

Chapter 2 – Kinematics of Particles – p1

- ◆ Kinematics of a Particle
- ◆ Rectilinear Motion
- ◆ Particle Motion Cases
- ◆ Kinematic Equations
- ◆ Relative Rectilinear Motion

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Introduction



Engineering Mechanics

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graph TD; A[Engineering Mechanics] --> B[Solid Mechanics]; A --> C[Fluid Mechanics]; B --> D[Rigid Bodies]; B --> E[Deformable Bodies]; D --> F[Statics]; D --> G[Dynamics];
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Solid Mechanics

Fluid Mechanics

Rigid Bodies

Deformable Bodies

Statics

Dynamics

What is Dynamics?

Scientific Definition

Dynamics is a scientific discipline that deals with **systems** undergoing changes in **state**.

Engineering Definition

Dynamics is a branch of Mechanics that deals with the relation between **forces** and the **motion** of *bodies*.

$$F = ma$$



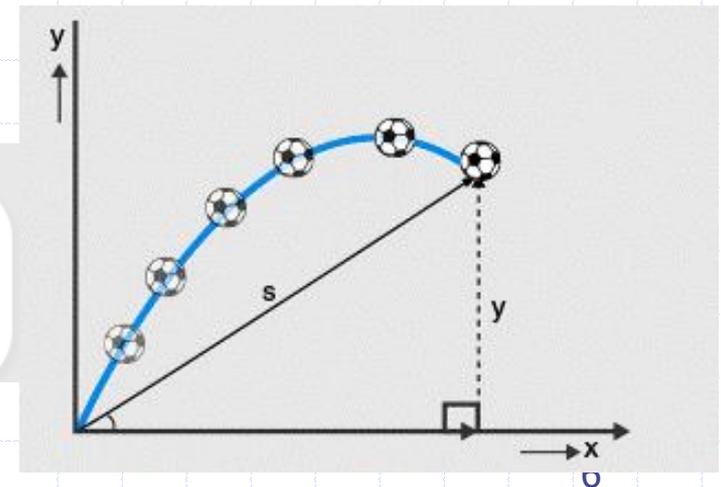
Dynamics is divided into two parts:

Kinematics – the study of motion without reference to any forces

- Study of the geometry of motion
- A purely mathematical construct

KINEMATICS

THE BRANCH OF MECHANICS CONCERNED WITH THE MOTION OF OBJECTS WITHOUT REFERENCE TO THE FORCES WHICH CAUSE THE MOTION.



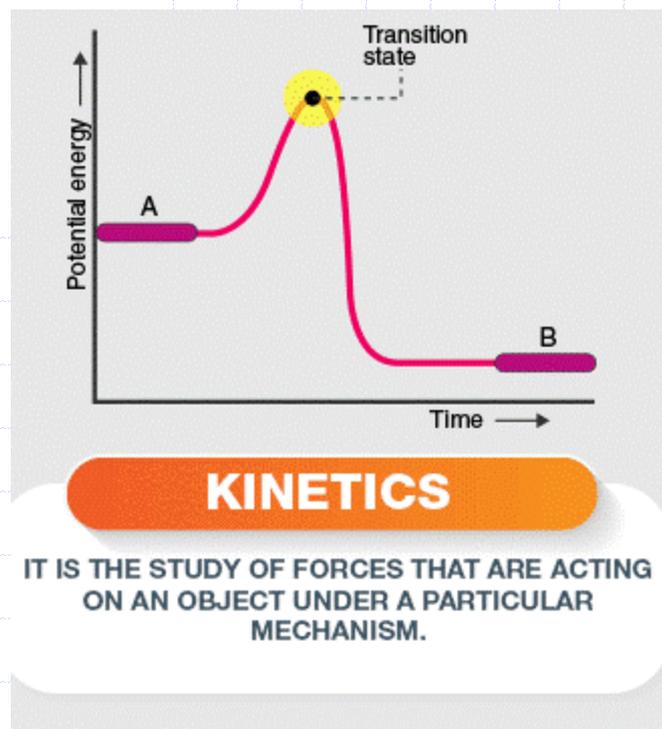
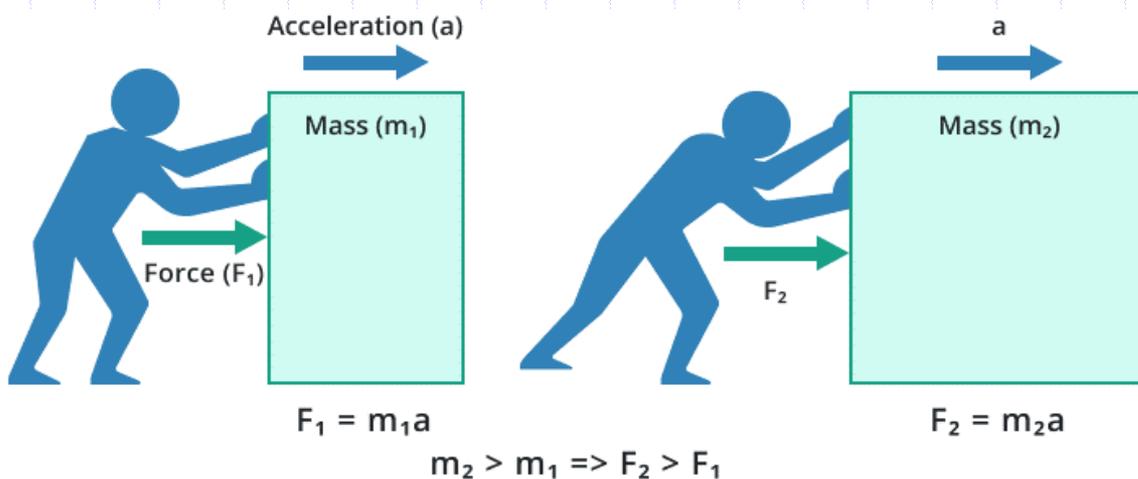
Kinetics – the study of motion that results from forces action on bodies

- Used to predict motion caused by given forces

- OR -

to determine the forces required to produce a given motion

- Based on physical law



1. Kinematics of a Particle



Kinematics of a Particle



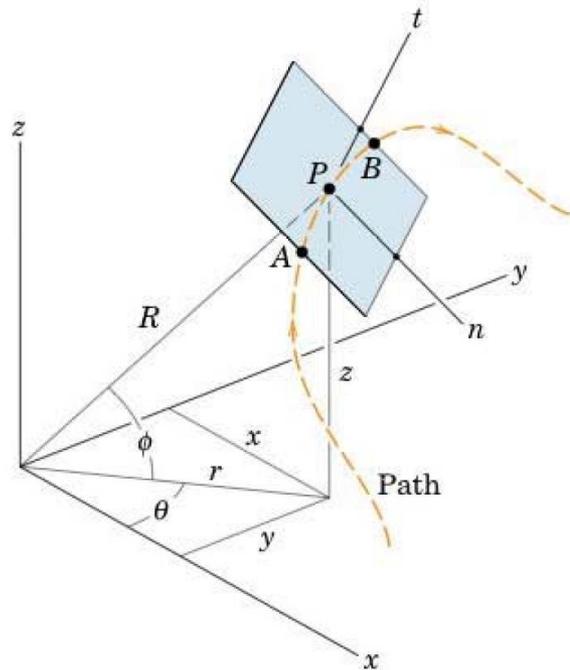
Although each of these planes is rather large, from a distance their motion can be modeled as if each plane were a particle

Kinematics of Particles

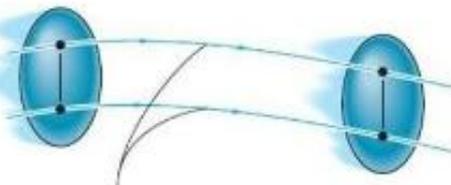
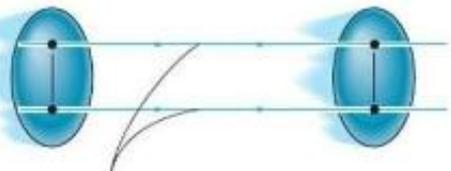
Relates motion and time w/o ref. to forces that cause the motion

Here, "motion" is quantified by

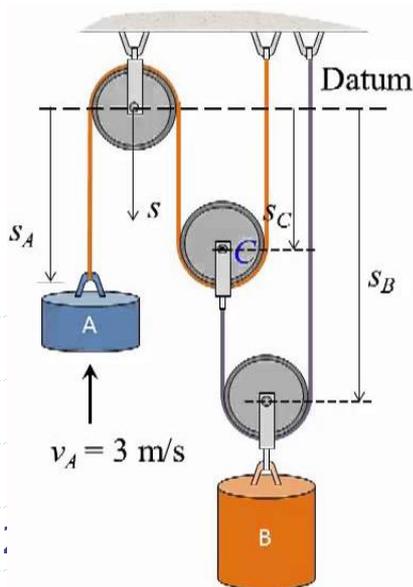
- Position \vec{r}
- Velocity \vec{v}
- Acceleration \vec{a}



Motion Types:



Path of curvilinear translation

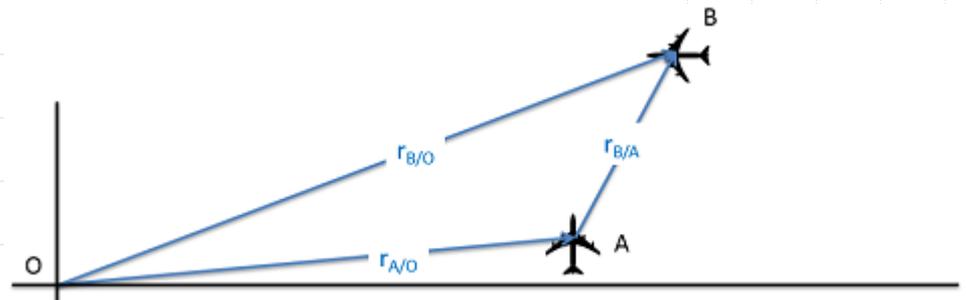


① Rectilinear

② Curvilinear

③ Relative

④ Dependent

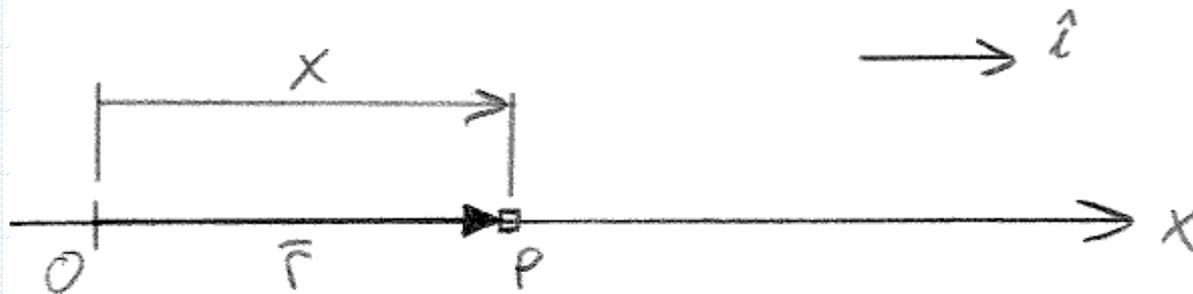


2. Rectilinear Motion



Rectilinear Motion of Particles

Particle is constrained to move in a straight line.



$$\vec{r} = \vec{OP} = x \hat{i}$$

Position Vector

Comments:

A Scalar

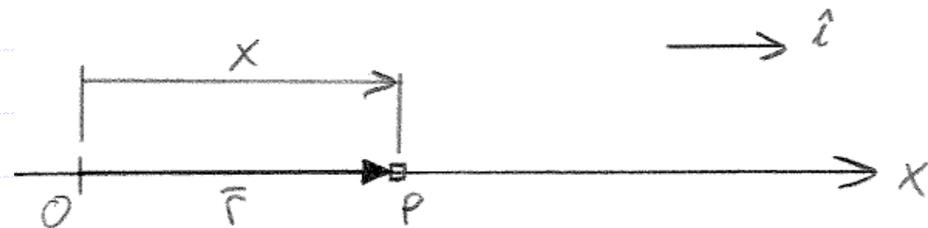
- Magnitude of x specifies distance of P away from O
- sign of x indicates the direction of P relative to O

Scalar notation suffices.
(Rectilinear Motion only!)

Position x

x = algebraic dist. from O to P

(m, ft, mi, etc.)



Velocity v

We define

$$v = \frac{dx}{dt}$$

(m/s,
ft/s,
mph,
etc.)

Instantaneous
velocity

Acceleration a

$$\text{Ave. Acc.} = \frac{\Delta v}{\Delta t} \quad \left(\frac{m/s}{s} = m/s^2, ft/s^2, \text{etc.} \right)$$

In the limit as $\Delta t \rightarrow 0$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Instantaneous
Acceleration

Shorthand:

$$(\dot{\quad}) = \frac{d(\quad)}{dt}$$

\Rightarrow

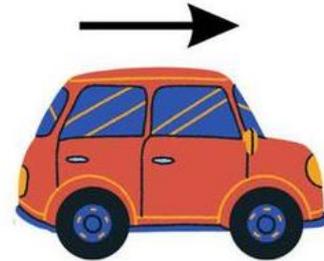
$$v = \dot{x}$$

$$a = \dot{v} = \ddot{x}$$

IMPORTANT POINTS

- ◆ Dynamics is concerned with bodies that have accelerated motion
- ◆ Kinematics is a study of the geometry of the motion
- ◆ Kinetics is a study of the forces that cause the motion
- ◆ Rectilinear kinematics refers to straight-line motion

Speed = 15 m/s
Velocity = 15 m/s
towards south



- ◆ Speed refers to the magnitude of velocity
- ◆ Average speed is the total distance traveled divided by the total time; this is different from the average velocity which is the displacement divided by the time

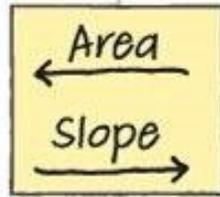
$$\text{average speed} = \frac{\text{tot. distance}}{\text{tot. time}}$$

$$v_{avg.} = \frac{\Delta x}{\Delta t}$$

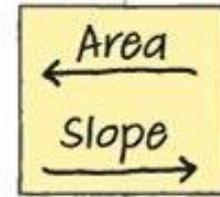
- ◆ The acceleration, $a = dv/dt$, is negative when the particle is slowing down or decelerating.
- ◆ A particle can have an acceleration and yet have zero velocity



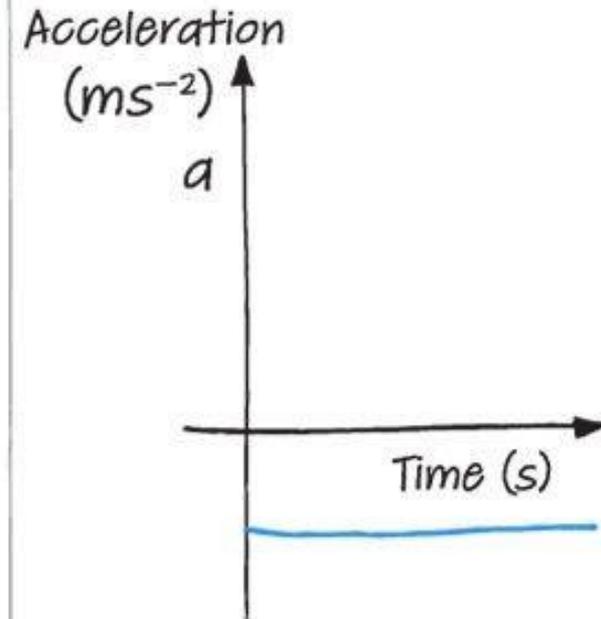
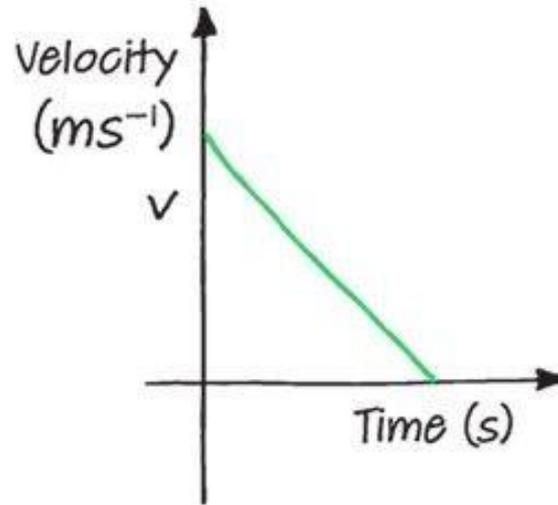
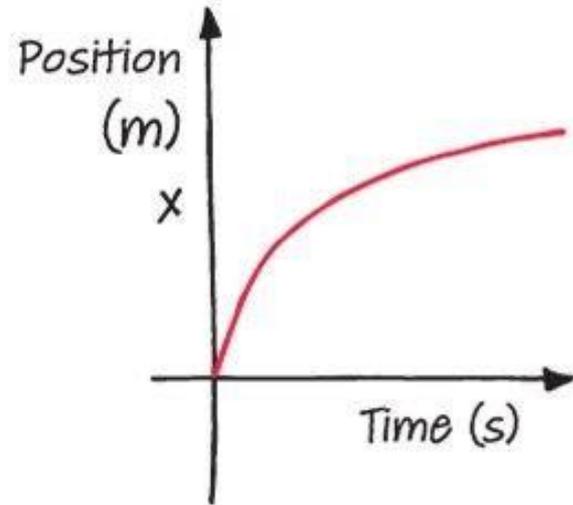
POSITION-TIME



VELOCITY-TIME



ACCELERATION-TIME



Example 1

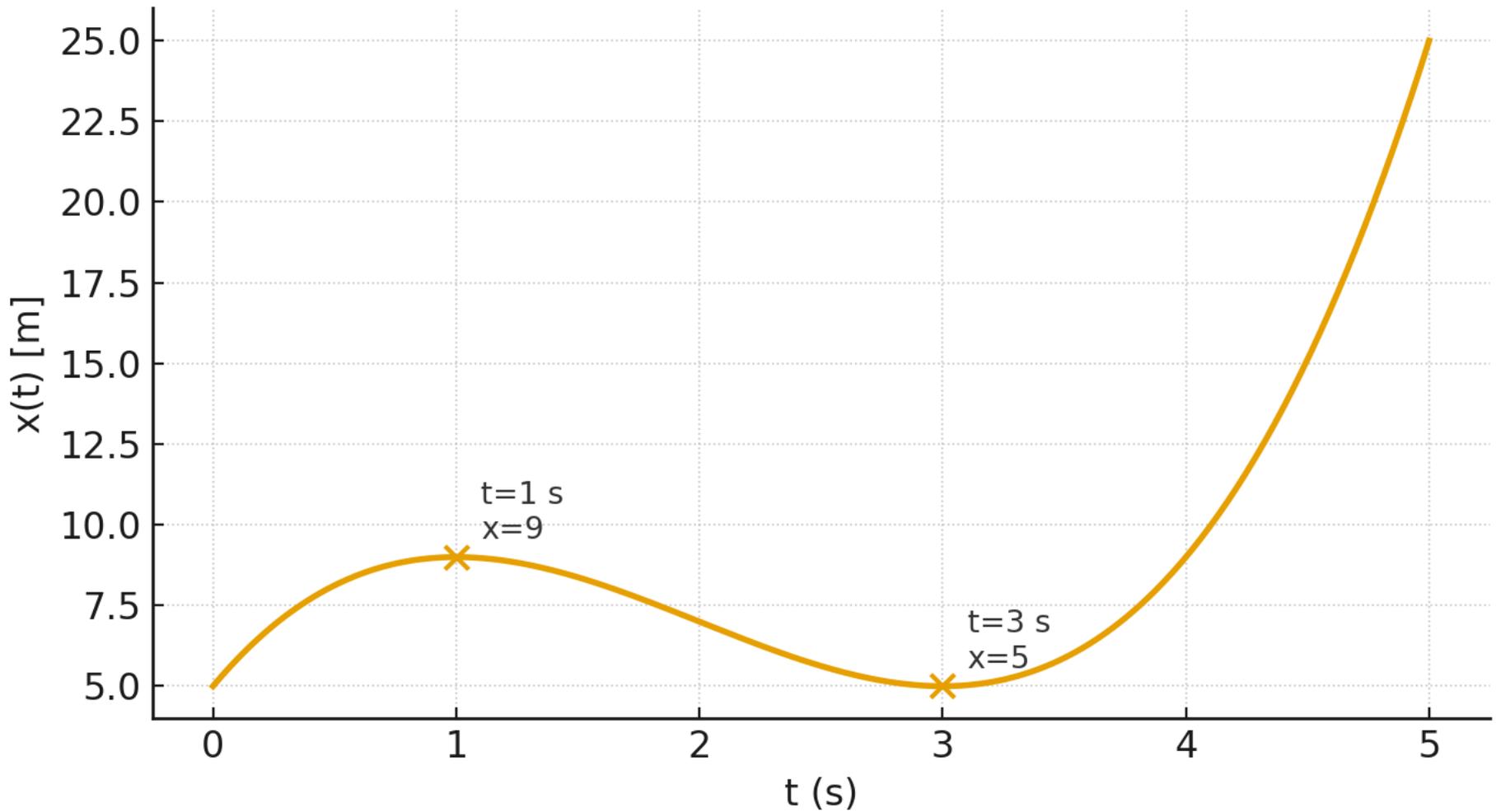
◆ A particle moves along a line with position:

$$\mathbf{x(t) = t^3 - 6t^2 + 9t + 5 \quad (m, t \text{ in s})}$$

◆ Tasks:

- a) Find the total distance traveled from $t=0$ to $t=5$ s.
- b) Determine acceleration at the critical times when velocity is zero.

Displacement vs Time

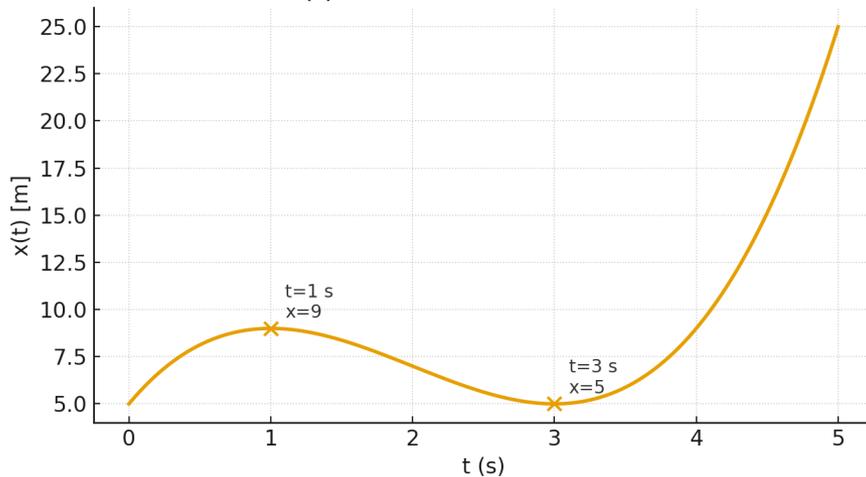
$$x(t) = t^3 - 6t^2 + 9t + 5$$


Step 1. Velocity

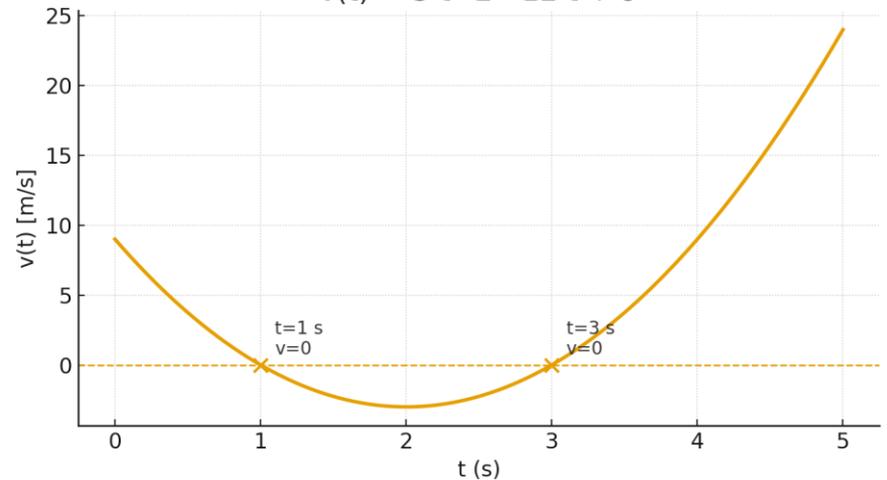
$$\begin{aligned} \diamond v(t) &= dx/dt = 3t^2 - 12t + 9 \\ &= 3(t^2 - 4t + 3) = 3(t-1)(t-3) \end{aligned}$$

\diamond Turning points where $v(t)=0$:
 $t = 1 \text{ s}, t = 3 \text{ s}$

Displacement vs Time
 $x(t) = t^3 - 6t^2 + 9t + 5$

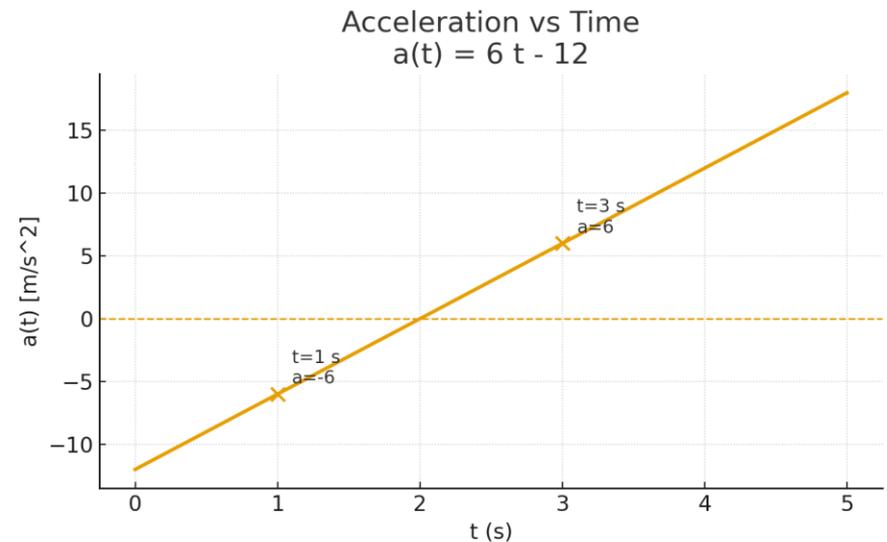
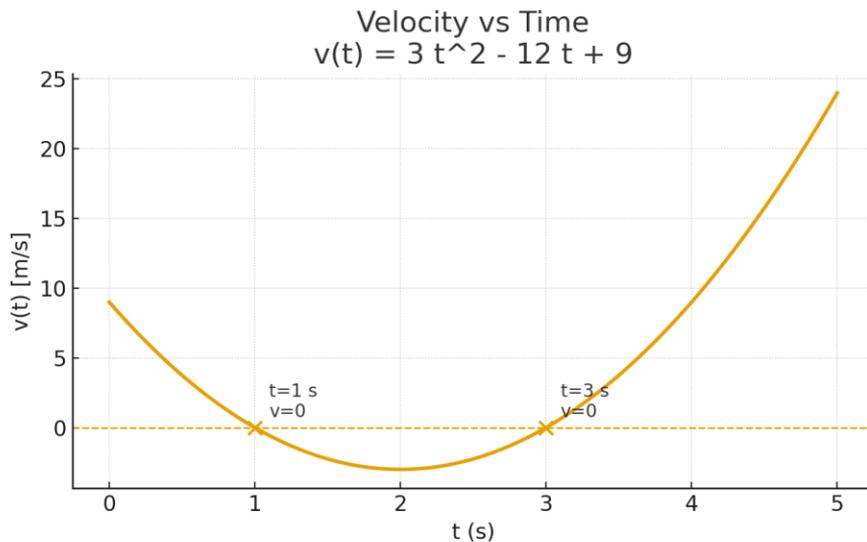


Velocity vs Time
 $v(t) = 3t^2 - 12t + 9$



Step 2. Acceleration

- ◆ $a(t) = dv/dt = 6t - 12$
- ◆ $a(1) = -6 \text{ m/s}^2$ (negative, local maximum)
- ◆ $a(3) = +6 \text{ m/s}^2$ (positive, local minimum)



Step 3. Positions

◆ Evaluate position at key times:

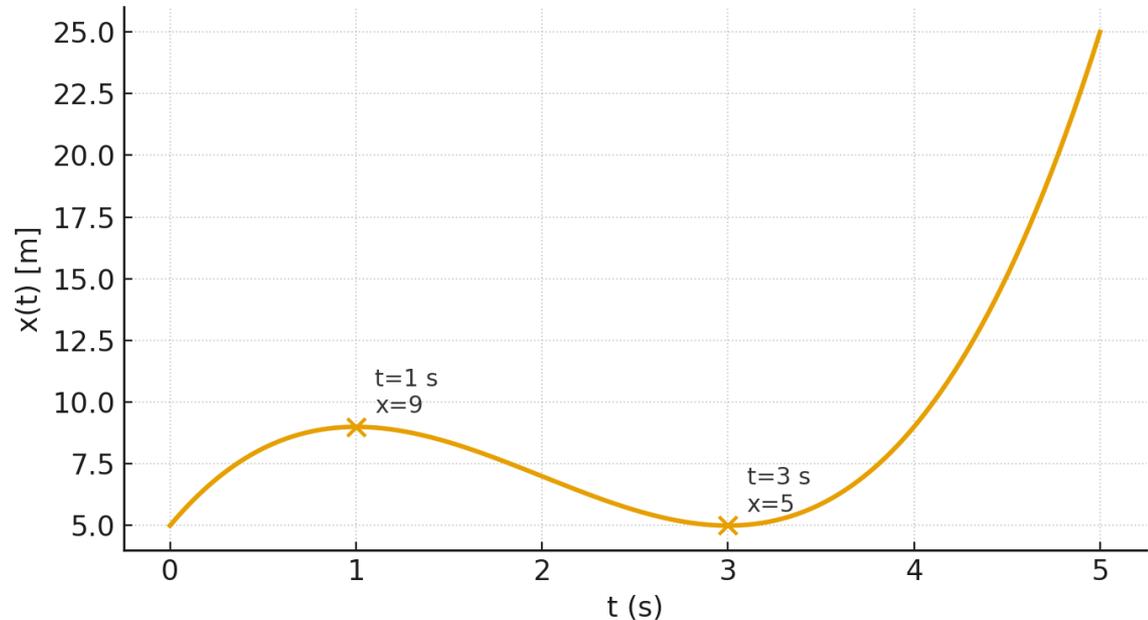
◆ $x(0) = 5 \text{ m}$

◆ $x(1) = 9 \text{ m}$

◆ $x(3) = 5 \text{ m}$

◆ $x(5) = 25 \text{ m}$

Displacement vs Time
 $x(t) = t^3 - 6t^2 + 9t + 5$



Step 4. Distances

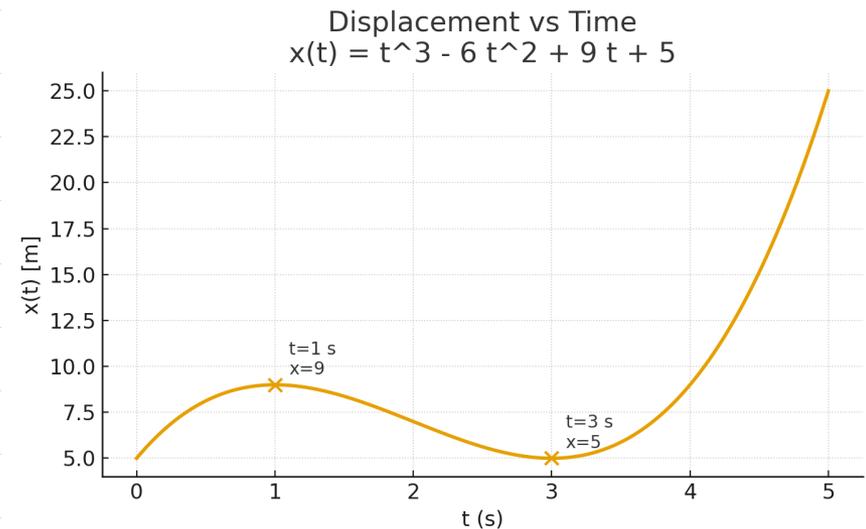
◆ Distance traveled in each interval:

◆ $0 \rightarrow 1 \text{ s}: |9 - 5| = 4 \text{ m}$

◆ $1 \rightarrow 3 \text{ s}: |5 - 9| = 4 \text{ m}$

◆ $3 \rightarrow 5 \text{ s}: |25 - 5| = 20 \text{ m}$

◆ **Total Distance
= 28 m (Answer)**

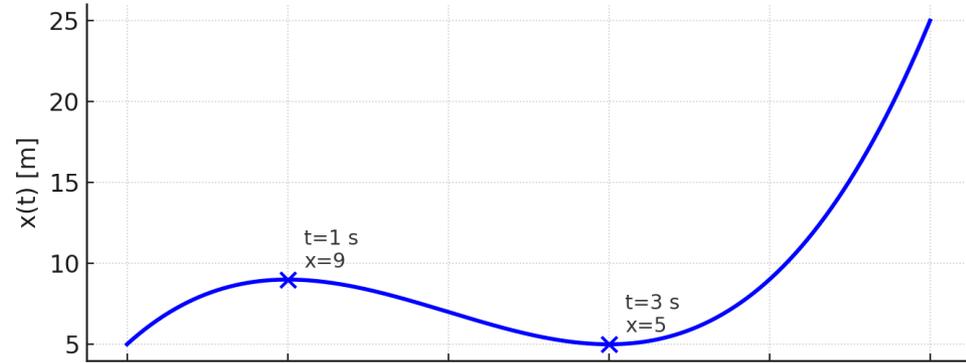


Notes

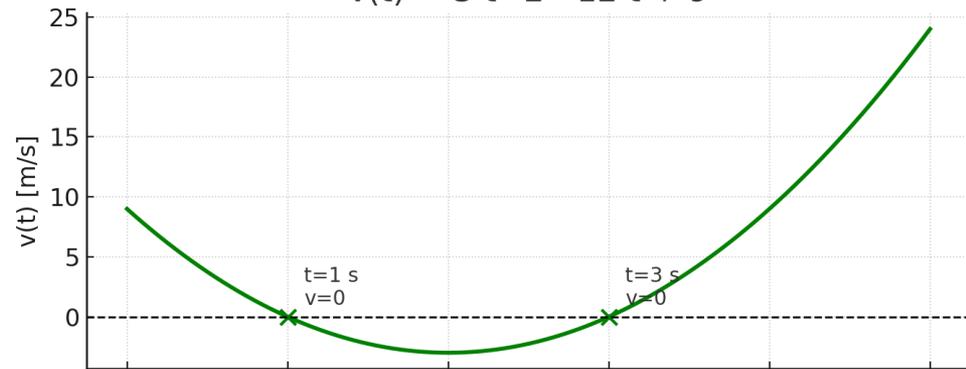
- Displacement vs Distance:
displacement = 20 m,
distance = 28 m
- At $t=1$ s and $t=3$ s: velocity = 0,
acceleration changes sign

Displacement, Velocity, and Acceleration

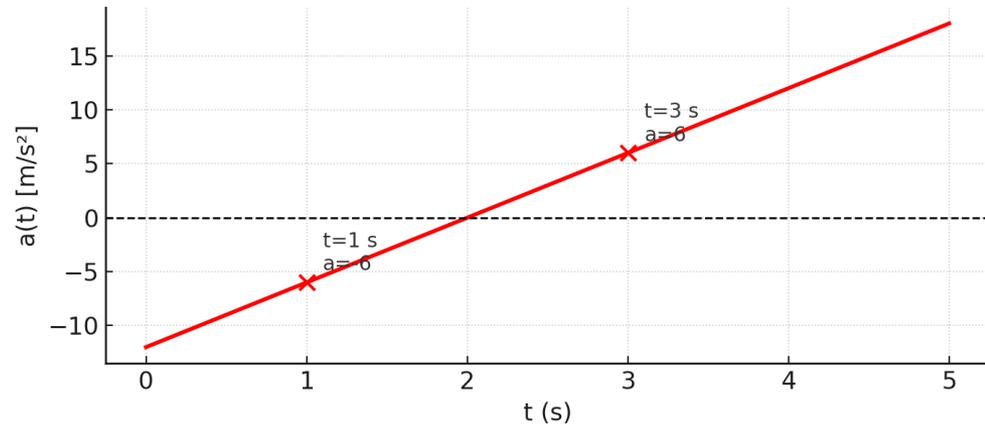
Displacement vs Time
 $x(t) = t^3 - 6t^2 + 9t + 5$



Velocity vs Time
 $v(t) = 3t^2 - 12t + 9$



Acceleration vs Time
 $a(t) = 6t - 12$



3. Particle Motion Cases



Determination of Particle Motions

- Typically acc. known (measured, via $\vec{F} = m\vec{a}$)
- Vel. & pos. found via integration

Common Cases:

① $a = a(t)$

② $a = a(x)$

③ $a = a(v)$

Integrate Directly
(Show by Example)

Special Cases:

$a = \text{const.}$

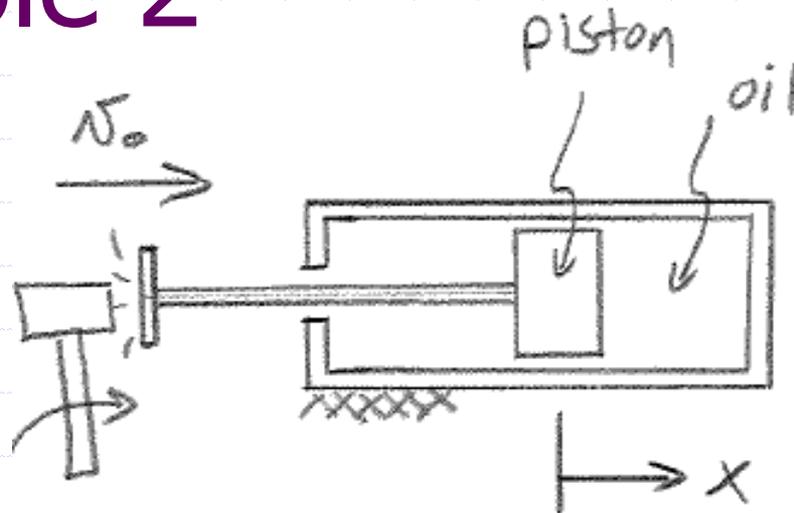
④ $a = 0$

⑤ $a = \text{const}$

}

lead to
Kinematic Equations

Example 2



$$\text{@ } t = 0, x = 0, v = v_0 > 0$$

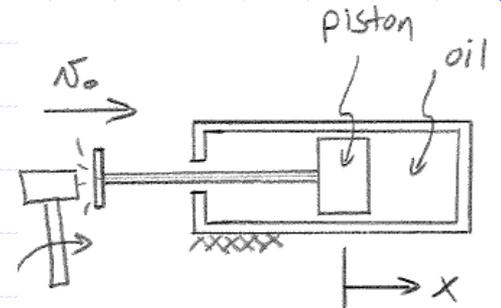
$$a(v) = -kv \quad (\text{Case ③})$$

$k > 0$ is constant

Express (A) $v = v(t)$

(B) $x = x(t)$

(C) $v = v(x)$



PART A: Find $v = v(t)$

$$a(v) = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{a(v)} = \frac{dv}{-Kv}$$

$$\Rightarrow \int_0^t dt = \frac{-1}{K} \int_{v_0}^v \frac{dv}{v} \quad \int \frac{1}{x} dx = \ln|x| + C$$

$$\Rightarrow t - 0 = \frac{-1}{k} (\ln |N| - \ln |N_0|)$$

$$= \frac{-1}{k} \ln \left(\frac{N}{N_0} \right)$$

$$\Rightarrow \ln \left(\frac{N}{N_0} \right) = -kt$$

$$y = e^x \Rightarrow x = \ln(y)$$

$$\Rightarrow \frac{N}{N_0} = e^{-kt}$$

$$\therefore N(t) = N_0 e^{-kt}$$

PART B: Find $x = x(t)$

$$v(t) = \frac{dx}{dt}$$

Known from (A)

$$\Rightarrow dx = v(t) dt = v_0 e^{-kt} dt$$

$$\Rightarrow \int_{x_0}^x dx = v_0 \int_0^t e^{-kt} dt$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\Rightarrow X - \cancel{X_0}^0 = \cancel{X_0} \cdot \frac{-1}{K} e^{-Kt} \Big|_0^t$$
$$= -\frac{\cancel{X_0}}{K} (e^{-Kt} - 1)$$

$$\therefore X(t) = \frac{\cancel{X_0}}{K} (1 - e^{-Kt})$$

PART C: Find $v = v(x)$

$$a(v) = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v$$

"chain rule"

Given

$$\Rightarrow dx = \frac{v dv}{a(v)} = \frac{v dv}{-kv} = -\frac{1}{k} dv$$

$$\Rightarrow \int_{x_0}^x dx = \frac{-1}{K} \int_{\nu_0}^{\nu} d\nu$$

$$\Rightarrow x - \cancel{x_0} = \frac{-1}{K} (\nu - \nu_0)$$

$$\Rightarrow \nu - \nu_0 = -Kx$$

$$\therefore \nu(x) = \nu_0 - Kx$$

4. Kinematic Equations



Uniform Rectilinear Motion

Case (4):

$$a = \frac{dv}{dt} = 0 \quad \Rightarrow \quad v = \text{const.}$$

$$\text{Then } v = \frac{dx}{dt} \quad \Rightarrow \quad dx = v dt$$

$$\Rightarrow x - x_0 = v(t - 0)$$

$$x = x_0 + vt$$

pos. - time

Uniformly Accelerated Rectilinear Motion

Case ⑤: $a = \text{const.}$

Then

$$a = \frac{dv}{dt} \Rightarrow dv = a dt$$

$$\Rightarrow v - v_0 = a(t - 0)$$

$$v = v_0 + at$$

vel. - time

Now consider

$$v = \frac{dx}{dt} \Rightarrow dx = v(t) dt$$

$v_0 + at$

$$\Rightarrow dx = (v_0 + at) dt$$

$$\Rightarrow x - x_0 = v_0 t + a \frac{t^2}{2}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

pos-time

Another relationship - Invoke chain rule

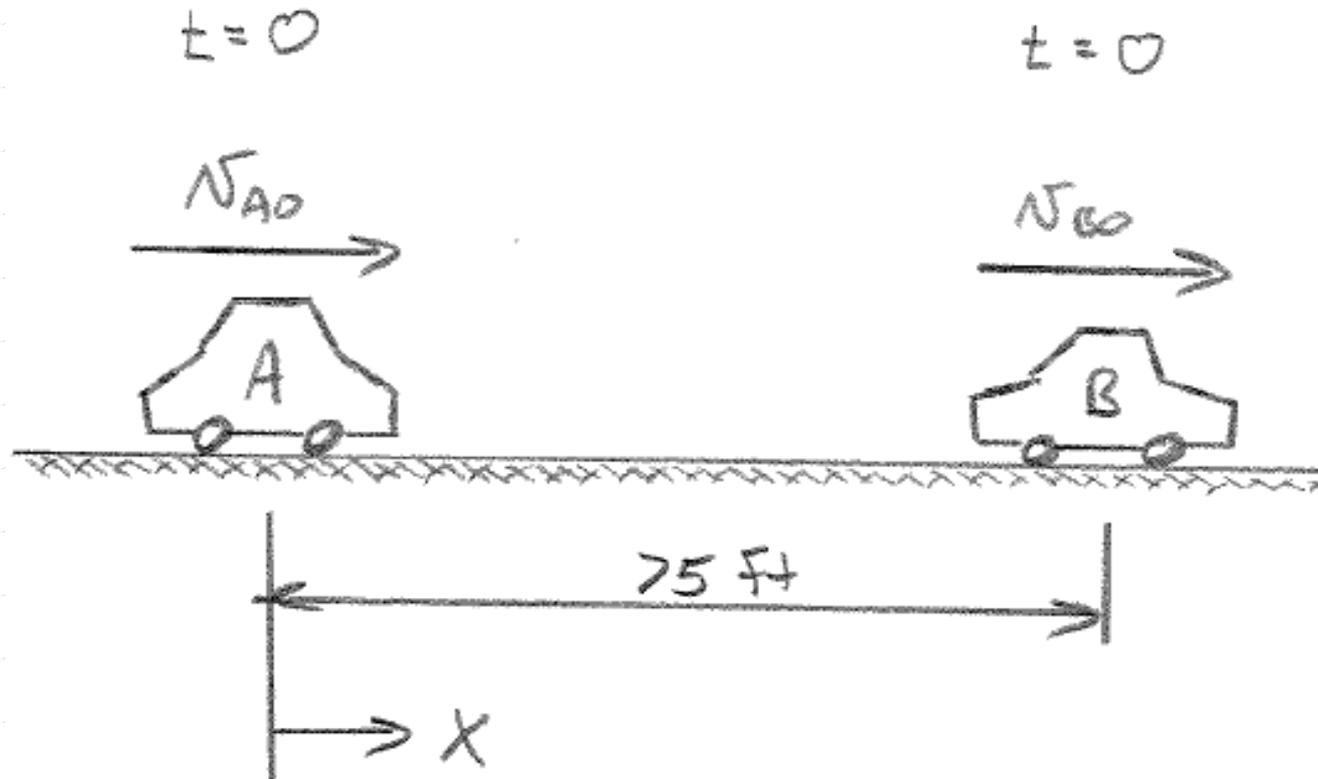
$$a = \frac{dv}{dt} = v \frac{dv}{dx} \Rightarrow v dv = a dx$$

$$\Rightarrow \frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

vel. - pos.

Example 3



Data:

$$v_A(t=0) = v_{A0} = 24 \text{ mph}, \quad a_A = 1.8 \text{ ft/s}^2$$

$$v_B(t=0) = v_{B0} = 36 \text{ mph}, \quad a_B = -1.2 \text{ ft/s}^2$$

Find: (A) When \dot{x} where A overtakes B

(B) corresponding speed of each car

PART A: Use kinematic eqns for $a = \text{const.}$

$$v = v_0 + at \text{ ————— } \textcircled{1}$$

$$x - x_0 = v_0 t + \frac{1}{2} at^2 \text{ — } \textcircled{2} \quad \leftarrow \text{pos. - time}$$

$$v^2 - v_0^2 = 2a(x - x_0) \text{ — } \textcircled{3}$$

$$\text{Car A: } x_A(t) = x_{A0} + v_{A0}t + \frac{1}{2} a_A t^2 \text{ — } \textcircled{4}$$

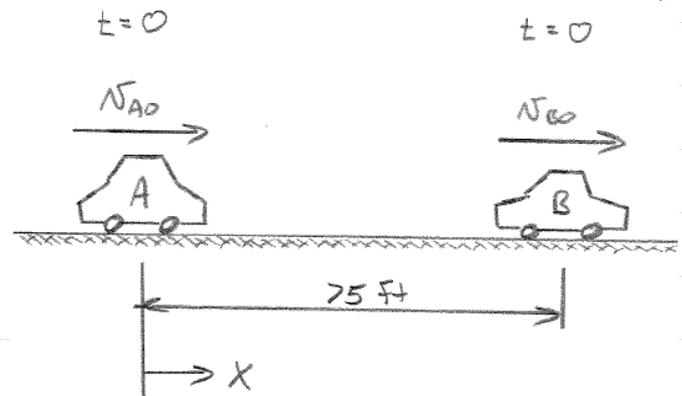
$$\text{Car B: } x_B(t) = x_{B0} + v_{B0}t + \frac{1}{2} a_B t^2$$

$$\text{Here, } x_{A0} = 0 \quad , \quad x_{B0} = 75 \text{ ft}$$

$$v_{A0} = \left(24 \frac{\text{mi}}{\text{hr}}\right) \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) \left(\frac{1 \text{ hr}}{3600 \text{ s}}\right) = 35.2 \text{ ft/s}$$

$$v_{B0} = 36 \text{ mph} = 52.8 \text{ ft/s}$$

a_A, a_B given



A overtakes B when $x_A(t) = x_B(t)$

$$\Rightarrow x_{A0} + v_{A0}t + \frac{1}{2}a_A t^2 = x_{B0} + v_{B0}t + \frac{1}{2}a_B t^2$$

$$\Rightarrow (a_A - a_B)t^2 + 2(v_{A0} - v_{B0})t + 2(x_{A0} - x_{B0}) = 0$$

$$\Rightarrow \left(3 \frac{\text{ft}}{\text{s}^2}\right)t^2 + \left(-35.2 \frac{\text{ft}}{\text{s}}\right)t + (-150 \text{ ft}) = 0$$

of form $at^2 + bt + c = 0$

so
$$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow t_1 = 15.05 \text{ s}$

$t_2 = -3.32 \text{ s} \leftarrow \text{nonsense}$

$\therefore t = t_1 = 15.1 \text{ s}$

For positions, plug $t = t_1$ into (4).

$x_p = 733.7 \text{ ft} \quad (\text{or } \sim 737 \text{ ft?})$

try this [tool](#)



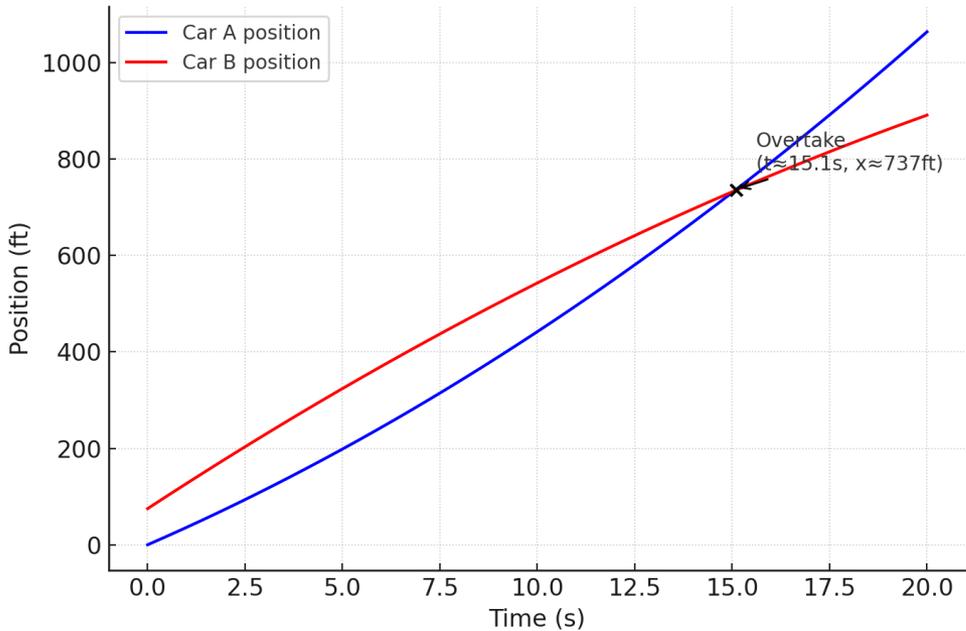
PART B: Find $v_A(t_1)$, $v_B(t_1)$

⊕ Easy! Use ①.

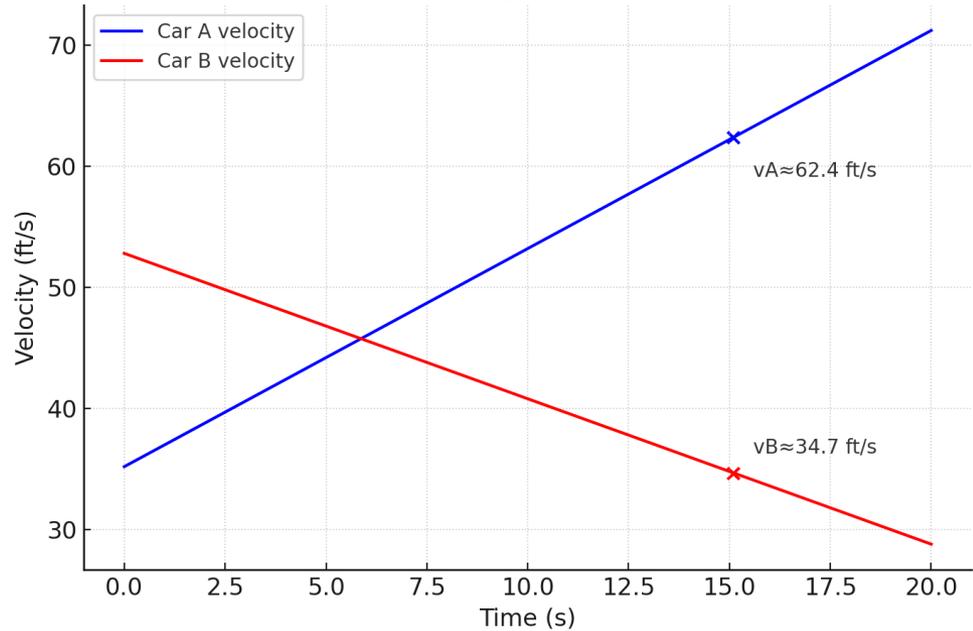
$$v_A(t_1) = v_{A0} + a_A t_1 = 62.3 \text{ ft/s} = 42.5 \text{ mph}$$

$$v_B(t_1) = v_{B0} + a_B t_1 = 34.7 \text{ ft/s} = 23.7 \text{ mph}$$

Position vs Time



Velocity vs Time



5. Relative Rectilinear Motion

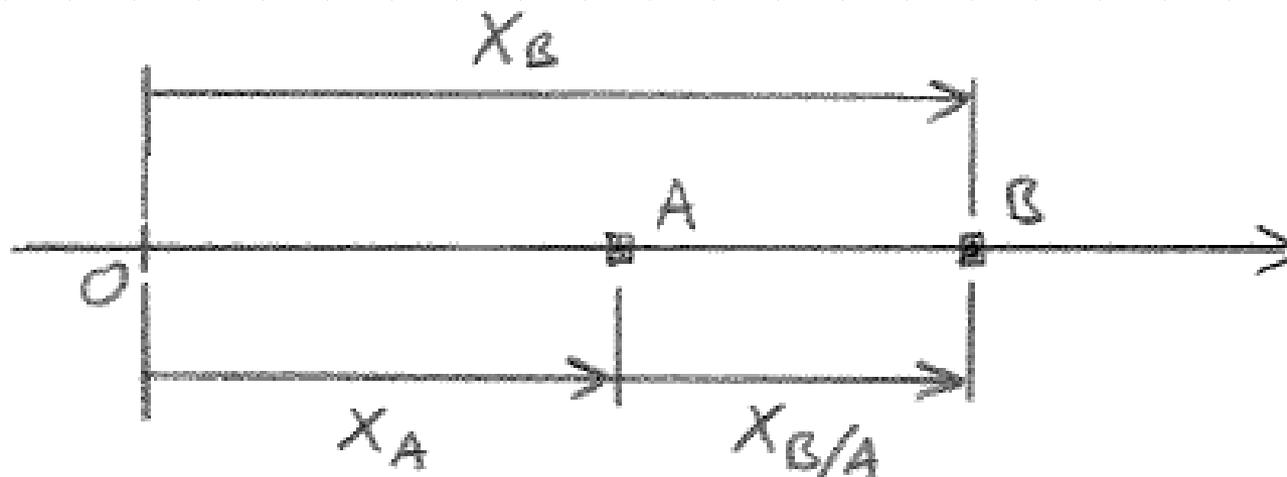


Relative Rectilinear Motion

↳ Relative motions are often more important than absolute motions.

We define $x_{B/A} = x_B - x_A$

B/A reads "B with respect to A"



$\frac{d}{dt}$ to obtain

$$v_{B/A} = v_B - v_A$$

$$a_{B/A} = a_B - a_A$$

SUMMARY



Rectilinear Motion

Velocity and Acceleration

$$v = \frac{dx}{dt} \quad a = \frac{dv}{dt}$$

Cases 1,2,3: $a = a(t)$, $a = a(x)$, $a = a(v)$

Integrate directly.

Rectilinear Motion

Velocity and Acceleration

$$v = \frac{dx}{dt} \quad a = \frac{dv}{dt}$$

Cases 4,5: $a = 0$, $a = \text{constant}$

$$x = x_0 + vt$$

$$v = v_0 + at$$

$$x - x_0 = v_0 t + \frac{1}{2} at^2$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

Additional Problems



Sample Problem 2/1

The position coordinate of a particle which is confined to move along a straight line is given by $s = 2t^3 - 24t + 6$, where s is measured in meters from a convenient origin and t is in seconds. Determine (a) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$, (b) the acceleration of the particle when $v = 30$ m/s, and (c) the net displacement of the particle during the interval from $t = 1$ s to $t = 4$ s.

Solution. The velocity and acceleration are obtained by successive differentiation of s with respect to the time. Thus,

$$[v = \dot{s}] \quad v = 6t^2 - 24 \text{ m/s}$$

$$[a = \dot{v}] \quad a = 12t \text{ m/s}^2$$

- ① **(a)** Substituting $v = 72 \text{ m/s}$ into the expression for v gives us $72 = 6t^2 - 24$, from which $t = \pm 4 \text{ s}$. The negative root describes a mathematical solution for t before the initiation of motion, so this root is of no physical interest. Thus, the desired result is

$$t = 4 \text{ s} \quad \text{Ans.}$$

- (b)** Substituting $v = 30 \text{ m/s}$ into the expression for v gives $30 = 6t^2 - 24$, from which the positive root is $t = 3 \text{ s}$, and the corresponding acceleration is

$$a = 12(3) = 36 \text{ m/s}^2 \quad \text{Ans.}$$

- (c)** The net displacement during the specified interval is

$$\Delta s = s_4 - s_1 \quad \text{or}$$

$$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6]$$

$$= 54 \text{ m} \quad \text{Ans.}$$

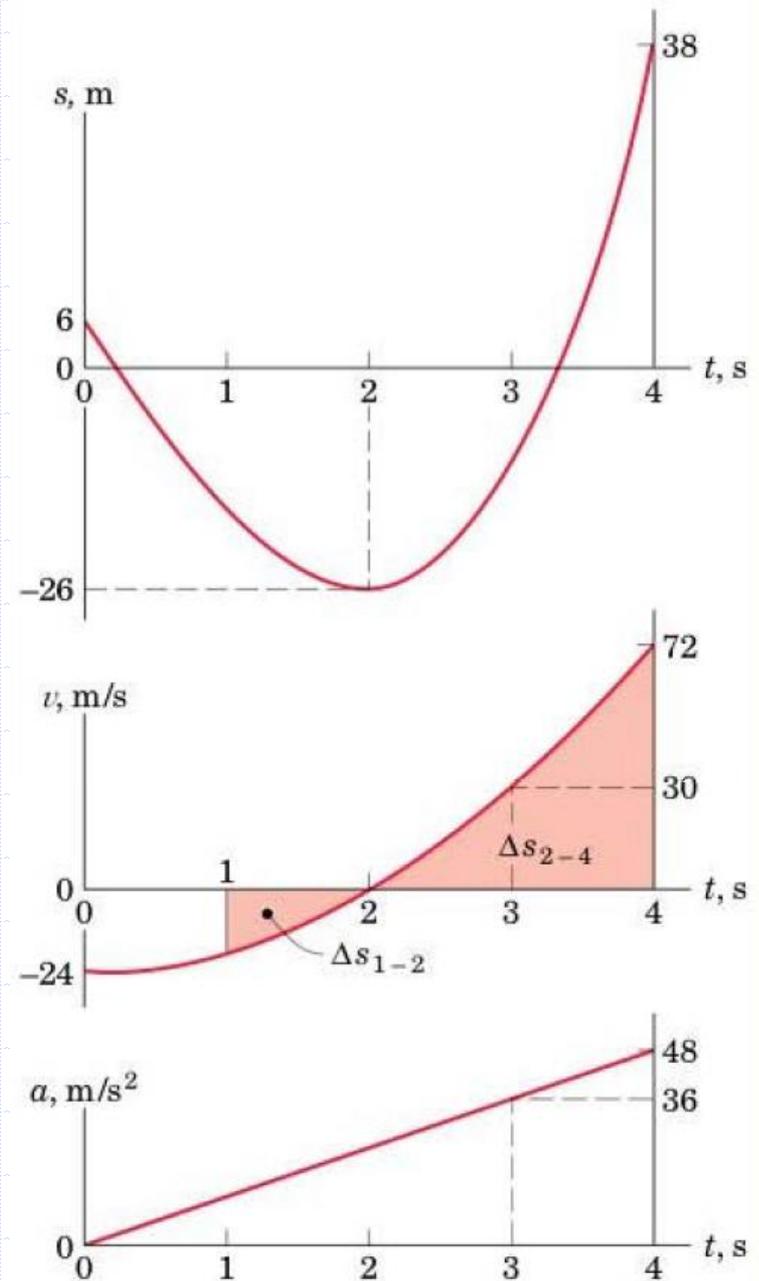
- ② which represents the net advancement of the particle along the s -axis from the position it occupied at $t = 1 \text{ s}$ to its position at $t = 4 \text{ s}$.

To help visualize the motion, the values of s , v , and a are plotted against the time t as shown. Because the area under the v - t curve represents displacement,

- ③ we see that the net displacement from $t = 1 \text{ s}$ to $t = 4 \text{ s}$ is the positive area Δs_{2-4} less the negative area Δs_{1-2} .

Helpful Hints

- ① Be alert to the proper choice of sign when taking a square root. When the situation calls for only one answer, the positive root is not always the one you may need.
- ② Note carefully the distinction between italic s for the position coordinate and the vertical s for seconds.
- ③ Note from the graphs that the values for v are the slopes (\dot{s}) of the s - t curve and that the values for a are the slopes (\dot{v}) of the v - t curve. *Suggestion:* Integrate $v dt$ for each of the two intervals and check the answer for Δs . Show that the total distance traveled during the interval $t = 1$ s to $t = 4$ s is 74 m.



Sample Problem 2/4

- ① A freighter is moving at a speed of 8 knots when its engines are suddenly stopped. If it takes 10 minutes for the freighter to reduce its speed to 4 knots, determine and plot the distance s in nautical miles moved by the ship and its speed v in knots as functions of the time t during this interval. The deceleration of the ship is proportional to the square of its speed, so that $a = -kv^2$.

Solution. The speeds and the time are given, so we may substitute the expression for acceleration directly into the basic definition $a = dv/dt$ and integrate. Thus,

$$-kv^2 = \frac{dv}{dt} \quad \frac{dv}{v^2} = -k dt \quad \int_8^v \frac{dv}{v^2} = -k \int_0^t dt$$

②

$$-\frac{1}{v} + \frac{1}{8} = -kt \quad v = \frac{8}{1 + 8kt}$$

Now we substitute the end limits of $v = 4$ knots and $t = \frac{10}{60} = \frac{1}{6}$ hour and get

$$4 = \frac{8}{1 + 8k(1/6)} \quad k = \frac{3}{4} \text{ mi}^{-1} \quad v = \frac{8}{1 + 6t} \quad \text{Ans.}$$

The speed is plotted against the time as shown.

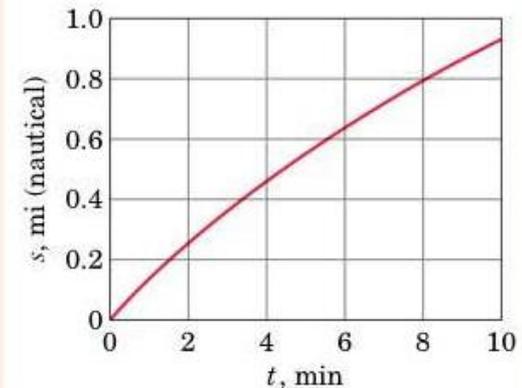
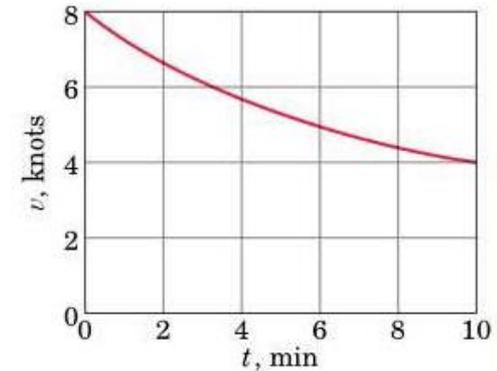
The distance is obtained by substituting the expression for v into the definition $v = ds/dt$ and integrating. Thus,

$$\frac{8}{1 + 6t} = \frac{ds}{dt} \quad \int_0^t \frac{8 dt}{1 + 6t} = \int_0^s ds \quad s = \frac{4}{3} \ln(1 + 6t) \quad \text{Ans.}$$

The distance s is also plotted against the time as shown, and we see that the ship has moved through a distance $s = \frac{4}{3} \ln(1 + \frac{6}{6}) = \frac{4}{3} \ln 2 = 0.924$ mi (nautical) during the 10 minutes.

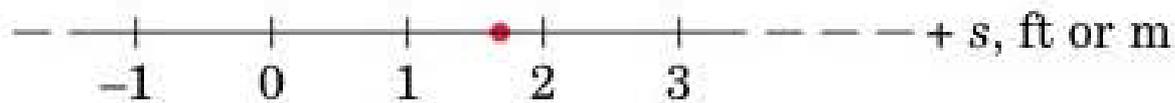
Helpful Hints

- ① Recall that one knot is the speed of one nautical mile (6076 ft) per hour. Work directly in the units of nautical miles and hours.
- ② We choose to integrate to a general value of v and its corresponding time t so that we may obtain the variation of v with t .

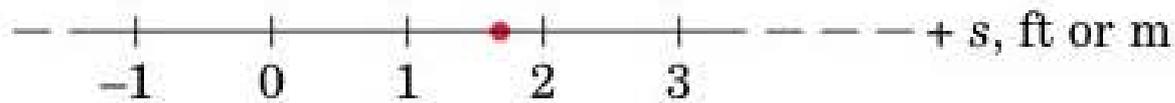


2/1 The velocity of a particle is given by $v = 25t^2 - 80t - 200$, where v is in feet per second and t is in seconds. Plot the velocity v and acceleration a versus time for the first 6 seconds of motion and evaluate the velocity when a is zero.

Ans. $v = -264$ ft/sec



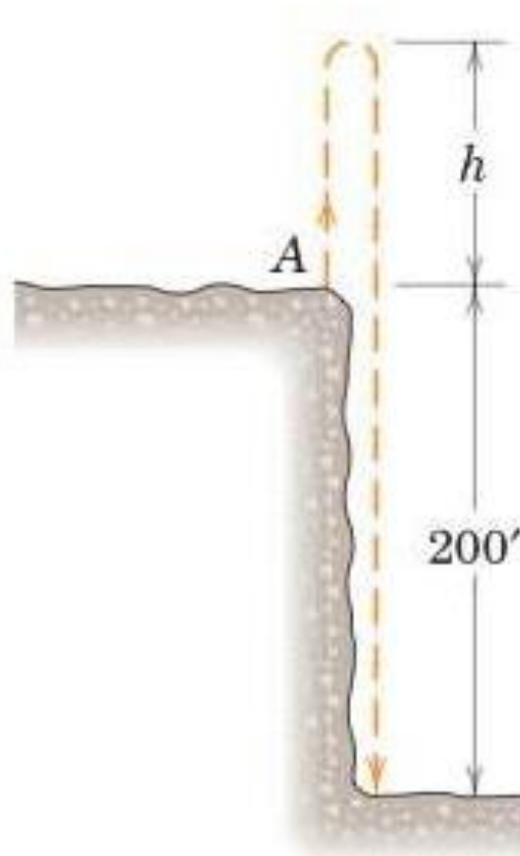
2/2 The position of a particle is given by $s = 2t^3 - 40t^2 + 200t - 50$, where s is in meters and t is in seconds. Plot the position, velocity, and acceleration as functions of time for the first 12 seconds of motion. Determine the time at which the velocity is zero.



2/6 The acceleration of a particle is given by $a = -ks^2$, where a is in meters per second squared, k is a constant, and s is in meters. Determine the velocity of the particle as a function of its position s . Evaluate your expression for $s = 5$ m if $k = 0.1 \text{ m}^{-1}\text{s}^{-2}$ and the initial conditions at time $t = 0$ are $s_0 = 3$ m and $v_0 = 10$ m/s.

2/8 The velocity of a particle moving in a straight line is decreasing at the rate of 3 m/s per meter of displacement at an instant when the velocity is 10 m/s. Determine the acceleration a of the particle at this instant.

2/10 A ball is thrown vertically up with a velocity of 80 ft/sec at the edge of a 200-ft cliff. Calculate the height h to which the ball rises and the total time t after release for the ball to reach the bottom of the cliff. Neglect air resistance and take the downward acceleration to be 32.2 ft/sec^2 .



2/11 A rocket is fired vertically up from rest. If it is designed to maintain a constant upward acceleration of $1.5g$, calculate the time t required for it to reach an altitude of 30 km and its velocity at that position.

Ans. $t = 63.9$ s, $v = 940$ m/s

2/32 A motorcycle patrolman starts from rest at A two seconds after a car, speeding at the constant rate of 120 km/h, passes point A. If the patrolman accelerates at the rate of 6 m/s^2 until he reaches his maximum permissible speed of 150 km/h, which he maintains, calculate the distance s from point A to the point at which he overtakes the car.

