# Chapter 2 – Kinematics of Particles – p1

- Kinematics of a Particle
- Rectilinear Motion
- Particle Motion Cases
- Kinematic Equations
- Relative Rectilinear Motion

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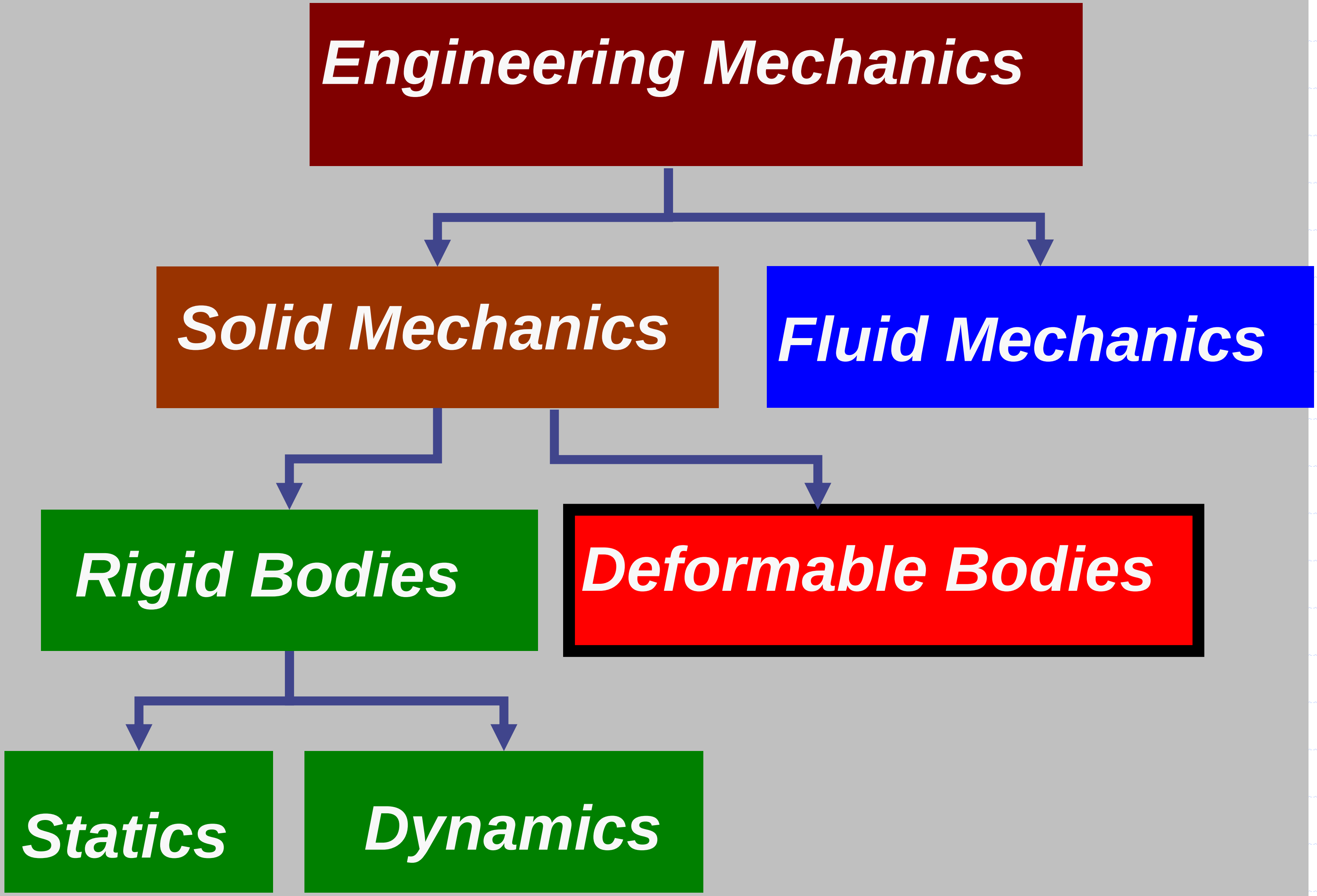

Engineering Mechanics
Solid Mechanics
Fluid Mechanics
Rigid Bodies
Deformable Bodies
Statics
Dynamics

# What is Dynamics?

**Scientific Definition**

Dynamics is a scientific discipline that deals with **systems** undergoing changes in **state**.

# Engineering Definition

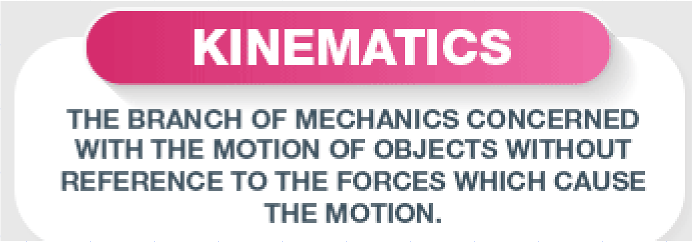
Dynamics is a branch of Mechanics that deals with the relation between **forces** and the **motion** of *bodies*.

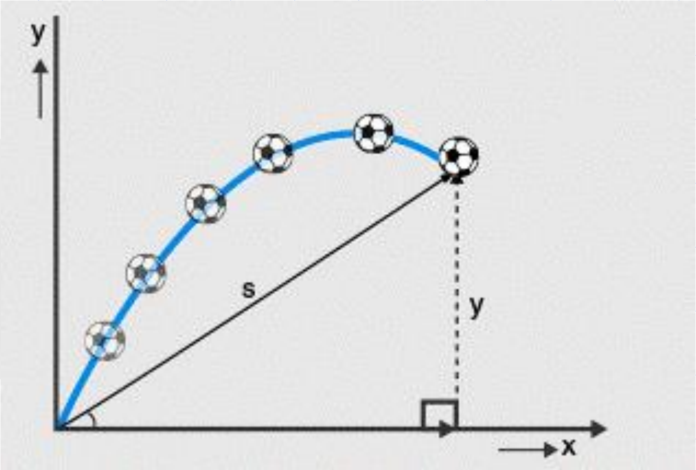
$$F = ma$$

# Dynamics is divided into two parts:

**<u>Kinematics</u>** – the study of motion without reference to any forces

- Study of the geometry of motion
- A purely mathematical construct



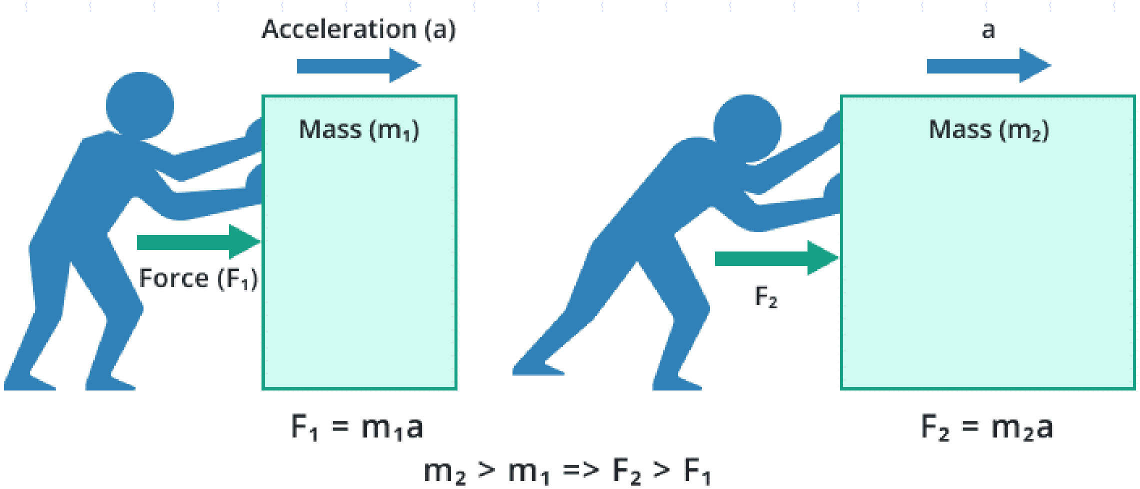
**Kinetics** – the study of motion that results from forces action on bodies

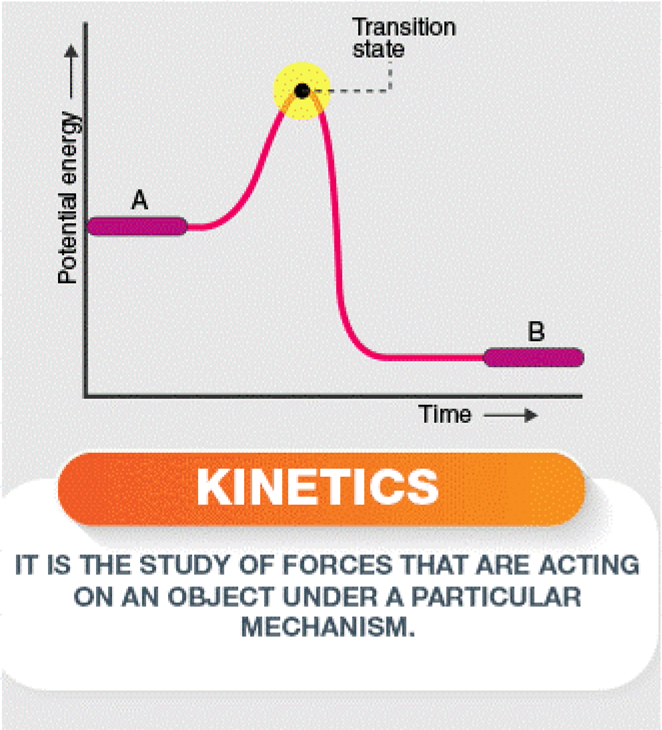
- Used to predict motion caused by given forces

  - OR -

  to determine the forces required to produce a given motion

- Based on physical law



# Kinematics of a Particle

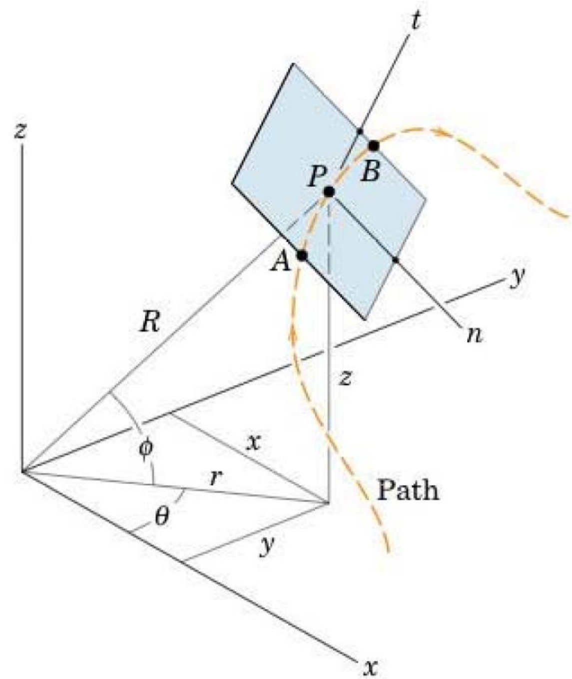
Although each of these planes is rather large, from a distance their motion can be modeled as if each plane were a particle

# Kinematics of Particles

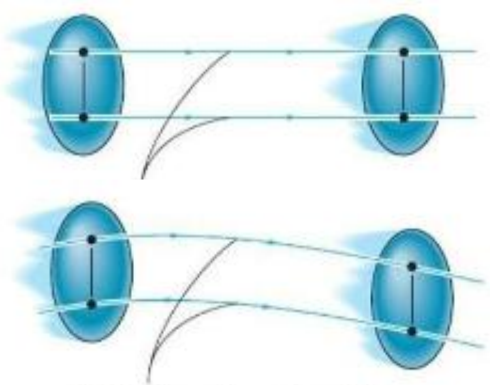
Relates motion and time w/o ref. to forces that cause the motion

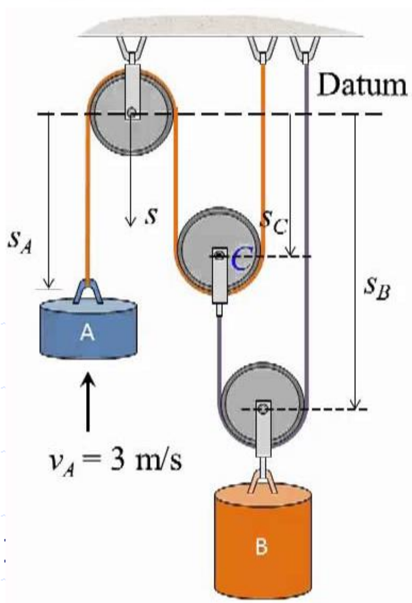
Here, "motion" is quantified by

- Position $\bar{r}$
- Velocity $\bar{v}$
- Acceleration $\bar{a}$

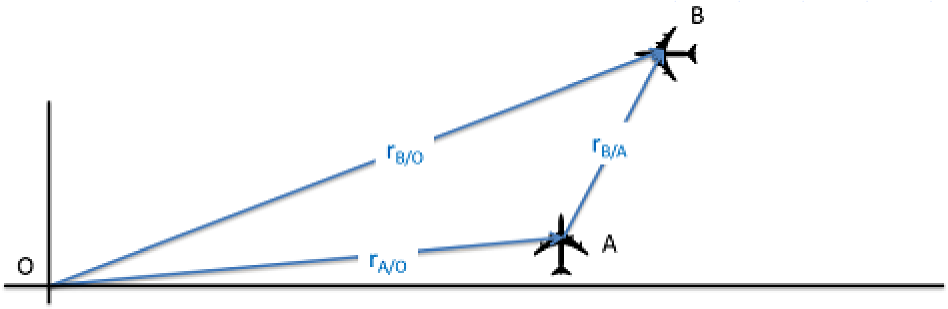


# Motion Types:


Path of curvilinear translation

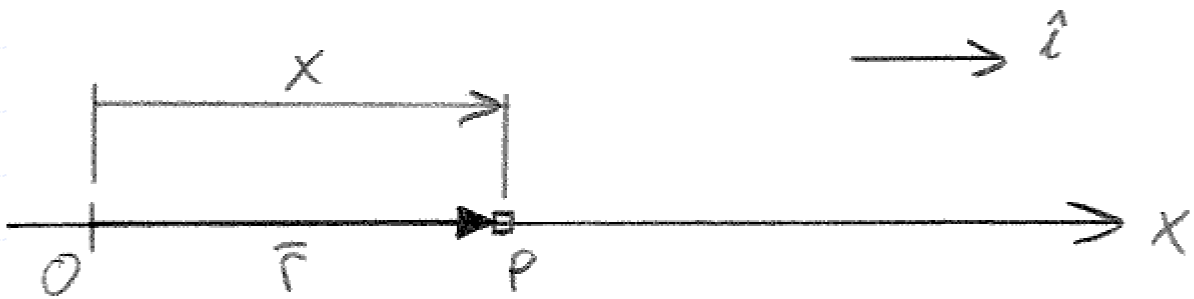



1. Rectilinear
2. Curvilinear
3. Relative
4. Dependent



# Rectilinear Motion of Particles

- Particle is constrained to move in a straight line.




$$\bar{r} = \overrightarrow{OP} = x\,\hat{i}$$

Position Vector

Comments:

A Scalar

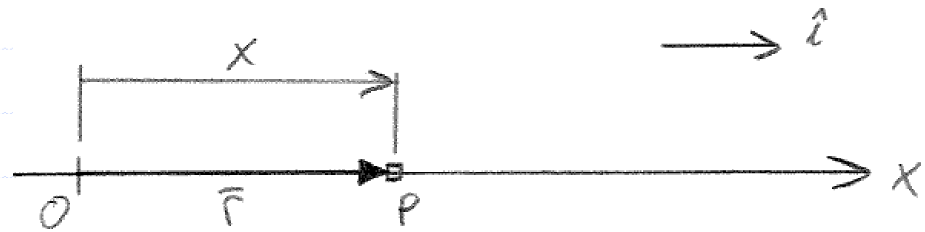
- Magnitude of $x$ specifies distance of P away from O
- sign of $x$ indicates the direction of P relative to O

Scalar notation suffices.
(Rectilinear Motion only!)

## Position x

$x$ = algebraic dist. from O to P

(m, ft, mi, etc.)




## Velocity v

We define

$$v = \frac{dx}{dt}$$

($m/s$, $ft/s$, $mph$, etc.)

Instantaneous Velocity

# Acceleration $a$

Ave. Acc. $= \dfrac{\Delta v}{\Delta t}$ $\left(\dfrac{m/s}{s} = m/s^2,\ ft/s^2,\ \text{etc.}\right)$

In the limit as $\Delta t \to 0$

$$\boxed{a = \frac{dv}{dt} = \frac{d^2x}{dt^2}}$$

Instantaneous Acceleration

Shorthand: $(\dot{\ }) = \dfrac{d(\ )}{dt}$

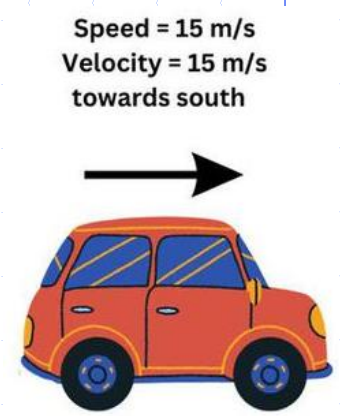
$\Rightarrow$ $v = \dot{x}$

$a = \dot{v} = \ddot{x}$

# IMPORTANT POINTS

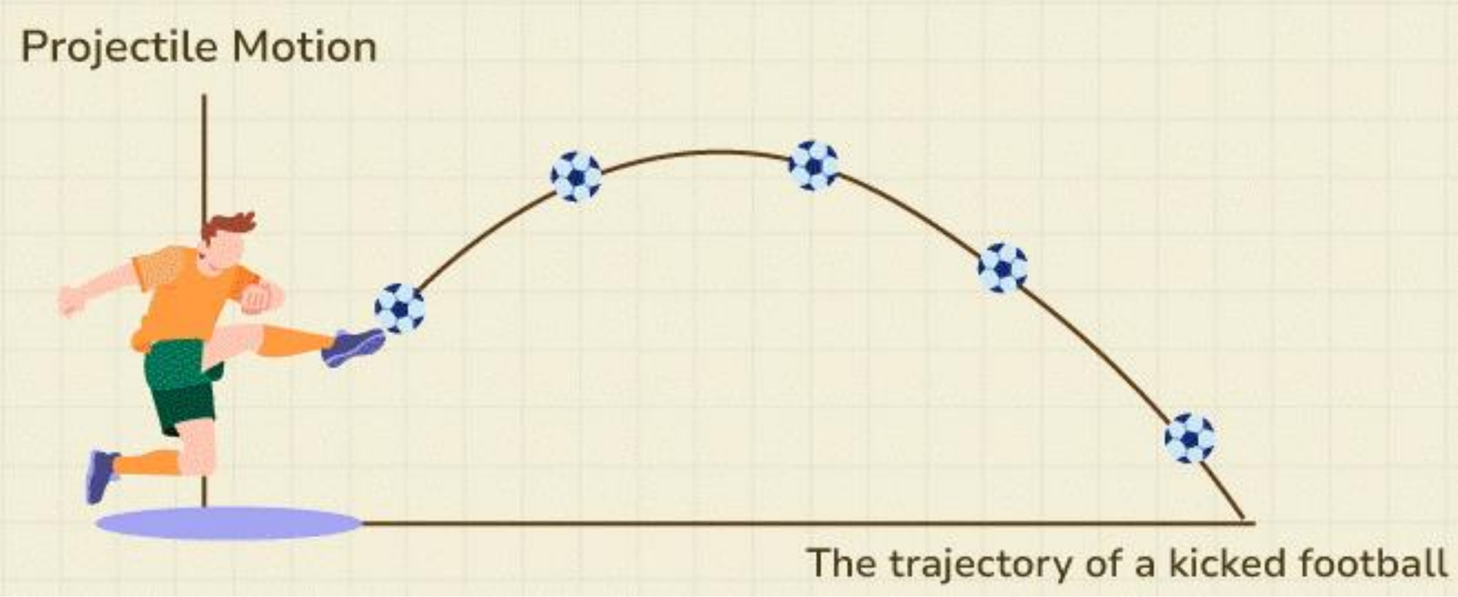
- Dynamics is concerned with bodies that have accelerated motion
- Kinematics is a study of the geometry of the motion
- Kinetics is a study of the forces that cause the motion
- Rectilinear kinematics refers to straight-line motion

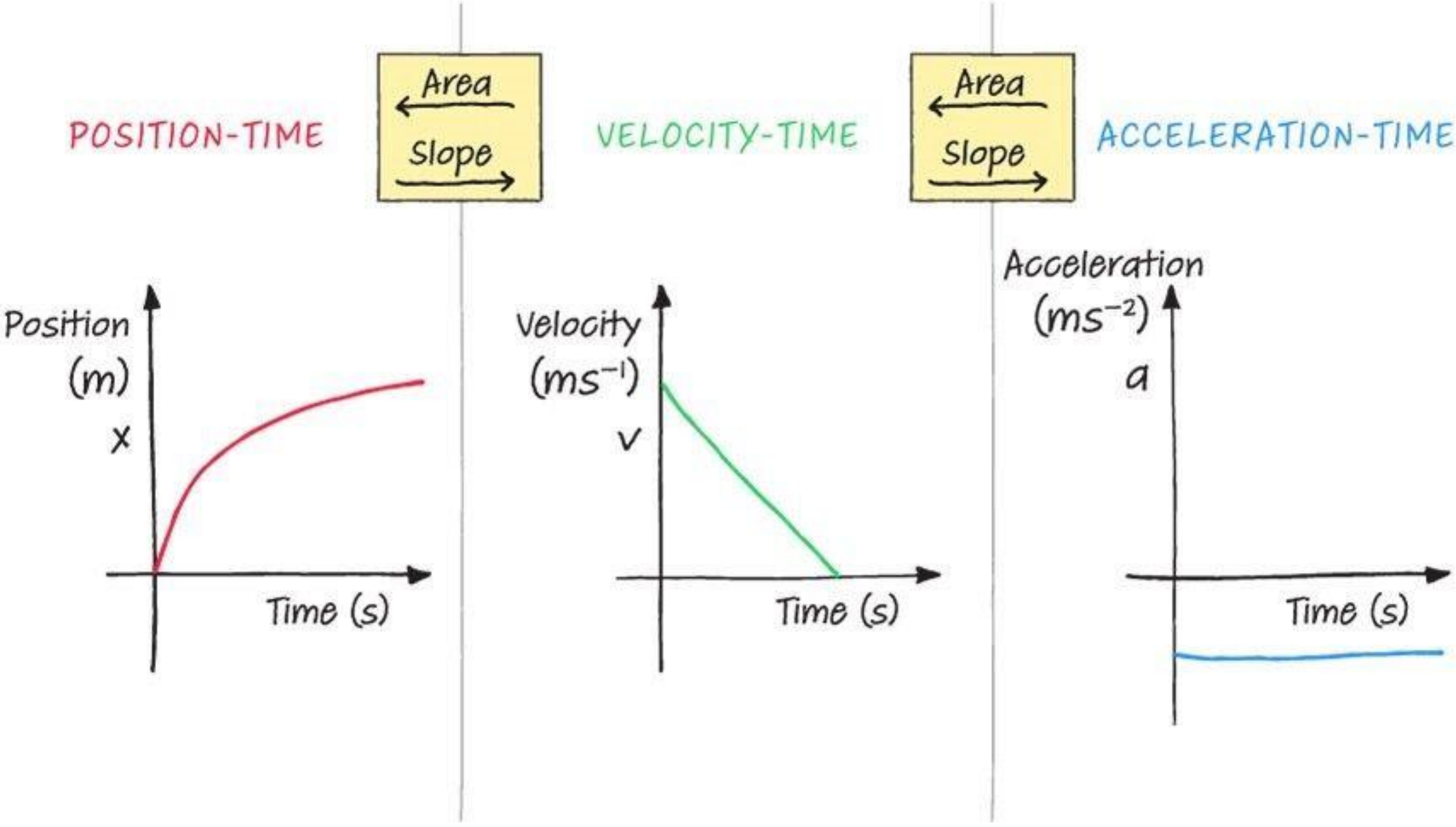
- Speed refers to the magnitude of velocity
- Average speed is the total distance traveled divided by the total time; this is different from the average velocity which is the displacement divided by the time




$$average\ speed = \frac{tot.distance}{tot.time} \qquad v_{avg.} = \frac{\Delta x}{\Delta t}$$

- The acceleration, $a = dv/dt$, is negative when the particle is slowing down or decelerating.
- A particle can have an acceleration and yet have zero velocity




POSITION-TIME
Area
Slope
VELOCITY-TIME
Area
Slope
ACCELERATION-TIME
Position
(m)
x
Time (s)
Velocity
(ms⁻¹)
v
Time (s)
Acceleration
(ms⁻²)
a
Time (s)

# Example 1

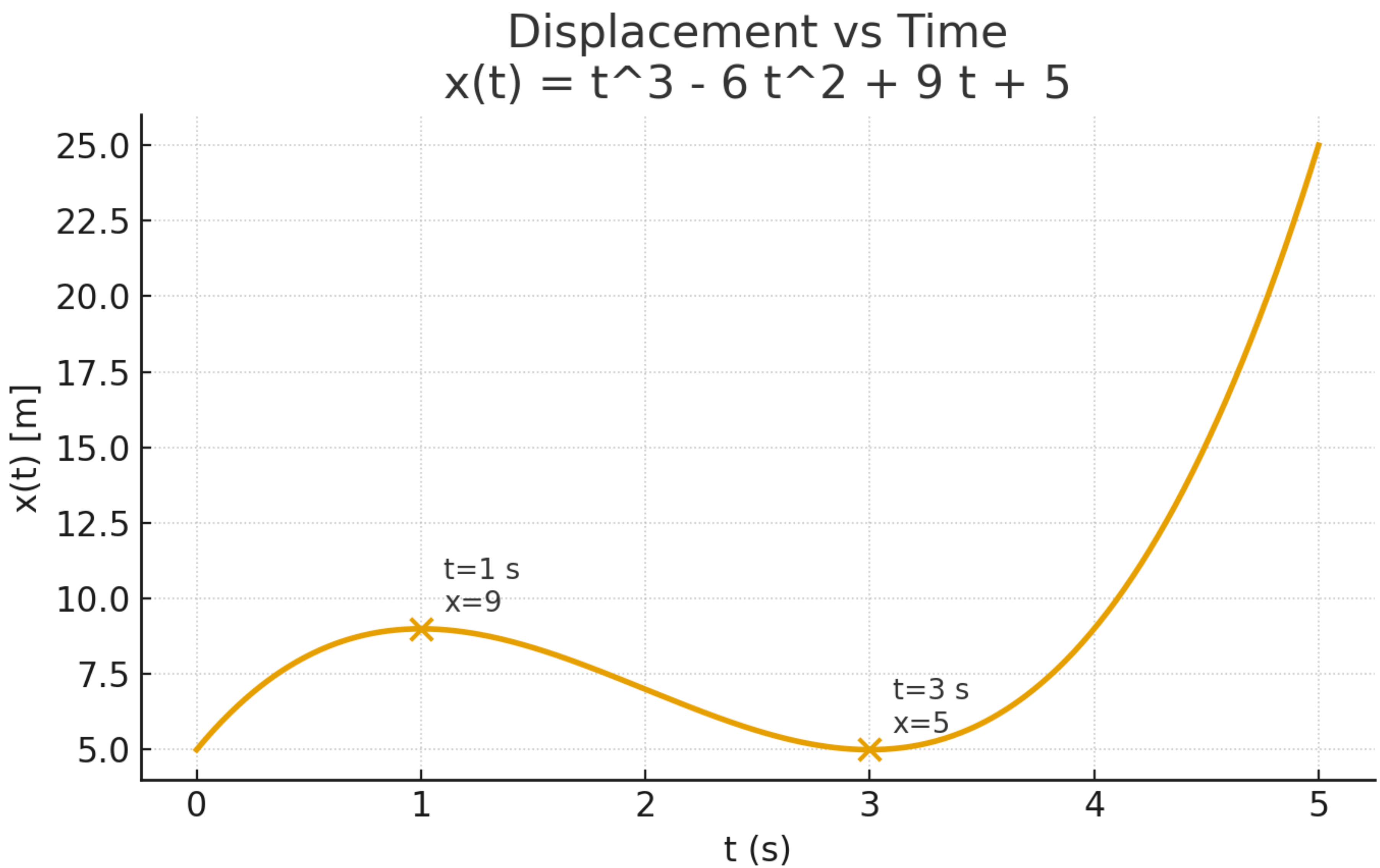
- A particle moves along a line with position:

  $$x(t) = t^3 - 6t^2 + 9t + 5 \quad (m, t \text{ in } s)$$

- Tasks:

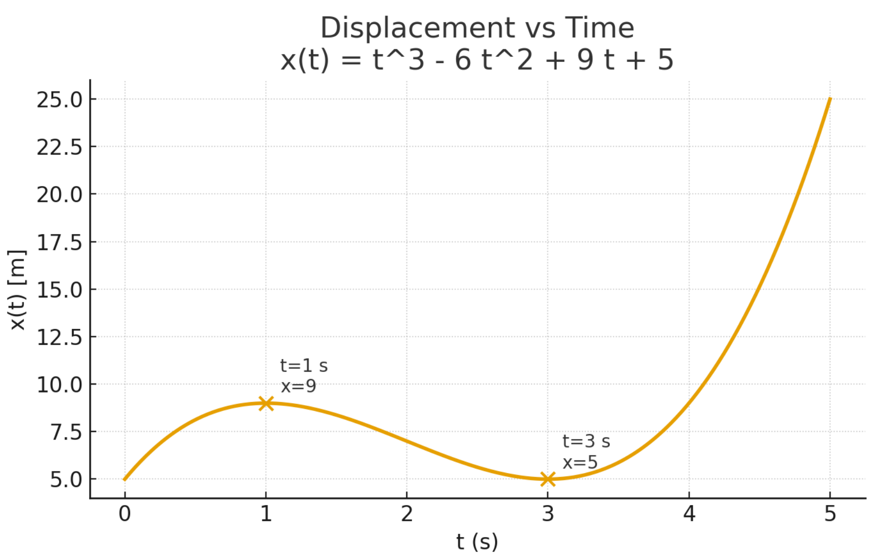
a) Find the total distance traveled from $t=0$ to $t=5$ s.

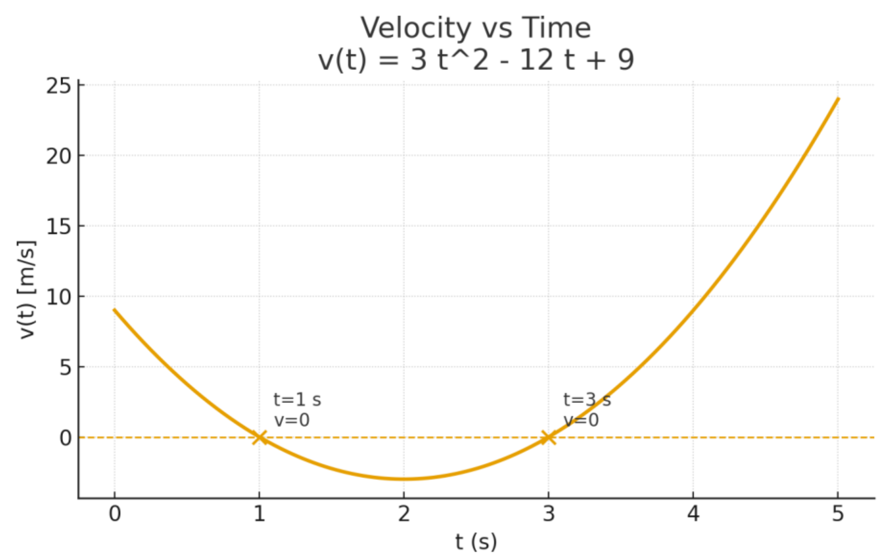
b) Determine acceleration at the critical times when velocity is zero.


Displacement vs Time
x(t) = t^3 - 6 t^2 + 9 t + 5
t=1 s
x=9
t=3 s
x=5
x(t) [m]
t (s)
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
0
1
2
3
4
5

# Step 1. Velocity

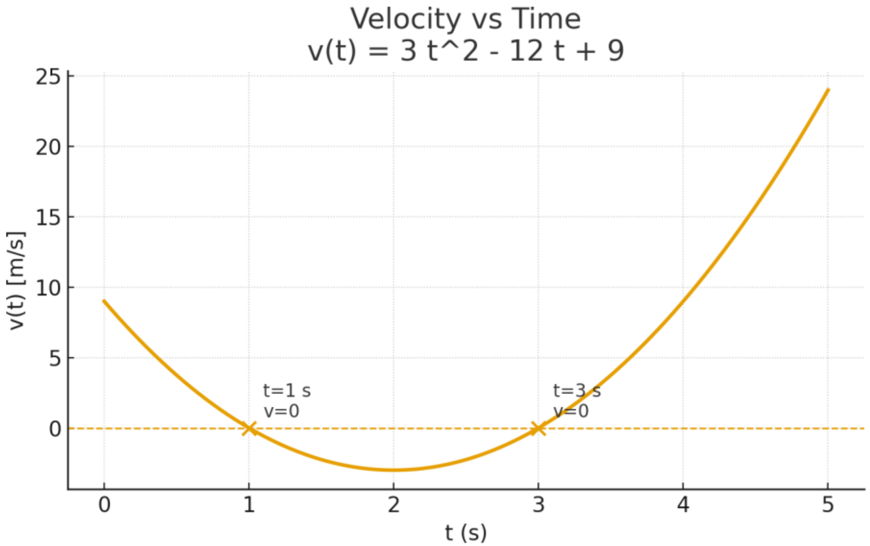
- $v(t) = dx/dt = 3t^2 - 12t + 9$
  $= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$
- Turning points where $v(t)=0$:
  $t = 1$ s, $t = 3$ s

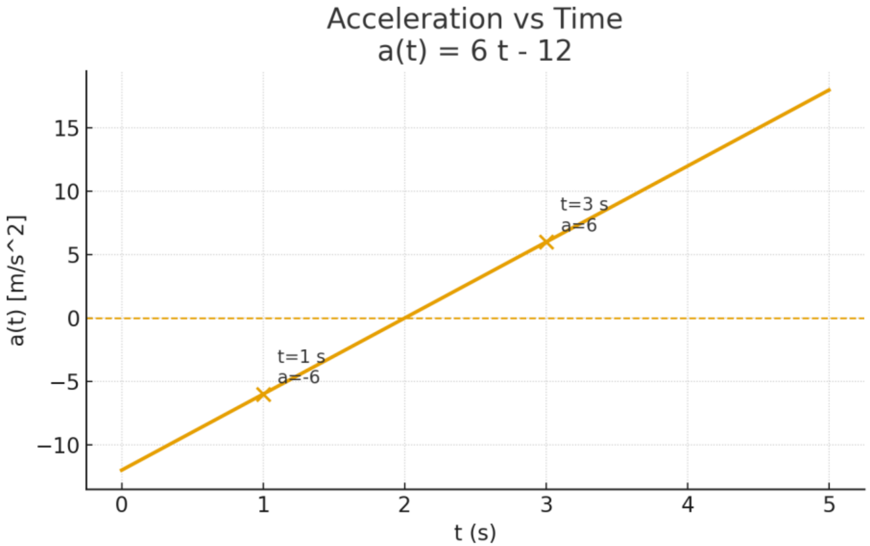


# Step 2. Acceleration

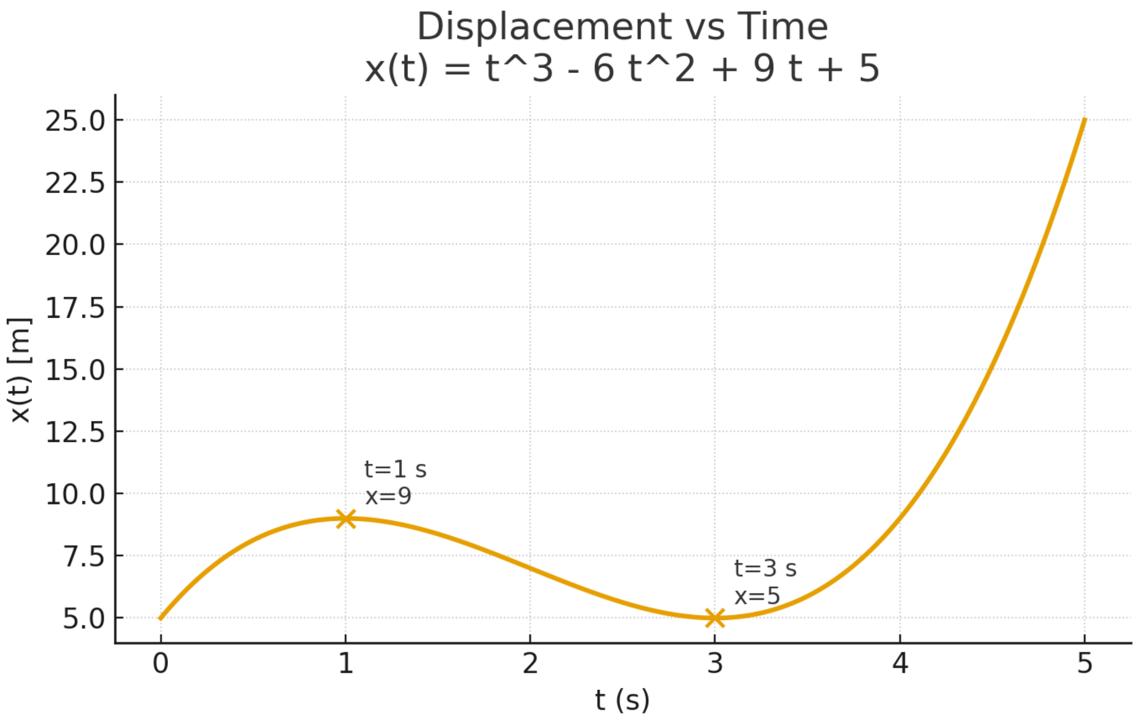
- $a(t) = dv/dt = 6t - 12$
- $a(1) = -6\ m/s^2$ (negative, local maximum)
- $a(3) = +6\ m/s^2$ (positive, local minimum)



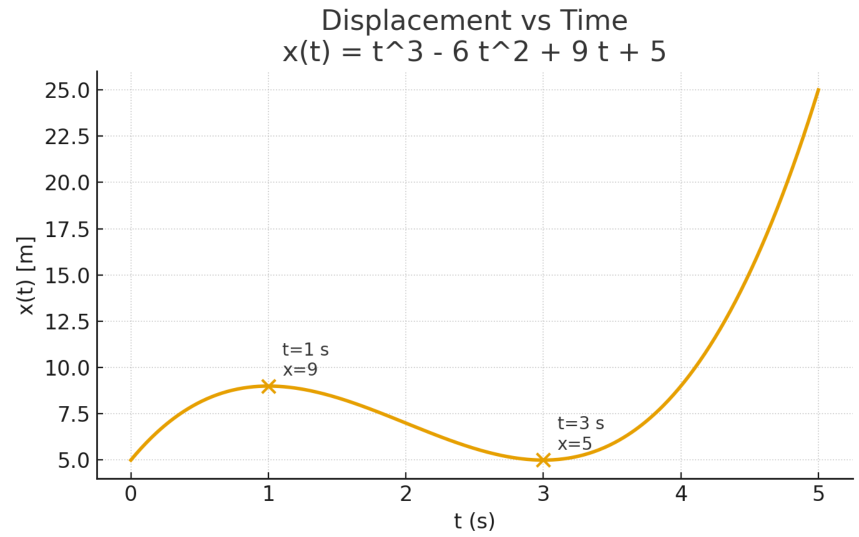
# Step 3. Positions

- Evaluate position at key times:
- $x(0) = 5$ m
- $x(1) = 9$ m
- $x(3) = 5$ m
- $x(5) = 25$ m



# Step 4. Distances

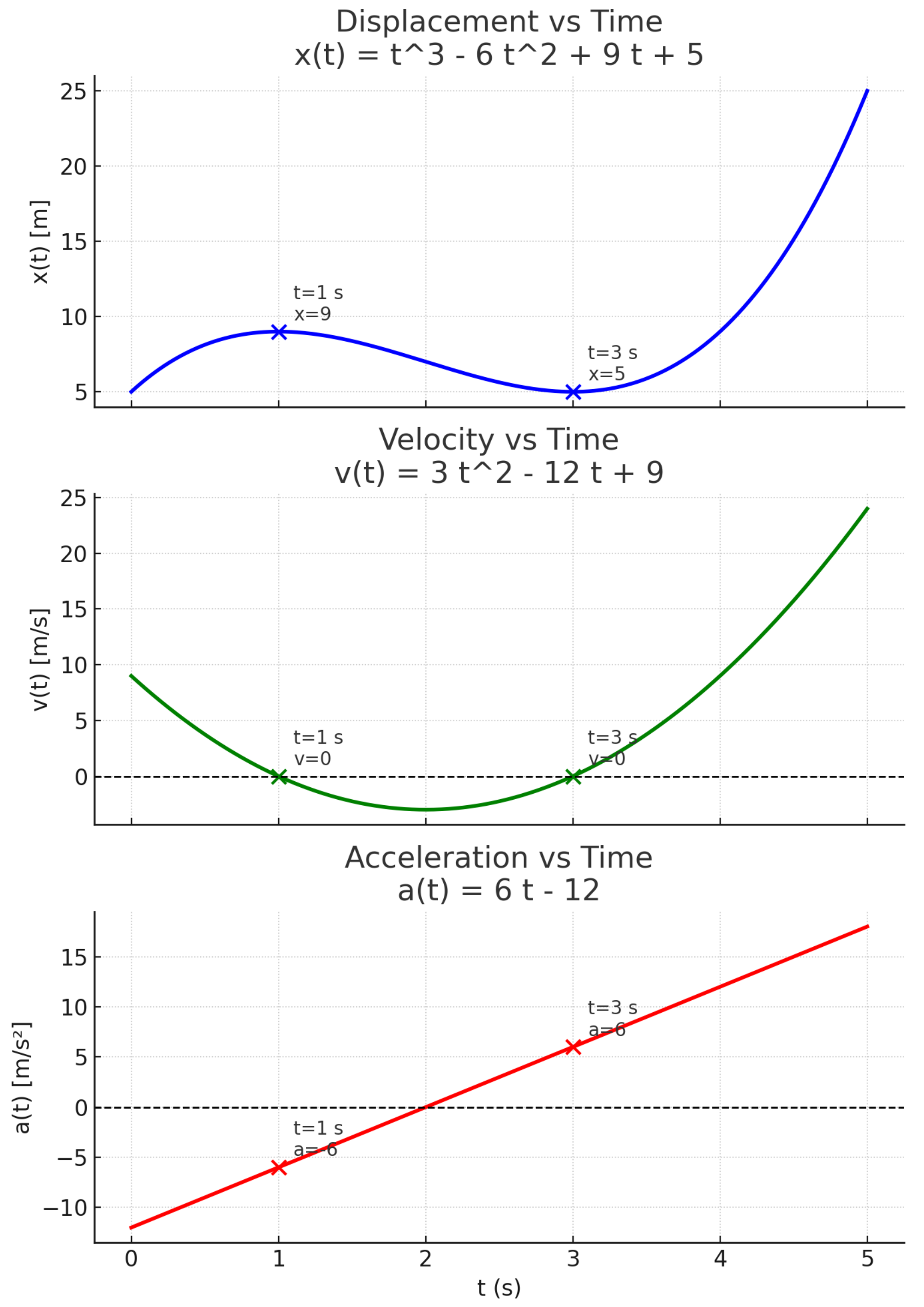
- Distance traveled in each interval:
- 0 → 1 s: |9 - 5| = 4 m
- 1 → 3 s: |5 - 9| = 4 m
- 3 → 5 s: |25 - 5| = 20 m
- **Total Distance = 28 m (Answer)**



# Notes

- Displacement vs Distance: displacement = 20 m, distance = 28 m
- At t=1 s and t=3 s: velocity = 0, acceleration changes sign

# Displacement, Velocity, and Acceleration



# Determination of Particle Motions

- Typically acc. known (measured, via $\bar{F} = m\bar{a}$)
- Vel. & pos. found via integration

Common Cases:

① $a = a(t)$

② $a = a(x)$

③ $a = a(v)$

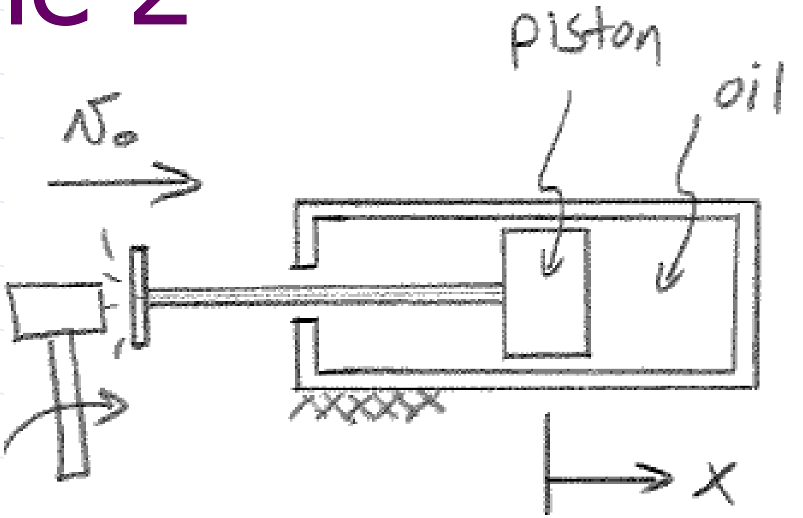
Integrate Directly (Show by Example)

Special Cases:

④ $a = 0$
⑤ $a = \text{const}$ }

lead to
Kinematic Equations

# Example 2




@ $t = 0$, $x = 0$, $v = v_0 > 0$

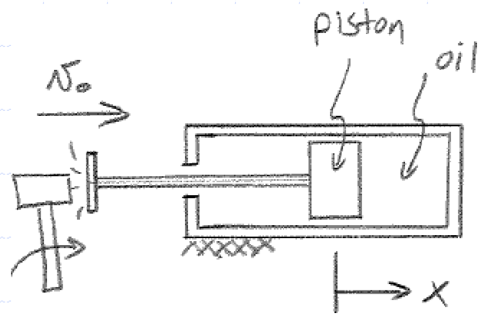
$a(v) = -Kv$ (Case ③)

$K > 0$ is constant

Express (A) $v = v(t)$

(B) $x = x(t)$

(C) $v = v(x)$




**PART A:** Find $v = v(t)$

$$a(v) = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{a(v)} = \frac{dv}{-Kv}$$

$$\Rightarrow \int_0^t dt = \frac{-1}{K}\int_{v_0}^{v} \frac{dv}{v}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\Rightarrow \quad t - 0 = \frac{-1}{K}\left(\ln|v| - \ln|v_0|\right)$$

$$= \frac{-1}{K}\ln\left(\frac{v}{v_0}\right)$$

$$\Rightarrow \quad \ln\left(\frac{v}{v_0}\right) = -Kt$$

$y = e^x \Rightarrow x = \ln(y)$

$$\Rightarrow \quad \frac{v}{v_0} = e^{-Kt}$$

$$\therefore \quad v(t) = v_0 e^{-Kt}$$

## PART B: Find $x = x(t)$

$$v(t) = \frac{dx}{dt}$$

↑ Known from (A)

$$\Rightarrow \quad dx = v(t)\,dt = v_0 e^{-Kt}\,dt$$

$$\Rightarrow \quad \int_{x_0}^{x} dx = v_0 \int_0^t e^{-Kt}\,dt$$

$$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + C$$

$$\Rightarrow \quad x - \cancelto{0}{x_0} = v_0 \cdot \frac{-1}{k} e^{-kt} \Big|_0^t$$

$$= \frac{-v_0}{k}\left(e^{-kt} - 1\right)$$

$$\therefore \quad x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)$$

PART C: Find $v = v(x)$

$$a(v) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

"chain rule" ($\frac{dx}{dt} \to v$)

Given: $a(v)$

$$\Rightarrow \quad dx = \frac{v\,dv}{a(v)} = \frac{v\,dv}{-Kv} = \frac{-1}{K}dv$$

$$\Rightarrow \quad \int_{x_0}^{x} dx = \frac{-1}{K} \int_{v_0}^{v} dv$$

$$\Rightarrow \quad x - \cancel{x_0}^{\,0} = \frac{-1}{K} (v - v_0)$$

$$\Rightarrow \quad v - v_0 = -Kx$$

$$\therefore \quad v(x) = v_0 - Kx$$

# Uniform Rectilinear Motion

Case ④:

$$a = \frac{dv}{dt} = 0 \quad \Rightarrow \quad v = \text{const.}$$

Then $v = \frac{dx}{dt} \Rightarrow dx = v\,dt$

$$\Rightarrow x - x_0 = v(t - 0)$$

$$\boxed{x = x_0 + vt}$$

pos. - time

# Uniformly Accelerated Rectilinear Motion

- Case ⑤: $a = \text{const.}$

Then

$$a = \frac{dv}{dt} \Rightarrow dv = a\,dt$$

$$\Rightarrow v - v_0 = a(t - 0)$$

$$\boxed{v = v_0 + at}$$

vel. – time

Now consider

$$v = \frac{dx}{dt} \Rightarrow dx = v(t)\,dt$$

$v(t) \to v_0 + at$

$$\Rightarrow dx = (v_0 + at)\,dt$$

$$\Rightarrow x - x_0 = v_0 t + a\frac{t^2}{2}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

pos - time

Another relationship - invoke chain rule

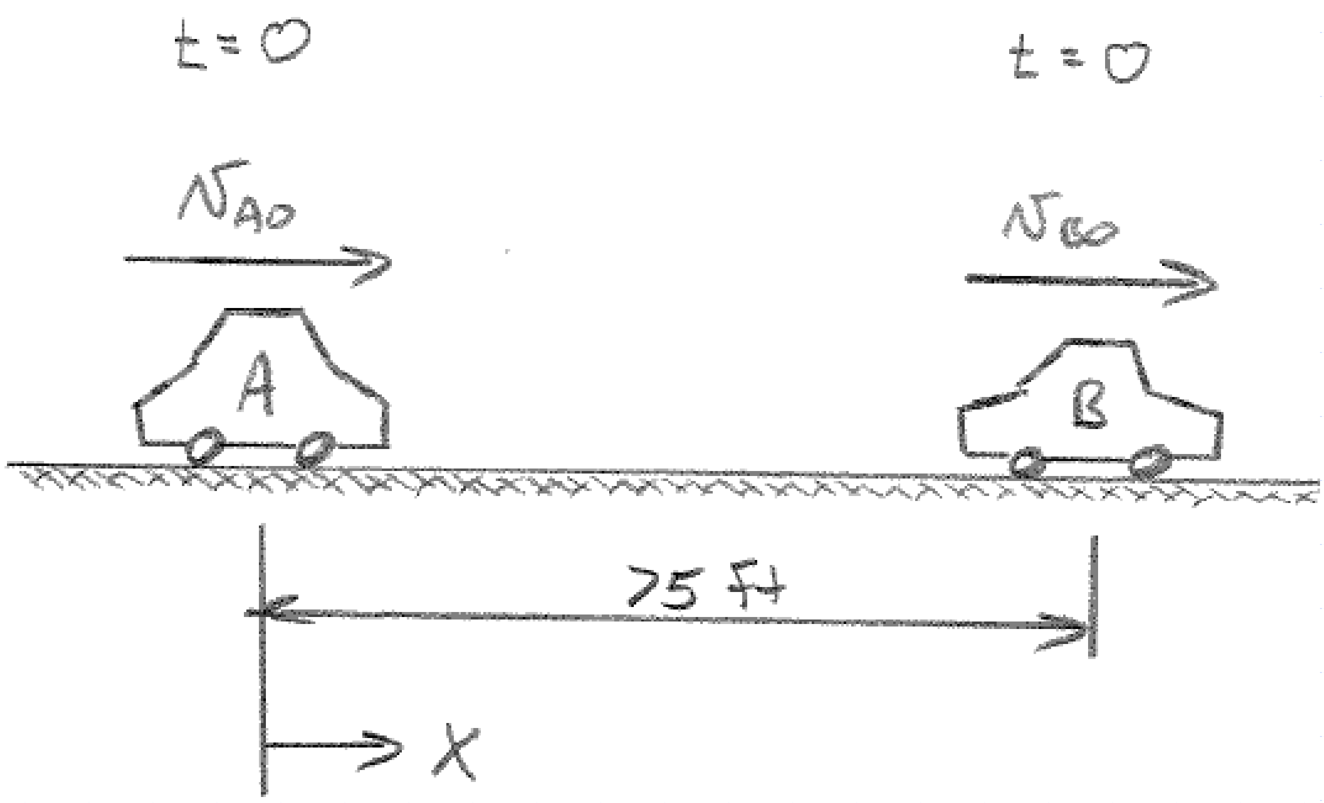
$$a = \frac{dv}{dt} = v\frac{dv}{dx} \Rightarrow v\,dv = a\,dx$$

$$\Rightarrow \frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

$$\boxed{v^2 - v_0^2 = 2a(x - x_0)}$$

vel. - pos.

# Example 3



Data:

$v_A(t=0) = v_{A0} = 24$ mph , $a_A = 1.8$ ft/s$^2$

$v_B(t=0) = v_{B0} = 36$ mph , $a_B = -1.2$ ft/s$^2$

Find: (A) When & where A overtakes B

(B) corresponding speed of each car

**PART A:** Use Kinematic eqns for $a =$ const.

$$v = v_0 + at \quad \text{——— ①}$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2 \quad \text{—— ②} \longleftarrow \text{pos. - time}$$

$$v^2 - v_0^2 = 2a(x - x_0) \quad \text{—— ③}$$

Car A: $x_A(t) = x_{A0} + v_{A0}t + \frac{1}{2}a_A t^2$

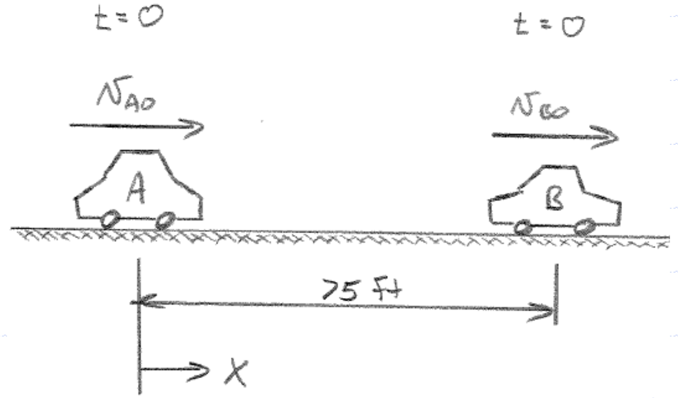
Car B: $x_B(t) = x_{B0} + v_{B0}t + \frac{1}{2}a_B t^2$ —— ④

Here, $x_{A0} = 0$ , $x_{B0} = 75\ \text{ft}$

$$v_{A0} = \left(24\ \frac{mi}{hr}\right)\left(\frac{5280\ ft}{1\ mi}\right)\left(\frac{1\ hr}{3600\ s}\right) = 35.2\ ft/s$$

$$v_{B0} = 36\ mph = 52.8\ ft/s$$

$a_A$, $a_B$ given




A overtakes B when $x_A(t) = x_B(t)$

$$\Rightarrow\ x_{A0} + v_{A0}t + \frac{1}{2}a_A t^2 = x_{B0} + v_{B0}t + \frac{1}{2}a_B t^2$$

$$\Rightarrow\ (a_A - a_B)t^2 + 2(v_{A0} - v_{B0})t + 2(x_{A0} - x_{B0}) = 0$$

$$\Rightarrow\ (3\ ft/s^2)t^2 + (-35.2\ ft/s)t + (-150\ ft) = 0$$

of form $at^2 + bt + c = 0$

try this tool

so $$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad t_1 = 15.05\ s$$

$$t_2 = -3.32\ s \quad \leftarrow \text{nonsense}$$

$$\therefore \quad t = t_1 = 15.1\ s$$

For positions, plug $t = t_1$ into ④.

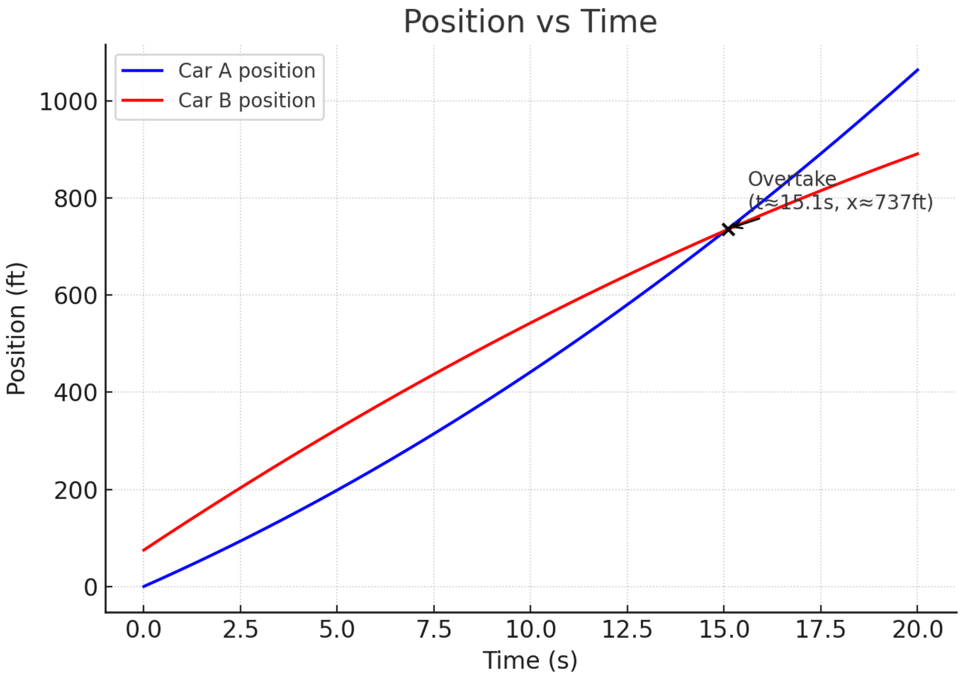
$$X_p = 733.7\ ft \qquad (\text{or} \sim 737\ ft?)$$

PART B: Find $v_A(t_1)$, $v_B(t_1)$

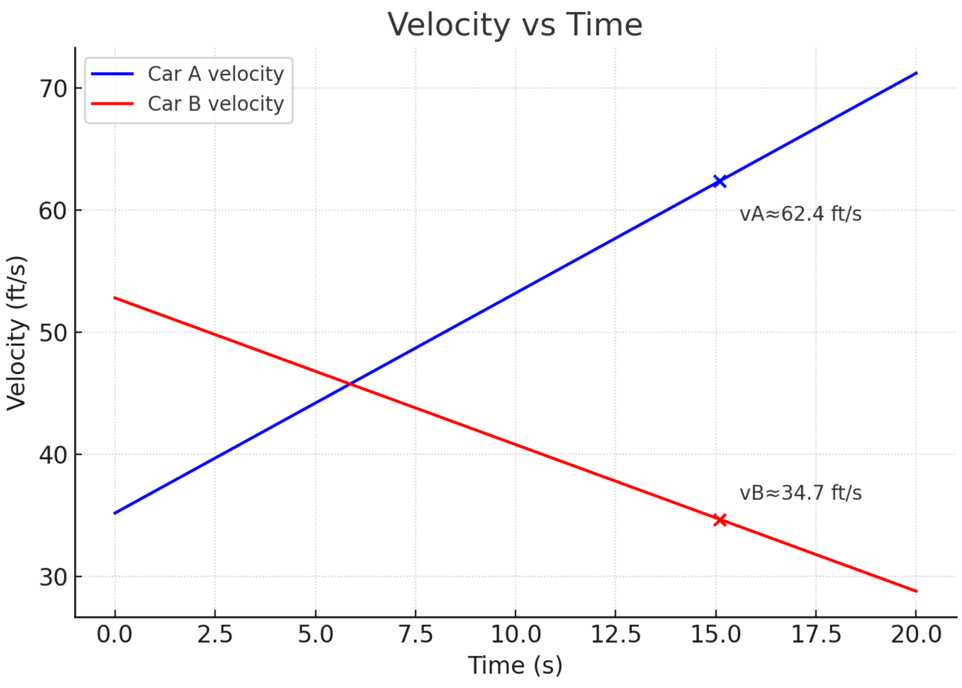
Easy! Use ①.

$$v_A(t_1) = v_{A0} + a_A t_1 = 62.3 \text{ ft/s} = 42.5 \text{ mph}$$

$$v_B(t_1) = v_{B0} + a_B t_1 = 34.7 \text{ ft/s} = 23.7 \text{ mph}$$



Position vs Time
Car A position
Car B position
Overtake
(t≈15.1s, x≈737ft)
Position (ft)
Time (s)
0
200
400
600
800
1000
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

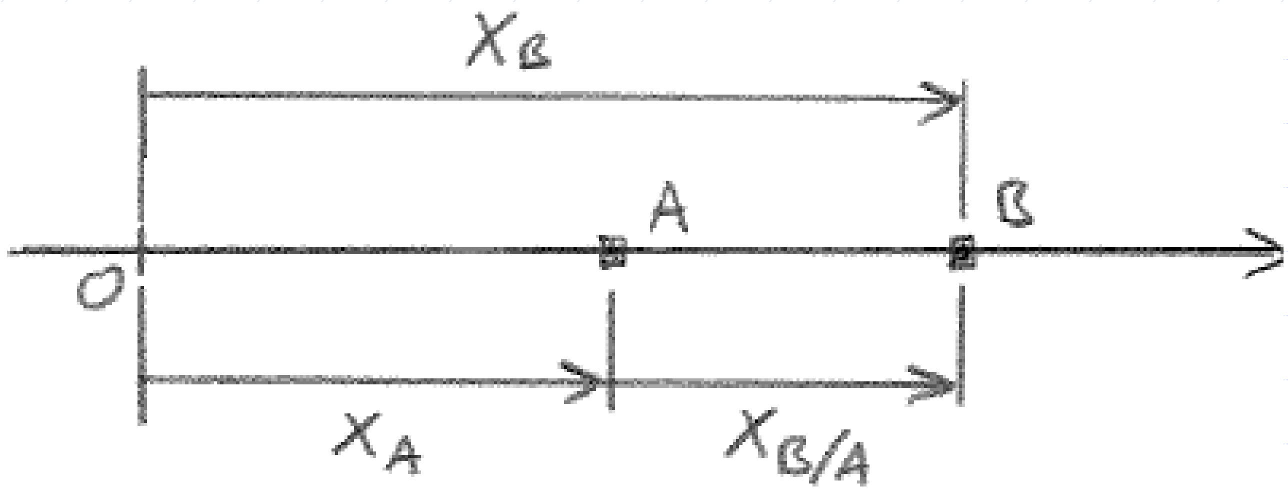
Velocity vs Time
Car A velocity
Car B velocity
vA≈62.4 ft/s
vB≈34.7 ft/s
Velocity (ft/s)
Time (s)
30
40
50
60
70
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

# Relative Rectilinear Motion

→ Relative motions are often more important than absolute motions.

We define $x_{B/A} = x_B - x_A$

B/A reads "B with respect to A"



$\frac{d}{dt}$ to obtain

$$v_{B/A} = v_B - v_A$$

$$a_{B/A} = a_B - a_A$$

# Rectilinear Motion

## *Velocity and Acceleration*

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt}$$

**Cases 1,2,3:** $a = a(t)$, $a = a(x)$, $a = a(v)$

Integrate directly.

# Rectilinear Motion

## *Velocity and Acceleration*

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt}$$

**Cases 4,5:** $a = 0$ , $a = \text{constant}$

$$x = x_o + vt$$

$$v = v_o + at$$

$$x - x_o = v_o t + \tfrac{1}{2} a t^2$$

$$v^2 - v_0^2 = 2a(x - x_o)$$

## Sample Problem 2/1

The position coordinate of a particle which is confined to move along a straight line is given by $s = 2t^3 - 24t + 6$, where $s$ is measured in meters from a convenient origin and $t$ is in seconds. Determine ($a$) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$, ($b$) the acceleration of the particle when $v = 30$ m/s, and ($c$) the net displacement of the particle during the interval from $t = 1$ s to $t = 4$ s.

**Solution.** The velocity and acceleration are obtained by successive differentiation of $s$ with respect to the time. Thus,

$[v = \dot{s}]$ $\qquad v = 6t^2 - 24$ m/s

$[a = \dot{v}]$ $\qquad a = 12t$ m/s$^2$

***(a)*** Substituting $v = 72$ m/s into the expression for $v$ gives us $72 = 6t^2 - 24$, from which $t = \pm 4$ s. The negative root describes a mathematical solution for $t$ before ① the initiation of motion, so this root is of no physical interest. Thus, the desired result is

$$t = 4 \text{ s} \qquad \textit{Ans.}$$

***(b)*** Substituting $v = 30$ m/s into the expression for $v$ gives $30 = 6t^2 - 24$, from which the positive root is $t = 3$ s, and the corresponding acceleration is

$$a = 12(3) = 36 \text{ m/s}^2 \qquad \textit{Ans.}$$

***(c)*** The net displacement during the specified interval is

$$\Delta s = s_4 - s_1 \qquad \text{or}$$

$$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6]$$

$$= 54 \text{ m} \qquad \textit{Ans.}$$

② which represents the net advancement of the particle along the $s$-axis from the position it occupied at $t = 1$ s to its position at $t = 4$ s.

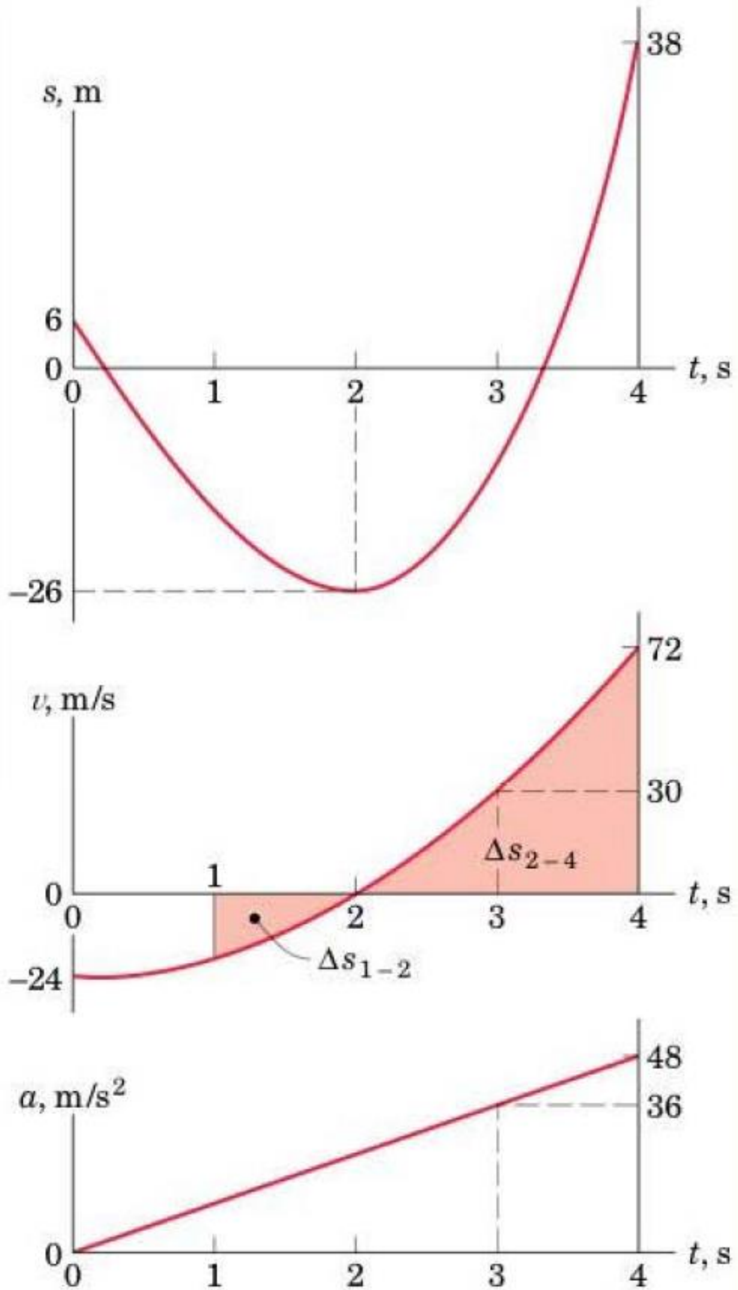
To help visualize the motion, the values of $s$, $v$, and $a$ are plotted against the time $t$ as shown. Because the area under the $v$-$t$ curve represents displacement, ③ we see that the net displacement from $t = 1$ s to $t = 4$ s is the positive area $\Delta s_{2-4}$ less the negative area $\Delta s_{1-2}$.

## Helpful Hints

① Be alert to the proper choice of sign when taking a square root. When the situation calls for only one answer, the positive root is not always the one you may need.

② Note carefully the distinction between italic *s* for the position coordinate and the vertical s for seconds.

③ Note from the graphs that the values for $v$ are the slopes ($\dot{s}$) of the $s$-$t$ curve and that the values for $a$ are the slopes ($\dot{v}$) of the $v$-$t$ curve. *Suggestion:* Integrate $v\,dt$ for each of the two intervals and check the answer for $\Delta s$. Show that the total distance traveled during the interval $t = 1$ s to $t = 4$ s is 74 m.



## Sample Problem 2/4

A freighter is moving at a speed of 8 knots when its engines are suddenly
① stopped. If it takes 10 minutes for the freighter to reduce its speed to 4 knots, determine and plot the distance $s$ in nautical miles moved by the ship and its speed $v$ in knots as functions of the time $t$ during this interval. The deceleration of the ship is proportional to the square of its speed, so that $a = -kv^2$.

**Solution.** The speeds and the time are given, so we may substitute the expression for acceleration directly into the basic definition $a = dv/dt$ and integrate. Thus,

$$-kv^2 = \frac{dv}{dt} \qquad \frac{dv}{v^2} = -k\,dt \qquad \int_8^v \frac{dv}{v^2} = -k\int_0^t dt$$

② $$-\frac{1}{v} + \frac{1}{8} = -kt \qquad v = \frac{8}{1+8kt}$$

Now we substitute the end limits of $v = 4$ knots and $t = \frac{10}{60} = \frac{1}{6}$ hour and get

$$4 = \frac{8}{1+8k(1/6)} \qquad k = \frac{3}{4}\,\text{mi}^{-1} \qquad v = \frac{8}{1+6t} \qquad \textit{Ans.}$$

The speed is plotted against the time as shown.

The distance is obtained by substituting the expression for $v$ into the definition $v = ds/dt$ and integrating. Thus,

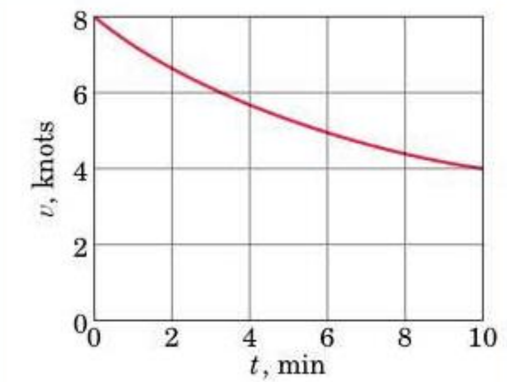
$$\frac{8}{1+6t} = \frac{ds}{dt} \qquad \int_0^t \frac{8\,dt}{1+6t} = \int_0^s ds \qquad s = \frac{4}{3}\ln(1+6t) \qquad \textit{Ans.}$$

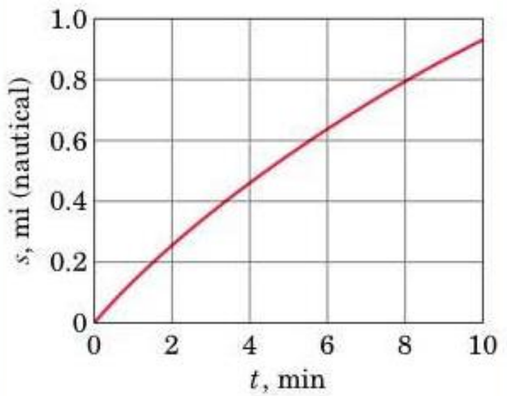
The distance $s$ is also plotted against the time as shown, and we see that the ship has moved through a distance $s = \frac{4}{3}\ln(1+\frac{6}{6}) = \frac{4}{3}\ln 2 = 0.924$ mi (nautical) during the 10 minutes.

**Helpful Hints**

① Recall that one knot is the speed of one nautical mile (6076 ft) per hour. Work directly in the units of nautical miles and hours.

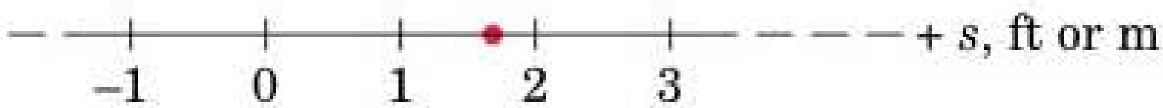
② We choose to integrate to a general value of $v$ and its corresponding time $t$ so that we may obtain the variation of $v$ with $t$.



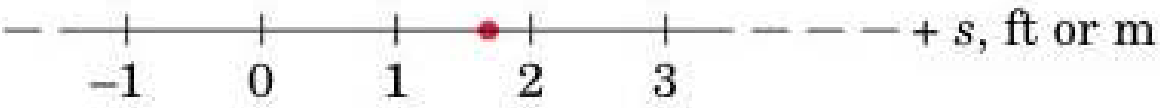



**2/1** The velocity of a particle is given by $v = 25t^2 - 80t - 200$, where $v$ is in feet per second and $t$ is in seconds. Plot the velocity $v$ and acceleration $a$ versus time for the first 6 seconds of motion and evaluate the velocity when $a$ is zero.

*Ans.* $v = -264$ ft/sec



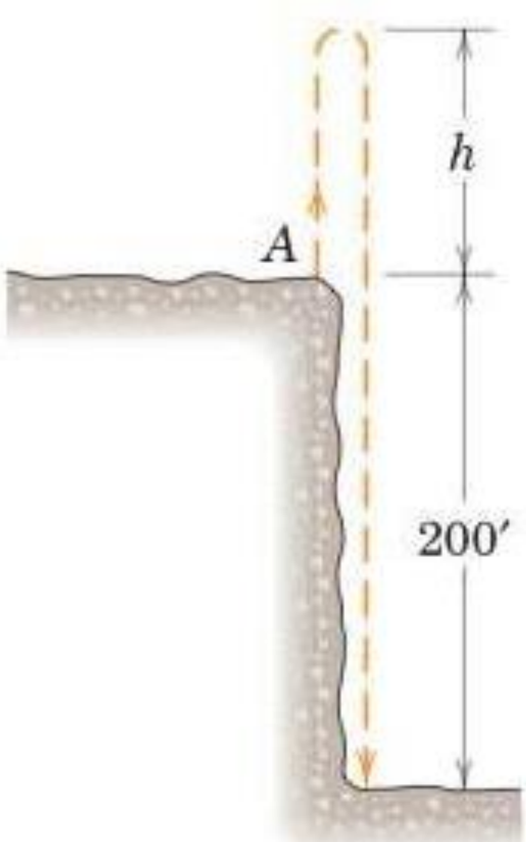
**2/2** The position of a particle is given by $s = 2t^3 - 40t^2 + 200t - 50$, where $s$ is in meters and $t$ is in seconds. Plot the position, velocity, and acceleration as functions of time for the first 12 seconds of motion. Determine the time at which the velocity is zero.



**2/6** The acceleration of a particle is given by $a = -ks^2$, where $a$ is in meters per second squared, $k$ is a constant, and $s$ is in meters. Determine the velocity of the particle as a function of its position $s$. Evaluate your expression for $s = 5$ m if $k = 0.1\ \text{m}^{-1}\text{s}^{-2}$ and the initial conditions at time $t = 0$ are $s_0 = 3$ m and $v_0 = 10$ m/s.

**2/8** The velocity of a particle moving in a straight line is decreasing at the rate of 3 m/s per meter of displacement at an instant when the velocity is 10 m/s. Determine the acceleration $a$ of the particle at this instant.

**2/10** A ball is thrown vertically up with a velocity of 80 ft/sec at the edge of a 200-ft cliff. Calculate the height $h$ to which the ball rises and the total time $t$ after release for the ball to reach the bottom of the cliff. Neglect air resistance and take the downward acceleration to be 32.2 ft/sec$^2$.



**2/11** A rocket is fired vertically up from rest. If it is designed to maintain a constant upward acceleration of $1.5g$, calculate the time $t$ required for it to reach an altitude of 30 km and its velocity at that position.

*Ans.* $t = 63.9$ s, $v = 940$ m/s

**2/32** A motorcycle patrolman starts from rest at $A$ two seconds after a car, speeding at the constant rate of 120 km/h, passes point $A$. If the patrolman accelerates at the rate of 6 $m/s^2$ until he reaches his maximum permissible speed of 150 km/h, which he maintains, calculate the distance $s$ from point $A$ to the point at which he overtakes the car.