Chapter 2 – Kinematics of Particles – p2

- Curvilinear Motion
- Projectile Motion (*x-y*)
- Relative Curvilinear Motion
- Normal & Tangential Components (*t-n*)
- Radial & Transverse Components (*r-θ*)

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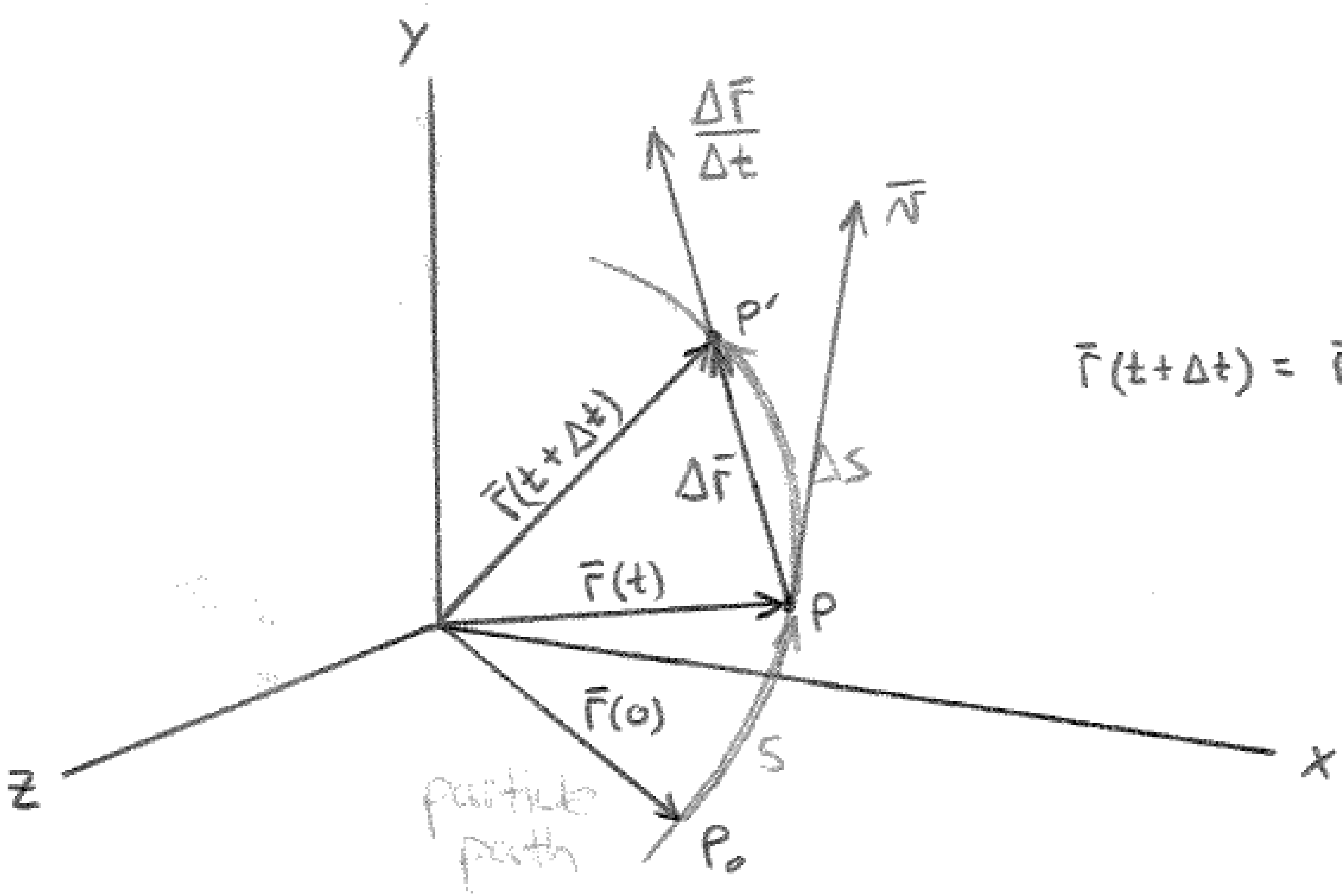
1. Curvilinear Motion of Particles

Curvilinear Motion of Particles

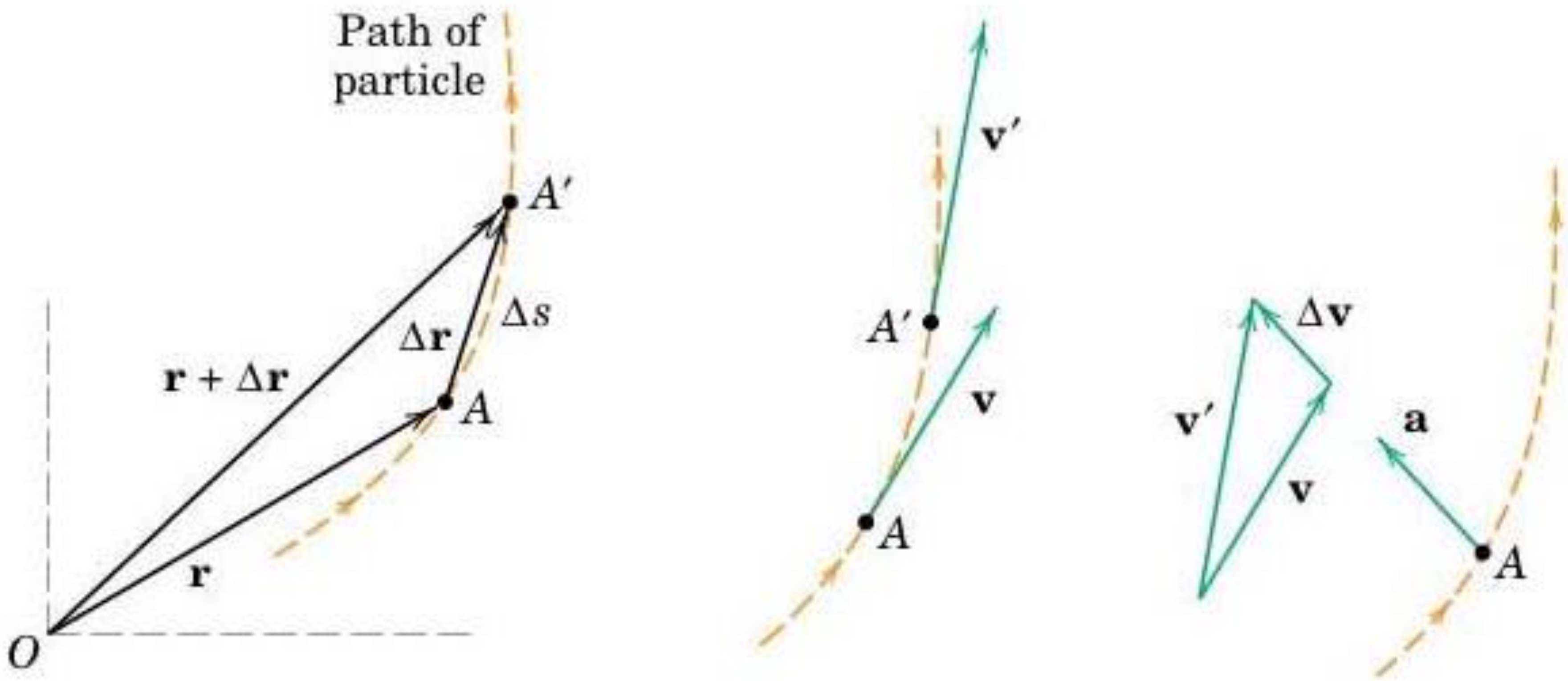
→ Particle is constrained to move along a curved path.

Velocity $\bar{v}$

→ Time-evolution of $\bar{r}$



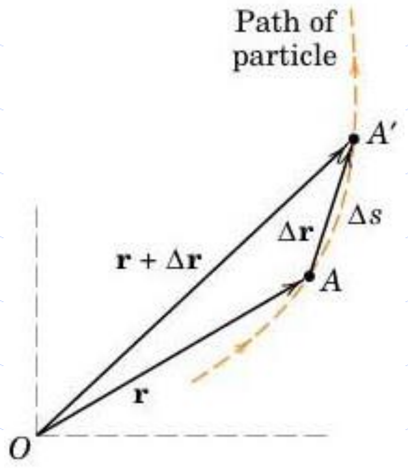
$$\bar{r}(t+\Delta t) = \bar{r}(t) + \Delta\bar{r}$$


Path of
particle
A′
Δs
Δr
r + Δr
A
r
O
v′
A′
v
A
Δv
v′
a
v
A

we define, Ave. Velocity $= \dfrac{\Delta \bar{r}}{\Delta t}$

Letting $\Delta t \to 0$,

$$\bar{v} = \lim_{\Delta t \to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t} = \frac{d\bar{r}}{dt}$$



$$\therefore \quad \boxed{\bar{v} = \frac{d\bar{r}}{dt}}$$

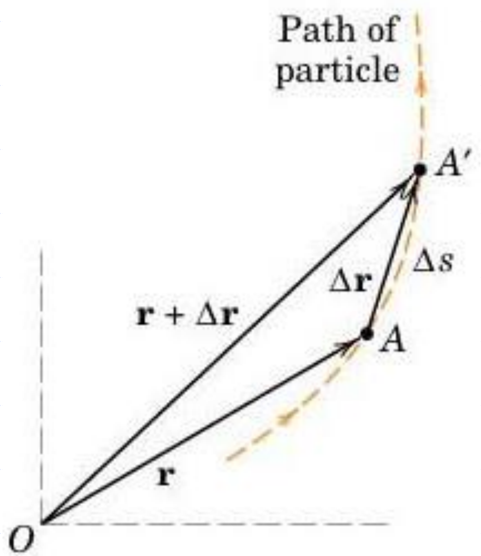
Instantaneous Velocity

Comments:

① $\Delta\bar{r}$ specifies dir. of Ave. Vel.

② As $\Delta t \to 0$, $\bar{v}$ becomes tangent to particle path.

The particle speed is



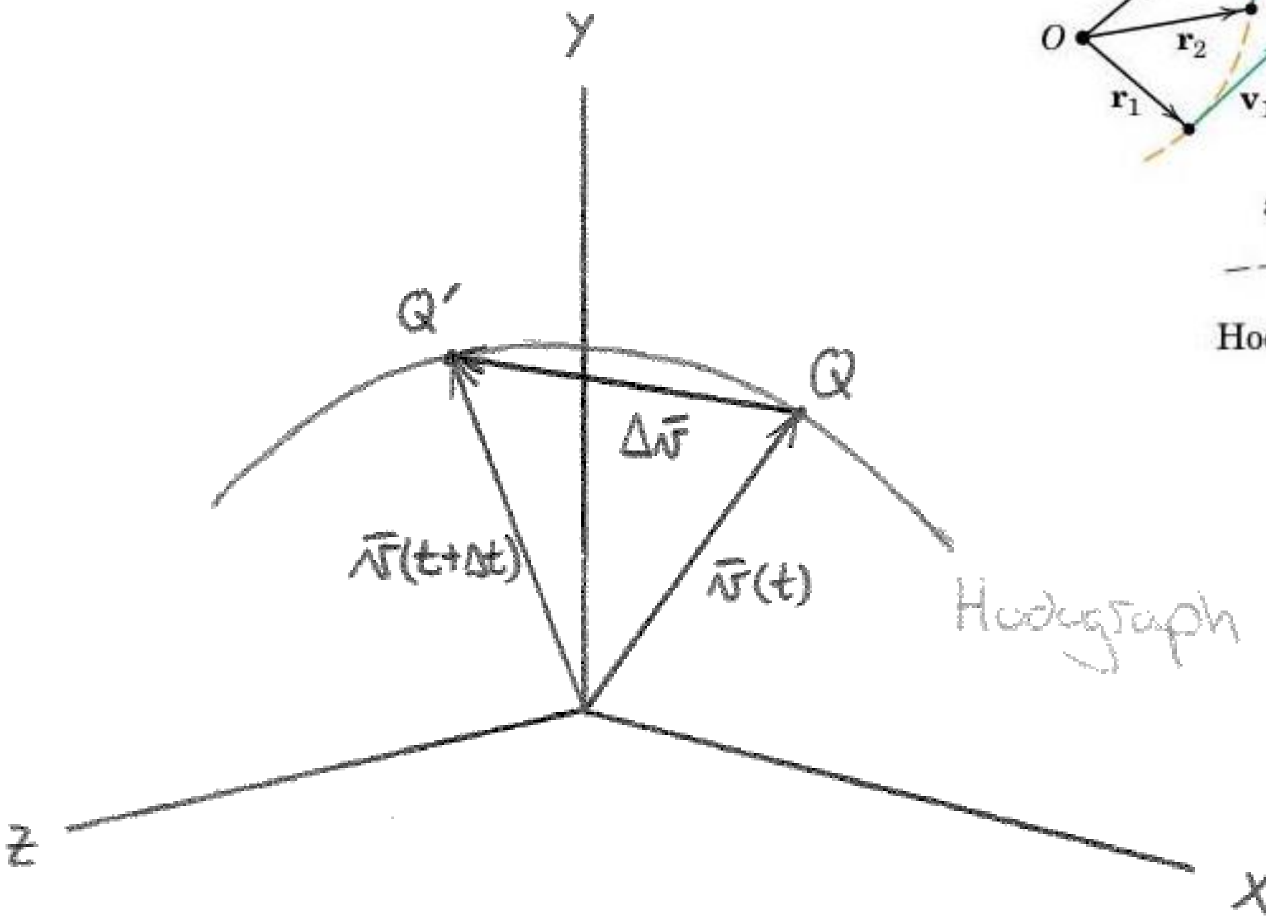
$$v = |\bar{v}| = \left| \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} \right| = \lim_{\Delta t \to 0} \left| \frac{\Delta \bar{r}}{\Delta t} \right|$$

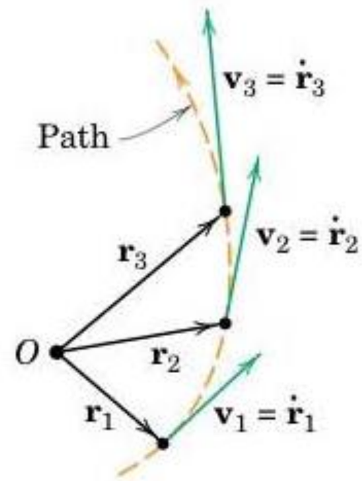
$$= \lim_{\Delta t \to 0} \frac{|\Delta \bar{r}|}{\Delta t} = \frac{ds}{dt}$$

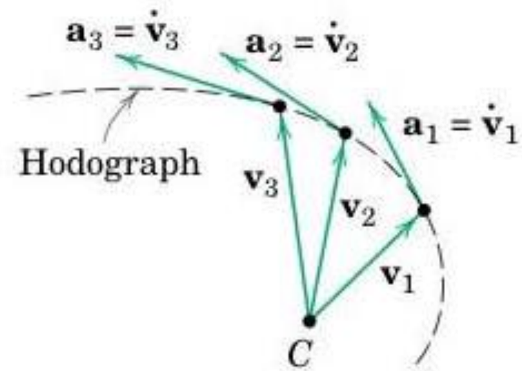
$$\therefore \boxed{v = \frac{ds}{dt}} \quad \text{Speed}$$

Acceleration $\bar{a}$

→ Time-evolution of $\bar{v}$







Hodograph (NOT path!)

We define,

$$\text{Ave. Acceleration} = \frac{\Delta \bar{v}}{\Delta t}$$

Letting $\Delta t \to 0$,

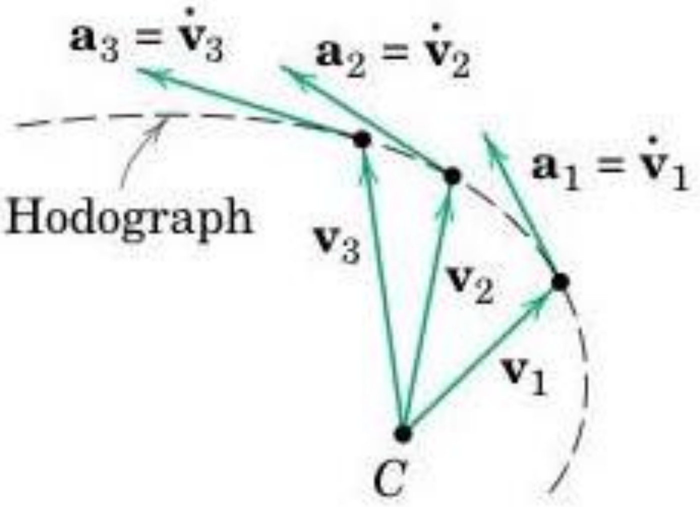
$$\bar{a} = \lim_{\Delta t \to 0} \frac{\bar{v}(t+\Delta t) - \bar{v}(t)}{\Delta t} = \frac{d\bar{v}}{dt}$$

$$\therefore \boxed{\bar{a} = \frac{d\bar{v}}{dt}}$$ Instantaneous Acceleration

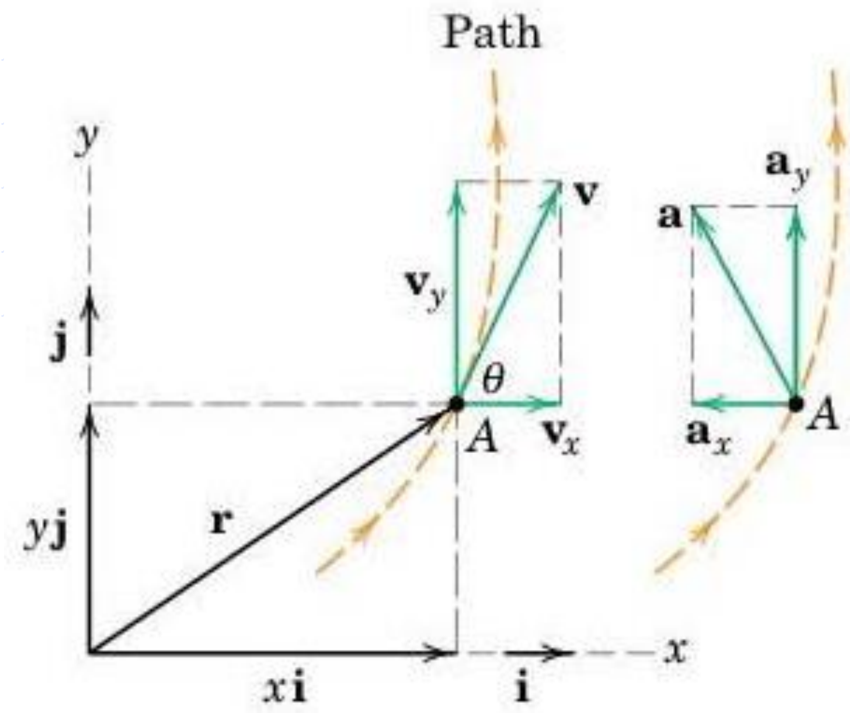
Comments:

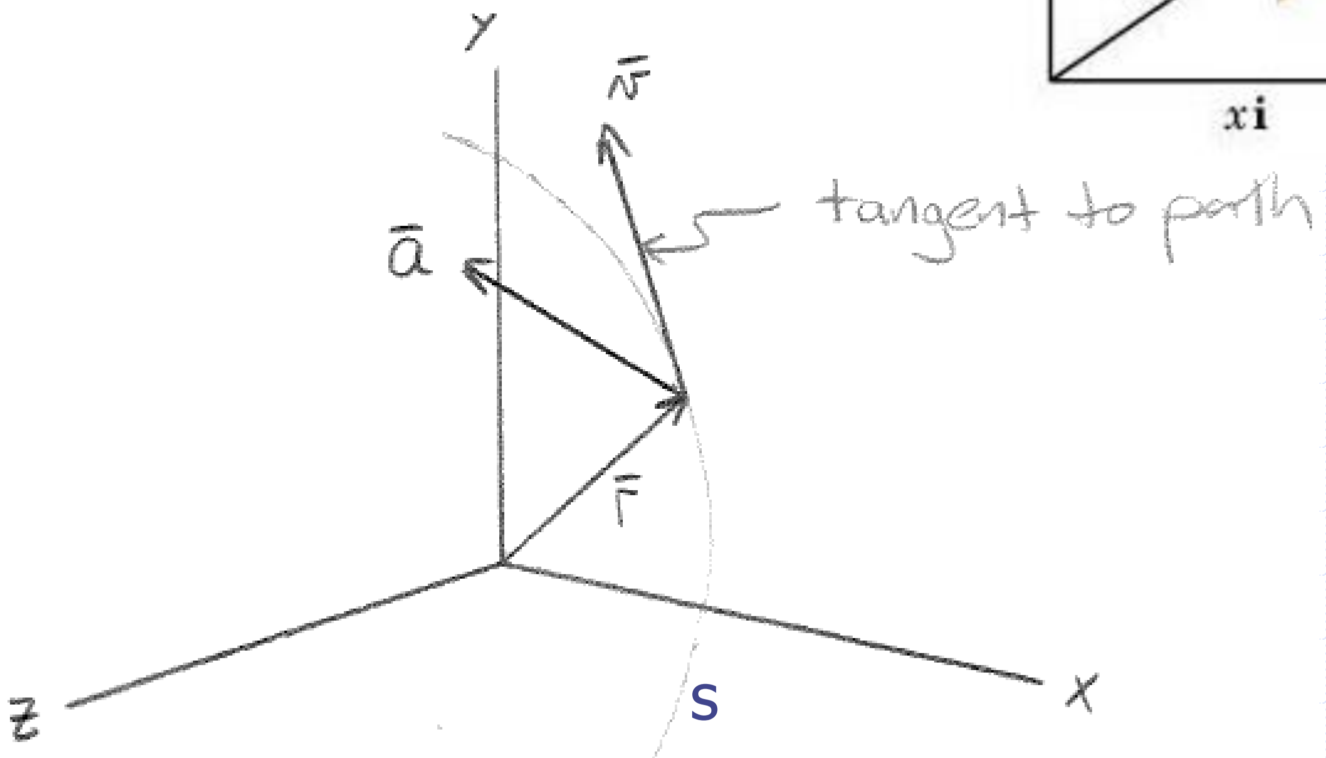
① $\Delta\bar{v}$ specifies dir. of Ave. Acc.

② As $\Delta t \to 0$, $\bar{a}$ becomes tangent to hodograph of motion, but NOT to particle path.



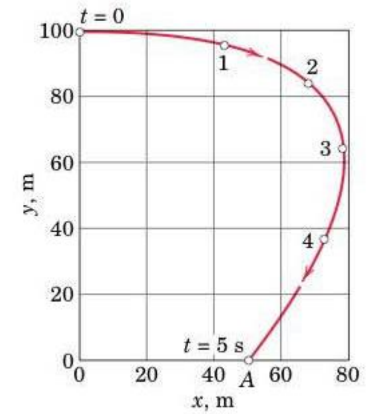
In summary:





Sample Problem 2/5

The curvilinear motion of a particle is defined by $v_x = 50 - 16t$ and $y = 100 - 4t^2$, where v_x is in meters per second, y is in meters, and t is in seconds. It is also known that $x = 0$ when $t = 0$. Plot the path of the particle and determine its velocity and acceleration when the position $y = 0$ is reached.



Solution. The x-coordinate is obtained by integrating the expression for v_x, and the x-component of the acceleration is obtained by differentiating v_x. Thus,

$$\left[\int dx = \int v_x\, dt\right] \qquad \int_0^x dx = \int_0^t (50 - 16t)\, dt \qquad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \qquad a_x = \frac{d}{dt}(50 - 16t) \qquad a_x = -16 \text{ m/s}^2$$

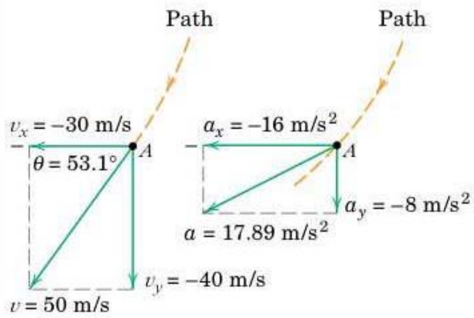
The y-components of velocity and acceleration are

$$[v_y = \dot{y}] \qquad v_y = \frac{d}{dt}(100 - 4t^2) \qquad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \qquad a_y = \frac{d}{dt}(-8t) \qquad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of x and y for various values of t and plot x against y to obtain the path as shown.

When $y = 0$, $0 = 100 - 4t^2$, so $t = 5$ s. For this value of the time, we have



$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where $y = 0$. Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

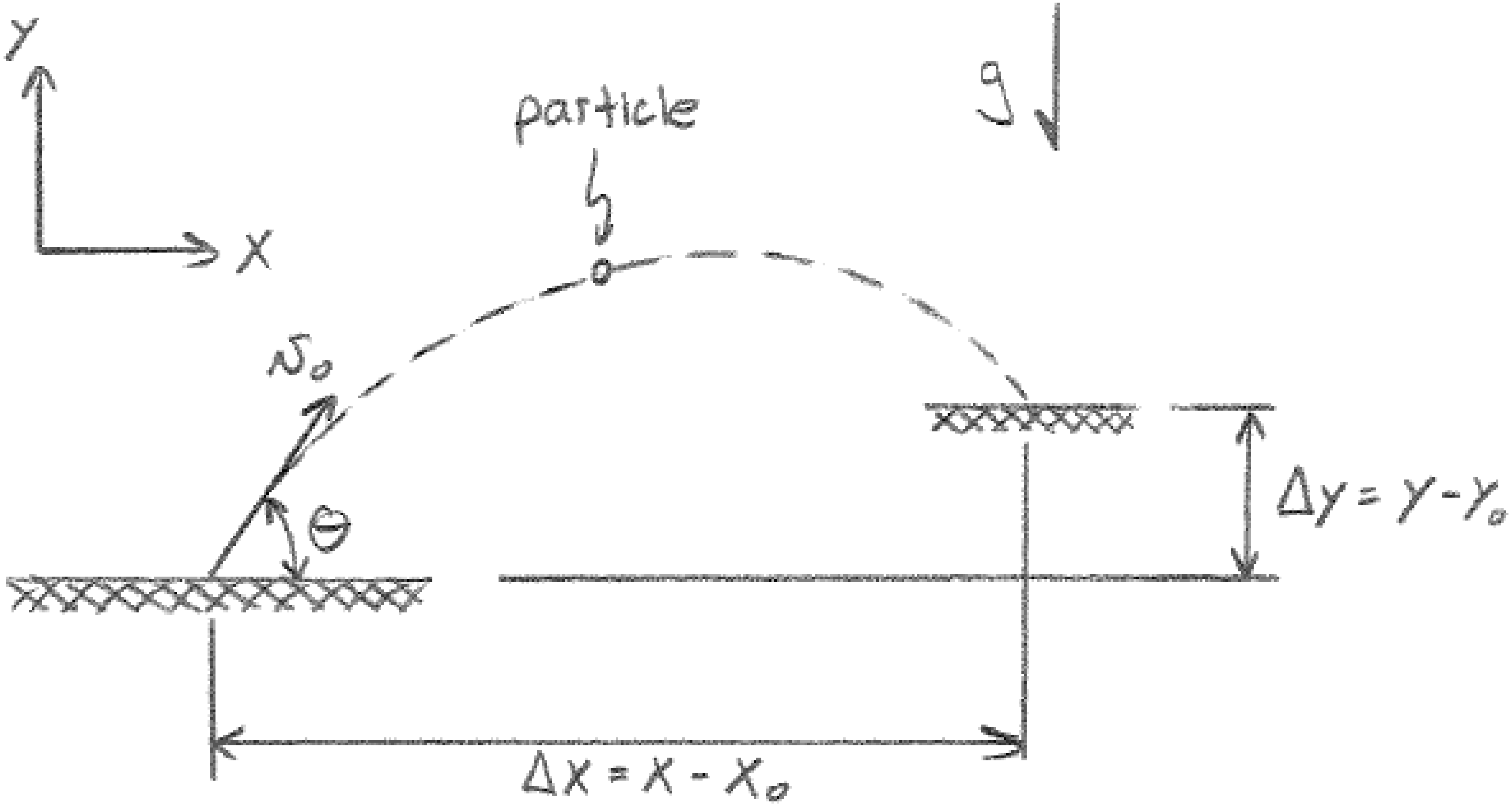
$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \qquad \textit{Ans.}$$

Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

2. Projectile Motion (*x*-*y*)

Projectile Motion

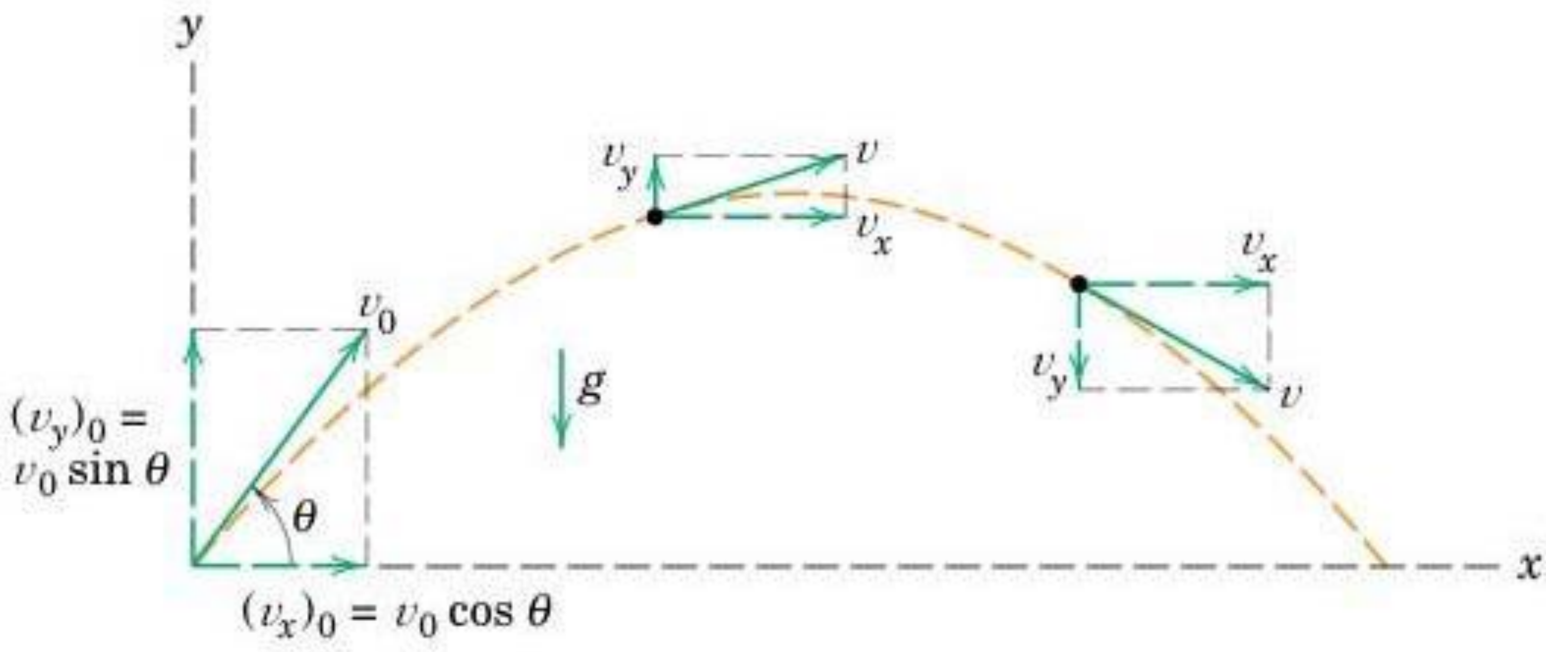


$a_z = 0 \iff$ motion remains in x-y plane

$a_y = -g \iff$ vertical motion is uniformly accelerated

$a_x = 0 \iff$ horizontal motion is uniform

Key: Analyze $\uparrow$ and $\longrightarrow$ motion separately.



$$a_x = 0 \qquad a_y = -g$$

$$v_x = (v_x)_0 \qquad v_y = (v_y)_0 - gt$$

$$x = x_0 + (v_x)_0 t \qquad y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2$$

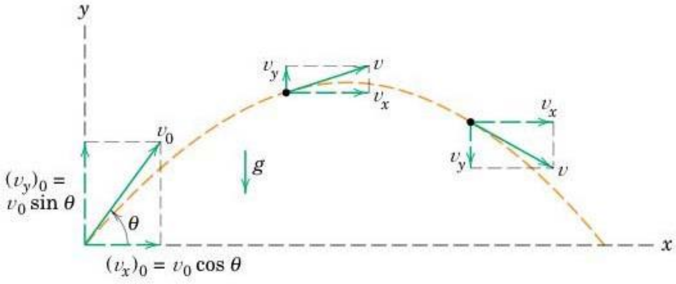
$$v_y{}^2 = (v_y)_0{}^2 - 2g(y - y_0)$$

Example 1 (See prev. diagram)

Find an expression that relates directly

$\Delta x, \Delta y$, v_0, θ

geometry — initial velocity



X-dynamics: $a_x = 0$, $v_{x0} = v_0 \cos\theta$

Invoke $x = x_0 + v_{x0} t$

$\Rightarrow \Delta x = v_0 \cos\theta \, t$

$$\Rightarrow t = \frac{\Delta x}{v_0 \cos\theta} \quad \text{——} \quad (*)$$

y-dynamics: $a_y = -g, \; v_{y0} = v_0 \sin\theta$

Invoke $y - y_0 = v_{y0} t + \frac{1}{2} a_y t^2$

$$\Rightarrow \Delta y = v_0 \sin\theta \, t - \frac{1}{2} g t^2 \quad \text{———} \quad (**)$$

$(*) \longrightarrow (**)$: $\Delta y = v_0 \sin\theta \cdot \frac{\Delta x}{v_0 \cos\theta} - \frac{g}{2} \cdot \frac{(\Delta x)^2}{v_0^2 \cos^2\theta}$

$$\therefore \quad \boxed{\Delta y = \Delta x \tan\theta - \frac{g(\Delta x)^2}{2 v_0^2 \cos^2\theta}} \quad \text{———} \quad (***)$$

(***) in a different form... substitute:

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

$$\boxed{(\tan\theta)^2 - \frac{2v_o^2}{g\Delta x}(\tan\theta) + \left[1 + \frac{2v_o^2\Delta y}{g(\Delta x)^2}\right] = 0}$$

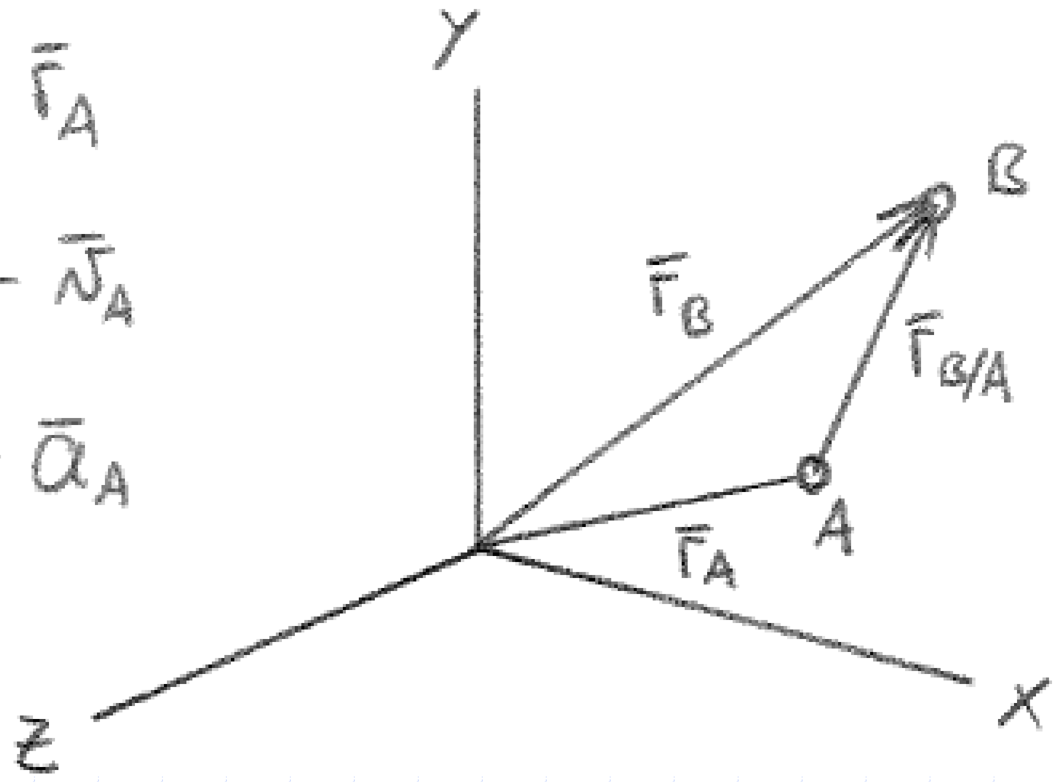
3. Relative Curvilinear Motion

Relative Curvilinear Motion

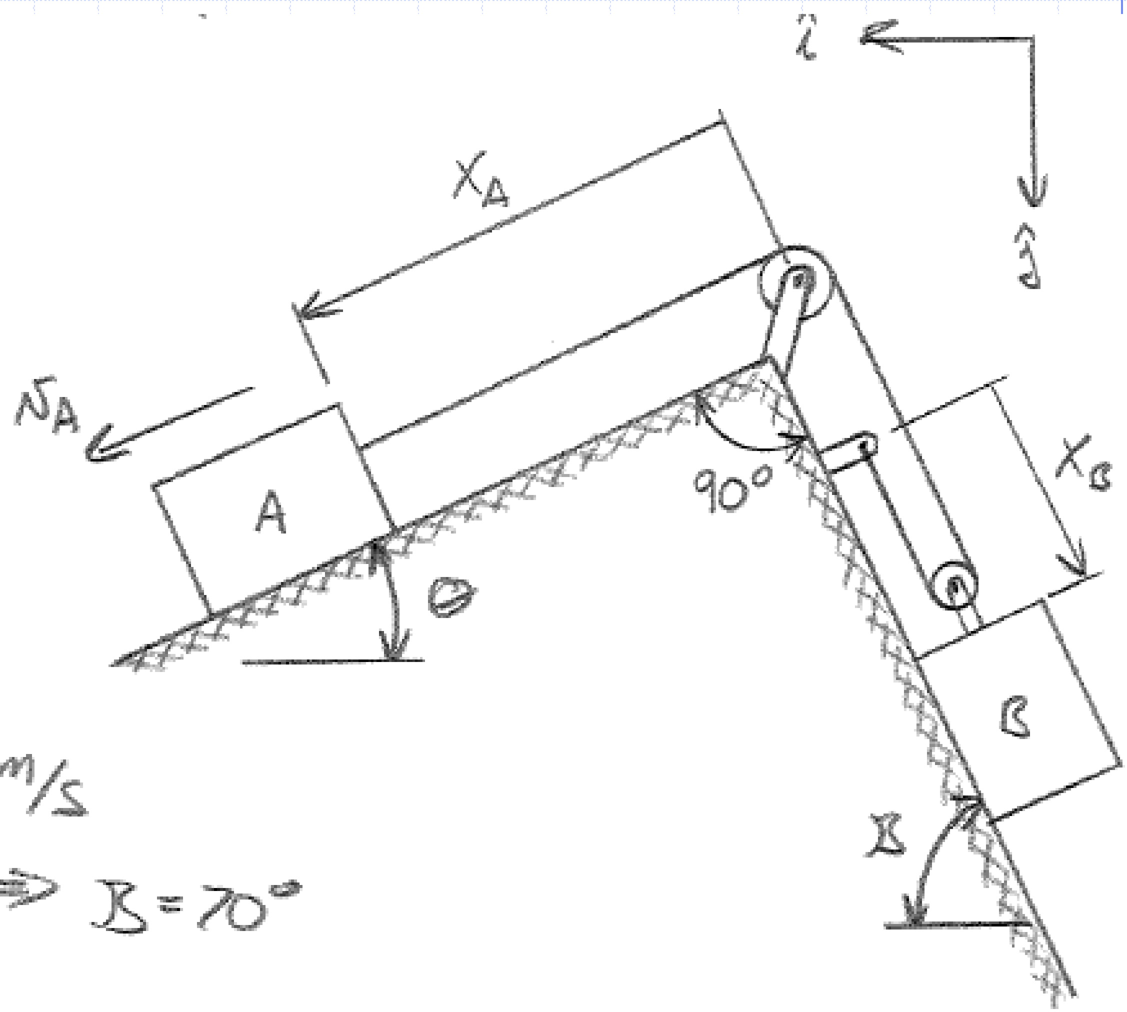
$$\bar{r}_{B/A} = \bar{r}_B - \bar{r}_A$$

$$\Rightarrow \bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

$$\Rightarrow \bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$$



Example 2



Data: $v_A = 80$ mm/s

$\theta = 20° \Rightarrow \beta = 70°$

Find: (A) $\bar{v}_B$

(B) $\bar{v}_{A/B}$

) Vectors!

PART A: Find $\bar{v}_B$

Kinematic Constraint: $2x_B + x_A = \text{const.}$

$$\Rightarrow \quad 2v_B + v_A = 0$$

$$\Rightarrow \quad v_B = -\frac{1}{2} v_A$$

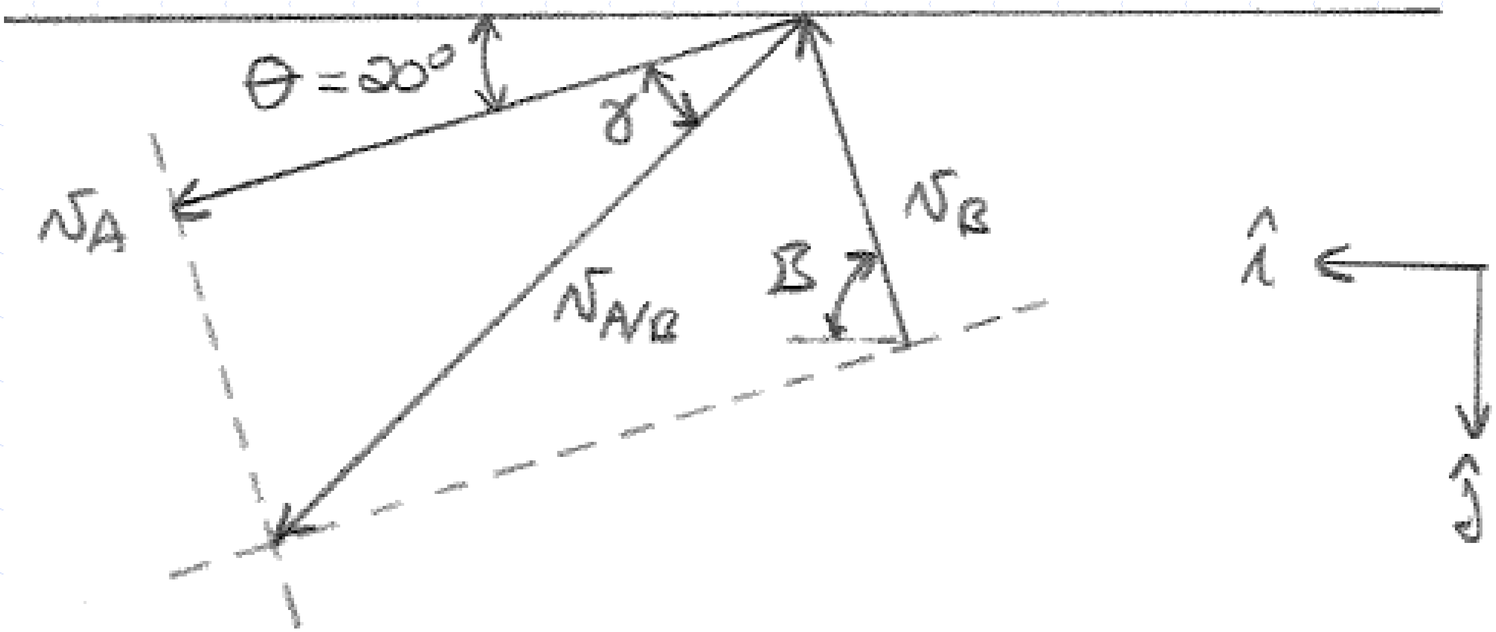
Thus

$$v_B = -\frac{1}{2}\left(80 \text{ mm/s}\right) = -40 \text{ mm/s}$$

$$\therefore \quad \bar{v}_B = 40 \text{ mm/s} \;\nwarrow\!\!\measuredangle\; 70^\circ$$

PART B: Find $\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B$

The geometry:



Two ways to solve:

① Geometrically — Here, easy! ... but won't be in general.

② Vectorially (rectangular components)

Method ②

Resolve

$$\bar{v}_A = v_A \cos\theta \,\hat{i} + v_A \sin\theta \,\hat{j}$$

$$\bar{v}_B = v_B \cos\beta \,\hat{i} - v_B \sin\beta \,\hat{j}$$

Thus $\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B$

$$= (v_A \cos\theta - v_B \cos\gamma)\hat{i} + (v_A \sin\theta + v_B \sin\gamma)\hat{j}$$

$$= (61.5 \text{ mm/s})\hat{i} + (64.9 \text{ mm/s})\hat{j}$$

$$|\vec{v}_{A/B}| = \sqrt{(61.5 \text{ mm/s})^2 + (64.9 \text{ mm/s})^2} = 89.4 \text{ mm/s}$$

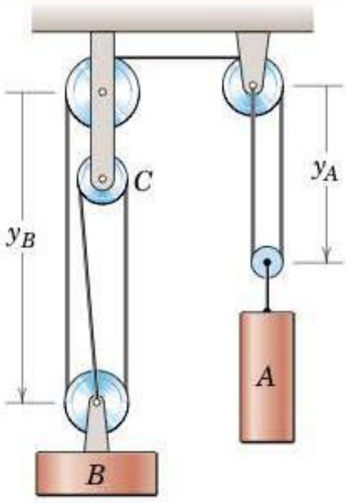
$$\gamma = \tan^{-1}\left(\frac{64.9 \text{ mm/s}}{61.5 \text{ mm/s}}\right) = 46.6^\circ$$

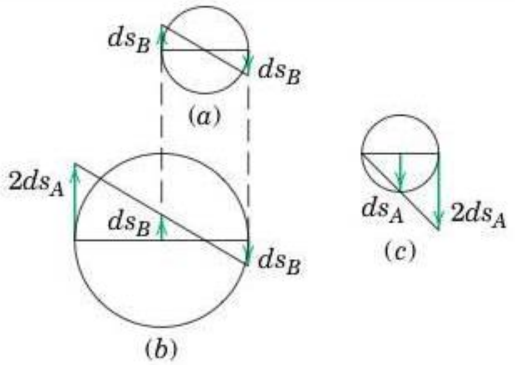
Error*

$$\therefore \quad \vec{v}_{A/B} = 89.4 \text{ mm/s} \swarrow \gamma = 66.6^\circ$$

Sample Problem 2/15

In the pulley configuration shown, cylinder A has a downward velocity of 0.3 m/s. Determine the velocity of B. Solve in two ways.





Solution (I). The centers of the pulleys at A and B are located by the coordinates y_A and y_B measured from fixed positions. The total constant length of cable in the pulley system is

$$L = 3y_B + 2y_A + \text{constants}$$

① where the constants account for the fixed lengths of cable in contact with the circumferences of the pulleys and the constant vertical separation between the two upper left-hand pulleys. Differentiation with time gives

$$0 = 3\dot{y}_B + 2\dot{y}_A$$

Substitution of $v_A = \dot{y}_A = 0.3$ m/s and $v_B = \dot{y}_B$ gives

② $$0 = 3(v_B) + 2(0.3) \qquad \text{or} \qquad v_B = -0.2 \text{ m/s} \qquad \textit{Ans.}$$

Solution (II). An enlarged diagram of the pulleys at A, B, and C is shown. During a differential movement ds_A of the center of pulley A, the left end of its horizontal diameter has no motion since it is attached to the fixed part of the cable. Therefore, the right-hand end has a movement of $2ds_A$ as shown. This movement is transmitted to the left-hand end of the horizontal diameter of the pulley at B. Further, from pulley C with its fixed center, we see that the displacements on each side are equal and opposite. Thus, for pulley B, the right-hand end of the diameter has a downward displacement equal to the upward displacement ds_B of its center. By inspection of the geometry, we conclude that

$$2ds_A = 3ds_B \qquad \text{or} \qquad ds_B = \tfrac{2}{3}ds_A$$

Dividing by dt gives

$$|v_B| = \tfrac{2}{3}v_A = \tfrac{2}{3}(0.3) = 0.2 \text{ m/s (upward)} \qquad \textit{Ans.}$$

Helpful Hints

① We neglect the small angularity of the cables between B and C.

② The negative sign indicates that the velocity of B is *upward*.

Sample Problem 2/16

The tractor A is used to hoist the bale B with the pulley arrangement shown. If A has a forward velocity v_A, determine an expression for the upward velocity v_B of the bale in terms of x.

Solution. We designate the position of the tractor by the coordinate x and the position of the bale by the coordinate y, both measured from a fixed reference. The total constant length of the cable is

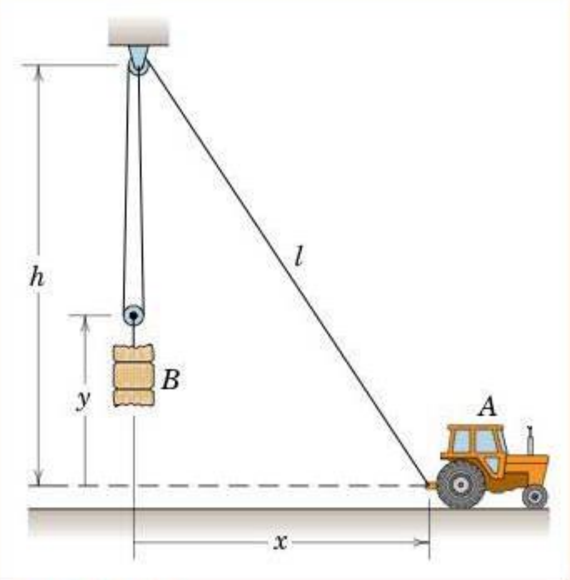
$$L = 2(h - y) + l = 2(h - y) + \sqrt{h^2 + x^2}$$

① Differentiation with time yields

$$0 = -2\dot{y} + \frac{x\dot{x}}{\sqrt{h^2 + x^2}}$$

Substituting $v_A = \dot{x}$ and $v_B = \dot{y}$ gives

$$v_B = \frac{1}{2}\frac{xv_A}{\sqrt{h^2 + x^2}} \qquad \textit{Ans.}$$

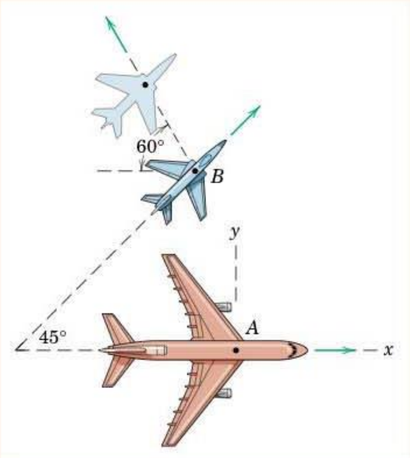


Helpful Hint

① Differentiation of the relation for a right triangle occurs frequently in mechanics.

Sample Problem 2/13

Passengers in the jet transport A flying east at a speed of 800 km/h observe a second jet plane B that passes under the transport in horizontal flight. Although the nose of B is pointed in the 45° northeast direction, plane B appears to the passengers in A to be moving away from the transport at the 60° angle as shown. Determine the true velocity of B.



Solution. The moving reference axes x-y are attached to A, from which the relative observations are made. We write, therefore,

① $$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Next we identify the knowns and unknowns. The velocity $\mathbf{v}_A$ is given in both magnitude and direction. The 60° direction of $\mathbf{v}_{B/A}$, the velocity which B appears to
② have to the moving observers in A, is known, and the true velocity of B is in the 45° direction in which it is heading. The two remaining unknowns are the magni-
③ tudes of $\mathbf{v}_B$ and $\mathbf{v}_{B/A}$. We may solve the vector equation in any one of three ways.

(I) Graphical. We start the vector sum at some point P by drawing $\mathbf{v}_A$ to a convenient scale and then construct a line through the tip of $\mathbf{v}_A$ with the known direction of $\mathbf{v}_{B/A}$. The known direction of $\mathbf{v}_B$ is then drawn through P, and the intersection C yields the unique solution enabling us to complete the vector triangle and scale off the unknown magnitudes, which are found to be

$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \qquad \textit{Ans.}$$

(II) Trigonometric. A sketch of the vector triangle is made to reveal the trigonometry, which gives

④ $$\frac{v_B}{\sin 60^\circ} = \frac{v_A}{\sin 75^\circ} \qquad v_B = 800\frac{\sin 60^\circ}{\sin 75^\circ} = 717 \text{ km/h} \qquad \textit{Ans.}$$

(III) Vector Algebra. Using unit vectors $\mathbf{i}$ and $\mathbf{j}$, we express the velocities in vector form as

$$\mathbf{v}_A = 800\mathbf{i} \text{ km/h} \qquad \mathbf{v}_B = (v_B \cos 45^\circ)\mathbf{i} + (v_B \sin 45^\circ)\mathbf{j}$$

$$\mathbf{v}_{B/A} = (v_{B/A} \cos 60^\circ)(-\mathbf{i}) + (v_{B/A} \sin 60^\circ)\mathbf{j}$$

Substituting these relations into the relative-velocity equation and solving separately for the $\mathbf{i}$ and $\mathbf{j}$ terms give

$$(\mathbf{i}\text{-terms}) \qquad v_B \cos 45^\circ = 800 - v_{B/A} \cos 60^\circ$$

$$(\mathbf{j}\text{-terms}) \qquad v_B \sin 45^\circ = v_{B/A} \sin 60^\circ$$

⑤ Solving simultaneously yields the unknown velocity magnitudes

$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \qquad \textit{Ans.}$$

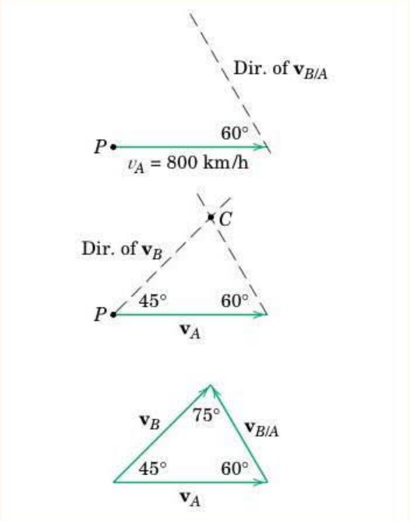
It is worth noting the solution of this problem from the viewpoint of an observer in B. With reference axes attached to B, we would write $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$. The apparent velocity of A as observed by B is then $\mathbf{v}_{A/B}$, which is the negative of $\mathbf{v}_{B/A}$.

Helpful Hints

① We treat each airplane as a particle.

② We assume no side slip due to cross wind.

③ Students should become familiar with all three solutions.

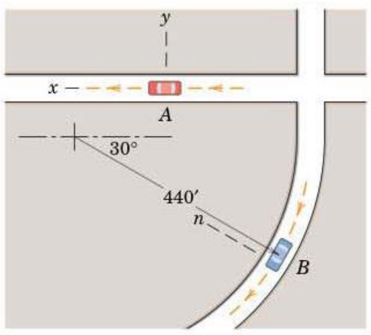


④ We must be prepared to recognize the appropriate trigonometric relation, which here is the law of sines.

⑤ We can see that the graphical or trigonometric solution is shorter than the vector algebra solution in this particular problem.

Sample Problem 2/14

Car A is accelerating in the direction of its motion at the rate of 3 ft/sec^2. Car B is rounding a curve of 440-ft radius at a constant speed of 30 mi/hr. Determine the velocity and acceleration which car B appears to have to an observer in car A if car A has reached a speed of 45 mi/hr for the positions represented.



Solution. We choose nonrotating reference axes attached to car A since the motion of B with respect to A is desired.

Velocity. The relative-velocity equation is

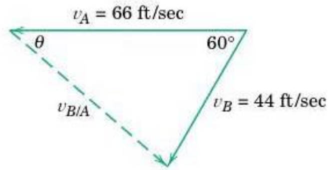
$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

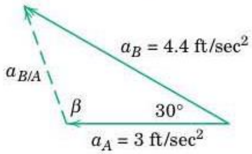
and the velocities of A and B for the position considered have the magnitudes

$$v_A = 45\frac{5280}{60^2} = 45\frac{44}{30} = 66 \text{ ft/sec} \qquad v_B = 30\frac{44}{30} = 44 \text{ ft/sec}$$

The triangle of velocity vectors is drawn in the sequence required by the equation, and application of the law of cosines and the law of sines gives

① $$v_{B/A} = 58.2 \text{ ft/sec} \qquad \theta = 40.9° \qquad \textit{Ans.}$$





Acceleration. The relative-acceleration equation is

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

The acceleration of A is given, and the acceleration of B is normal to the curve in the n-direction and has the magnitude

$$[a_n = v^2/\rho] \qquad a_B = (44)^2/440 = 4.4 \text{ ft/sec}^2$$

The triangle of acceleration vectors is drawn in the sequence required by the equation as illustrated. Solving for the x- and y-components of $\mathbf{a}_{B/A}$ gives us

$$(a_{B/A})_x = 4.4 \cos 30° - 3 = 0.810 \text{ ft/sec}^2$$

$$(a_{B/A})_y = 4.4 \sin 30° = 2.2 \text{ ft/sec}^2$$

from which $a_{B/A} = \sqrt{(0.810)^2 + (2.2)^2} = 2.34$ ft/sec^2 *Ans.*

The direction of $\mathbf{a}_{B/A}$ may be specified by the angle β which, by the law of sines, becomes

② $$\frac{4.4}{\sin\beta} = \frac{2.34}{\sin 30°} \qquad \beta = \sin^{-1}\left(\frac{4.4}{2.34}0.5\right) = 110.2° \qquad \textit{Ans.}$$

Helpful Hints

① Alternatively, we could use either a graphical or a vector algebraic solution.

② Be careful to choose between the two values 69.8° and 180 − 69.8 = 110.2°.

Suggestion: To gain familiarity with the manipulation of vector equations, it is suggested that the student rewrite the relative-motion equations in the form $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$ and $\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$ and redraw the vector polygons to conform with these alternative relations.

Caution: So far we are only prepared to handle motion relative to *nonrotating* axes. If we had attached the reference axes rigidly to car B, they would rotate with the car, and we would find that the velocity and acceleration terms relative to the rotating axes are *not* the negative of those measured from the nonrotating axes moving with A. Rotating axes are treated in Art. 5/7.

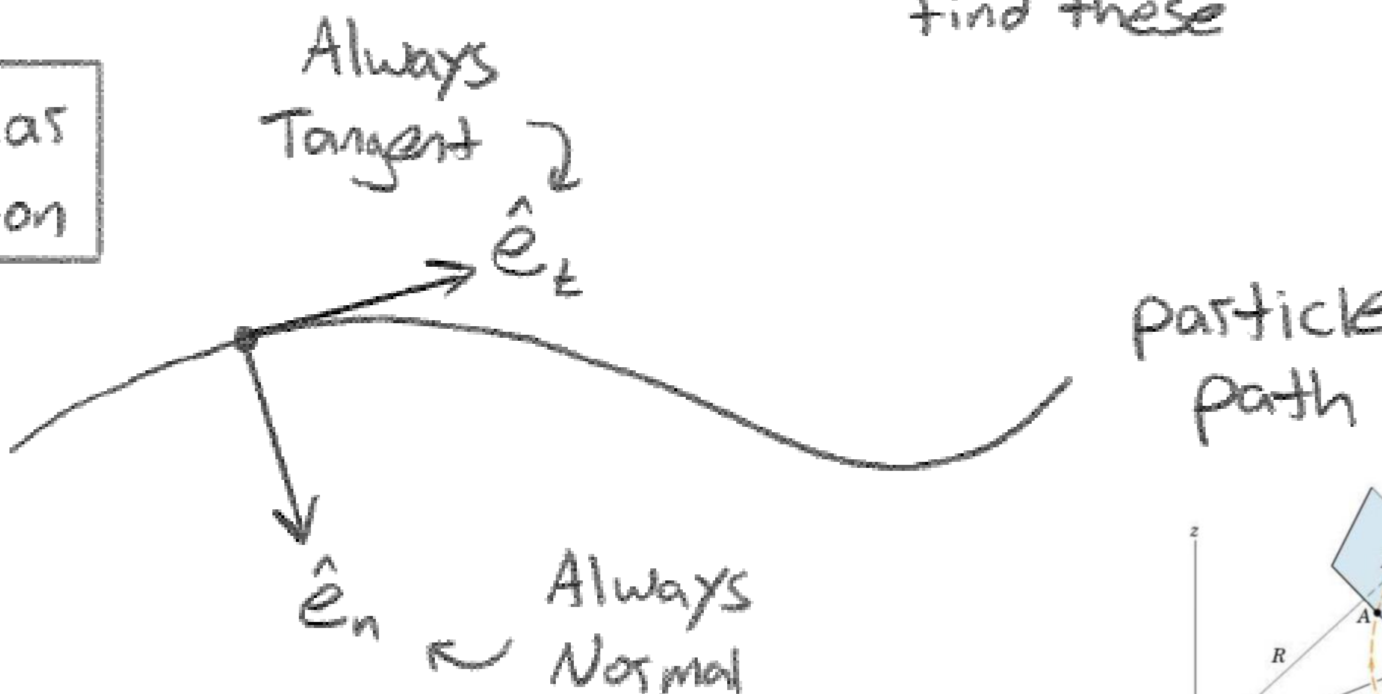
4. Normal & Tangential Components (*t*-*n*)

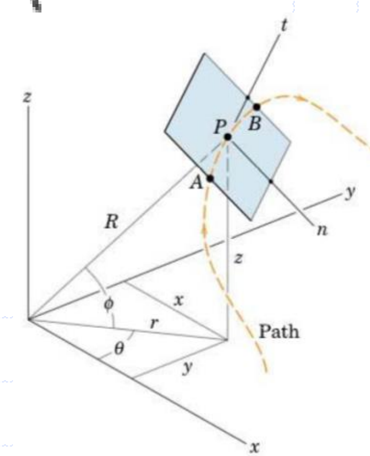
Normal & Tangential Components

The goal: Resolve $\bar{a} = a_t \hat{e}_t + a_n \hat{e}_n$

↑ Find these ↑

Planar Motion





Velocity is always tangent to path.

Then $\therefore \quad \bar{v} = v\hat{e}_t$

Needs work

$$\bar{a} = \frac{d(v\hat{e}_t)}{dt} = \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \qquad (*)$$

v (speed)

$$\frac{d\hat{e}_t}{dt} = \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt} \qquad (A)$$

θ to be defined...

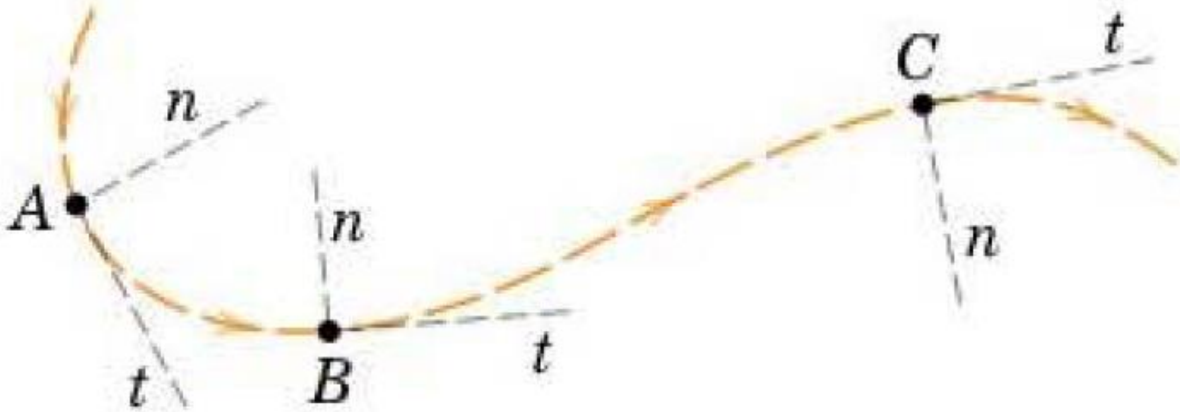
$$\bar{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

Change in speed — Change in direction

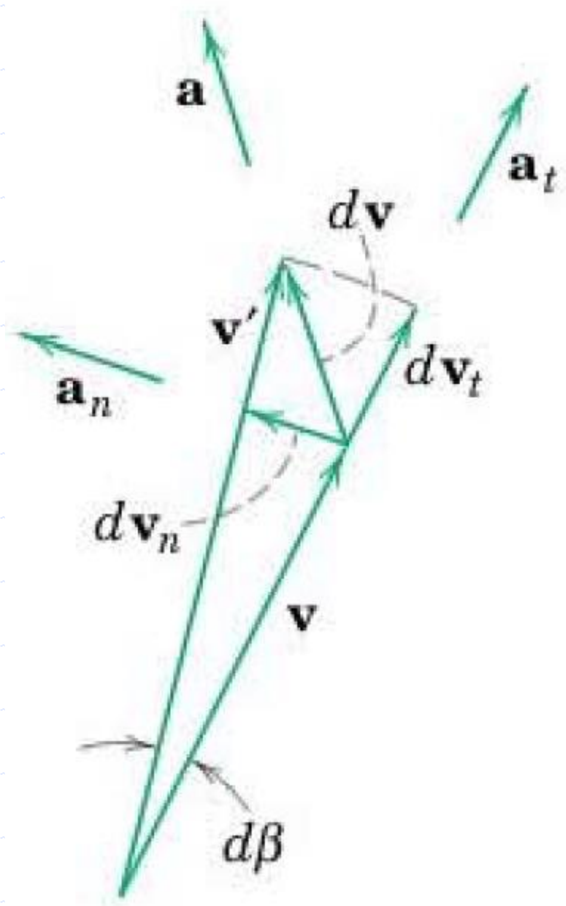
Here,

$$a_t = \frac{dv}{dt}$$ tangential acc.

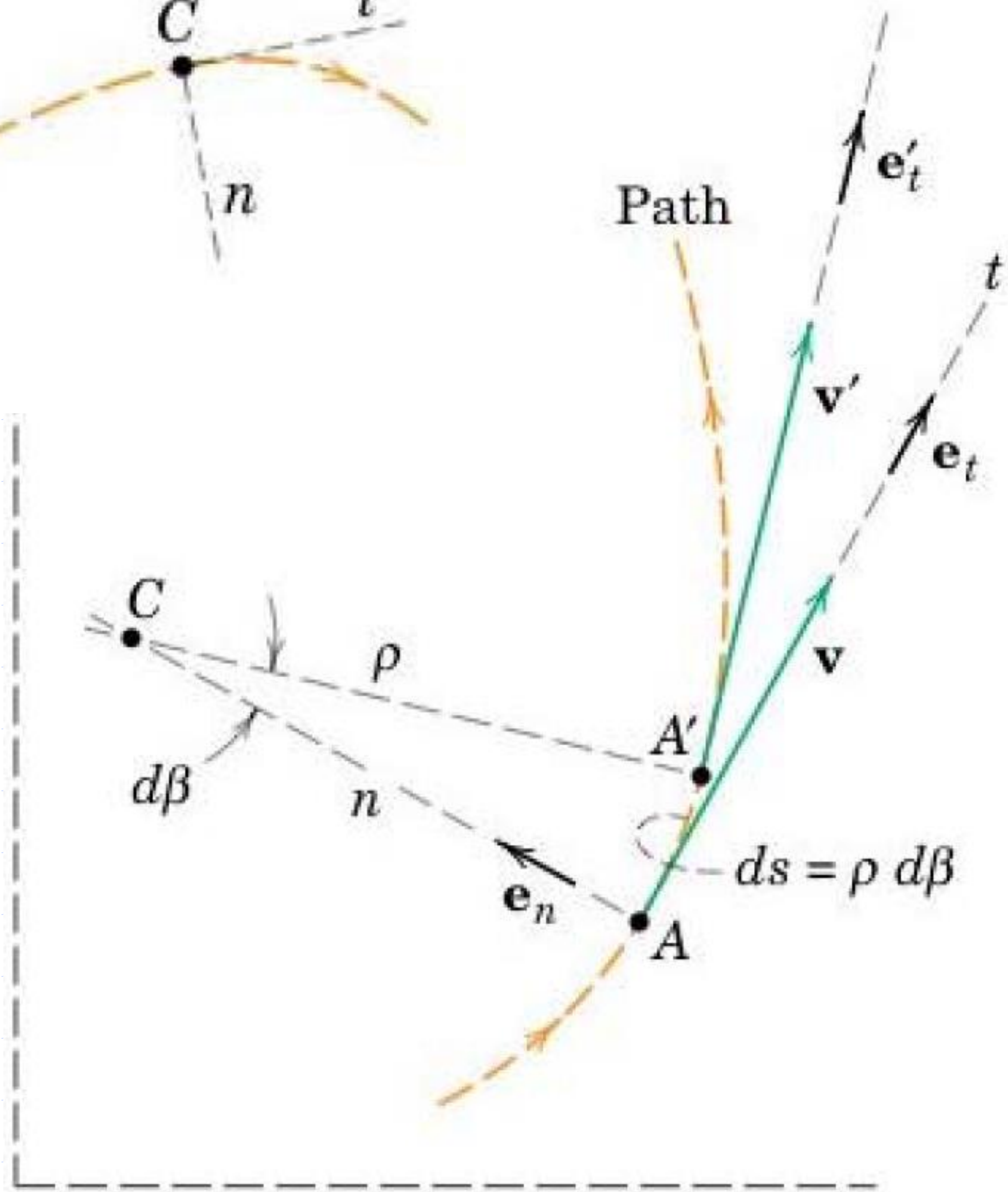
$$a_n = \frac{v^2}{\rho}$$ normal acc.



A
n
t
B
n
t
C
t
n



a
$\mathbf{a}_t$
$d\mathbf{v}$
$\mathbf{v}'$
$d\mathbf{v}_t$
$\mathbf{a}_n$
$d\mathbf{v}_n$
$\mathbf{v}$
$d\beta$



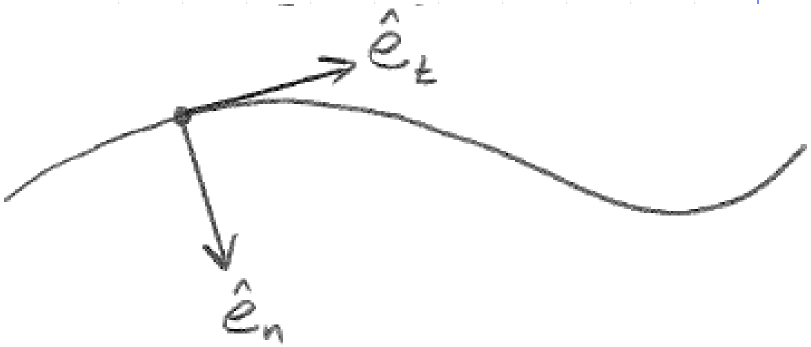
$\mathbf{e}_t'$
Path
t
$\mathbf{v}'$
$\mathbf{e}_t$
C
ρ
$\mathbf{v}$
A′
$d\beta$
n
$ds = \rho\, d\beta$
$\mathbf{e}_n$
A

- Velocity is tangent to the path

$$\mathbf{v} = v\,\hat{\mathbf{e}}_t$$

Differentiate Velocity

$$\mathbf{a} = \frac{d}{dt}(v\hat{e}_t) = \frac{dv}{dt}\,\hat{e}_t + v\,\frac{d\hat{e}_t}{dt}$$



Derivative of Tangent Vector

Chain rule:

$$\frac{d\hat{e}_t}{dt} = \frac{d\hat{e}_t}{d\theta}\cdot\frac{d\theta}{ds}\cdot\frac{ds}{dt}$$

- $\frac{d\hat{e}_t}{d\theta} = \hat{e}_n$
- $\frac{d\theta}{ds} = 1/\rho$ (curvature)
- $\frac{ds}{dt} = v$

So:

$$\frac{d\hat{e}_t}{dt} = \frac{v}{\rho}\,\hat{e}_n$$

»

$$\mathbf{a} = \frac{dv}{dt}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$$

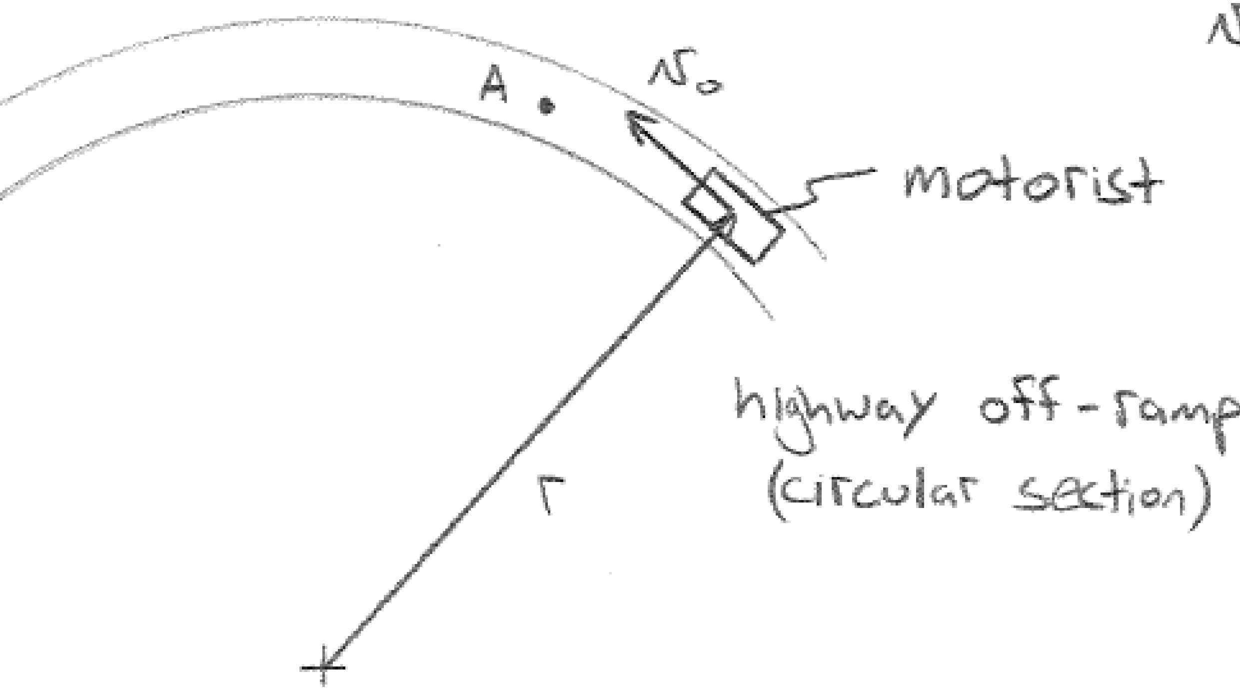
- Tangential arrow along path = $\frac{dv}{dt}\hat{e}_t$.
- Normal arrow pointing inward = $\frac{v^2}{\rho}\hat{e}_n$.
- Combined acceleration vector shown as sum of both.

Example 3

Data: $r = 350\,m$ (const.)

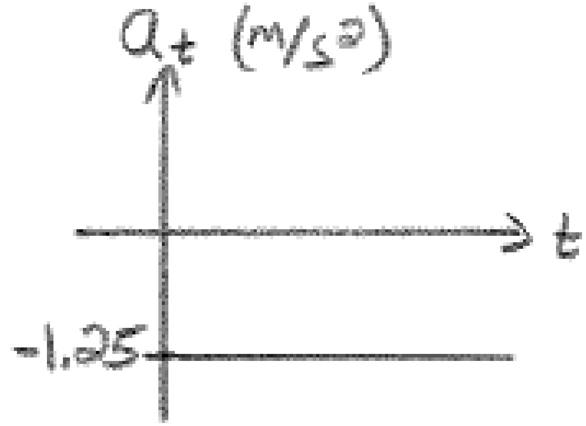
$v_0 = 20\ m/s$

↳ initially, before reaching A



Brakes applied @ A causing vehicle to slow down at a rate of $1.25\ m/s^2$.

Determine $a = |\bar{a}|$

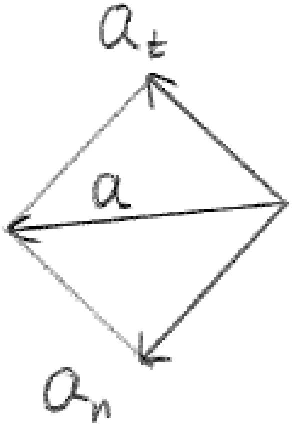


Ⓐ immediately after brakes are applied

Ⓑ 4s later

PART A:

$$a = \sqrt{a_t^2 + a_n^2}$$



where $a_t = -1.25 \ m/s^2$

$$a_n = \frac{v_0^2}{r} = \frac{(20 \ m/s)^2}{(350m)} = 1.143 \ m/s^2$$

Thus $a = \sqrt{(-1.25\ m/s^2)^2 + (1.143\ m/s^2)^2}$

$= 1.694\ m/s^2$

$\therefore \quad a = |\bar{a}| = 1.69\ m/s^2$

PART B: a_t same, $a_n = v^2/r$ diff.

$$a_t = \frac{dv}{dt} \Rightarrow dv = a_t\, dt$$

(↑ a_t: constant)

$$\Rightarrow v - v_0 = a_t t$$

Thus $v(t=4s) = (20\ m/s) + (-1.25\ m/s^2)(4s)$

$= 15.0\ m/s$

so $a_n = \frac{(15.0\ m/s)^2}{(350m)} = 0.643\ m/s^2$

and $a = \sqrt{(-1.25\ m/s^2)^2 + (0.643\ m/s^2)^2}$

$= 1.406\ m/s^2$

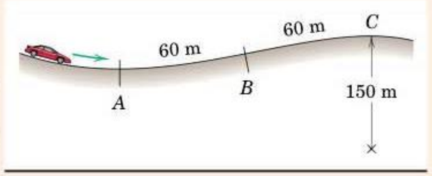
$\therefore\ a = |\bar{a}| = 1.41\ m/s^2$

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

$$a_t = \frac{dv}{dt} = v\frac{dv}{ds} \implies a_t\,ds = v\,dv$$

Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom A of the dip and 50 km/h at the top C of the hump, which is 120 m along the road from A. If the passengers experience a total acceleration of 3 m/s^2 at A and if the radius of curvature of the hump at C is 150 m, calculate (*a*) the radius of curvature ρ at A, (*b*) the acceleration at the inflection point B, and (*c*) the total acceleration at C.



Solution. The dimensions of the car are small compared with those of the ① path, so we will treat the car as a particle. The velocities are

$$v_A = \left(100\,\frac{\text{km}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)\left(1000\,\frac{\text{m}}{\text{km}}\right) = 27.8 \text{ m/s}$$

$$v_C = 50\,\frac{1000}{3600} = 13.89 \text{ m/s}$$

We find the constant deceleration along the path from

$$\left[\int v\,dv = \int a_t\,ds\right] \qquad \int_{v_A}^{v_C} v\,dv = a_t\int_0^s ds$$

$$a_t = \frac{1}{2s}(v_C{}^2 - v_A{}^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2$$

(a) Condition at A. With the total acceleration given and a_t determined, we can easily compute a_n and hence ρ from

$$[a^2 = a_n{}^2 + a_t{}^2] \qquad a_n{}^2 = 3^2 - (2.41)^2 = 3.19 \qquad a_n = 1.785 \text{ m/s}^2$$

$$[a_n = v^2/\rho] \qquad \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m} \qquad \textit{Ans.}$$

(b) Condition at B. Since the radius of curvature is infinite at the inflection point, $a_n = 0$ and

$$a = a_t = -2.41 \text{ m/s}^2 \qquad \textit{Ans.}$$

(c) Condition at C. The normal acceleration becomes

$$[a_n = v^2/\rho] \qquad a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2$$

With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ in the n- and t-directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t \text{ m/s}^2$$

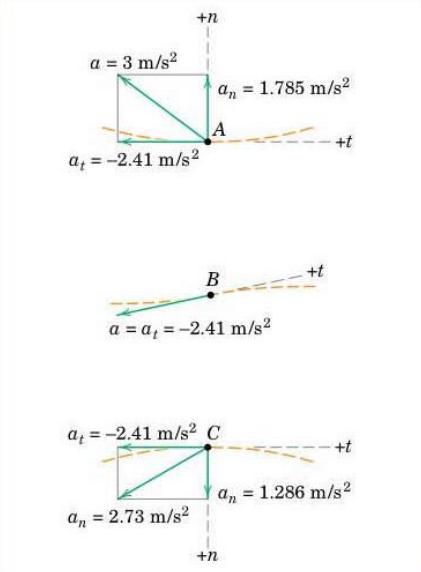
where the magnitude of $\mathbf{a}$ is

$$\left[a = \sqrt{a_n{}^2 + a_t{}^2}\right] \qquad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2 \qquad \textit{Ans.}$$

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

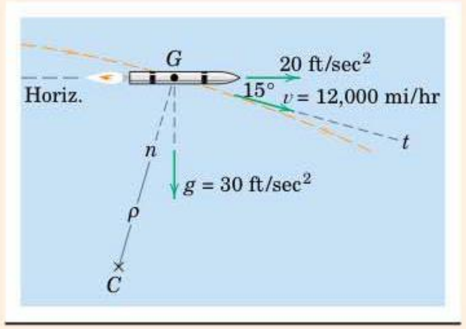
Helpful Hint

① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.



Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec², and the downward acceleration component is the acceleration due to gravity at that altitude, which is $g = 30$ ft/sec². At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is 12,000 mi/hr. For this position determine (*a*) the radius of curvature of the flight trajectory, (*b*) the rate at which the speed v is increasing, (*c*) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C, and (*d*) the vector expression for the total acceleration **a** of the rocket.



Solution. We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use n- and t-coordinates to describe the motion of G. The n- and t-components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their n- and t-components and then combining. From the figure we get ①

$$a_n = 30 \cos 15° - 20 \sin 15° = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15° + 20 \cos 15° = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

② $[a_n = v^2/\rho]$ $\qquad \rho = \dfrac{v^2}{a_n} = \dfrac{[(12{,}000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft}$ *Ans.*

(b) The rate at which v is increasing is simply the t-component of acceleration.

$[\dot{v} = a_t]$ $\qquad \dot{v} = 27.1 \text{ ft/sec}^2$ *Ans.*

(c) The angular rate $\dot{\beta}$ of line GC depends on v and ρ and is given by

$[v = \rho\dot{\beta}]$ $\qquad \dot{\beta} = v/\rho = \dfrac{12{,}000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec}$ *Ans.*

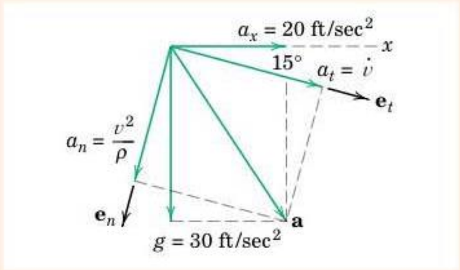
(d) With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ for the n- and t-directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2$$ *Ans.*

Helpful Hints

① Alternatively, we could find the resultant acceleration and then resolve it into n- and t-components.

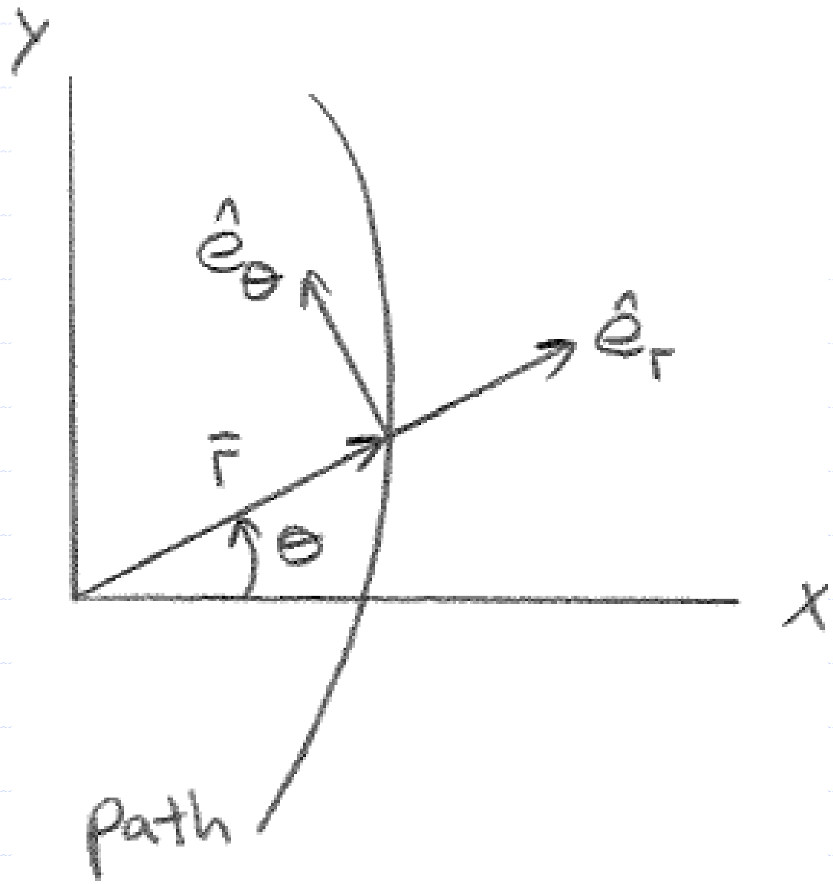
② To convert from mi/hr to ft/sec, multiply by $\dfrac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \dfrac{44 \text{ ft/sec}}{30 \text{ mi/hr}}$ which is easily remembered, as 30 mi/hr is the same as 44 ft/sec.



5. Radial & Transverse Components (r-θ)

Radial & Transverse Components

(aka Polar Coordinates)



$$\bar{v} = (\quad)\hat{e}_r + (\quad)\hat{e}_\theta$$

$$\bar{a} = (\quad)\hat{e}_r + (\quad)\hat{e}_\theta$$

Here, $\bar{r} = r\hat{e}_r$

So $\bar{v} = \dfrac{d\bar{r}}{dt} = \dfrac{d(r\hat{e}_r)}{dt} = \dfrac{dr}{dt}\hat{e}_r + r\dfrac{d\hat{e}_r}{dt}$

$\dfrac{dr}{dt} \rightarrow \dot{r}$

Dissect $\dfrac{d\hat{e}_r}{dt} = \dfrac{d\hat{e}_r}{d\theta}\dfrac{d\theta}{dt} = \dot{\theta}\hat{e}_\theta$

$\dfrac{d\theta}{dt} \rightarrow \dot{\theta}$

$\dfrac{d\hat{e}_r}{d\theta}$: can show $= \hat{e}_\theta$

$\therefore$

$$\boxed{\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} \quad \left(\frac{d\hat{e}_r}{dt} \rightarrow \dot{\theta}\hat{e}_\theta\right)$$

$$+ \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

Dissect $\frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{e}_r$ $\quad \left(\frac{d\theta}{dt} \rightarrow \dot{\theta}\right)$

($\frac{d\hat{e}_\theta}{d\theta}$) can show $= -\hat{e}_r$

Plug it all in ...

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

radial acc. + transverse acc.

1) Start from the polar velocity

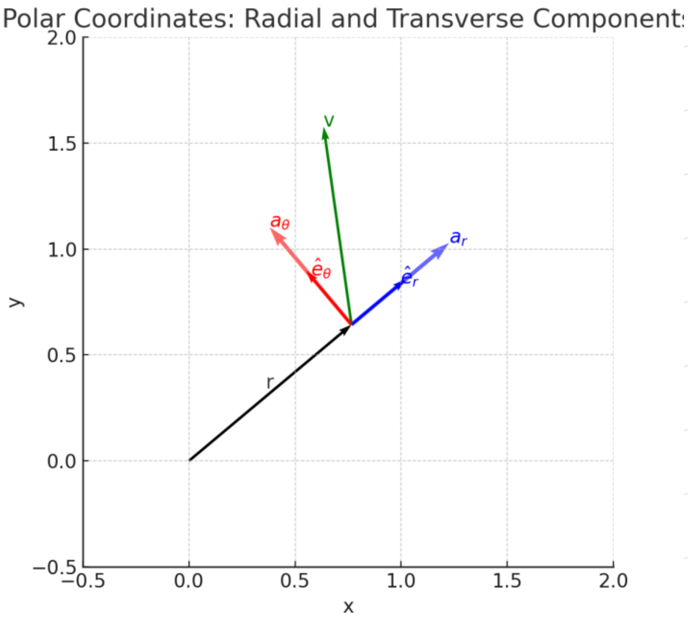
$$\mathbf{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\,\dot{\theta}\,\hat{\mathbf{e}}_\theta$$

where $\hat{\mathbf{e}}_r$ points outward from the origin and $\hat{\mathbf{e}}_\theta$ is perpendicular to $\hat{\mathbf{e}}_r$ in the direction of increasing θ.

2) Time derivatives of the unit vectors

Using $\hat{\mathbf{e}}_r = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ and
$\hat{\mathbf{e}}_\theta = -(\sin\theta)\,\hat{\mathbf{i}} + (\cos\theta)\,\hat{\mathbf{j}}$,
differentiate with respect to t:

$$\dot{\hat{\mathbf{e}}}_r = \dot{\theta}\,\hat{\mathbf{e}}_\theta, \qquad \dot{\hat{\mathbf{e}}}_\theta = -\dot{\theta}\,\hat{\mathbf{e}}_r.$$



3) Differentiate the velocity (product rule)

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r}\,\hat{\mathbf{e}}_r) + \frac{d}{dt}(r\,\dot{\theta}\,\hat{\mathbf{e}}_\theta).$$

Work each term:

$$\frac{d}{dt}(\dot{r}\,\hat{\mathbf{e}}_r) = \ddot{r}\,\hat{\mathbf{e}}_r + \dot{r}\,\dot{\hat{\mathbf{e}}}_r = \ddot{r}\,\hat{\mathbf{e}}_r + \dot{r}\,(\dot{\theta}\,\hat{\mathbf{e}}_\theta),$$

$$\frac{d}{dt}(r\,\dot{\theta}\,\hat{\mathbf{e}}_\theta) = (\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\hat{\mathbf{e}}_\theta + r\,\dot{\theta}\,\dot{\hat{\mathbf{e}}}_\theta = (\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\hat{\mathbf{e}}_\theta + r\,\dot{\theta}\,(-\dot{\theta}\,\hat{\mathbf{e}}_r).$$

4) Collect $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ parts

Radial:

$$\hat{\mathbf{e}}_r: \quad \ddot{r} - r\,\dot{\theta}^2.$$

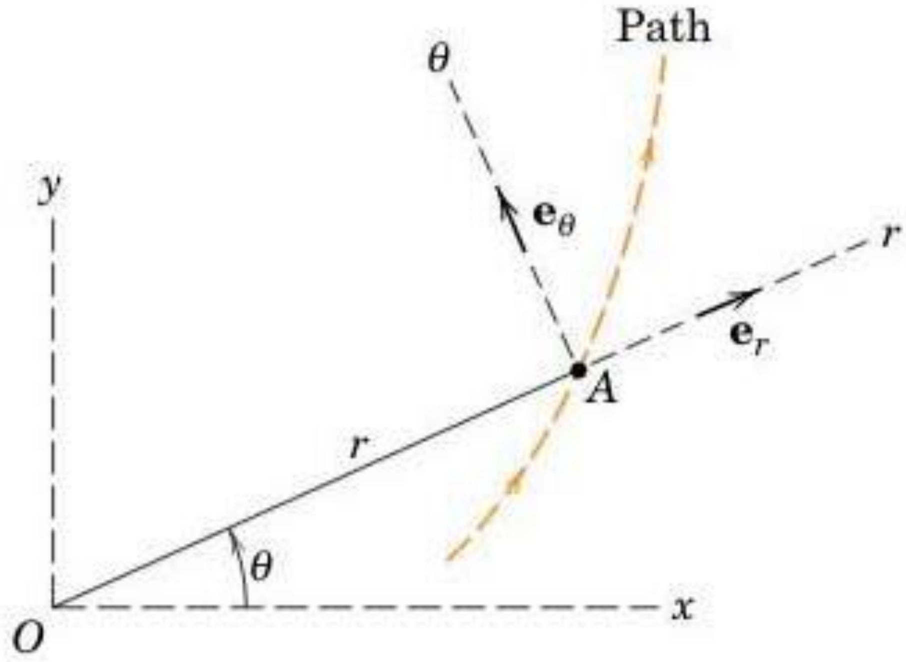
Transverse:

$$\hat{\mathbf{e}}_\theta: \quad \dot{r}\,\dot{\theta} + (\dot{r}\,\dot{\theta} + r\,\ddot{\theta}) = 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}.$$

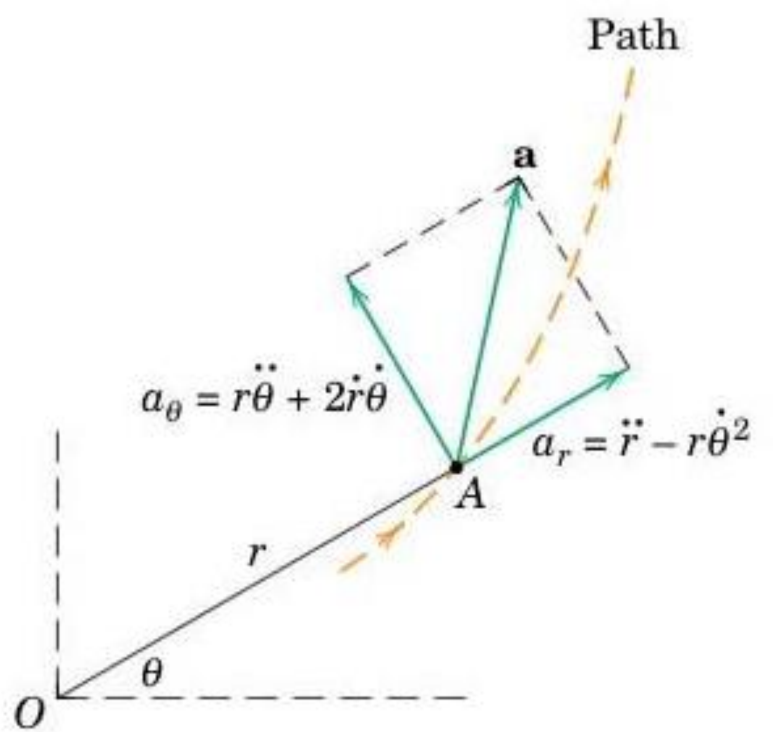
5) Final polar acceleration

$$\mathbf{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{\mathbf{e}}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{\mathbf{e}}_\theta$$

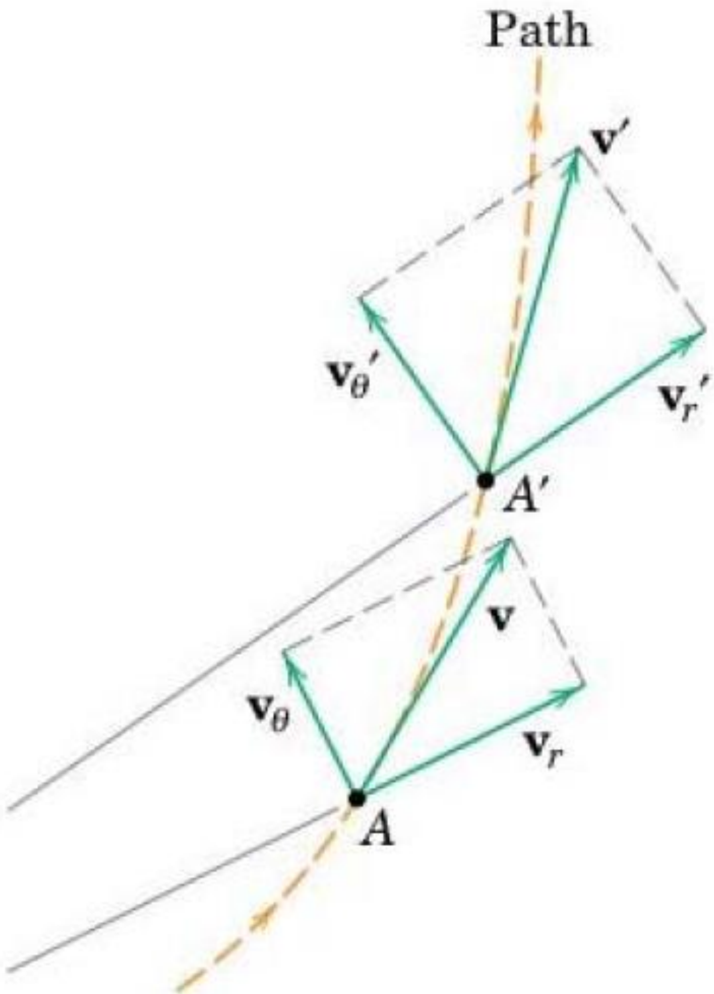
radial acc. + transverse acc.



Path
θ
y
$\mathbf{e}_\theta$
r
$\mathbf{e}_r$
A
r
θ
O
x

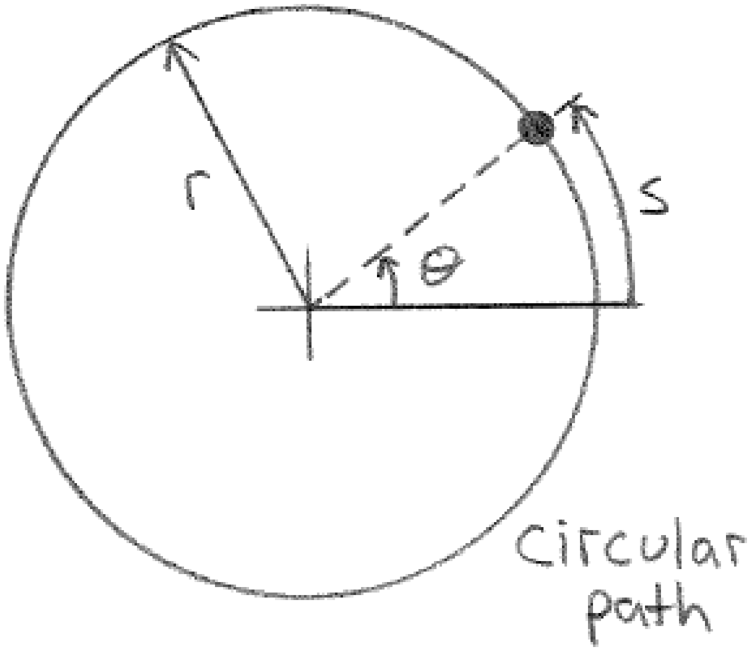


Path
a
$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$a_r = \ddot{r} - r\dot{\theta}^2$
A
r
θ
O



Path
$\mathbf{v}'$
$\mathbf{v}_\theta{}'$
$\mathbf{v}_r{}'$
A′
$\mathbf{v}$
$\mathbf{v}_\theta$
$\mathbf{v}_r$
A

Example 4



The Geometry:

$$s = r\theta \quad , \quad r = \text{const.}$$

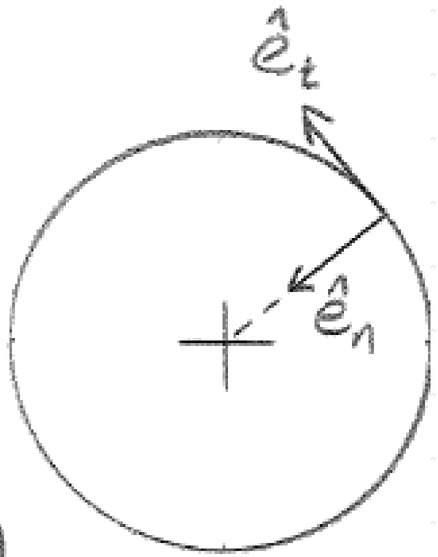
$$\Rightarrow \dot{s} = r\dot{\theta} \equiv v$$

Normal & Tangential Components

$$\bar{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

$$= \frac{d(r\dot{\theta})}{dt}\hat{e}_t + \frac{(r\dot{\theta})^2}{r}\hat{e}_n$$

$$= r\ddot{\theta}\,\hat{e}_t + r\dot{\theta}^2\hat{e}_n \qquad \text{———} \quad (*)$$

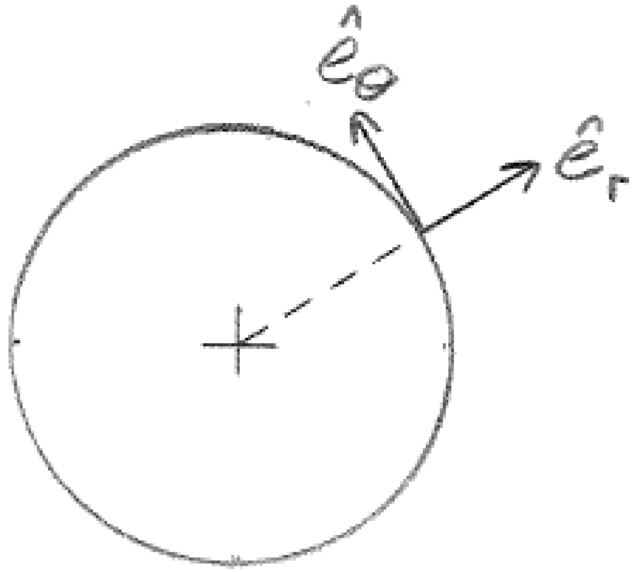


Radial & Transverse Components

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

where $r = \text{const.} \Rightarrow \dot{r} = \ddot{r} = 0$

$$\bar{a} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta \quad \text{———} \quad (**)$$



Relating the two...

For circular motion,

$$\hat{e}_r = -\hat{e}_n$$
$$\hat{e}_\theta = \hat{e}_t$$

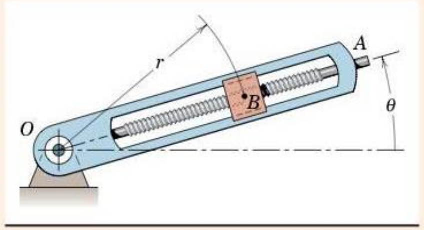
Then from (**),

$$\bar{a} = -r\dot{\theta}^2(-\hat{e}_n) + r\ddot{\theta}\hat{e}_t$$
$$= r\dot{\theta}^2\hat{e}_n + r\ddot{\theta}\hat{e}_t$$

... Same as (*) !

Sample Problem 2/9

Rotation of the radially slotted arm is governed by $\theta = 0.2t + 0.02t^3$, where θ is in radians and t is in seconds. Simultaneously, the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^2$, where r is in meters and t is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when $t = 3$ s.



Solution. The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for $t = 3$ s. ①

$$r = 0.2 + 0.04t^2 \qquad r_3 = 0.2 + 0.04(3^2) = 0.56 \text{ m}$$

$$\dot{r} = 0.08t \qquad \dot{r}_3 = 0.08(3) = 0.24 \text{ m/s}$$

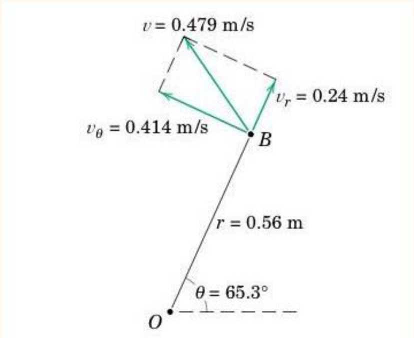
$$\ddot{r} = 0.08 \qquad \ddot{r}_3 = 0.08 \text{ m/s}^2$$

$$\theta = 0.2t + 0.02t^3 \qquad \theta_3 = 0.2(3) + 0.02(3^3) = 1.14 \text{ rad}$$

$$\text{or } \theta_3 = 1.14(180/\pi) = 65.3°$$

$$\dot{\theta} = 0.2 + 0.06t^2 \qquad \dot{\theta}_3 = 0.2 + 0.06(3^2) = 0.74 \text{ rad/s}$$

$$\ddot{\theta} = 0.12t \qquad \ddot{\theta}_3 = 0.12(3) = 0.36 \text{ rad/s}^2$$



The velocity components are obtained from Eq. 2/13 and for $t = 3$ s are

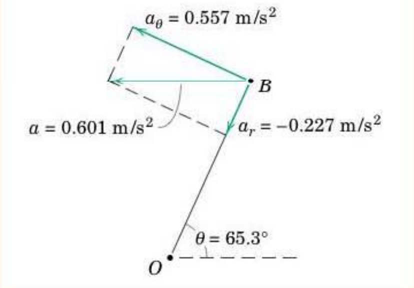
$$[v_r = \dot{r}] \qquad v_r = 0.24 \text{ m/s}$$

$$[v_\theta = r\dot{\theta}] \qquad v_\theta = 0.56(0.74) = 0.414 \text{ m/s}$$

$$[v = \sqrt{{v_r}^2 + {v_\theta}^2}] \qquad v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s} \qquad \textit{Ans.}$$

The velocity and its components are shown for the specified position of the arm.

The acceleration components are obtained from Eq. 2/14 and for $t = 3$ s are



$$[a_r = \ddot{r} - r\dot{\theta}^2] \qquad a_r = 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2$$

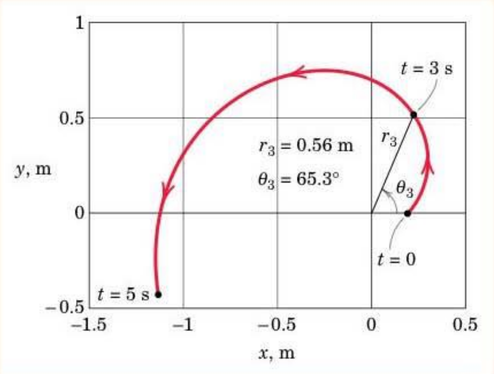
$$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] \qquad a_\theta = 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2$$

$$[a = \sqrt{{a_r}^2 + {a_\theta}^2}] \qquad a = \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2 \qquad \textit{Ans.}$$

The acceleration and its components are also shown for the 65.3° position of the arm.

Plotted in the final figure is the path of the slider B over the time interval $0 \le t \le 5$ s. This plot is generated by varying t in the given expressions for r and θ. Conversion from polar to rectangular coordinates is given by

$$x = r\cos\theta \qquad y = r\sin\theta$$

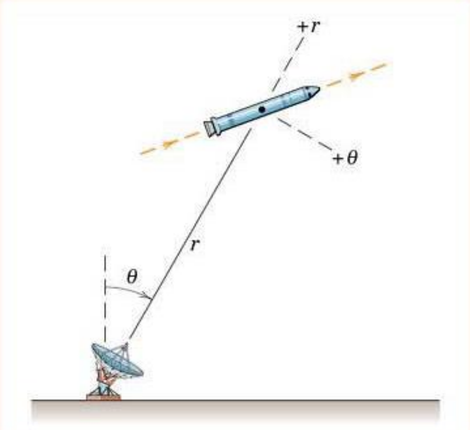


Helpful Hint

① We see that this problem is an example of constrained motion where the center B of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.

Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when $\theta = 30°$, the tracking data give $r = 25(10^4)$ ft, $\dot{r} = 4000$ ft/sec, and $\dot{\theta} = 0.80$ deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 31.4 ft/sec^2 vertically down. For these conditions determine the velocity v of the rocket and the values of $\ddot{r}$ and $\ddot{\theta}$.



Solution. The components of velocity from Eq. 2/13 are

$[v_r = \dot{r}]$ $\qquad v_r = 4000 \text{ ft/sec}$

① $[v_\theta = r\dot{\theta}]$ $\qquad v_\theta = 25(10^4)(0.80)\left(\frac{\pi}{180}\right) = 3490 \text{ ft/sec}$

$[v = \sqrt{v_r^2 + v_\theta^2}]$ $\qquad v = \sqrt{(4000)^2 + (3490)^2} = 5310 \text{ ft/sec}$ *Ans.*

Since the total acceleration of the rocket is $g = 31.4$ ft/sec^2 down, we can easily find its r- and θ-components for the given position. As shown in the figure, they are

② $$a_r = -31.4 \cos 30° = -27.2 \text{ ft/sec}^2$$

$$a_\theta = 31.4 \sin 30° = 15.70 \text{ ft/sec}^2$$

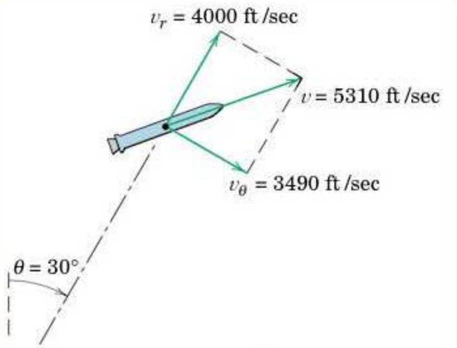
We now equate these values to the polar-coordinate expressions for a_r and a_θ which contain the unknowns $\ddot{r}$ and $\ddot{\theta}$. Thus, from Eq. 2/14

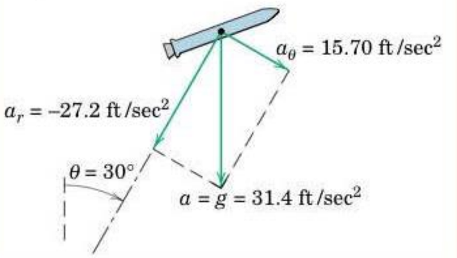
③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad -27.2 = \ddot{r} - 25(10^4)\left(0.80\frac{\pi}{180}\right)^2$

$$\ddot{r} = 21.5 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad 15.70 = 25(10^4)\ddot{\theta} + 2(4000)\left(0.80\frac{\pi}{180}\right)$

$$\ddot{\theta} = -3.84(10^{-4}) \text{ rad/sec}^2 \qquad \textit{Ans.}$$





Helpful Hints

① We observe that the angle θ in polar coordinates need not always be taken positive in a counterclockwise sense.

② Note that the r-component of acceleration is in the negative r-direction, so it carries a minus sign.

③ We must be careful to convert $\dot{\theta}$ from deg/sec to rad/sec.

SUMMARY

Curvilinear Motion

Velocity and Acceleration

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$v = \frac{ds}{dt}$$

Projectile Motion

$$\Delta y = \Delta x \tan\theta - \frac{g(\Delta x)^2}{2v_0^2\cos^2\theta}$$

$$(\tan\theta)^2 - \frac{2v_o^2}{g\Delta x}(\tan\theta) + \left(1 + \frac{2v_o^2\Delta y}{g(\Delta x)^2}\right) = 0$$

Curvilinear Motion

Relative Motions

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A$$

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$$

$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$$

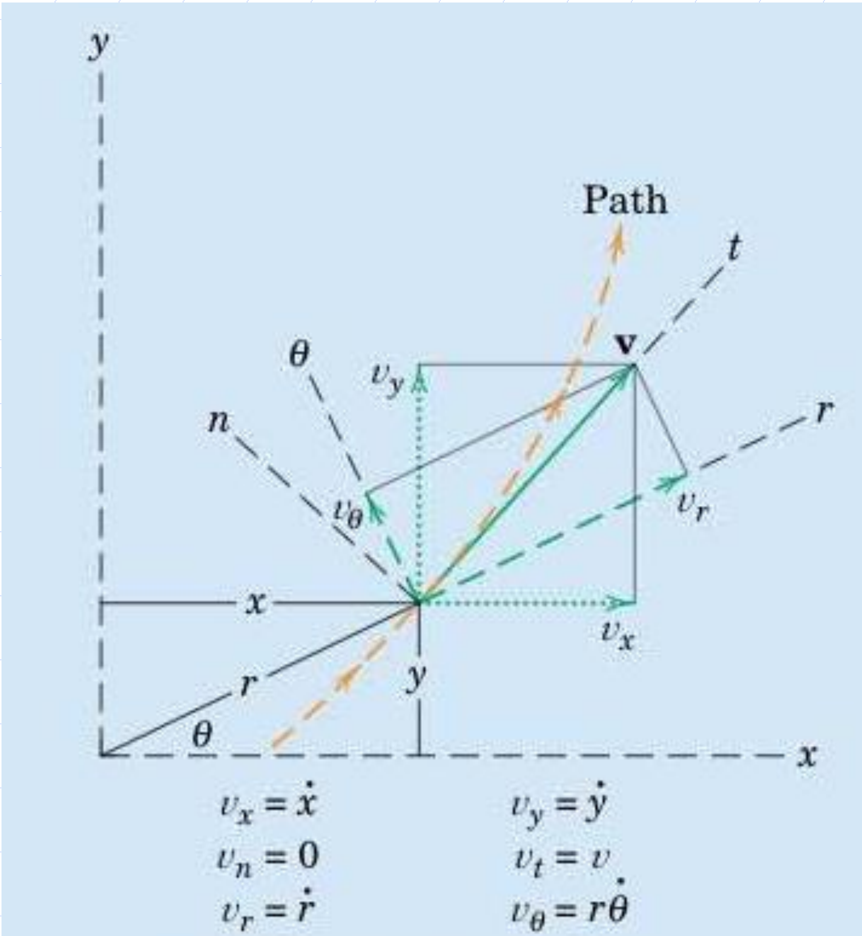
Radial/Transverse Components

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

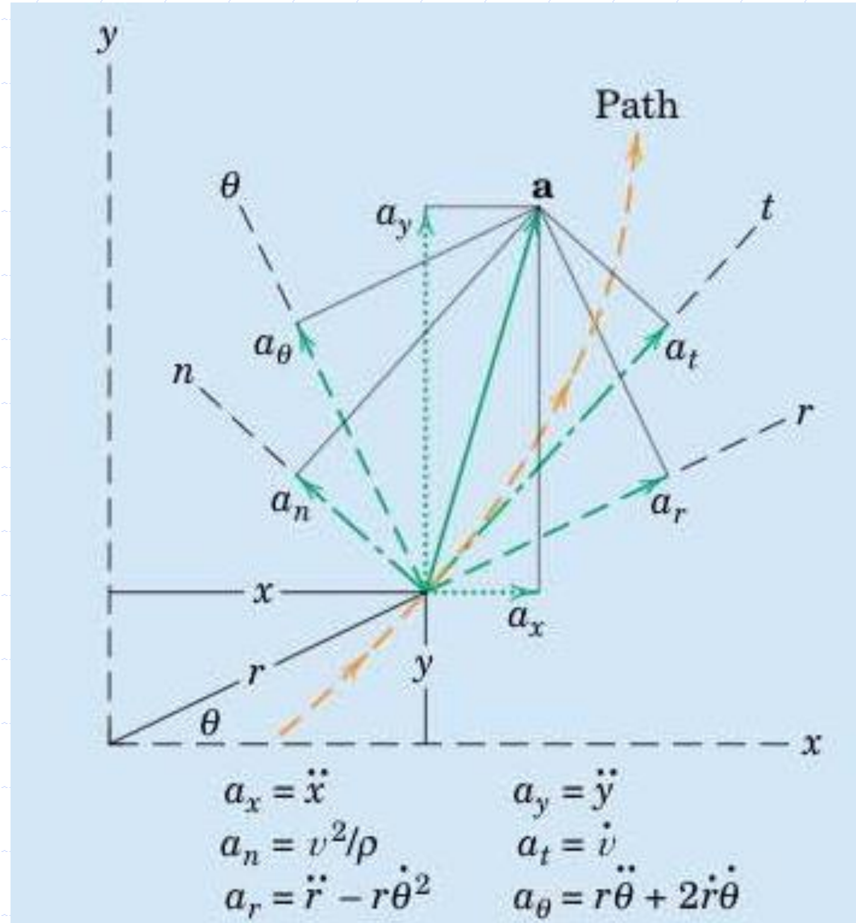
$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

Normal/Tangential Components

$$\bar{v} = v\hat{e}_t \qquad \mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$



(a) Velocity components



(b) Acceleration components