calculate magnitude and direction

$$a_{B/A} = \sqrt{0.811^2 + 2.20^2} = 2.34 \text{ ft/s}^2$$

$$\alpha = \tan^{-1} \left( \frac{2.20}{0.811} \right) = 69.8^\circ \text{ above the leftward horizontal}.$$

Using the angle marked  $\beta$  in the textbook sketch:

$$\beta = 180^\circ - 69.8^\circ = 110.2^\circ.$$



## ② Apply the law of cosines

Construct the acceleration triangle using

$$\mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_B.$$

The internal angle between the sides of lengths 3 and 4.4 is  $30^\circ$ . Therefore,

$$a_{B/A}^2 = a_A^2 + a_B^2 - 2a_A a_B \cos 30^\circ$$

$$a_{B/A} = \sqrt{3^2 + 4.4^2 - 2(3)(4.4) \cos 30^\circ} = 2.34 \text{ ft/s}^2.$$

## ③ Apply the law of sines

Using the angle  $\beta$  shown in the textbook's acceleration triangle:

$$\frac{\overbrace{4.4}^{\text{opposite } \beta}}{\sin \beta} = \frac{\overbrace{a_{B/A}}^{\text{opposite } 30^\circ}}{\sin 30^\circ}.$$

Using the unrounded value of  $a_{B/A}$ ,

$$\sin \beta = \frac{4.4 \sin 30^\circ}{a_{B/A}} \approx 0.9383.$$

The two possible angles are

$$\beta = 69.8^\circ \quad \text{or} \quad \beta = 180^\circ - 69.8^\circ = 110.2^\circ.$$

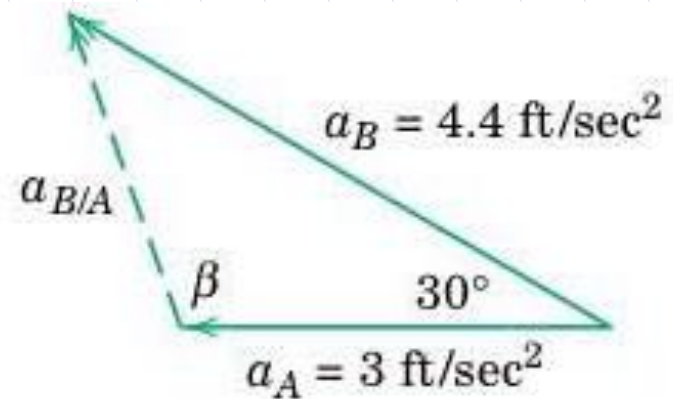
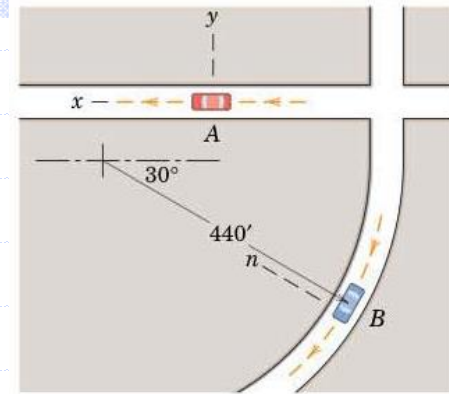
the angle opposite the side 4.4 must be **obtuse**. Hence,

$$\beta = 110.2^\circ.$$

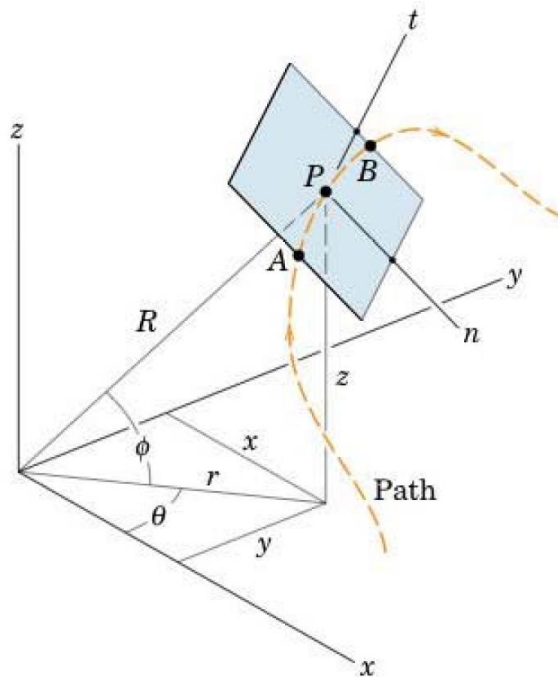
Final answer

$$a_{B/A} = 2.34 \text{ ft/s}^2, \quad 69.8^\circ \text{ above the leftward horizontal}$$

The physical direction angle is  $180^\circ - \beta = 69.8^\circ$ ;  $\beta$  is the internal angle marked in the triangle.



## 4. Normal & Tangential Components ( $t$ - $n$ )

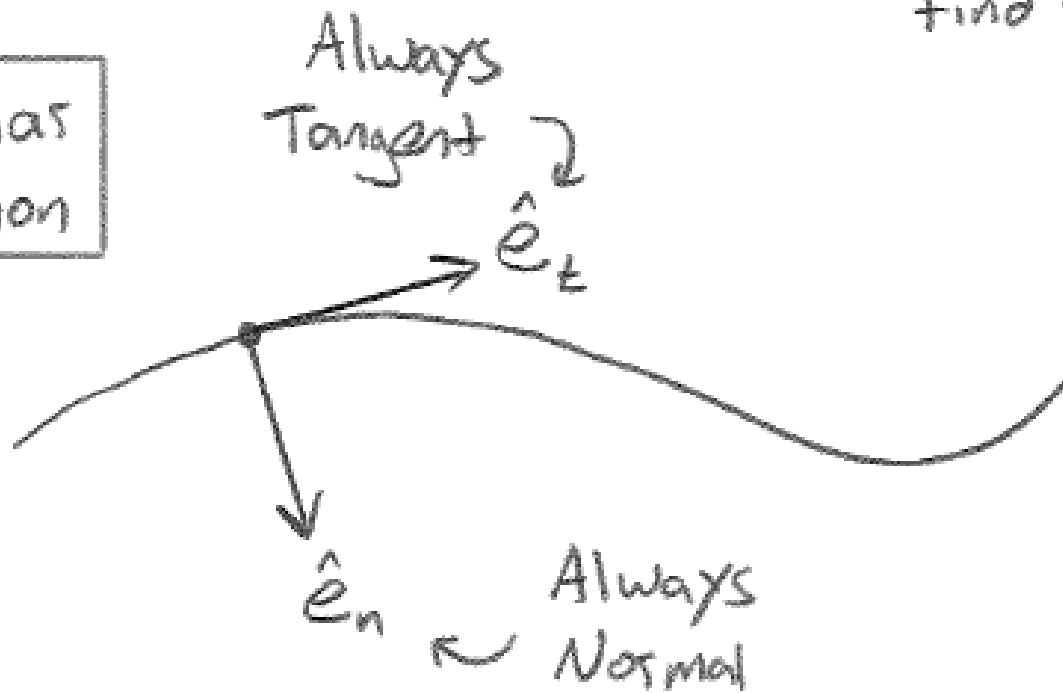


# Normal & Tangential Components

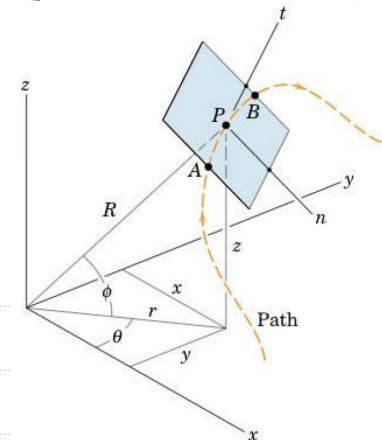
The goal: Resolve  $\bar{a} = a_t \hat{e}_t + a_n \hat{e}_n$

$\uparrow \qquad \qquad \uparrow$   
Find these

Planar Motion



particle path



Velocity is always tangent to path.

$$\therefore \bar{v} = v \hat{e}_t$$

Then

Needs work

$$\bar{a} = \frac{d(v \hat{e}_t)}{dt} = \frac{dv}{dt} \hat{e}_t + v \left( \frac{d\hat{e}_t}{dt} \right) \quad (*)$$

$$\frac{d\hat{e}_t}{dt} = \frac{d\hat{e}_t}{d\theta} \frac{d\theta}{ds} \frac{ds}{dt} \quad \text{--- } v \text{ (speed)} \quad \text{--- } \theta \text{ to be defined ...} \quad (A)$$

$$\bar{a} = \frac{dv}{dt} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n$$

Change in  
Speed

Change in  
direction

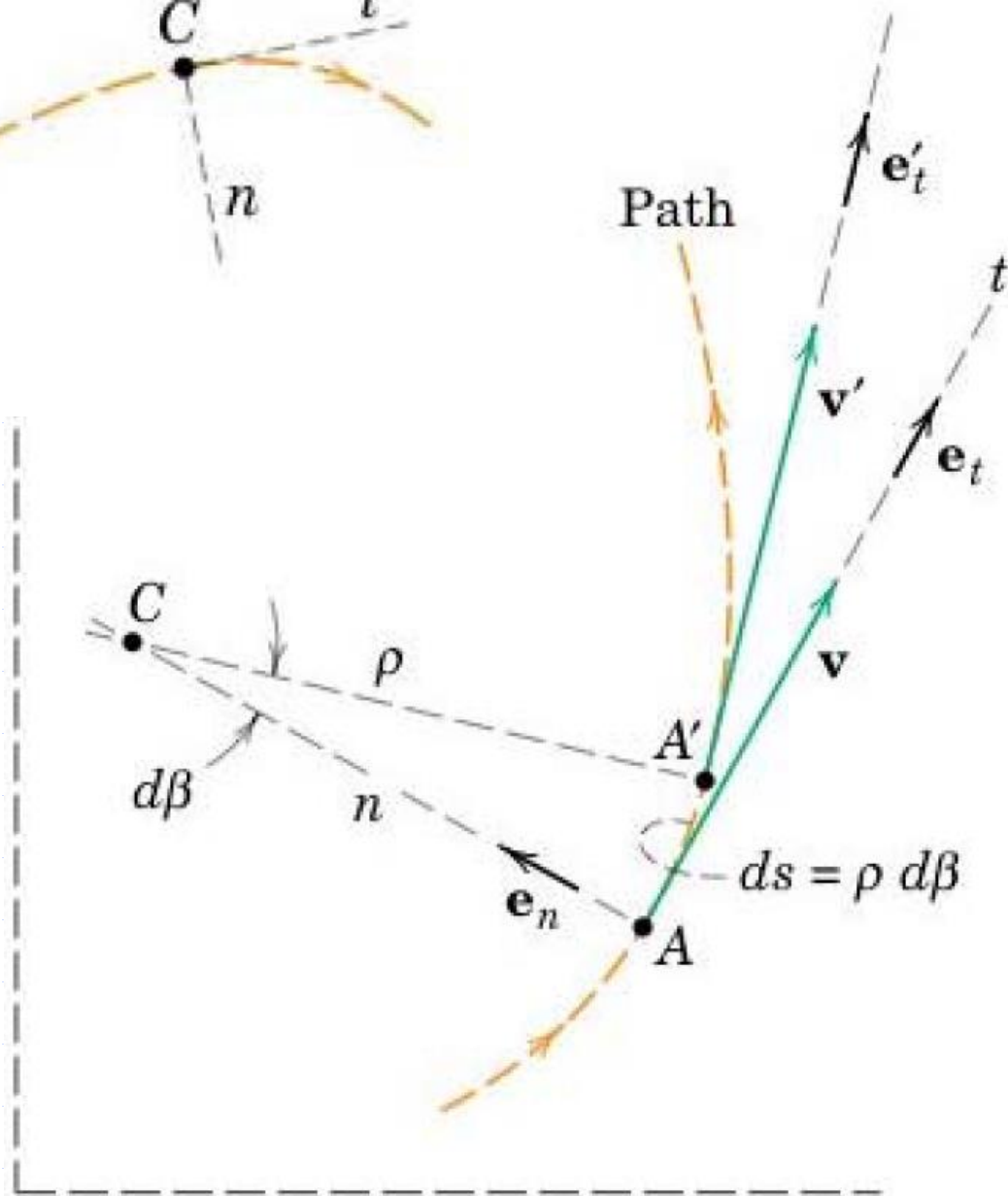
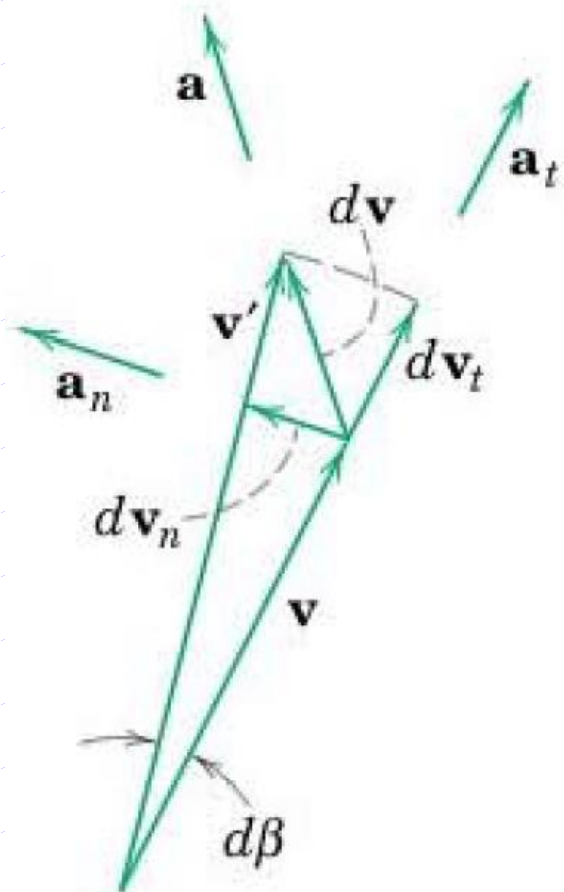
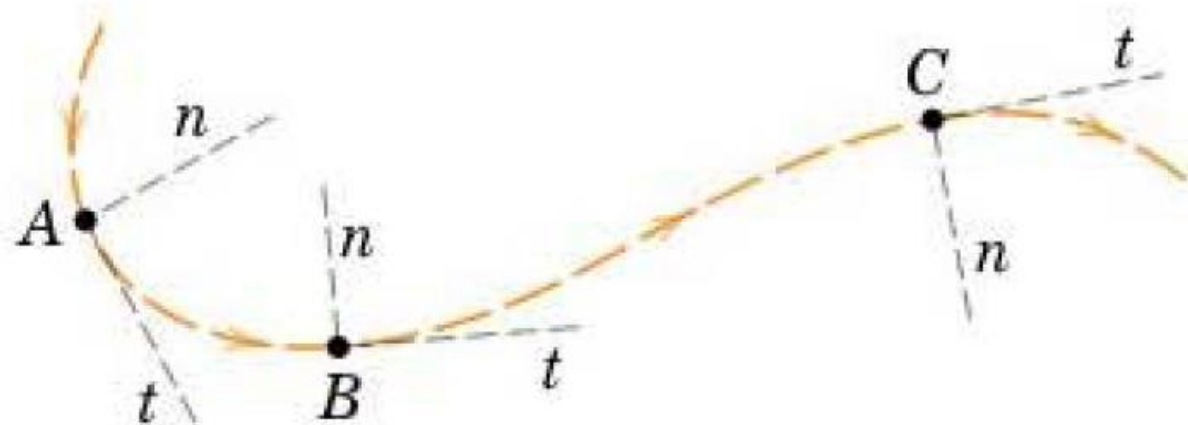
Here,

$$a_t = \frac{dv}{dt}$$

tangential acc.

$$a_n = \frac{v^2}{\rho}$$

normal acc.

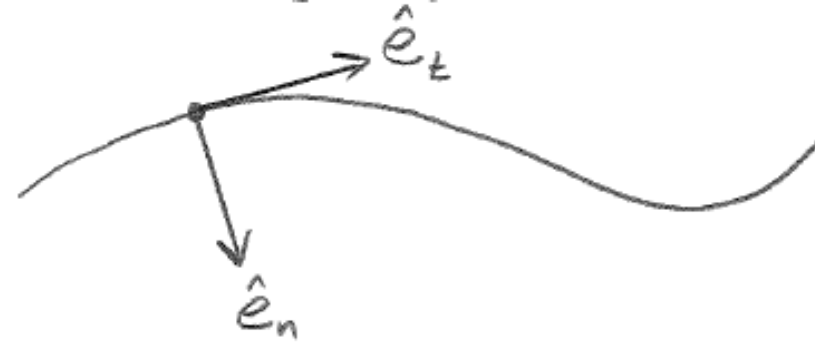


- Velocity is tangent to the path

$$\mathbf{v} = v \hat{\mathbf{e}}_t$$

## Differentiate Velocity

$$\mathbf{a} = \frac{d}{dt}(v \hat{\mathbf{e}}_t) = \frac{dv}{dt} \hat{\mathbf{e}}_t + v \frac{d\hat{\mathbf{e}}_t}{dt}$$



## Derivative of Tangent Vector

Chain rule:

$$\frac{d\hat{\mathbf{e}}_t}{dt} = \frac{d\hat{\mathbf{e}}_t}{d\theta} \cdot \frac{d\theta}{ds} \cdot \frac{ds}{dt}$$

- $\frac{d\hat{\mathbf{e}}_t}{d\theta} = \hat{\mathbf{e}}_n$
- $\frac{d\theta}{ds} = 1/\rho$  (curvature)
- $\frac{ds}{dt} = v$

So:

$$\frac{d\hat{\mathbf{e}}_t}{dt} = \frac{v}{\rho} \hat{\mathbf{e}}_n$$

»

$$\mathbf{a} = \frac{dv}{dt} \hat{\mathbf{e}}_t + \frac{v^2}{\rho} \hat{\mathbf{e}}_n$$

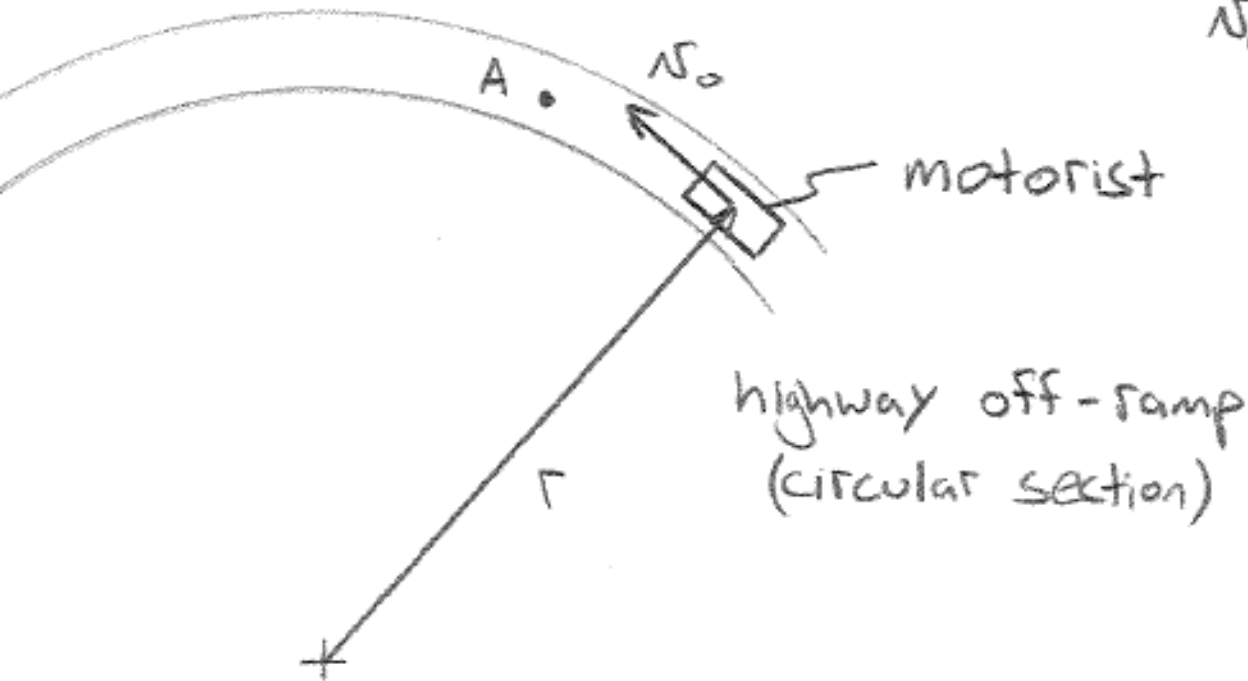
- Tangential arrow along path =  $\frac{dv}{dt} \hat{\mathbf{e}}_t$ .
- Normal arrow pointing inward =  $\frac{v^2}{\rho} \hat{\mathbf{e}}_n$ .
- Combined acceleration vector shown as sum of both.

# Example 3

Data:  $r = 350\text{m}$  (const.)

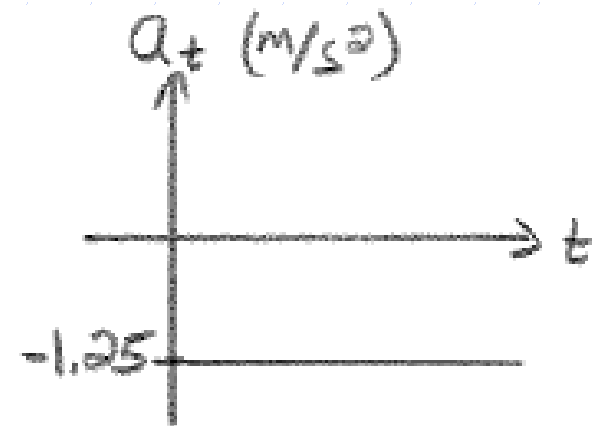
$$v_0 = 20\text{ m/s}$$

↳ initially, before reaching A



Brakes applied @ A causing vehicle to slow down at a rate of  $1.25\text{ m/s}^2$ .

Determine  $a = |\bar{a}|$

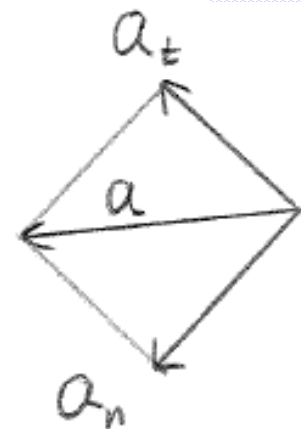


(A) immediately after brakes are applied

(B) 4s later

PART A:

$$a = \sqrt{a_t^2 + a_n^2}$$



where  $a_t = -1.25 \text{ m/s}^2$

$$a_n = \frac{v^2}{r} = \frac{(20 \text{ m/s})^2}{(350 \text{ m})} = 1.143 \text{ m/s}^2$$

Thus  $a = \sqrt{(-1.25 \text{ m/s}^2)^2 + (1.143 \text{ m/s}^2)^2}$   
 $= 1.694 \text{ m/s}^2$

$\therefore a = |\vec{a}| = 1.69 \text{ m/s}^2$

PART B:  $a_t$  same,  $a_n = v^2/r$  diff.

$$a_t = \frac{dv}{dt} \quad \Rightarrow \quad dv = \underset{\substack{\uparrow \\ \text{constant}}}{a_t} dt$$

$$\Rightarrow v - v_0 = a_t t$$

Thus 
$$v(t=4s) = (20 \text{ m/s}) + (-1.25 \text{ m/s}^2)(4s)$$
$$= 15.0 \text{ m/s}$$

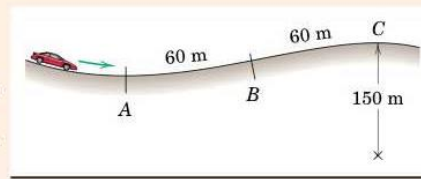
so 
$$a_n = \frac{(15.0 \text{ m/s})^2}{(350 \text{ m})} = 0.643 \text{ m/s}^2$$

and 
$$a = \sqrt{(-1.25 \text{ m/s}^2)^2 + (0.643 \text{ m/s}^2)^2}$$
$$= 1.406 \text{ m/s}^2$$

$$\therefore a = |\bar{a}| = 1.41 \text{ m/s}^2$$

## Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom *A* of the dip and 50 km/h at the top *C* of the hump, which is 120 m along the road from *A*. If the passengers experience a total acceleration of  $3 \text{ m/s}^2$  at *A* and if the radius of curvature of the hump at *C* is 150 m, calculate (a) the radius of curvature  $\rho$  at *A*, (b) the acceleration at the inflection point *B*, and (c) the total acceleration at *C*.



**Solution.** The dimensions of the car are small compared with those of the path, so we will treat the car as a particle. The velocities are

$$v_A = \left(100 \frac{\text{km}}{\text{h}}\right) \left(\frac{1 \text{ h}}{3600 \text{ s}}\right) \left(1000 \frac{\text{m}}{\text{km}}\right) = 27.8 \text{ m/s}$$

$$v_C = 50 \frac{1000}{3600} = 13.89 \text{ m/s}$$

We find the constant deceleration along the path from

$$\left[ \int v \, dv = \int a_t \, ds \right] \quad \int_{v_A}^{v_C} v \, dv = a_t \int_0^s ds$$

$$a_t = \frac{1}{2s} (v_C^2 - v_A^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2$$

**(a) Condition at A.** With the total acceleration given and  $a_t$  determined, we can easily compute  $a_n$  and hence  $\rho$  from

$$[a^2 = a_n^2 + a_t^2] \quad a_n^2 = 3^2 - (2.41)^2 = 3.19 \quad a_n = 1.785 \text{ m/s}^2$$

$$[a_n = v^2/\rho] \quad \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m} \quad \text{Ans.}$$

**(b) Condition at B.** Since the radius of curvature is infinite at the inflection point,  $a_n = 0$  and

$$a = a_t = -2.41 \text{ m/s}^2 \quad \text{Ans.}$$

**(c) Condition at C.** The normal acceleration becomes

$$[a_n = v^2/\rho] \quad a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2$$

With unit vectors  $\mathbf{e}_n$  and  $\mathbf{e}_t$  in the *n*- and *t*-directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t \text{ m/s}^2$$

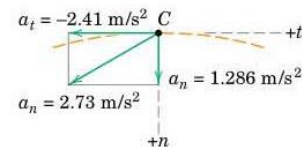
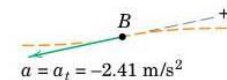
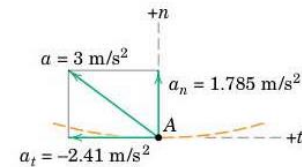
where the magnitude of  $\mathbf{a}$  is

$$[a = \sqrt{a_n^2 + a_t^2}] \quad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2 \quad \text{Ans.}$$

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

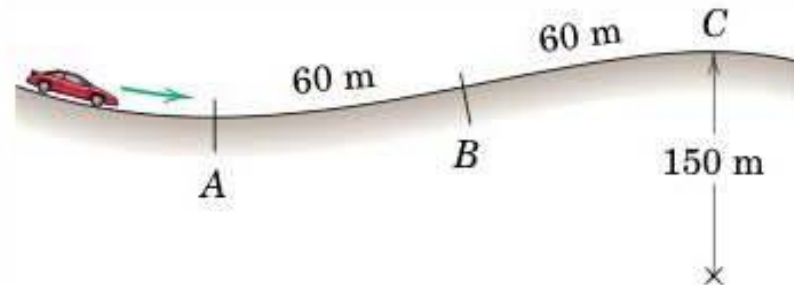
### Helpful Hint

- ① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.



## Given

- Uniform deceleration along the path:  $a_t$  is constant.
- $v_A = 100 \text{ km/h}$ ,  $v_C = 50 \text{ km/h}$ .
- Distance along the road:  $s_{AC} = 120 \text{ m}$ .
- Total acceleration at  $A$ :  $a_A = 3 \text{ m/s}^2$ .
- Radius of curvature at  $C$ :  $\rho_C = 150 \text{ m}$ .
- $B$  is an inflection point.



## Required

- (a) Radius of curvature  $\rho_A$ .
- (b) Acceleration at  $B$ .
- (c) Total acceleration at  $C$ .

## Strategy

Choose  $+t$  along the motion and  $+n$  toward the center of curvature.

$$\mathbf{a} = \underbrace{a_t \mathbf{e}_t}_{\text{change in speed}} + \underbrace{a_n \mathbf{e}_n}_{\text{change in direction}}$$

- ① Find the constant  $a_t$  from the speeds and distance.
- ② Use  $a_n = v^2/\rho$  at each point.
- ③ Combine perpendicular components using  $a = \sqrt{a_t^2 + a_n^2}$ .

## Determine the tangential acceleration

### ① Convert the speeds

$$v_A = \frac{100}{3.6} = 27.778 \text{ m/s}, \quad v_C = \frac{50}{3.6} = 13.889 \text{ m/s}.$$

### ② Connect acceleration to distance using the chain rule

We know the distance traveled, rather than the elapsed time:

$$a_t = \frac{dv}{dt} = \underbrace{\frac{dv}{ds} \frac{ds}{dt}}_{\text{chain rule}} = \frac{dv}{ds} v.$$

Therefore,

$$a_t ds = v dv.$$

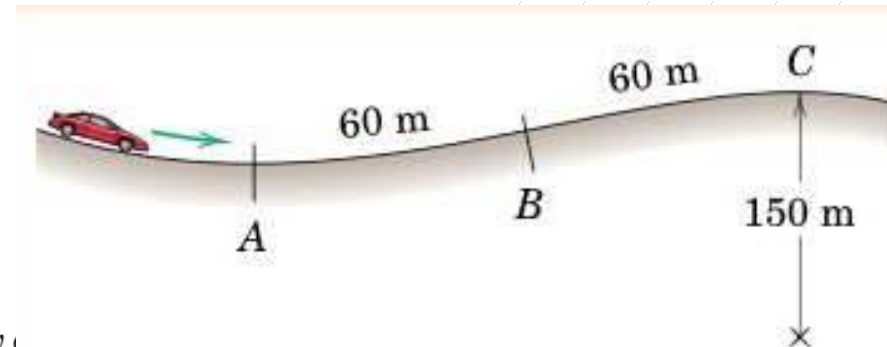
### ③ Integrate from $A$ to $C$

Set  $s = 0$  at  $A$ . Since  $a_t$  is constant,

$$a_t \int_0^{120} ds = \int_{v_A}^{v_C} v dv$$

$$120a_t = \frac{1}{2} (v_C^2 - v_A^2).$$

$$a_t = \frac{13.889^2 - 27.778^2}{2(120)} = -2.411 \text{ m/s}^2$$



The negative sign means that the tangential acceleration acts **opposite the motion**, at all three points.

(a) Find the radius of curvature at  $A$

The total acceleration is  $3 \text{ m/s}^2$ . Because  $a_t$  and  $a_n$  are perpendicular

$$a_A^2 = a_t^2 + a_{nA}^2.$$

Hence,

$$a_{nA} = \sqrt{3^2 - (-2.411)^2} = 1.785 \text{ m/s}^2$$

Using  $a_{nA} = v_A^2 / \rho_A$ ,

$$\rho_A = \frac{v_A^2}{a_{nA}} = \frac{27.778^2}{1.785} \approx 432 \text{ m}$$

At the bottom of the dip,  $a_{nA}$  points **upward**.

(b) Find the acceleration at  $B$

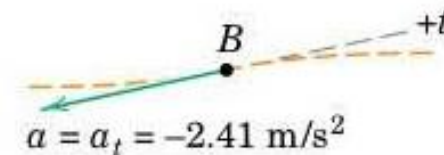
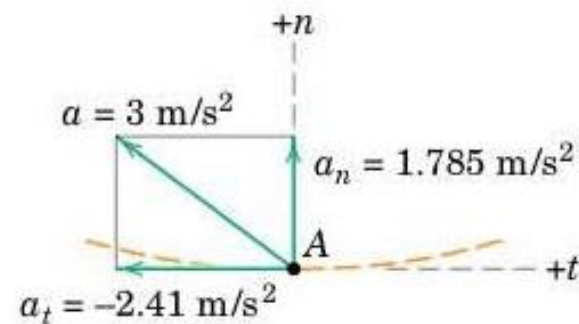
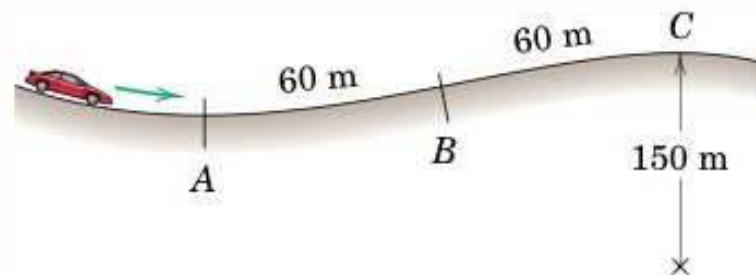
At the inflection point, the curvature is zero:

$$\frac{1}{\rho_B} = 0 \Rightarrow a_{nB} = \frac{v_B^2}{\rho_B} = 0.$$

Only tangential acceleration remains:

$$\mathbf{a}_B = -2.411 \mathbf{e}_t \text{ m/s}^2.$$

Magnitude:  $2.41 \text{ m/s}^2$ , backward along the tangent.



Part (c): Total acceleration at  $C$

① Determine the components

The tangential acceleration is unchanged:

$$a_{tC} = -2.411 \text{ m/s}^2.$$

The normal acceleration is

$$a_{nC} = \frac{v_C^2}{\rho_C} = \frac{13.889^2}{150} = 1.286 \text{ m/s}^2.$$

At the top of the hump, the center of curvature is **below** the car, so  $+n$  points downward.

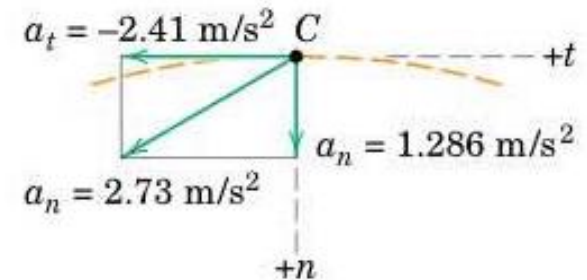
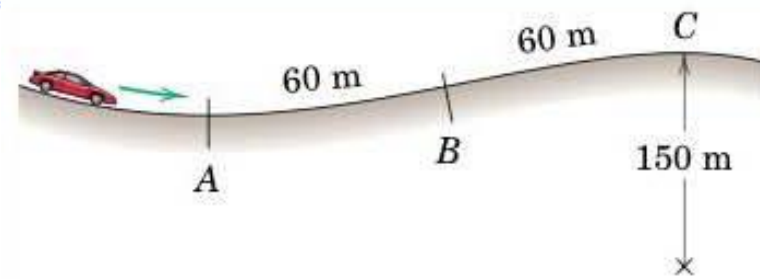
② Write the acceleration vector

$$\mathbf{a}_C = \underbrace{-2.411 \mathbf{e}_t}_{\text{backward}} + \underbrace{1.286 \mathbf{e}_n}_{\text{downward}} \text{ m/s}^2$$

③ Find the magnitude

$$a_C = \sqrt{a_{tC}^2 + a_{nC}^2} = \sqrt{(-2.411)^2 + (1.286)^2}$$

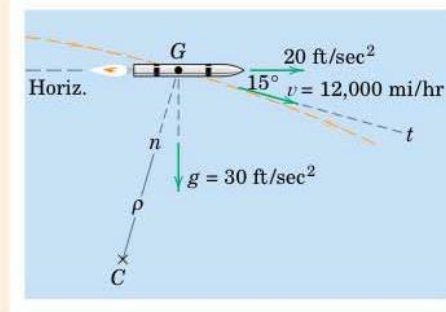
$$a_C = 2.73 \text{ m/s}^2$$



**Remember:** The tangential acceleration is constant, but the total acceleration changes because the normal component changes.

## Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of  $20 \text{ ft/sec}^2$ , and the downward acceleration component is the acceleration due to gravity at that altitude, which is  $g = 30 \text{ ft/sec}^2$ . At the instant represented, the velocity of the mass center  $G$  of the rocket along the  $15^\circ$  direction of its trajectory is  $12,000 \text{ mi/hr}$ . For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed  $v$  is increasing, (c) the angular rate  $\dot{\beta}$  of the radial line from  $G$  to the center of curvature  $C$ , and (d) the vector expression for the total acceleration  $\mathbf{a}$  of the rocket.



**Solution.** We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use  $n$ - and  $t$ -coordinates to describe the motion of  $G$ . The  $n$ - and  $t$ -components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their  $n$ - and  $t$ -components and then combining. From the figure we get

$$a_n = 30 \cos 15^\circ - 20 \sin 15^\circ = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15^\circ + 20 \cos 15^\circ = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

$$\textcircled{1} [a_n = v^2/\rho] \quad \rho = \frac{v^2}{a_n} = \frac{[(12,000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft} \quad \text{Ans.}$$

(b) The rate at which  $v$  is increasing is simply the  $t$ -component of acceleration.

$$[\dot{v} = a_t] \quad \dot{v} = 27.1 \text{ ft/sec}^2 \quad \text{Ans.}$$

(c) The angular rate  $\dot{\beta}$  of line  $GC$  depends on  $v$  and  $\rho$  and is given by

$$[\dot{\beta} = v/\rho] \quad \dot{\beta} = v/\rho = \frac{12,000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec} \quad \text{Ans.}$$

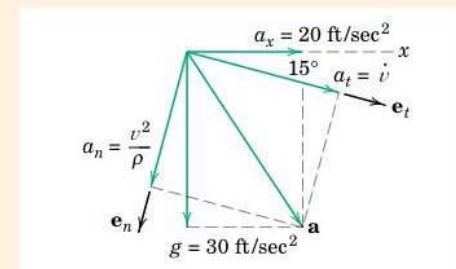
(d) With unit vectors  $\mathbf{e}_n$  and  $\mathbf{e}_t$  for the  $n$ - and  $t$ -directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2 \quad \text{Ans.}$$

### Helpful Hints

① Alternatively, we could find the resultant acceleration and then resolve it into  $n$ - and  $t$ -components.

② To convert from mi/hr to ft/sec, multiply by  $\frac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \frac{44}{30} \text{ ft/sec}$  which is easily remembered, as  $30 \text{ mi/hr}$  is the same as  $44 \text{ ft/sec}$ .



### Given

- The rocket maintains a horizontal attitude.
- Horizontal acceleration:  $a_x = 20 \text{ ft/s}^2$  to the right.
- Downward acceleration:  $g = 30 \text{ ft/s}^2$ .
- Speed:  $v = 12,000 \text{ mi/h}$ .
- The velocity is directed  $15^\circ$  below the horizontal.

### Required

- (a) Radius of curvature,  $\rho$ .
- (b) Rate of increase of speed,  $\dot{v}$ .
- (c) Angular rate,  $\dot{\beta}$ , of the line  $GC$ .
- (d) Total acceleration in  $n$ - $t$  vector form.

### Strategy

Choose  $+t$  along the velocity and  $+n$  toward the center of curvature  $C$ .

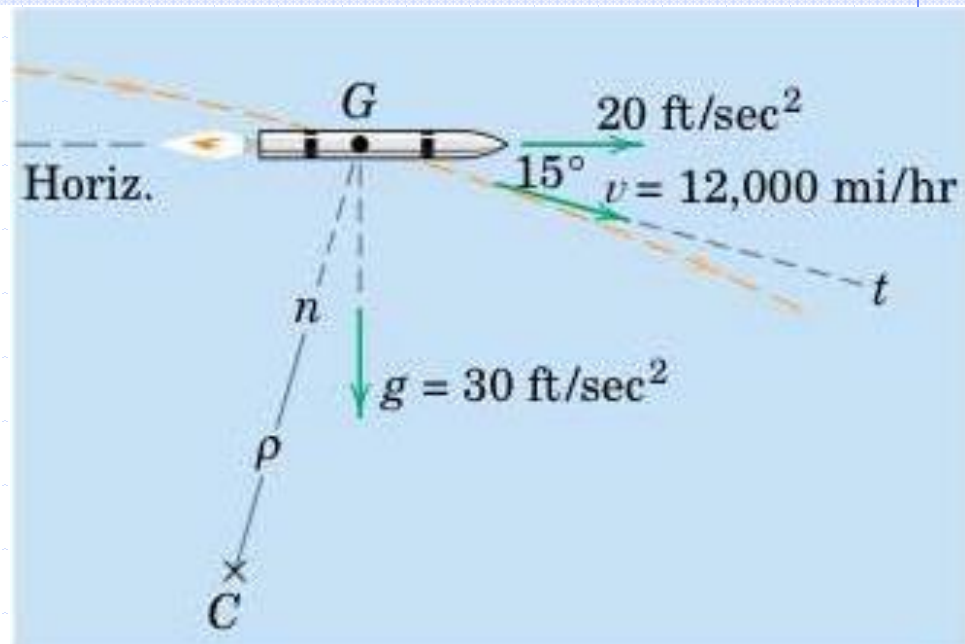
- ① Resolve the known acceleration into  $a_t$  and  $a_n$ .
- ② Use the normal component to determine the curvature:

$$a_n = \frac{v^2}{\rho}.$$

- ③ Use the tangential component to determine the speed change:

$$a_t = \dot{v}.$$

- ④ Relate the turning rate to  $v$  and  $\rho$ , then assemble  $\mathbf{a}$ .



Determine the normal and tangential accelerations

① Establish the local directions

- $+t$ : forward,  $15^\circ$  below the horizontal.
- $+n$ : perpendicular to  $+t$ , pointing downward and left toward  $C$ .

② Resolve along the tangent

Both given acceleration components have a projection in the positive tangential direction:

$$a_t = \underbrace{20 \cos 15^\circ}_{\text{horizontal contribution}} + \underbrace{30 \sin 15^\circ}_{\text{downward contribution}}.$$

$$a_t = 27.1 \text{ ft/s}^2$$

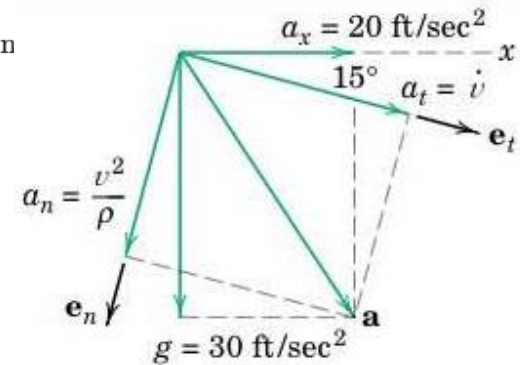
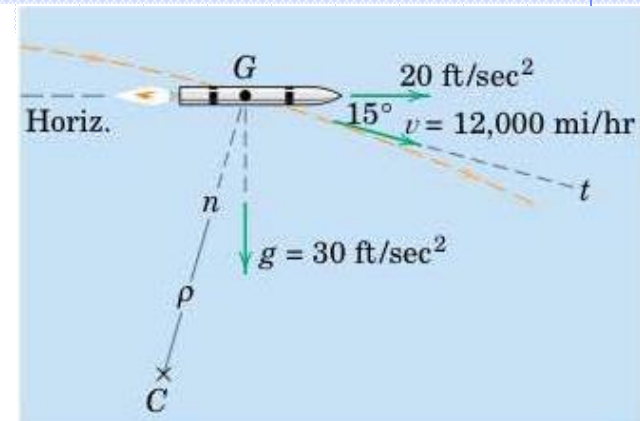
Since  $a_t > 0$ , the rocket's speed is increasing.

③ Resolve along the inward normal

Gravity contributes toward  $C$ ; the horizontal acceleration contributes opposite  $+n$ :

$$a_n = \underbrace{30 \cos 15^\circ}_{\text{toward } C} - \underbrace{20 \sin 15^\circ}_{\text{opposite the inward normal}}.$$

$$a_n = 23.8 \text{ ft/s}^2$$



**Key distinction:** The rocket's axis is horizontal, but the **tangent follows its velocity**, which is inclined.

① Convert the speed

$$v = 12,000 \left( \frac{5280}{3600} \right) = \boxed{17,600 \text{ ft/s}}.$$

(a) Find the radius of curvature

Starting with

$$a_n = \frac{v^2}{\rho},$$

rearrange:

$$\rho = \frac{v^2}{a_n}.$$

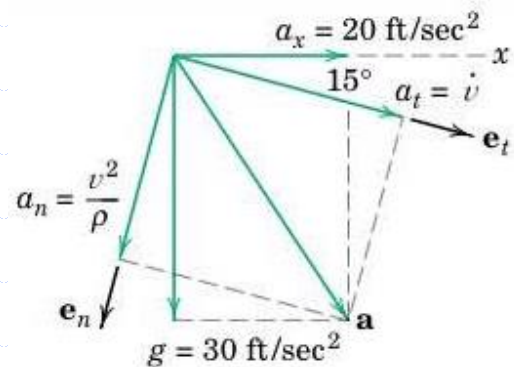
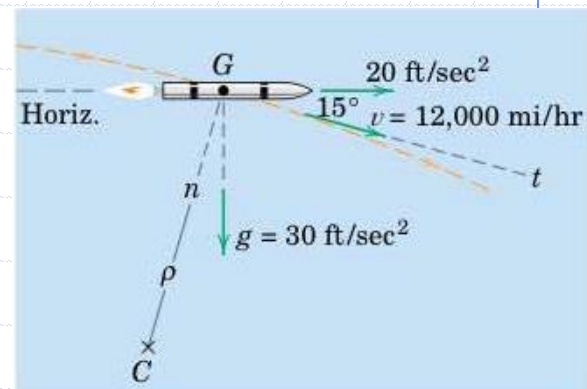
Thus,

$$\rho \approx \frac{(17,600)^2}{23.8} \approx 13.0 \times 10^6 \text{ ft}$$

(b) Find the rate of increase of speed

The tangential acceleration is the rate of change of speed:

$$\dot{v} = a_t = 27.1 \text{ ft/s}^2.$$



**Interpretation:** At this instant, the speed increases at a rate of 27.1 ft/s per second.

(c) Find the angular rate of line  $GC$

For a small change in trajectory direction, measured in radians,

$$ds = \rho d\beta.$$

Divide by  $dt$ :

$$\underbrace{\frac{ds}{dt}}_v = \rho \underbrace{\frac{d\beta}{dt}}_{\dot{\beta}} \Rightarrow \boxed{\dot{\beta} = \frac{v}{\rho}}.$$

Therefore, the angular-rate magnitude is

$$\dot{\beta} \approx \frac{17,600}{13.0 \times 10^6} = 1.35 \times 10^{-3} \text{ rad/s}$$

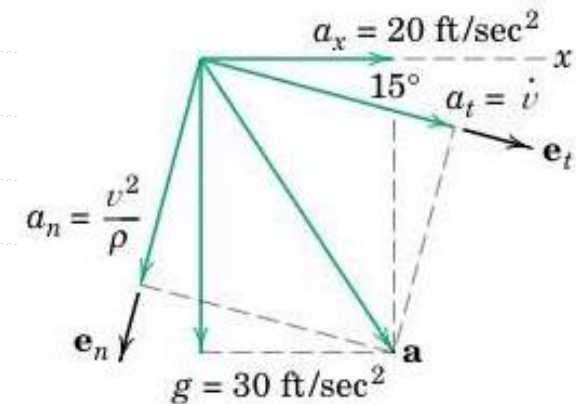
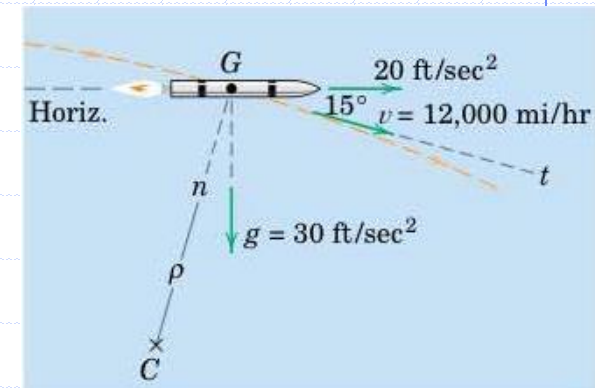
The trajectory tangent and its perpendicular normal line turn **clockwise** at this instant.

(d) Write the total acceleration

$$\mathbf{a} = a_n \mathbf{e}_n + a_t \mathbf{e}_t$$

$$\mathbf{a} = \underbrace{23.8 \mathbf{e}_n}_{\text{changes direction}} + \underbrace{27.1 \mathbf{e}_t}_{\text{changes speed}} \text{ ft/s}^2$$

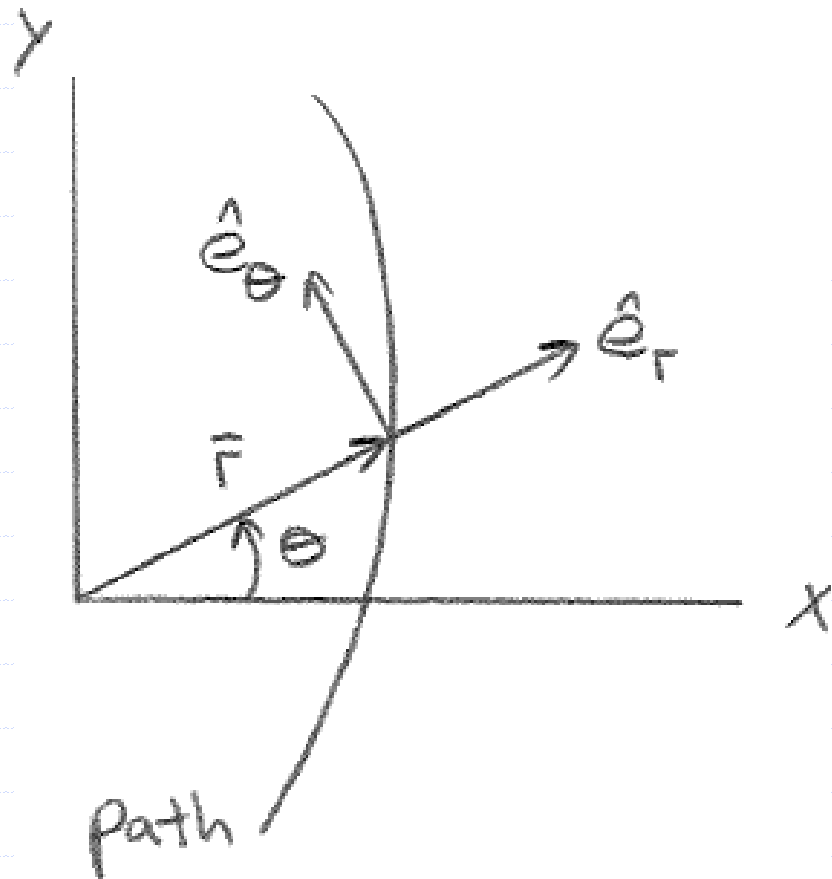
**Looking ahead:** An angular rate measures how quickly a direction rotates. We will use the same idea in rigid-body kinematics, commonly written as  $\omega = \dot{\theta}$ . Here,  $\dot{\beta}$  describes the turning of the **trajectory's normal line**; the rocket itself maintains a horizontal attitude.



[illegible]

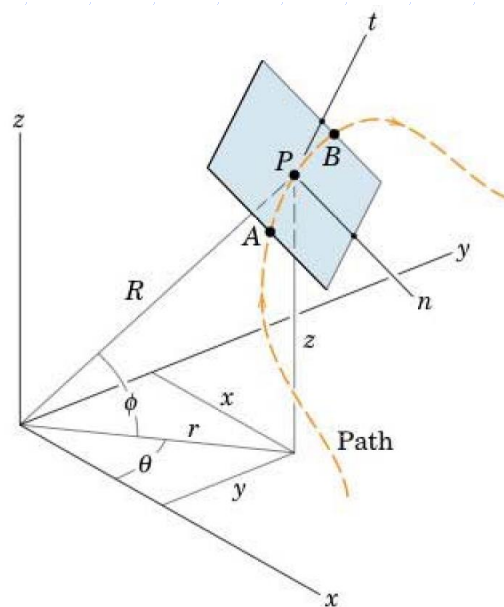
# Radial & Transverse Components

⊙ (aka Polar Coordinates)



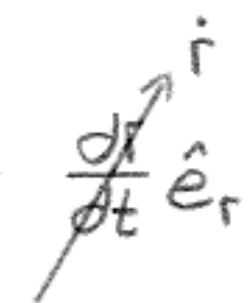
$$\vec{v} = ( \quad ) \hat{e}_r + ( \quad ) \hat{e}_\theta$$

$$\vec{a} = ( \quad ) \hat{e}_r + ( \quad ) \hat{e}_\theta$$

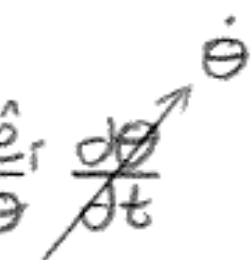


Here,  $\vec{r} = r\hat{e}_r$

so  $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\hat{e}_r)}{dt} = \cancel{\frac{dr}{dt}}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$



Dissect  $\frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta} \cancel{\frac{d\theta}{dt}} = \dot{\theta}\hat{e}_\theta$



can show  $= \hat{e}_\theta$

$\therefore$

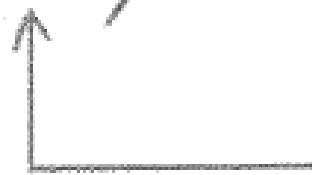
$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt} (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$$

$$= \ddot{r}\hat{e}_r + \cancel{\dot{r}\frac{d\hat{e}_r}{dt}} + \dot{\theta}\hat{e}_\theta$$

$$+ (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

Dissect  $\frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta} \frac{d\theta}{dt} = -\dot{\theta}\hat{e}_r$



can show =  $-\hat{e}_r$

+ Plug it all in ...

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

Radial  
acc.                      +                      transverse  
acc.

## 1) Start from the polar velocity

$$\mathbf{v} = \dot{r} \hat{\mathbf{e}}_r + r \dot{\theta} \hat{\mathbf{e}}_\theta$$

where  $\hat{\mathbf{e}}_r$  points outward from the origin and  $\hat{\mathbf{e}}_\theta$  is perpendicular to  $\hat{\mathbf{e}}_r$  in the direction of increasing  $\theta$ .

## 2) Time derivatives of the unit vectors

Using  $\hat{\mathbf{e}}_r = (\cos \theta) \hat{\mathbf{i}} + (\sin \theta) \hat{\mathbf{j}}$  and

$$\hat{\mathbf{e}}_\theta = -(\sin \theta) \hat{\mathbf{i}} + (\cos \theta) \hat{\mathbf{j}},$$

differentiate with respect to  $t$ :

$$\dot{\hat{\mathbf{e}}}_r = \dot{\theta} \hat{\mathbf{e}}_\theta, \quad \dot{\hat{\mathbf{e}}}_\theta = -\dot{\theta} \hat{\mathbf{e}}_r.$$

## 3) Differentiate the velocity (product rule)

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r} \hat{\mathbf{e}}_r) + \frac{d}{dt}(r \dot{\theta} \hat{\mathbf{e}}_\theta).$$

Work each term:

$$\frac{d}{dt}(\dot{r} \hat{\mathbf{e}}_r) = \ddot{r} \hat{\mathbf{e}}_r + \dot{r} \dot{\hat{\mathbf{e}}}_r = \ddot{r} \hat{\mathbf{e}}_r + \dot{r} (\dot{\theta} \hat{\mathbf{e}}_\theta),$$

$$\frac{d}{dt}(r \dot{\theta} \hat{\mathbf{e}}_\theta) = (\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\mathbf{e}}_\theta + r \dot{\theta} \dot{\hat{\mathbf{e}}}_\theta = (\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\mathbf{e}}_\theta + r \dot{\theta} (-\dot{\theta} \hat{\mathbf{e}}_r).$$

## 4) Collect $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ parts

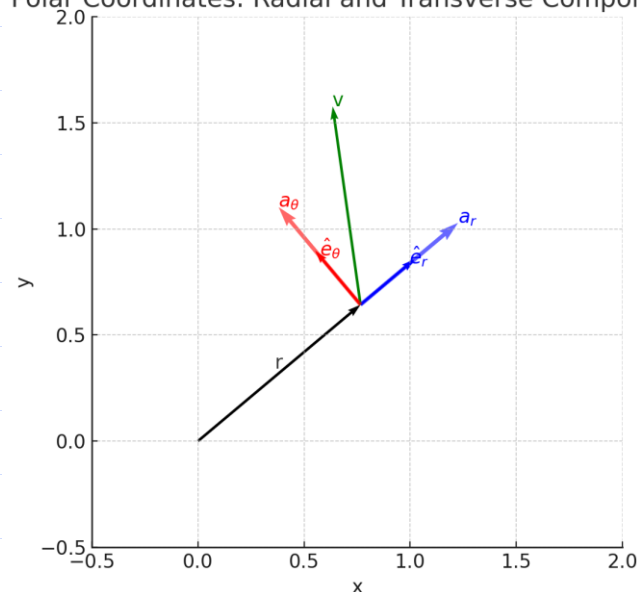
Radial:

$$\hat{\mathbf{e}}_r : \quad \ddot{r} - r \dot{\theta}^2.$$

Transverse:

$$\hat{\mathbf{e}}_\theta : \quad \dot{r} \dot{\theta} + (r \ddot{\theta} + r \dot{\theta}^2) = 2 \dot{r} \dot{\theta} + r \ddot{\theta}.$$

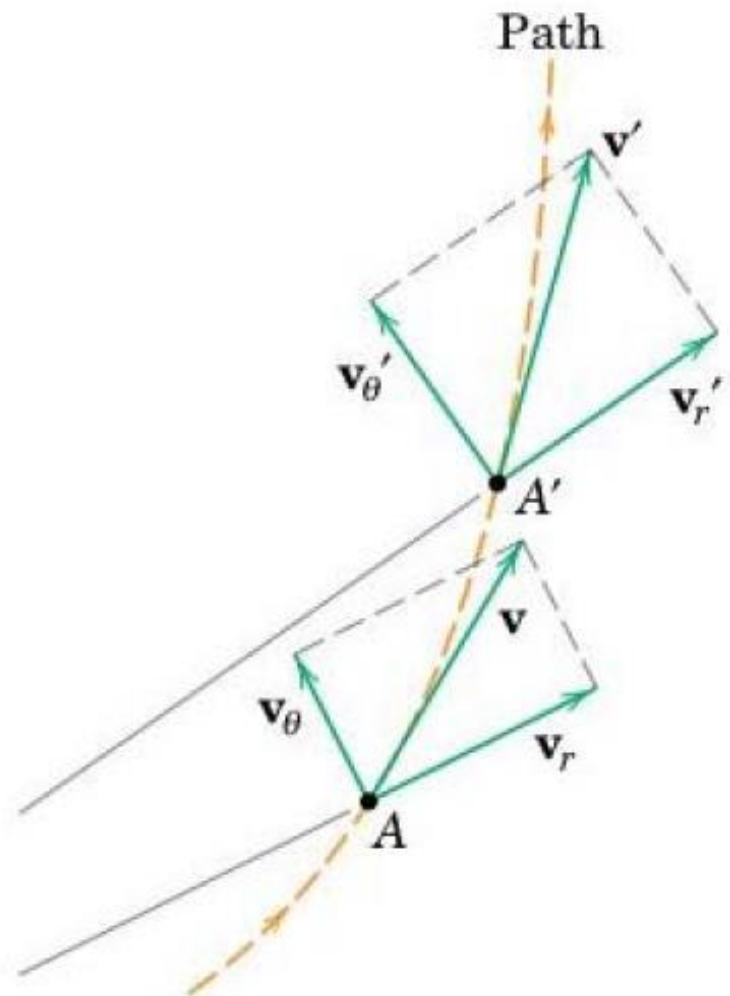
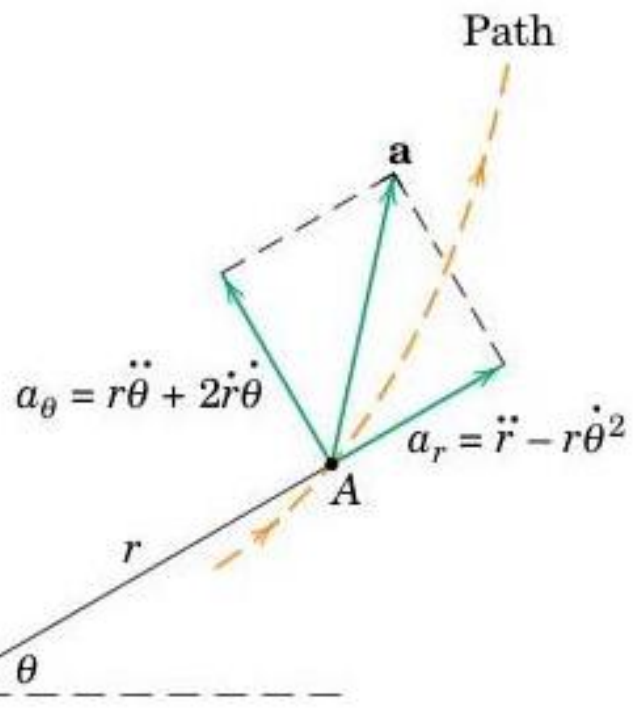
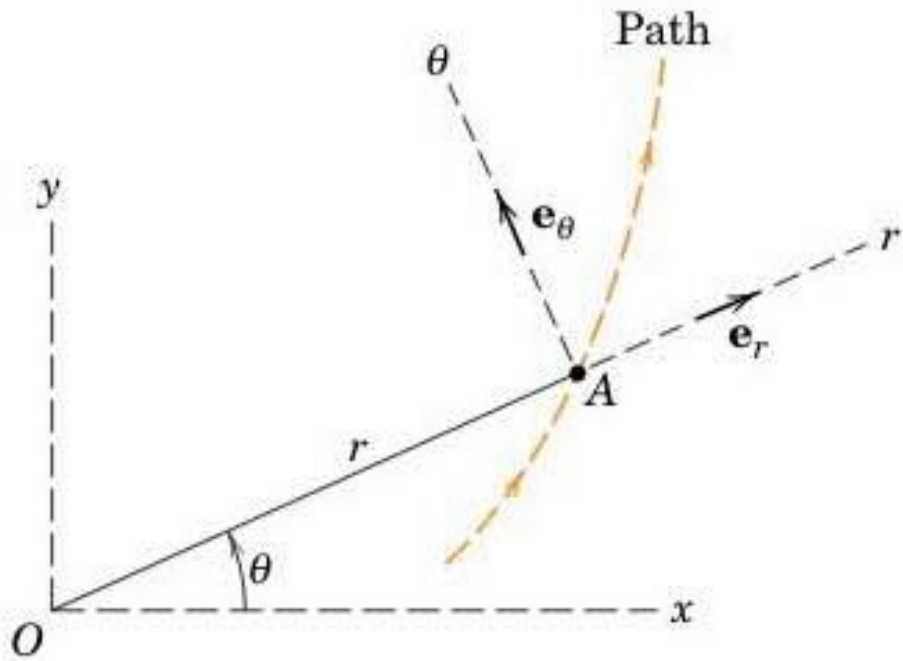
Polar Coordinates: Radial and Transverse Component:



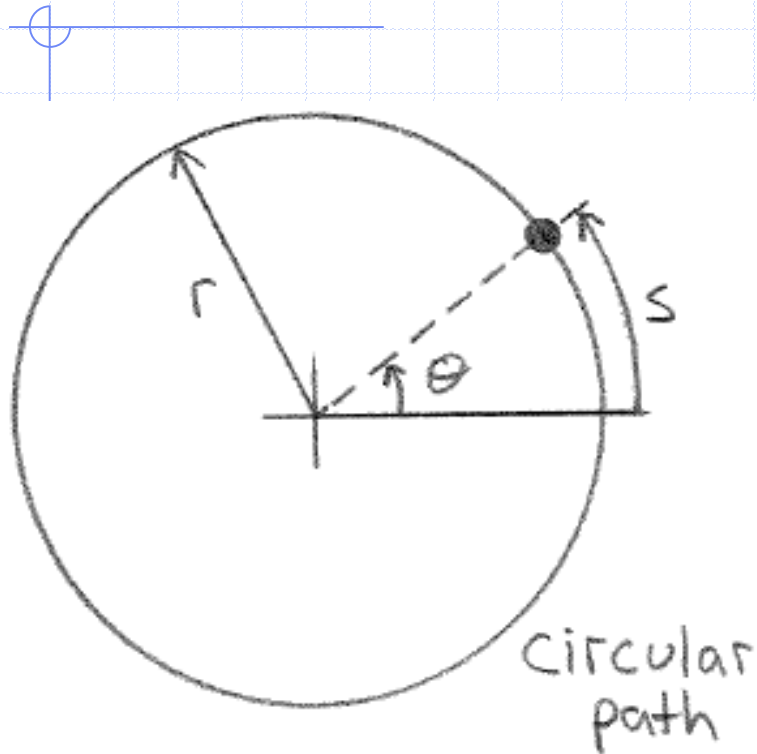
## 5) Final polar acceleration

$$\mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{e}}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\mathbf{e}}_\theta$$

Radial acc. + Transverse acc.



# Example 4



The Geometry:

$$s = r\theta, \quad r = \text{const.}$$

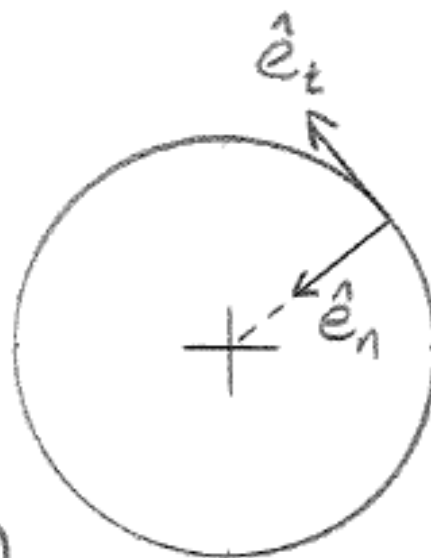
$$\Rightarrow \dot{s} = r\dot{\theta} \equiv v$$

## Normal & Tangential Components

$$\bar{a} = \frac{dv}{dt} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n$$

$$= \frac{d(r\dot{\theta})}{dt} \hat{e}_t + \frac{(r\dot{\theta})^2}{r} \hat{e}_n$$

$$= r\ddot{\theta} \hat{e}_t + r\dot{\theta}^2 \hat{e}_n \quad \text{--- (*)}$$

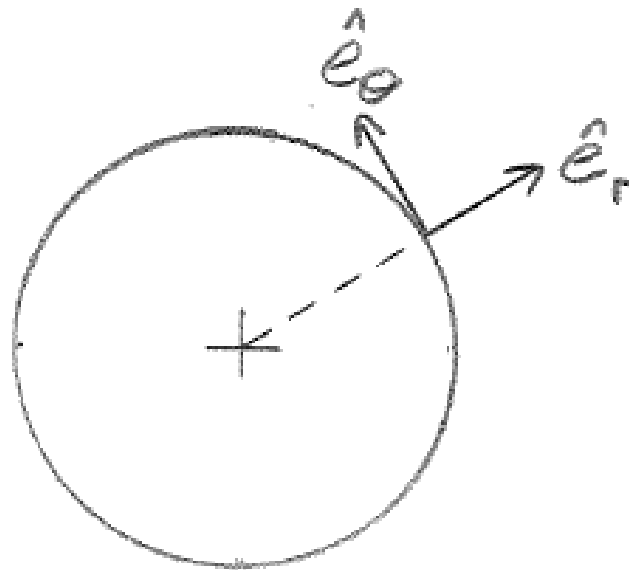


## Radial & Transverse Components

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

where  $r = \text{const.} \Rightarrow \dot{r} = \ddot{r} = 0$

$$\bar{a} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta \quad \text{--- (**)}$$

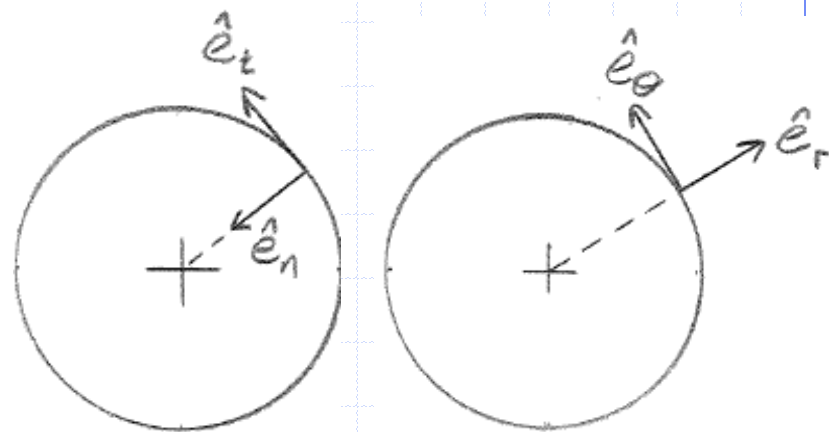


## Relating the two...

For circular motion,

$$\hat{e}_r = -\hat{e}_n$$

$$\hat{e}_\theta = \hat{e}_t$$



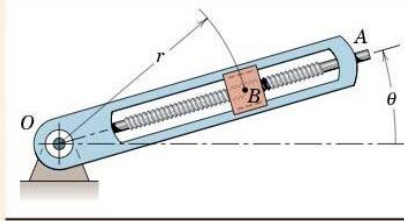
Then from (\*\*),  $\bar{a} = -r\ddot{\theta}^2 \hat{e}_r + r\ddot{\theta} \hat{e}_\theta$  ——— (\*\*)

$$\begin{aligned}\bar{a} &= -r\dot{\theta}^2 (-\hat{e}_n) + r\ddot{\theta} \hat{e}_t \\ &= r\dot{\theta}^2 \hat{e}_n + r\ddot{\theta} \hat{e}_t\end{aligned}$$

$r\ddot{\theta} \hat{e}_t + r\dot{\theta}^2 \hat{e}_n$  ——— (\*) ... Same as (\*) !

## Sample Problem 2/9

Rotation of the radially slotted arm is governed by  $\theta = 0.2t + 0.02t^3$ , where  $\theta$  is in radians and  $t$  is in seconds. Simultaneously, the power screw in the arm engages the slider  $B$  and controls its distance from  $O$  according to  $r = 0.2 + 0.04t^2$ , where  $r$  is in meters and  $t$  is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when  $t = 3$  s.



- Solution.** The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for  $t = 3$  s.

$$\begin{aligned} r &= 0.2 + 0.04t^2 & r_3 &= 0.2 + 0.04(3^2) = 0.56 \text{ m} \\ \dot{r} &= 0.08t & \dot{r}_3 &= 0.08(3) = 0.24 \text{ m/s} \\ \ddot{r} &= 0.08 & \ddot{r}_3 &= 0.08 \text{ m/s}^2 \\ \theta &= 0.2t + 0.02t^3 & \theta_3 &= 0.2(3) + 0.02(3^3) = 1.14 \text{ rad} \\ & & \text{or } \theta_3 &= 1.14(180/\pi) = 65.3^\circ \\ \dot{\theta} &= 0.2 + 0.06t^2 & \dot{\theta}_3 &= 0.2 + 0.06(3^2) = 0.74 \text{ rad/s} \\ \ddot{\theta} &= 0.12t & \ddot{\theta}_3 &= 0.12(3) = 0.36 \text{ rad/s}^2 \end{aligned}$$

The velocity components are obtained from Eq. 2/13 and for  $t = 3$  s are

$$\begin{aligned} [v_r = \dot{r}] & & v_r &= 0.24 \text{ m/s} \\ [v_\theta = r\dot{\theta}] & & v_\theta &= 0.56(0.74) = 0.414 \text{ m/s} \\ [v = \sqrt{v_r^2 + v_\theta^2}] & & v &= \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s} \quad \text{Ans.} \end{aligned}$$

The velocity and its components are shown for the specified position of the arm.

The acceleration components are obtained from Eq. 2/14 and for  $t = 3$  s are

$$\begin{aligned} [a_r = \ddot{r} - r\dot{\theta}^2] & & a_r &= 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2 \\ [a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] & & a_\theta &= 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2 \\ [a = \sqrt{a_r^2 + a_\theta^2}] & & a &= \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2 \quad \text{Ans.} \end{aligned}$$

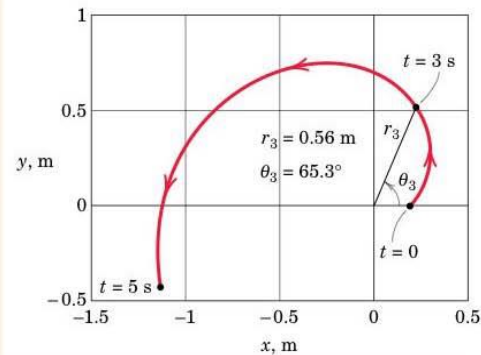
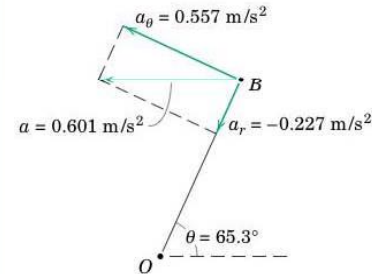
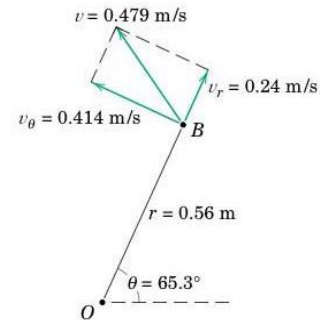
The acceleration and its components are also shown for the  $65.3^\circ$  position of the arm.

Plotted in the final figure is the path of the slider  $B$  over the time interval  $0 \leq t \leq 5$  s. This plot is generated by varying  $t$  in the given expressions for  $r$  and  $\theta$ . Conversion from polar to rectangular coordinates is given by

$$x = r \cos \theta \quad y = r \sin \theta$$

### Helpful Hint

- ① We see that this problem is an example of constrained motion where the center  $B$  of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.



## Given

The slider's distance from  $O$  is

$$r(t) = 0.2 + 0.04t^2 \quad (\text{m}),$$

and the arm's angular position is

$$\theta(t) = 0.2t + 0.02t^3 \quad (\text{rad}),$$

where  $t$  is in seconds.

## Required

Find the magnitudes of the slider's **velocity** and **acceleration** at  $t = 3$  s.

## Coordinate directions

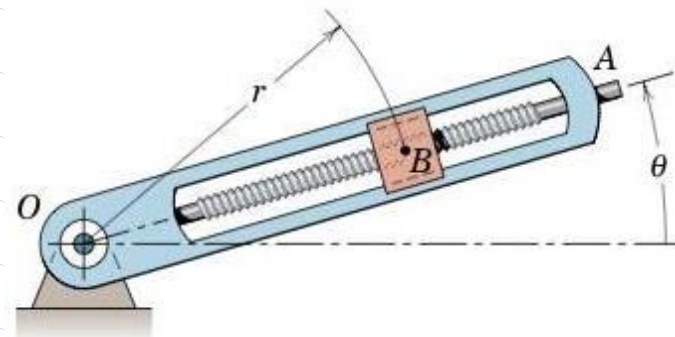
- $+\mathbf{e}_r$ : outward from  $O$ , along the arm.
- $+\mathbf{e}_\theta$ : perpendicular to the arm, in the direction of increasing  $\theta$ .

## Solution plan

- ① Differentiate  $r(t)$  and  $\theta(t)$ , then evaluate at  $t = 3$  s.
- ② Calculate the radial and transverse velocity components.
- ③ Calculate the radial and transverse acceleration components.
- ④ Convert selected positions to  $x, y$  coordinates to visualize the path.

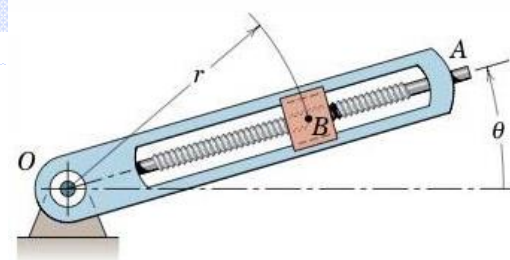
$$\mathbf{v} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{e}_\theta$$



## Position coordinates and their time derivatives

① Differentiate first; then substitute  $t = 3$  s



| Quantity        | General expression | Evaluation at $t = 3$ s                         |
|-----------------|--------------------|-------------------------------------------------|
| $r$             | $0.2 + 0.04t^2$    | $0.2 + 0.04(3)^2 = \boxed{0.56 \text{ m}}$      |
| $\dot{r}$       | $0.08t$            | $0.08(3) = \boxed{0.24 \text{ m/s}}$            |
| $\ddot{r}$      | $0.08$             | $\boxed{0.08 \text{ m/s}^2}$                    |
| $\theta$        | $0.2t + 0.02t^3$   | $0.2(3) + 0.02(3)^3 = \boxed{1.14 \text{ rad}}$ |
| $\dot{\theta}$  | $0.2 + 0.06t^2$    | $0.2 + 0.06(3)^2 = \boxed{0.74 \text{ rad/s}}$  |
| $\ddot{\theta}$ | $0.12t$            | $0.12(3) = \boxed{0.36 \text{ rad/s}^2}$        |

For labeling the sketch,

$$\theta = 1.14 \left( \frac{180^\circ}{\pi} \right) = \boxed{65.3^\circ}.$$

**Interpretation:**  $\dot{r} > 0$  means the slider moves outward along the arm;  $\dot{\theta} > 0$  means the arm rotates counterclockwise.

**Reminder:** Use angular rates in **radians per second** in the polar-coordinate formulas.

② Calculate the velocity components

$$v_r = \dot{r} = 0.240 \text{ m/s}$$

$$v_\theta = r\dot{\theta} = (0.56)(0.74) = 0.4144 \text{ m/s}.$$

Since the components are perpendicular,

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{0.240^2 + 0.4144^2} = 0.479 \text{ m/s}$$

③ Calculate the acceleration components

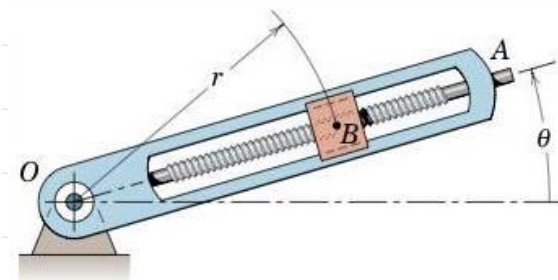
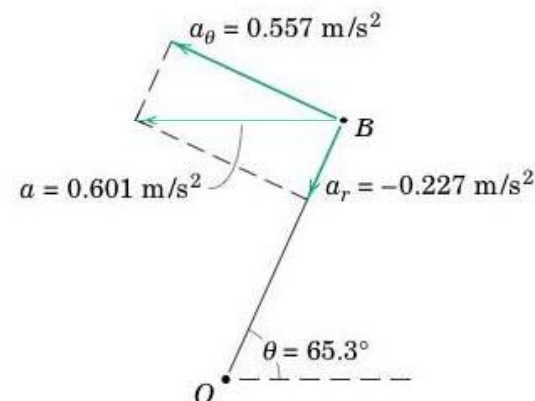
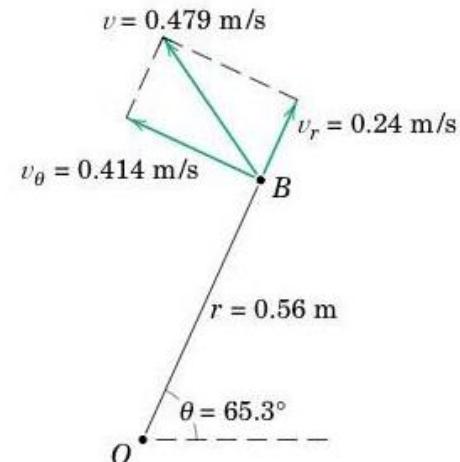
$$a_r = \underbrace{\ddot{r}}_{0.08} - \underbrace{r\dot{\theta}^2}_{0.56(0.74)^2} = -0.2267 \text{ m/s}^2$$

$$a_\theta = \underbrace{r\ddot{\theta}}_{0.56(0.36)} + \underbrace{2\dot{r}\dot{\theta}}_{2(0.24)(0.74)} = 0.5568 \text{ m/s}^2$$

$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(-0.2267)^2 + 0.5568^2} = 0.601 \text{ m/s}^2$$

Interpret the signs:  $a_r < 0$  points inward toward  $O$ ;  $a_\theta > 0$  points in the positive transverse direction.

Key observation: Even though  $\ddot{r} > 0$ , the radial acceleration is negative because  $a_r = \ddot{r} - r\dot{\theta}^2$ .



④ Convert from polar to rectangular coordinates

$$x = r \cos \theta, \quad y = r \sin \theta$$

Therefore,

$$x(t) = (0.2 + 0.04t^2) \cos(0.2t + 0.02t^3)$$

$$y(t) = (0.2 + 0.04t^2) \sin(0.2t + 0.02t^3).$$

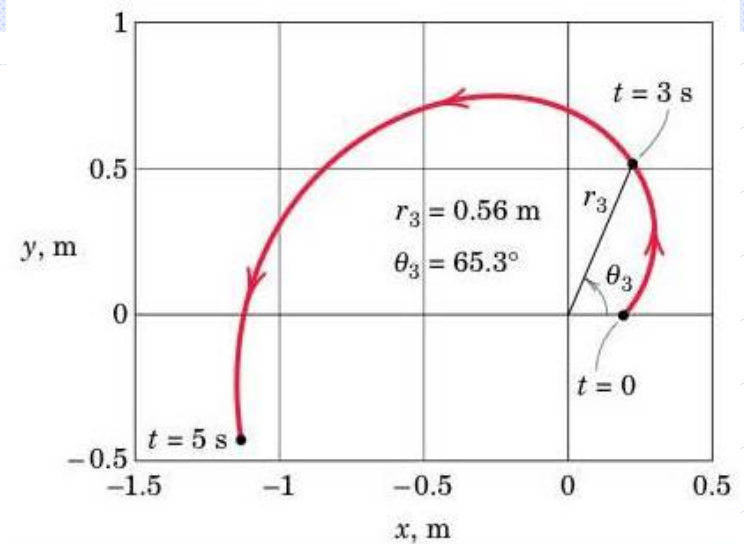
Use **radian mode** when evaluating the trigonometric functions.

| $t$ (s) | $r$ (m) | $\theta$ (rad) | $x$ (m) | $y$ (m) |
|---------|---------|----------------|---------|---------|
| 0       | 0.200   | 0.000          | 0.200   | 0.000   |
| 1       | 0.240   | 0.220          | 0.234   | 0.052   |
| 2       | 0.360   | 0.560          | 0.305   | 0.191   |
| 3       | 0.560   | 1.140          | 0.234   | 0.509   |
| 4       | 0.840   | 2.080          | -0.409  | 0.733   |
| 5       | 1.200   | 3.500          | -1.124  | -0.421  |

For example, at  $t = 3$  s,

$$x = 0.56 \cos(1.14) = 0.234 \text{ m}, \quad y = 0.56 \sin(1.14) = 0.509 \text{ m}.$$

Plot  $y$  against  $x$ , marking these points on a smooth curve with arrows in increasing time.



**Teaching connection:** The slider moves along the arm **relative to the arm**, but its path relative to the ground is curved because the arm rotates simultaneously.

## Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when  $\theta = 30^\circ$ , the tracking data give  $r = 25(10^4)$  ft,  $\dot{r} = 4000$  ft/sec, and  $\dot{\theta} = 0.80$  deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is  $31.4$  ft/sec<sup>2</sup> vertically down. For these conditions determine the velocity  $v$  of the rocket and the values of  $\ddot{r}$  and  $\ddot{\theta}$ .

**Solution.** The components of velocity from Eq. 2/13 are

$$[v_r = \dot{r}] \quad v_r = 4000 \text{ ft/sec}$$

$$\textcircled{1} [v_\theta = r\dot{\theta}] \quad v_\theta = 25(10^4)(0.80)\left(\frac{\pi}{180}\right) = 3490 \text{ ft/sec}$$

$$[v = \sqrt{v_r^2 + v_\theta^2}] \quad v = \sqrt{(4000)^2 + (3490)^2} = 5310 \text{ ft/sec} \quad \text{Ans.}$$

Since the total acceleration of the rocket is  $g = 31.4$  ft/sec<sup>2</sup> down, we can easily find its  $r$ - and  $\theta$ -components for the given position. As shown in the figure, they are

$$\textcircled{2} \quad a_r = -31.4 \cos 30^\circ = -27.2 \text{ ft/sec}^2$$

$$a_\theta = 31.4 \sin 30^\circ = 15.70 \text{ ft/sec}^2$$

We now equate these values to the polar-coordinate expressions for  $a_r$  and  $a_\theta$  which contain the unknowns  $\ddot{r}$  and  $\ddot{\theta}$ . Thus, from Eq. 2/14

$$\textcircled{3} [a_r = \ddot{r} - r\dot{\theta}^2] \quad -27.2 = \ddot{r} - 25(10^4)\left(0.80 \frac{\pi}{180}\right)^2$$

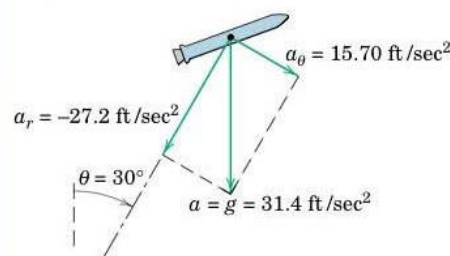
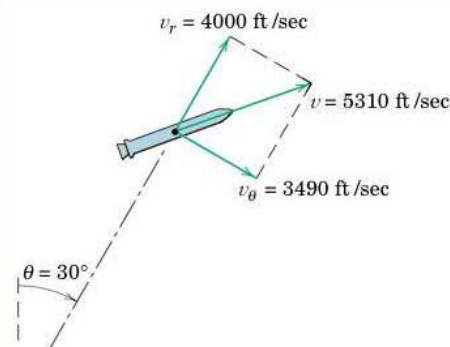
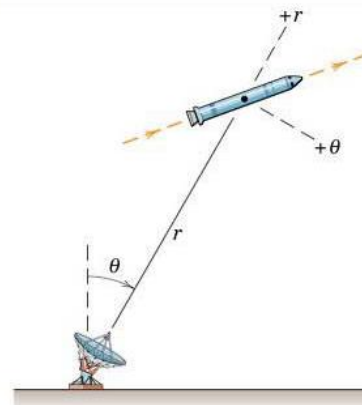
$$\ddot{r} = 21.5 \text{ ft/sec}^2$$

$$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] \quad 15.70 = 25(10^4)\ddot{\theta} + 2(4000)\left(0.80 \frac{\pi}{180}\right)$$

$$\ddot{\theta} = -3.84(10^{-4}) \text{ rad/sec}^2$$

Ans.

Ans.



### Helpful Hints

- ① We observe that the angle  $\theta$  in polar coordinates need not always be taken positive in a counterclockwise sense.
- ② Note that the  $r$ -component of acceleration is in the negative  $r$ -direction, so it carries a minus sign.
- ③ We must be careful to convert  $\dot{\theta}$  from deg/sec to rad/sec.

Given

At the instant shown,

$$\theta = 30^\circ, \quad r = 25(10^4) = 250,000 \text{ ft},$$

$$\dot{r} = 4000 \text{ ft/s}, \quad \dot{\theta} = 0.80^\circ/\text{s}.$$

The rocket is in unpowered flight above the atmosphere. Its acceleration is  $31.4 \text{ ft/s}^2$  **vertically downward**, due only to gravity.

Required

- ① Rocket velocity,  $\mathbf{v}$ , and its magnitude.
- ② Radial-coordinate second derivative,  $\ddot{r}$ .
- ③ Angular acceleration,  $\ddot{\theta}$ .

Coordinate directions

- $+\mathbf{e}_r$ : outward from the radar toward the rocket.
- $+\mathbf{e}_\theta$ : in the direction of increasing  $\theta$ , **clockwise in this diagram**.
- Here,  $\theta$  is measured from the **vertical**.

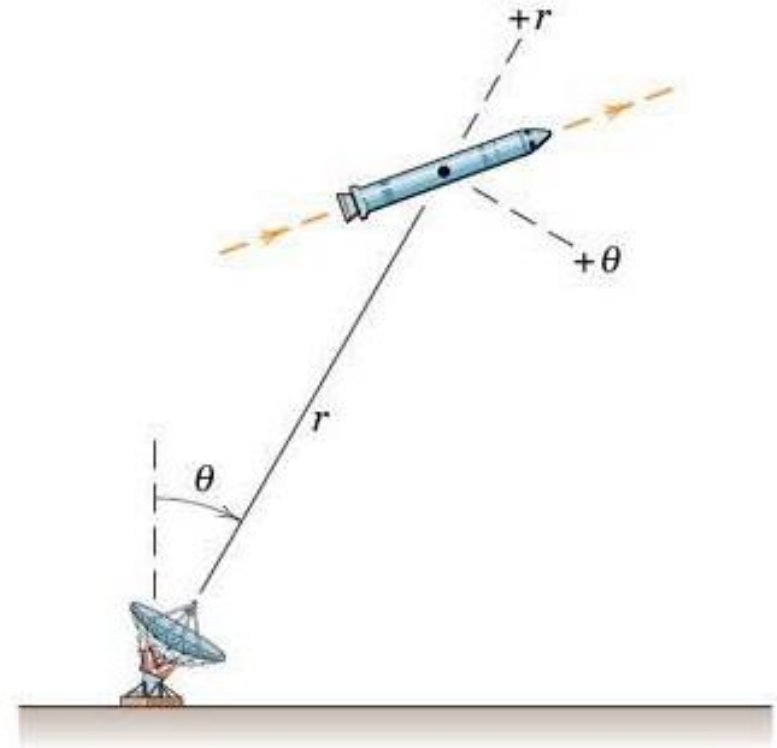
Strategy

Find velocity using

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}.$$

Resolve gravity into  $a_r, a_\theta$ , then solve

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}.$$



① Convert the angular rate to radians per second

$$\dot{\theta} = 0.80 \left( \frac{\pi}{180} \right) = \boxed{0.01396 \text{ rad/s}}.$$

② Calculate the radial and transverse components

$$v_r = \dot{r} = \boxed{4000 \text{ ft/s}}$$

$$v_\theta = r\dot{\theta} = 250,000 \left( 0.80 \frac{\pi}{180} \right) = \boxed{3491 \text{ ft/s}}.$$

Therefore,

$$\mathbf{v} = 4000 \mathbf{e}_r + 3491 \mathbf{e}_\theta \text{ ft/s}.$$

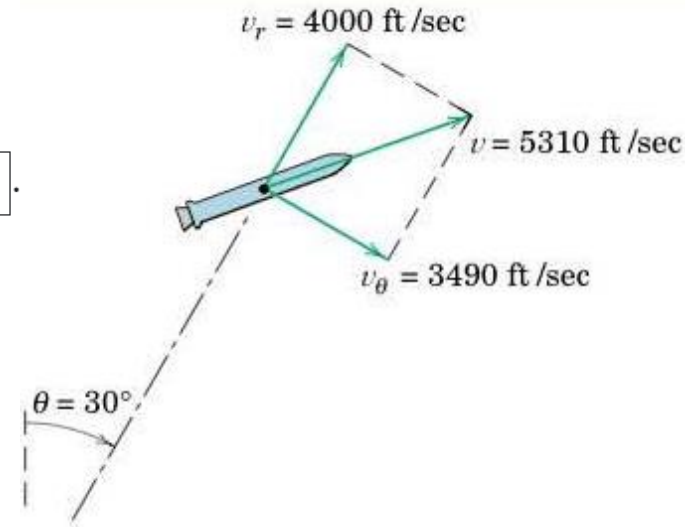
③ Calculate the speed

Since the components are perpendicular,

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{4000^2 + 3491^2}.$$

$$\boxed{v \approx 5310 \text{ ft/s}}$$

**Interpretation:** The rocket moves outward from the radar while its line of sight rotates clockwise. Its velocity therefore has **both radial and transverse components**.



① Resolve the known downward acceleration

Because  $\theta$  is measured from the vertical,

$$a_r = \underbrace{-31.4 \cos 30^\circ}_{\text{opposite} + e_r} = \boxed{-27.2 \text{ ft/s}^2}$$

$$a_\theta = \underbrace{31.4 \sin 30^\circ}_{\text{along} + e_\theta} = \boxed{15.7 \text{ ft/s}^2}$$

② Use the radial equation to find  $\ddot{r}$

$$a_r = \ddot{r} - r\dot{\theta}^2 \Rightarrow \ddot{r} = a_r + r\dot{\theta}^2.$$

$$\ddot{r} = -31.4 \cos 30^\circ + 250,000 \left( 0.80 \frac{\pi}{180} \right)^2$$

$$\boxed{\ddot{r} = 21.5 \text{ ft/s}^2}$$

③ Use the transverse equation to find  $\ddot{\theta}$

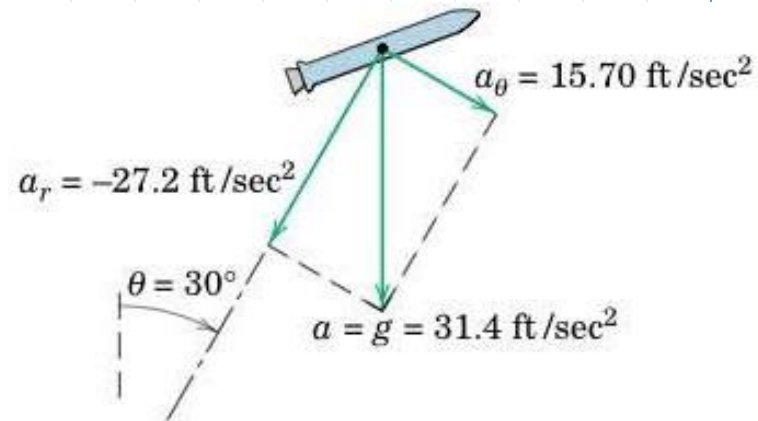
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \Rightarrow \ddot{\theta} = \frac{a_\theta - 2\dot{r}\dot{\theta}}{r}.$$

$$\ddot{\theta} = \frac{15.7 - 2(4000) \left( 0.80 \frac{\pi}{180} \right)}{250,000}$$

$$\boxed{\ddot{\theta} = -3.84 \times 10^{-4} \text{ rad/s}^2}$$

**Interpret the results:**  $\ddot{r} > 0$  means the outward range rate is increasing. Since  $\dot{\theta} > 0$  but  $\ddot{\theta} < 0$ , the clockwise angular rate is decreasing.

**Reminder:**  $\ddot{r} \neq a_r$  and  $r\ddot{\theta} \neq a_\theta$ ; the polar directions themselves rotate.



# SUMMARY



# Curvilinear Motion

## *Velocity and Acceleration*

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$v = \frac{ds}{dt}$$

## *Projectile Motion*

$$\Delta y = \Delta x \tan \theta - \frac{g(\Delta x)^2}{2v_0^2 \cos^2 \theta}$$

$$(\tan \theta)^2 - \frac{2v_0^2}{g\Delta x} (\tan \theta) + \left( 1 + \frac{2v_0^2 \Delta y}{g(\Delta x)^2} \right) = 0$$

# Curvilinear Motion

## Relative Motions

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A$$

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$$

$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$$

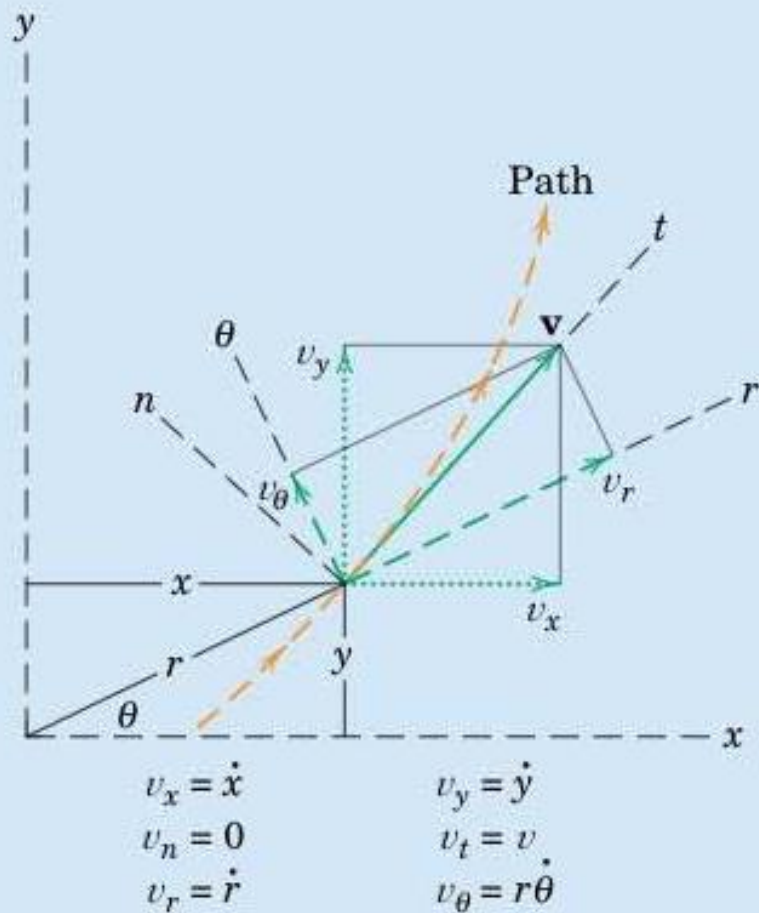
## Radial/Transverse Components

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

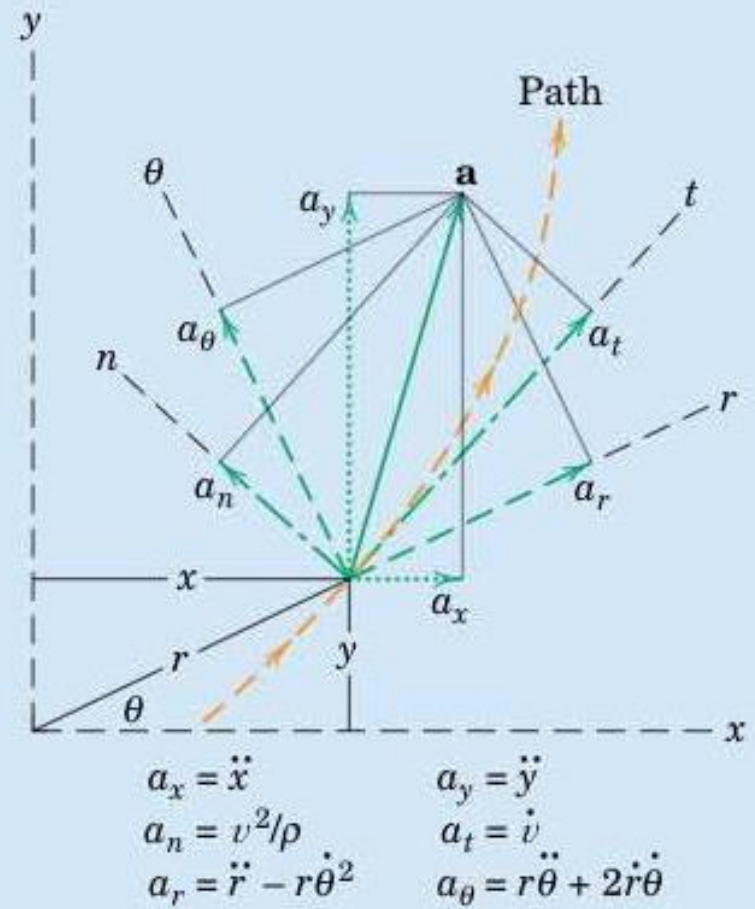
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

## Normal/Tangential Components

$$\mathbf{v} = v\hat{\mathbf{e}}_t \quad \mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$



(a) Velocity components

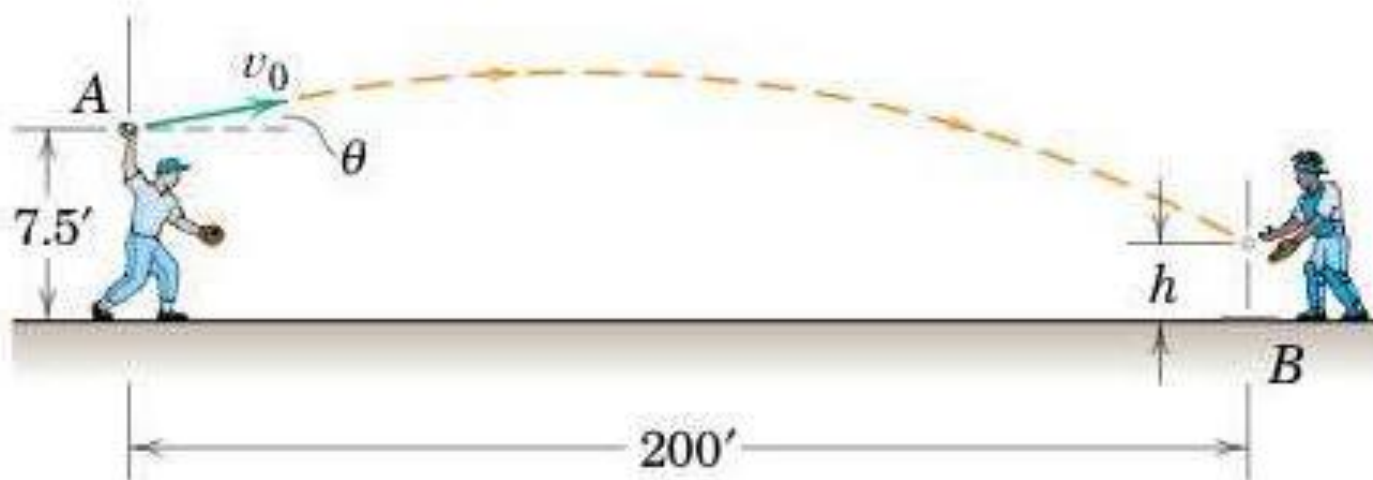


(b) Acceleration components

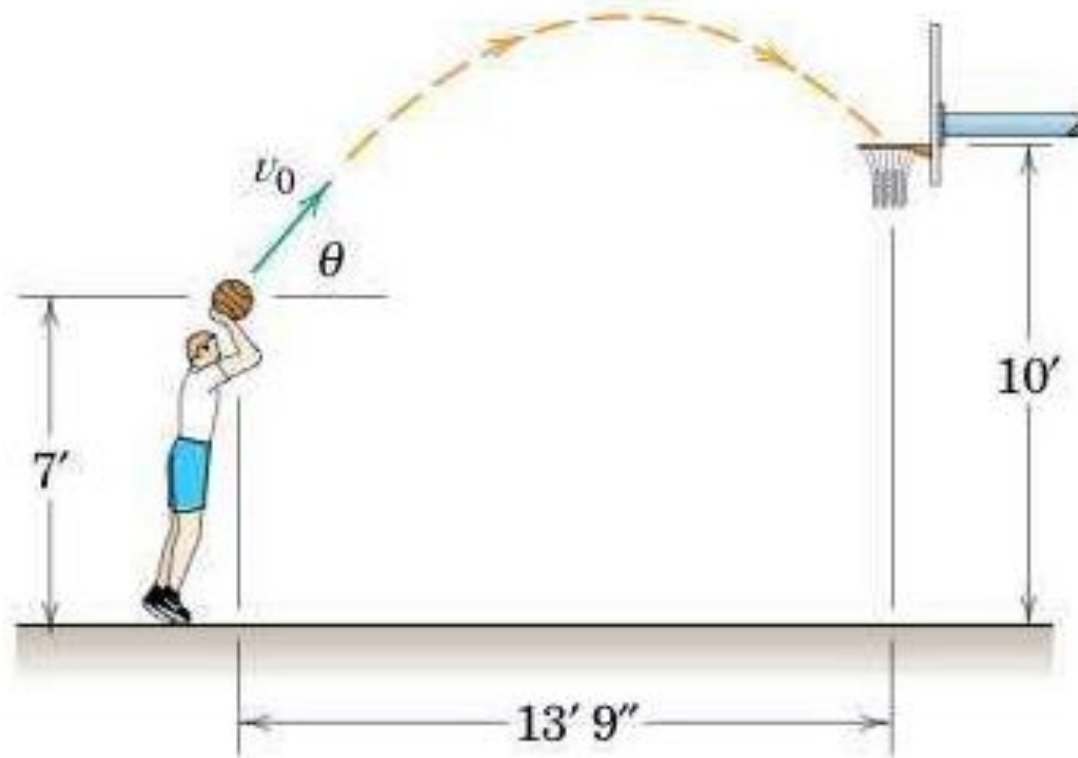
# Additional Problems



**2/80** An outfielder experiments with two different trajectories for throwing to home plate from the position shown: (a)  $v_0 = 140$  ft/sec with  $\theta = 8^\circ$  and (b)  $v_0 = 120$  ft/sec with  $\theta = 12^\circ$ . For each set of initial conditions, determine the time  $t$  required for the baseball to reach home plate and the altitude  $h$  as the ball crosses the plate.



**2/94** The basketball player likes to release his foul shots at an angle  $\theta = 50^\circ$  to the horizontal as shown. What initial speed  $v_0$  will cause the ball to pass through the center of the rim?



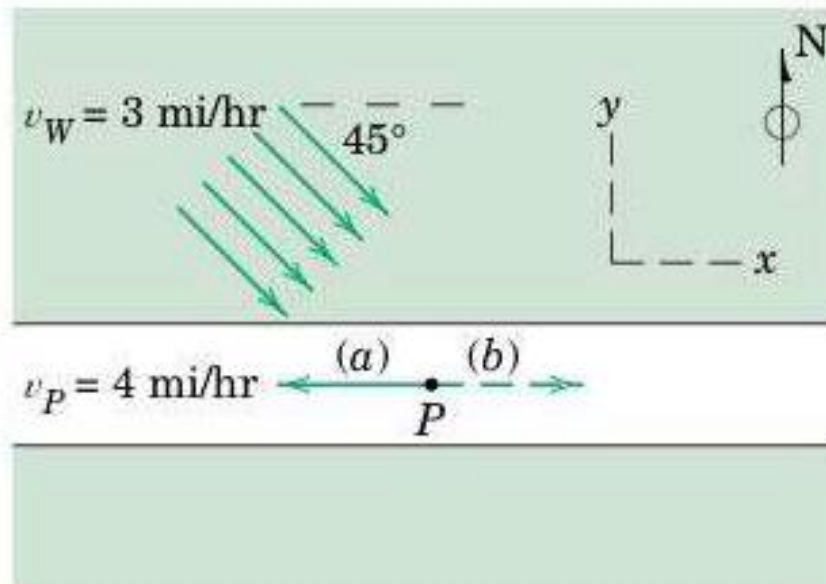
**2/191** A woman  $P$  walks on an east-west street at a speed of 4 mi/hr. The wind blows out of the northwest as shown at a speed of 3 mi/hr. Determine the velocity of the wind relative to the woman if she (a) walks west and (b) walks east on the street. Express your results both in terms of unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  and as magnitudes and compass directions.

*Ans.* (a)  $\mathbf{v}_{w/p} = 6.12\mathbf{i} - 2.12\mathbf{j}$  mi/hr

$v_{w/p} = 6.48$  mi/hr at  $19.11^\circ$  south of east

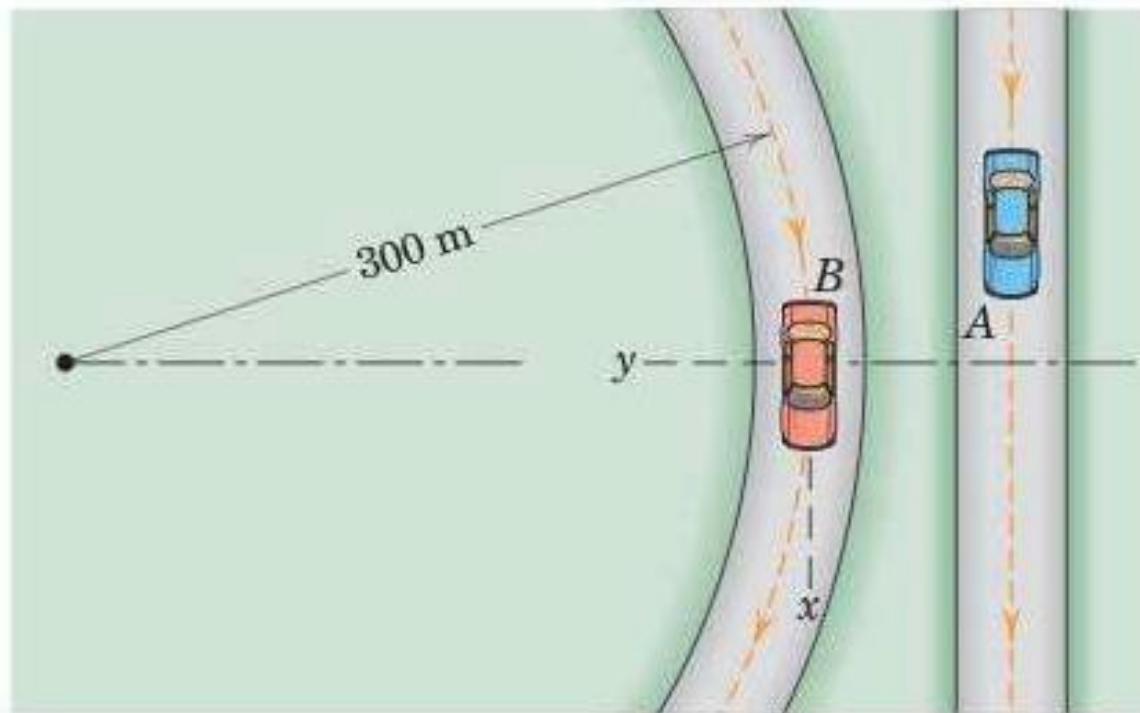
(b)  $\mathbf{v}_{w/p} = -1.879\mathbf{i} - 2.12\mathbf{j}$  mi/hr

$v_{w/p} = 2.83$  mi/hr at  $48.5^\circ$  south of west

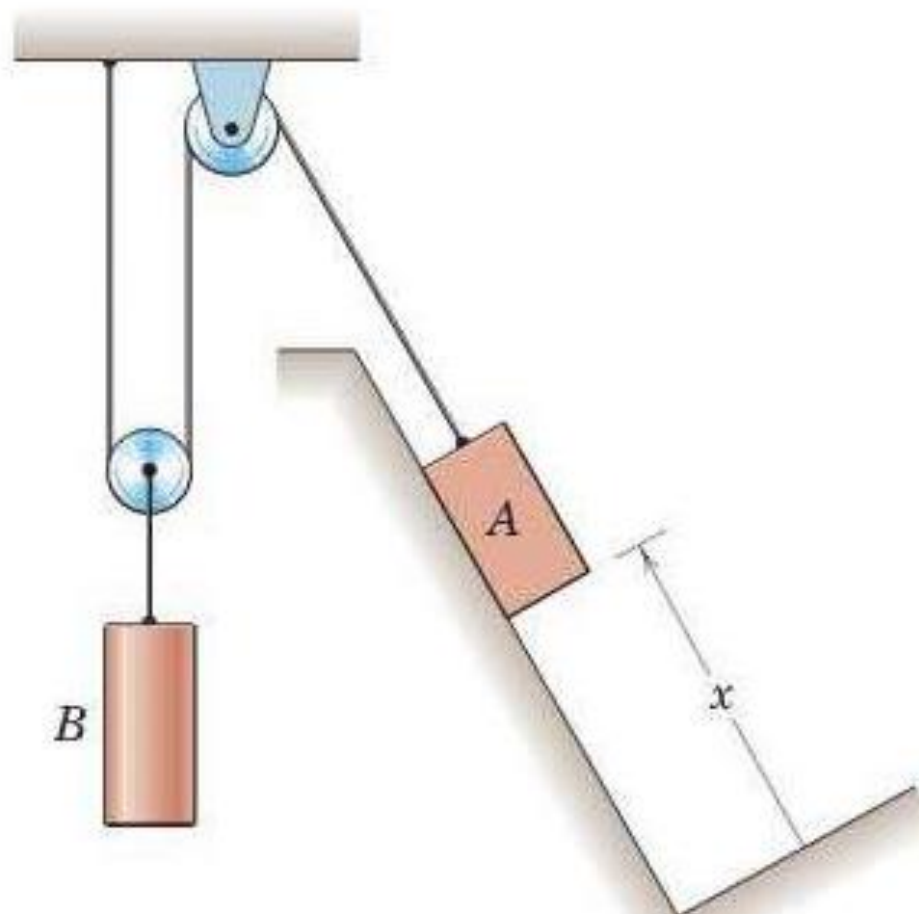


**2/193** For the instant represented, car *A* has a speed of 100 km/h, which is increasing at the rate of 8 km/h each second. Simultaneously, car *B* also has a speed of 100 km/h as it rounds the turn and is slowing down at the rate of 8 km/h each second. Determine the acceleration that car *B* appears to have to an observer in car *A*.

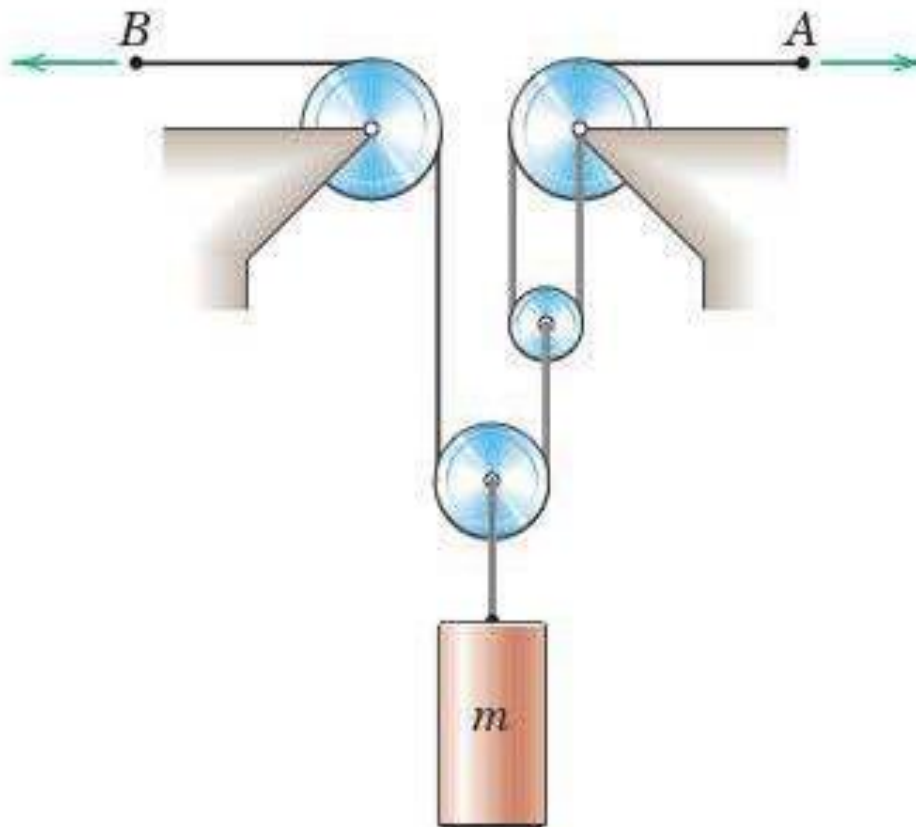
*Ans.*  $\mathbf{a}_{B/A} = -4.44\mathbf{i} + 2.57\mathbf{j} \text{ m/s}^2$



**2/214** If the velocity  $\dot{x}$  of block A up the incline is increasing at the rate of 0.044 m/s each second, determine the acceleration of B.



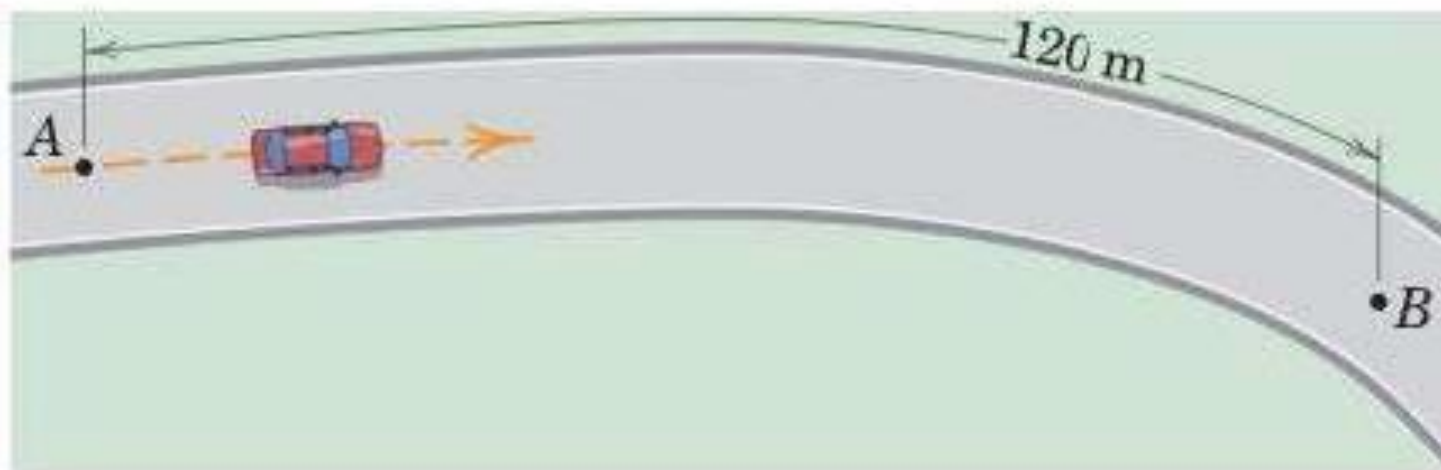
**2/218** For the pulley system shown, each of the cables at  $A$  and  $B$  is given a velocity of  $2 \text{ m/s}$  in the direction of the arrow. Determine the upward velocity  $v$  of the load  $m$ .



**2/103** A particle moves in a circular path of 0.3-m radius. Calculate the magnitude  $a$  of the acceleration of the particle (a) if its speed is constant at 0.6 m/s and (b) if its speed is 0.6 m/s but is increasing at the rate of 0.9 m/s each second.

*Ans. (a)  $a = 1.2 \text{ m/s}^2$ , (b)  $a = 1.5 \text{ m/s}^2$*

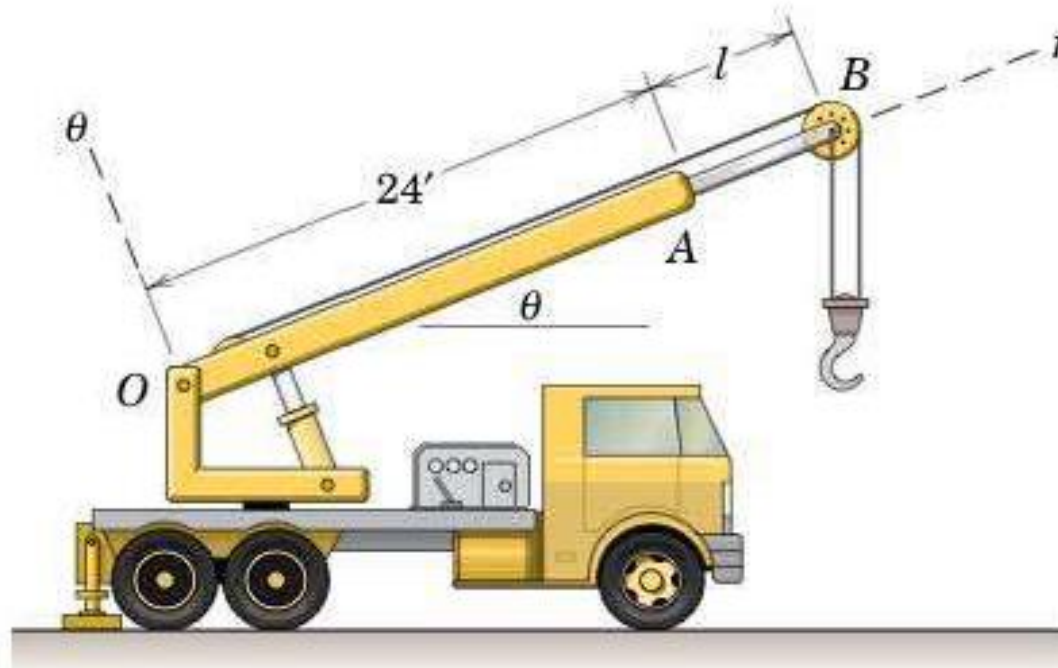
**2/116** A car travels along the level curved road with a speed which is decreasing at the constant rate of  $0.6 \text{ m/s}$  each second. The speed of the car as it passes point  $A$  is  $16 \text{ m/s}$ . Calculate the magnitude of the total acceleration of the car as it passes point  $B$  which is  $120 \text{ m}$  along the road from  $A$ . The radius of curvature of the road at  $B$  is  $60 \text{ m}$ .



**2/139** The boom  $OAB$  pivots about point  $O$ , while section  $AB$  simultaneously extends from within section  $OA$ . Determine the velocity and acceleration of the center  $B$  of the pulley for the following conditions:  $\theta = 20^\circ$ ,  $\dot{\theta} = 5 \text{ deg/sec}$ ,  $\ddot{\theta} = 2 \text{ deg/sec}^2$ ,  $l = 7 \text{ ft}$ ,  $\dot{l} = 1.5 \text{ ft/sec}$ ,  $\ddot{l} = -4 \text{ ft/sec}^2$ . The quantities  $\dot{l}$  and  $\ddot{l}$  are the first and second time derivatives, respectively, of the length  $l$  of section  $AB$ .

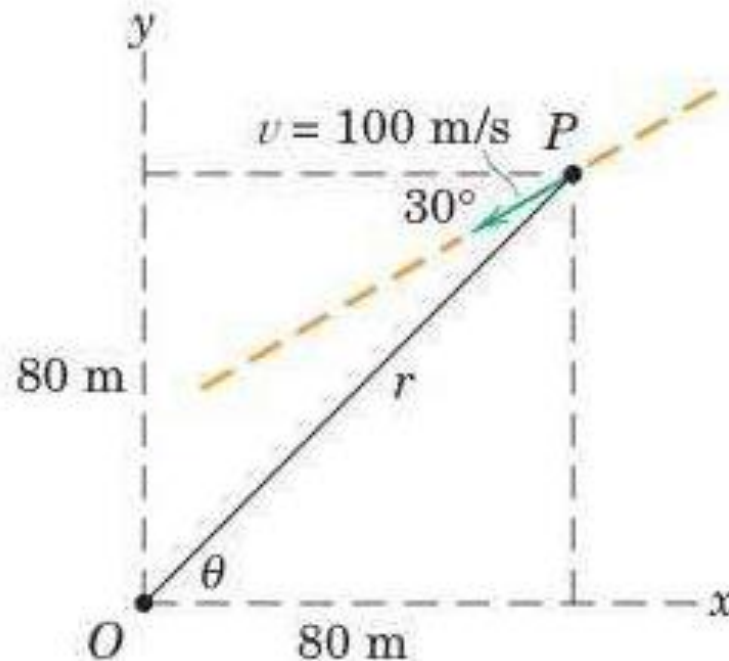
*Ans.*  $\mathbf{v} = 1.5\mathbf{e}_r + 2.71\mathbf{e}_\theta \text{ ft/sec}$

$\mathbf{a} = -4.24\mathbf{e}_r + 1.344\mathbf{e}_\theta \text{ ft/sec}^2$



**2/143** As it passes the position shown, the particle  $P$  has a constant speed  $v = 100$  m/s along the straight line shown. Determine the corresponding values of  $\dot{r}$ ,  $\dot{\theta}$ ,  $\ddot{r}$ , and  $\ddot{\theta}$ .

$$\text{Ans. } \dot{r} = -96.6 \text{ m/s}, \dot{\theta} = 0.229 \text{ rad/s} \\ \ddot{r} = 5.92 \text{ m/s}^2, \ddot{\theta} = -0.391 \text{ rad/s}^2$$



**2/162** The robot arm is elevating and extending simultaneously. At a given instant,  $\theta = 30^\circ$ ,  $\dot{\theta} = 10 \text{ deg/s} = \text{constant}$ ,  $l = 0.5 \text{ m}$ ,  $\dot{l} = 0.2 \text{ m/s}$ , and  $\ddot{l} = -0.3 \text{ m/s}^2$ . Compute the magnitudes of the velocity  $\mathbf{v}$  and acceleration  $\mathbf{a}$  of the gripped part  $P$ . In addition, express  $\mathbf{v}$  and  $\mathbf{a}$  in terms of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ .

