Chapter 2 – Part 2
Moment – Couple – Resultant

STATICS, AGE-1330
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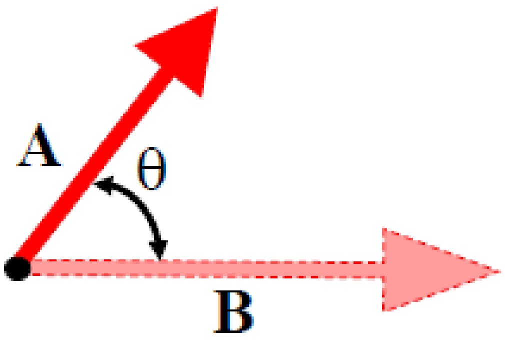
Product of 2 Vectors: Dot Product

- **Dot Product** *(Scalar product)*
  - $\mathbf{A}.\mathbf{B} = AB \cos \theta$



  - **Applications**
    - Determination of the angle between two vectors

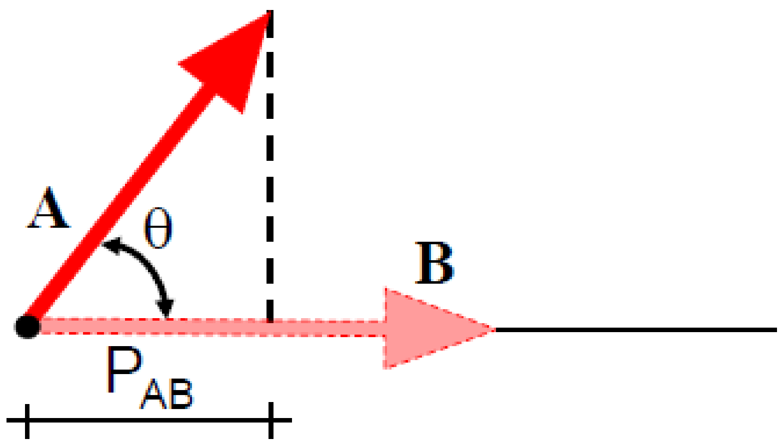
$$\mathbf{A}.\mathbf{B} = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}).(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}) = A_xB_x+A_yB_y+A_zB_z$$

$$\mathbf{A}.\mathbf{B} = AB\cos\theta$$

Obtain θ

Product of 2 Vectors: Dot Product

- **Applications**
  - Determination of the projection of a vector on a given axis

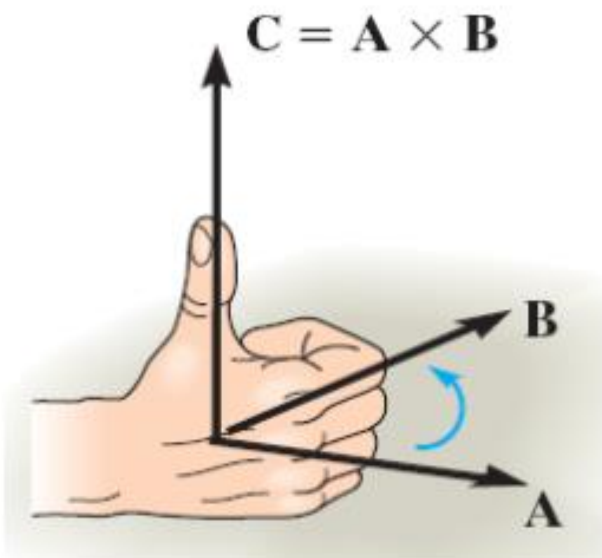


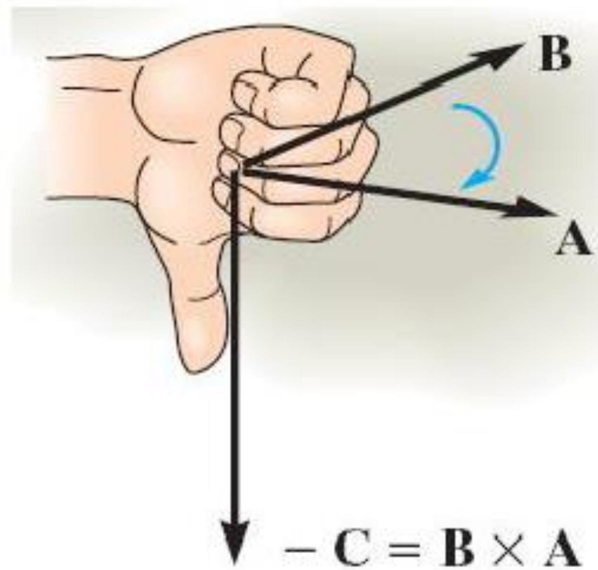
$\mathbf{A}.\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$

$P_{AB} = A\cos\theta = (\mathbf{A}.\mathbf{B})/B$

Product of 2 Vectors: Cross Product

- **Cross Product** *(Vector Product)*
  - $\mathbf{C} = \mathbf{A} \times \mathbf{B}$
    - $C = AB \sin\theta$

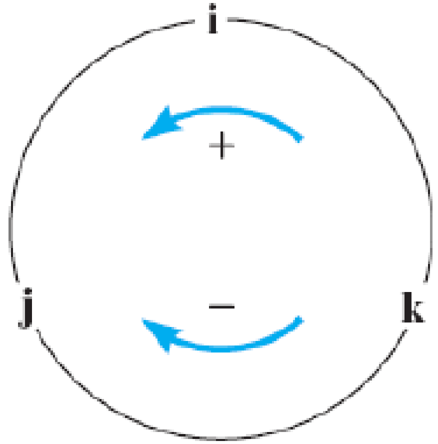


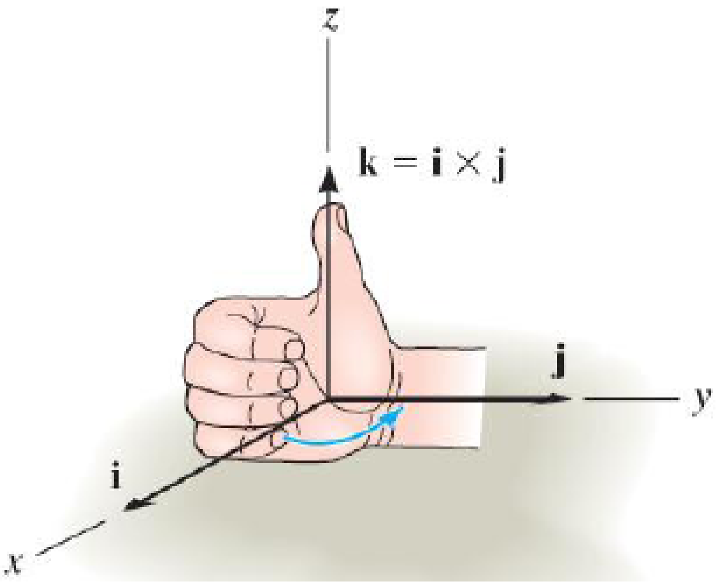


Product of 2 Vectors: Cross Product

- **Cross Product**

Cartesian Vector





$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \mathbf{i} \times \mathbf{i} = 0$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{j} \times \mathbf{j} = 0$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{k} \times \mathbf{k} = 0$$

Product of 2 Vectors: Cross Product

- Cross Product
  - Distributive property
    - $\mathbf{C} \times (\mathbf{A}+\mathbf{B}) = \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}$

$$\mathbf{A}\times\mathbf{B} = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}) \times (B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})$$

$$= (A_yB_z - A_zB_y)\mathbf{i} + (\ldots.)\mathbf{j} + (\ldots.)\mathbf{k}$$

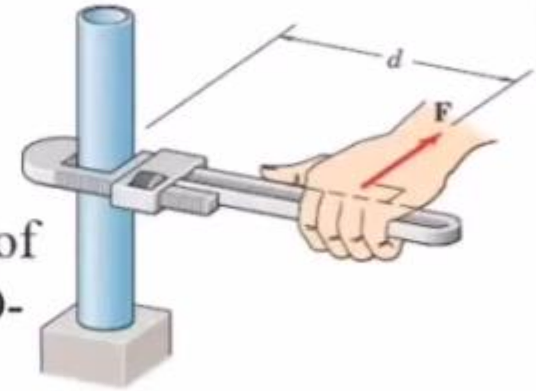
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \end{vmatrix}$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \quad \mathbf{i}\times\mathbf{k}=-\mathbf{j} \quad \mathbf{i}\times\mathbf{i}=\mathbf{0}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i} \quad \mathbf{j}\times\mathbf{i}=-\mathbf{k} \quad \mathbf{j}\times\mathbf{j}=\mathbf{0}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j} \quad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \quad \mathbf{k}\times\mathbf{k}=\mathbf{0}$$

2- Moment

A force can rotate a body about an axis

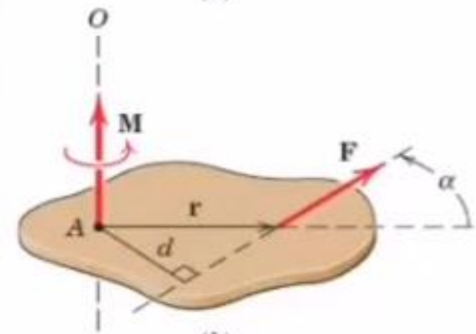
The magnitude of the moment or tendency of the force to rotate the body about the axis O-O perpendicular to the plane of the body is proportional both to ***the magnitude of the force and to the moment arm d.***



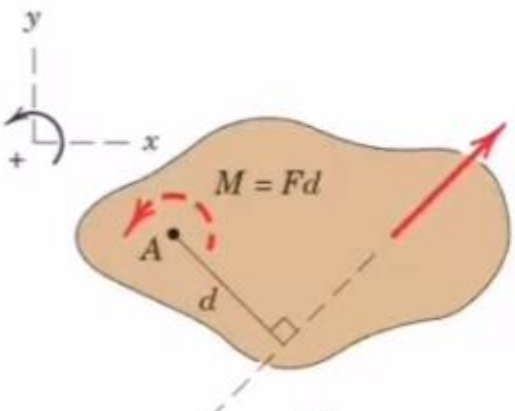
(a)

$$M = Fd$$

(N.m)



(b)

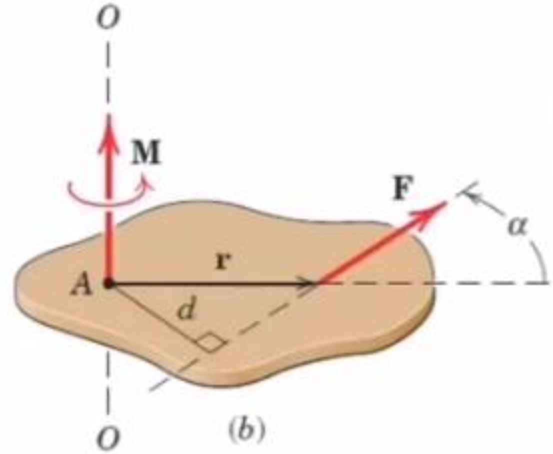


(d)

NOTE: The moment is a vector **M** perpendicular to the plane of the body

The Cross Product

A vector approach for moment calculations. The moment of **F** about point A of may be represented by the cross-product expression.



$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where **r** is a **position vector** which runs from the moment reference point A to any point on the line of action of **F**.

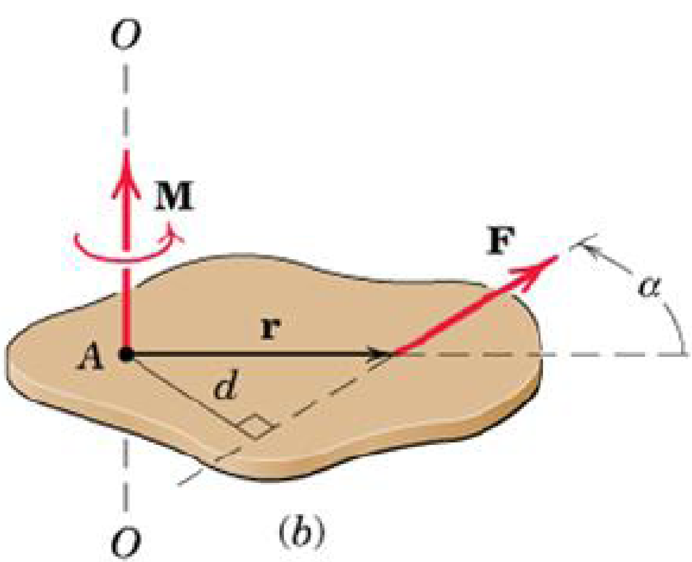
The magnitude

$$M = Fr \sin \alpha = Fd$$

NOTE: $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ $\quad$ $\mathbf{F} \times \mathbf{r} = -\mathbf{M}$

Moment of a Force

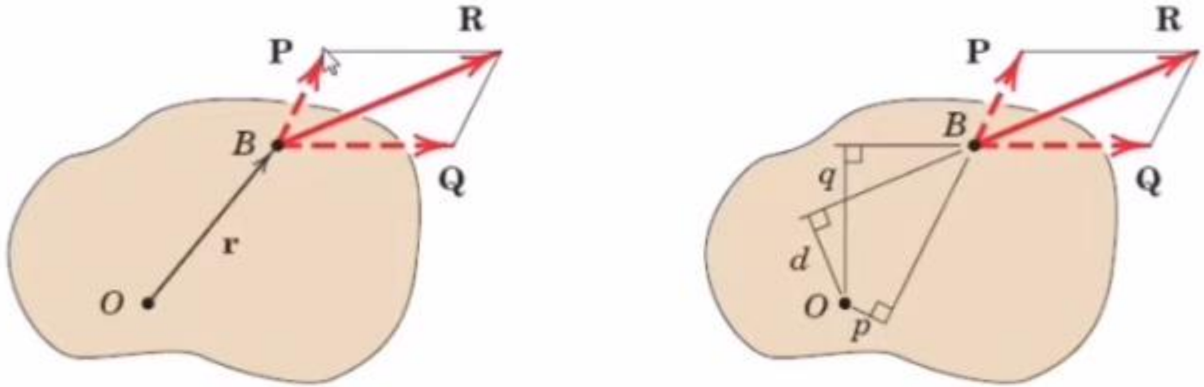
- Characteristic
  - Moment arm ($d = r\,sin\alpha$) does not depend on the particular point on the line of action of $\mathbf{F}$ to which the vector $\mathbf{r}$ is directed
    - **Sliding vector**
      - Line of action same as the moment axis



(b)

Varignon's Theorem

The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.



(a) (b)

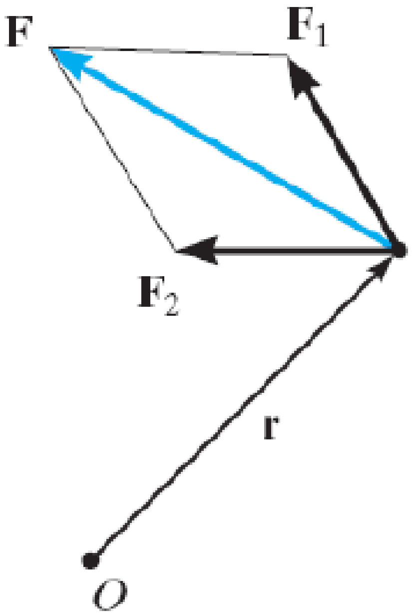
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q}) = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$$

The magnitude

$$M_O = Rd = -pP + qQ$$

Moment of a Force

- **Varignon's Theorem**
  - **Moment** of a **force** about **a point** is equal to the **sum of the moments** of the **components of the force** about the **same point**

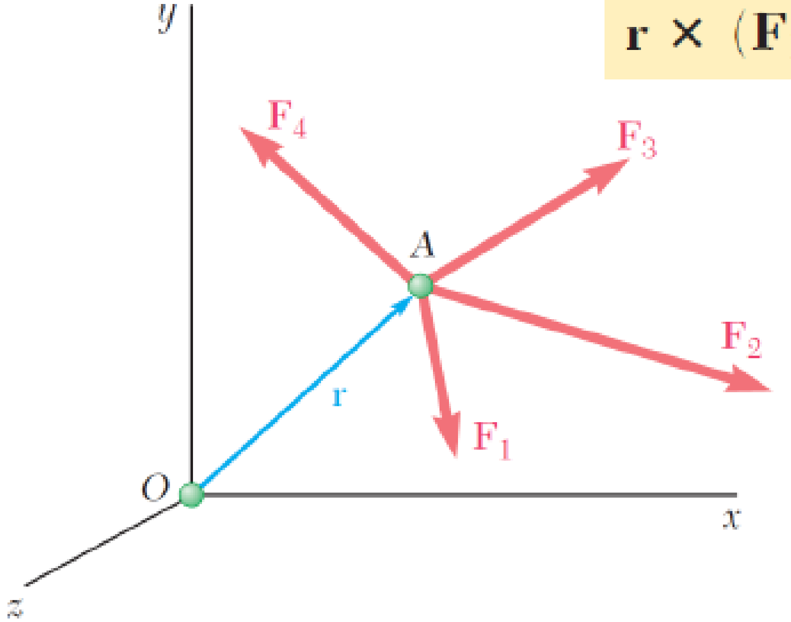


$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

Moment of a System of Concurrent Forces

- **Varignon's Theorem**
  - **Moment** of the **resultant** of a system of **concurrent forces** about **a point** is equal to the **sum of the moments** of the **of the individual forces** about the **same point**

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \cdots) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \cdots$$



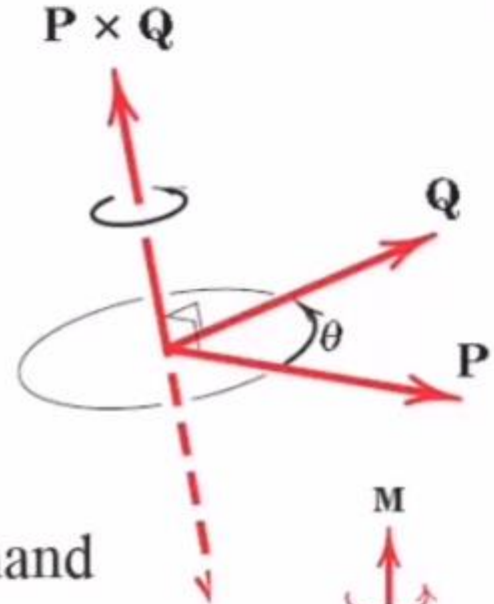
IMPORTANT RELATIONS

1. Cross or vector product.

The cross product **P x Q** of the two vectors P and Q is defined as a vector with a magnitude:

$$|\mathbf{P} \times \mathbf{Q}| = PQ \sin \theta$$

and a direction specified by the right-hand rule as shown



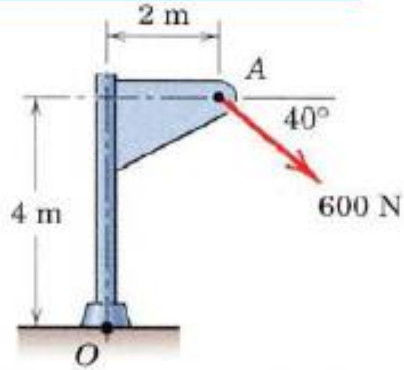
$$\mathbf{Q} \times \mathbf{P} = -\mathbf{P} \times \mathbf{Q}$$

Distributive law

$$\mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = \mathbf{P} \times \mathbf{Q} + \mathbf{P} \times \mathbf{R}$$

Moment: Example

Calculate the magnitude of the moment about the base point O of the 600-N force in five different ways.

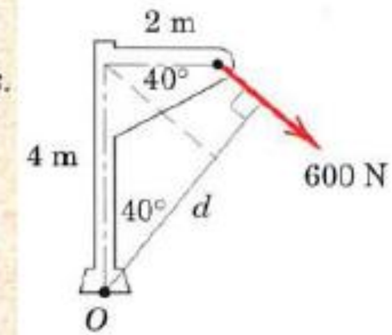


Solution. (*I*) The moment arm to the 600-N force is

$$d = 4 \cos 40^\circ + 2 \sin 40^\circ = 4.35 \text{ m}$$

By $M = Fd$ the moment is clockwise and has the magnitude

$$M_O = 600(4.35) = 2610 \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$

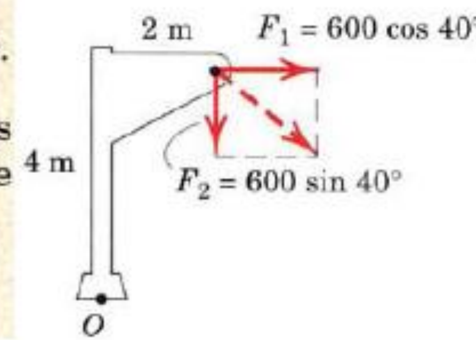


(*II*) Replace the force by its rectangular components at A

$$F_1 = 600 \cos 40^\circ = 460 \text{ N}, \qquad F_2 = 600 \sin 40^\circ = 386 \text{ N}$$

By Varignon's theorem, the moment becomes

$$M_O = 460(4) + 386(2) = 2610 \text{ N} \cdot \text{m} \qquad \textit{Ans.}$$



(*III*) By the principle of transmissibility, move the 600-N force along its line of action to point B, which eliminates the moment of the component F_2. The moment arm of F_1 becomes

$$d_1 = 4 + 2 \tan 40^\circ = 5.68 \text{ m}$$

Moment: Example

and the moment is

$$M_O = 460(5.68) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

(IV) Moving the force to point C eliminates the moment of the component F_1. The moment arm of F_2 becomes

$$d_2 = 2 + 4 \cot 40^\circ = 6.77 \text{ m}$$

and the moment is

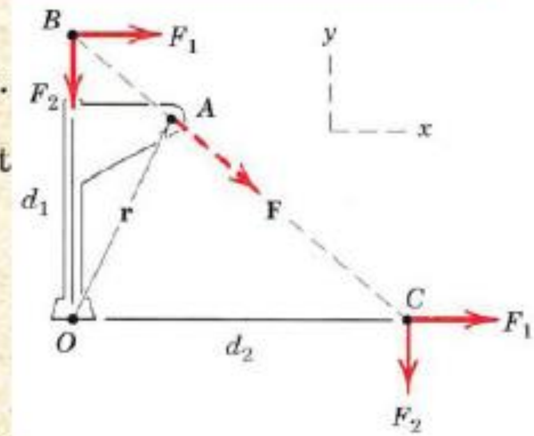
$$M_O = 386(6.77) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

(V) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have

$$\begin{aligned}\mathbf{M}_O &= \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i} \cos 40^\circ - \mathbf{j} \sin 40^\circ) \\ &= -2610\mathbf{k} \text{ N}\cdot\text{m}\end{aligned}$$

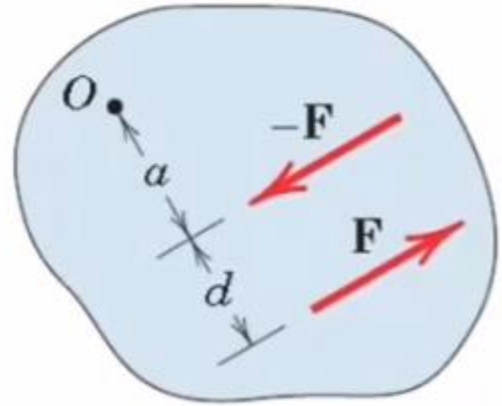
The minus sign indicates that the vector is in the negative z-direction. The magnitude of the vector expression is

$$M_O = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$



3- Couple

Couple is the moment produced by two equal, opposite, and non-collinear forces.



This couple has a magnitude M *through any point O*:

$$M = F(a + d) - Fa$$

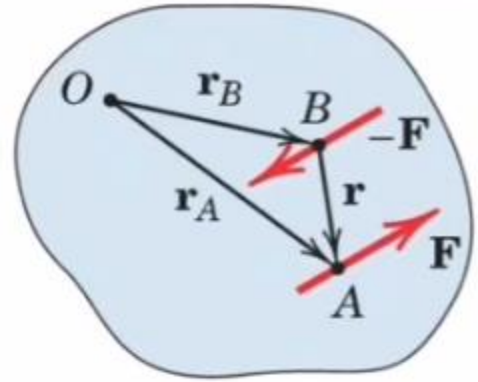
OR

$$M = Fd$$

NOTE: the moment of a couple has the **same value** for all moment centers.

Vector Algebra Method

With the cross-product, the combined moment about point ***O*** of the forces:



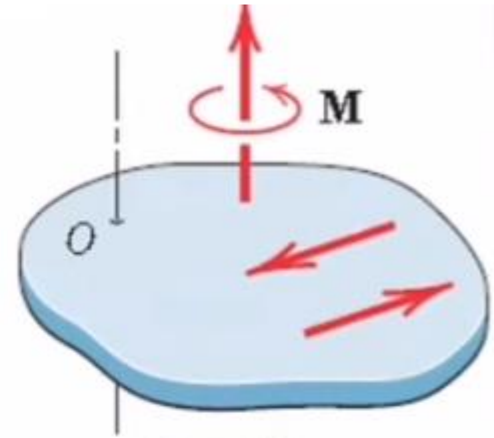
$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

Because $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, we can express **M** as:

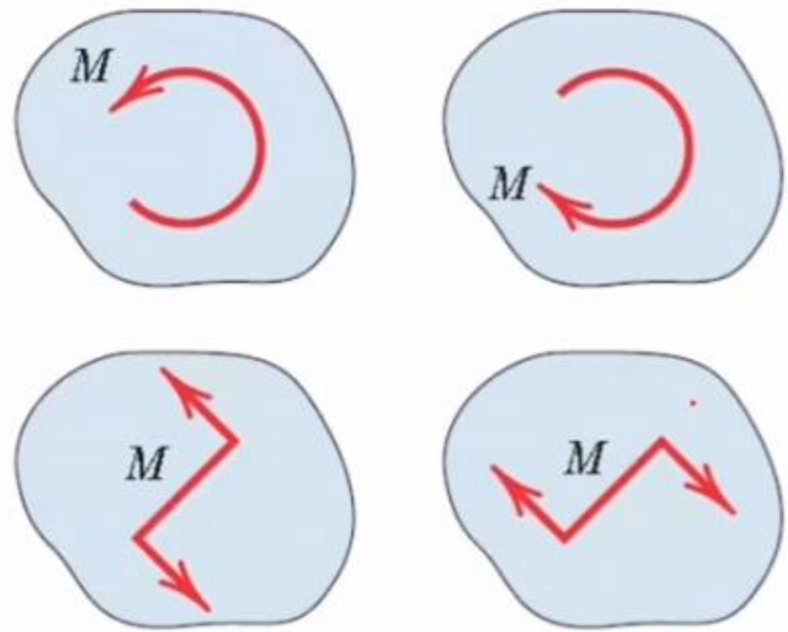
$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

NOTE: the moment expression contains no reference to the moment center O and, therefore, is the same for all moment centers.

Thus, we may represent **M** by **a free vector**, where the direction of **M** is normal to the plane of the couple and the sense of **M** is established by the right-hand rule.



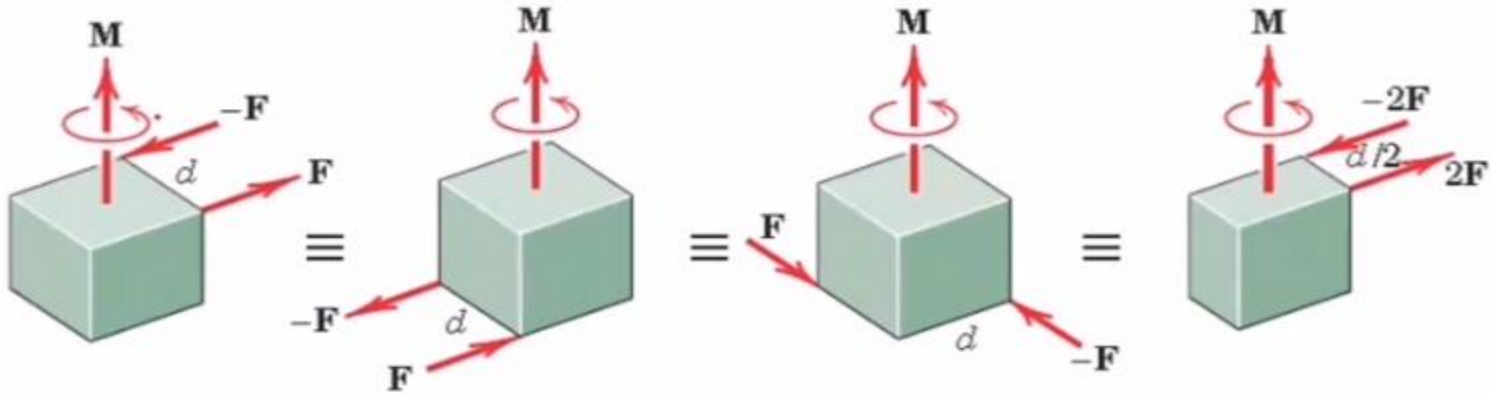
We may represent the direction of moment or couple vector as : clockwise or counterclockwise



Counterclockwise couple

Clockwise couple

Equivalent Couples



In each of the four cases, the couples are equivalent and are described by the same free vector which represents the identical tendencies to rotate the bodies.

$$M = Fd$$

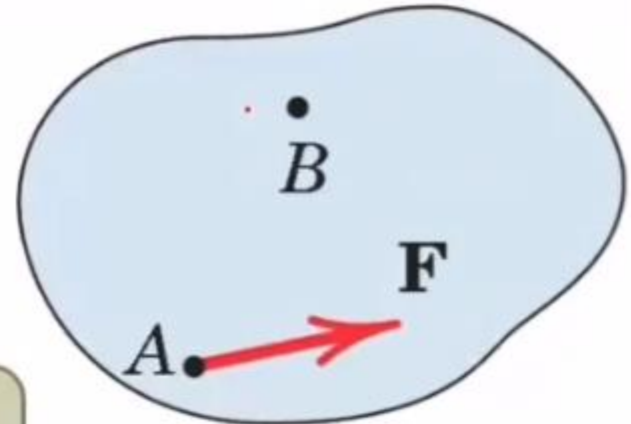
Moment directions may be accounted for by using a stated sign convention, such as a **plus sign (+) for counterclockwise** moments and a **minus sign (-) for clockwise moments**, or vice versa.

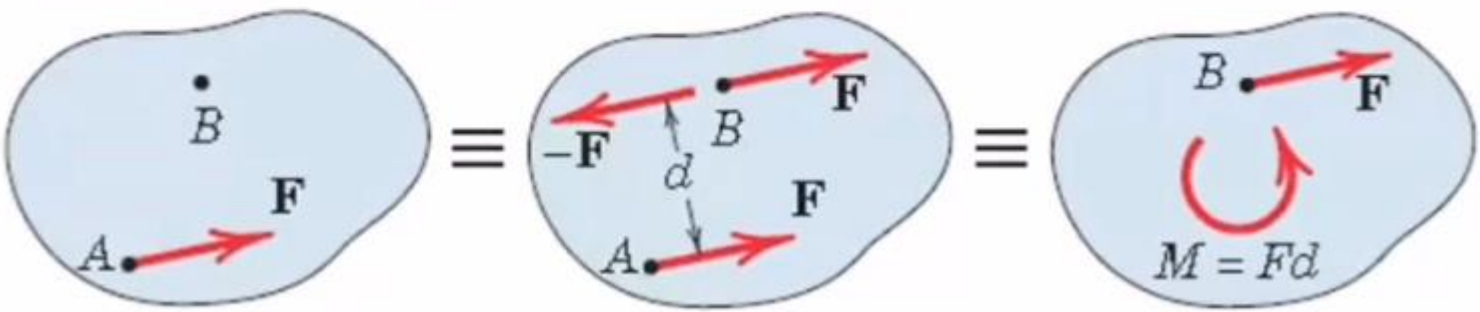
Force–Couple Systems

A force acting on a body

+ any fixed axis which does not intersect the line of the force.

= The same force at the fixed point **+** couple moment ($M=Fd$)



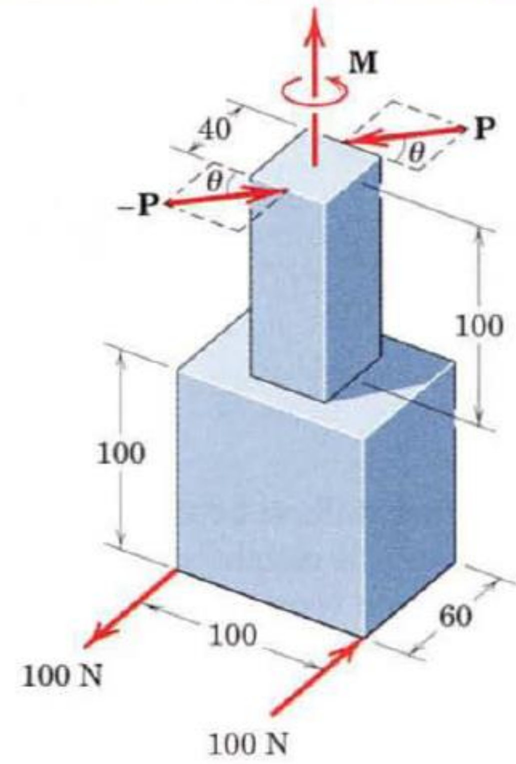


The combination of the force and couple in the right-hand is referred to as *a force–couple system.*

➢ *We may reverse this process.*

Sample Problem 2/6

The rigid structural member is subjected to a couple consisting of the two 100-N forces. Replace this couple by an equivalent couple consisting of the two forces **P** and −**P**, each of which has a magnitude of 400 N. Determine the proper angle θ.



Dimensions in millimeters

Solution. The original couple is counterclockwise when the plane of the forces is viewed from above, and its magnitude is

$[M = Fd]$ $$M = 100(0.1) = 10\ \text{N}\cdot\text{m}$$

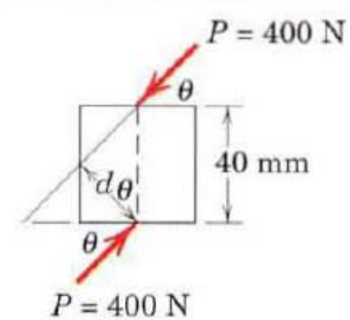
The forces **P** and −**P** produce a counterclockwise couple

$$M = 400(0.040)\cos\theta$$

Equating the two expressions gives

$$10 = 400(0.040)\cos\theta$$

$$\theta = \cos^{-1}\frac{10}{16} = 51.3^\circ \qquad \textit{Ans.}$$



Helpful Hint

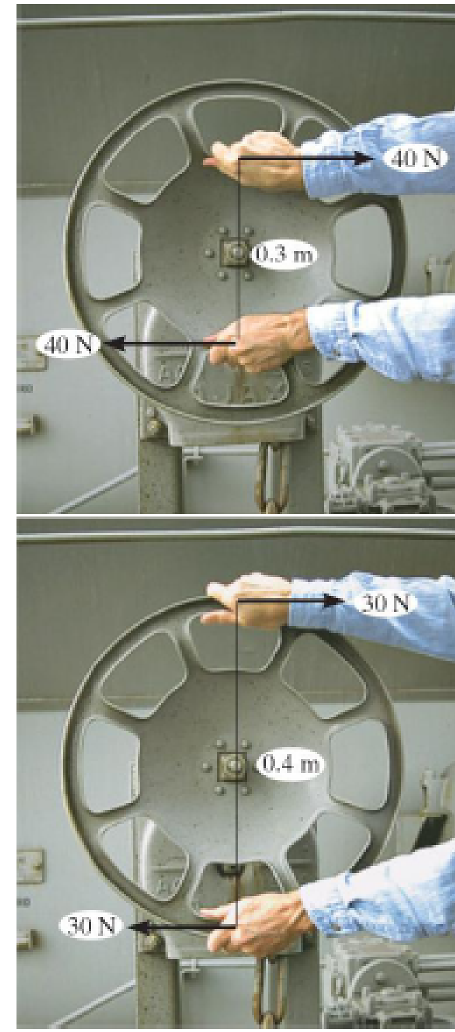
① Since the two equal couples are parallel free vectors, the only dimensions which are relevant are those which give the perpendicular distances between the forces of the couples.

Couple: Example

Moment reqd to turn the shaft connected at center of the wheel = 12 Nm

- First case: Couple Moment produced by 40 N forces = 12 Nm
- Second case: Couple Moment produced by 30 N forces = 12 Nm

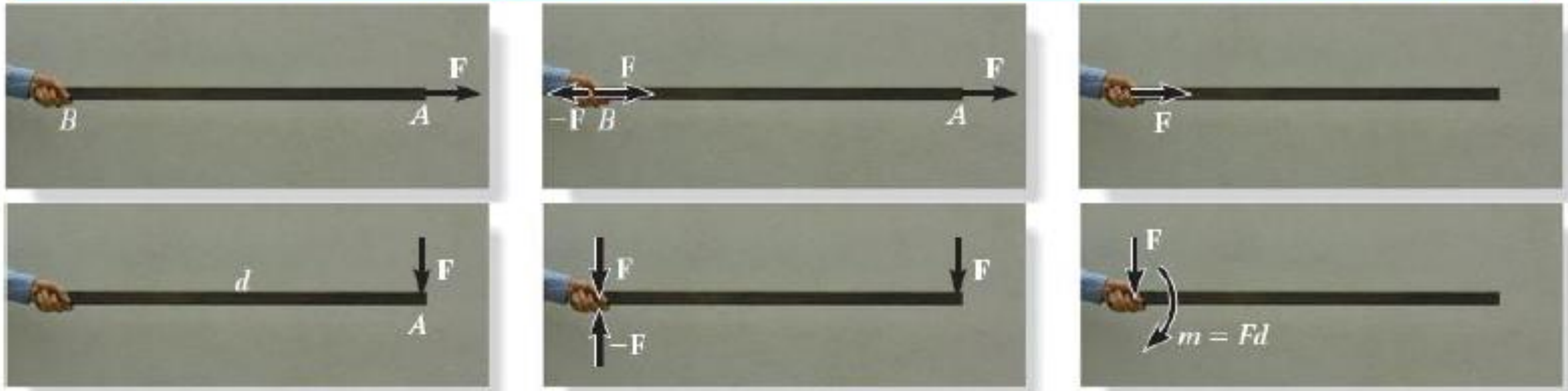
  If only One hand is used → F = 60N

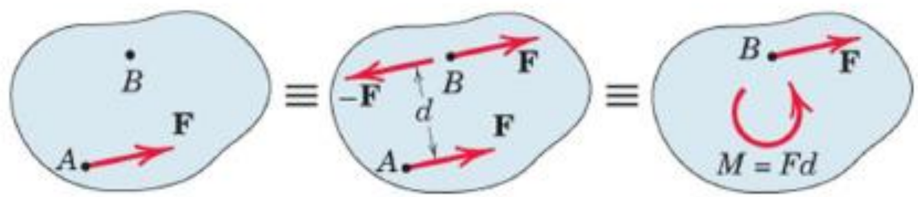


Same couple moment will be produced even if the shaft is not connected at the center of the wheel

→ Couple Moment is a Free Vector

Equivalent Systems (Force-Couple Systems)



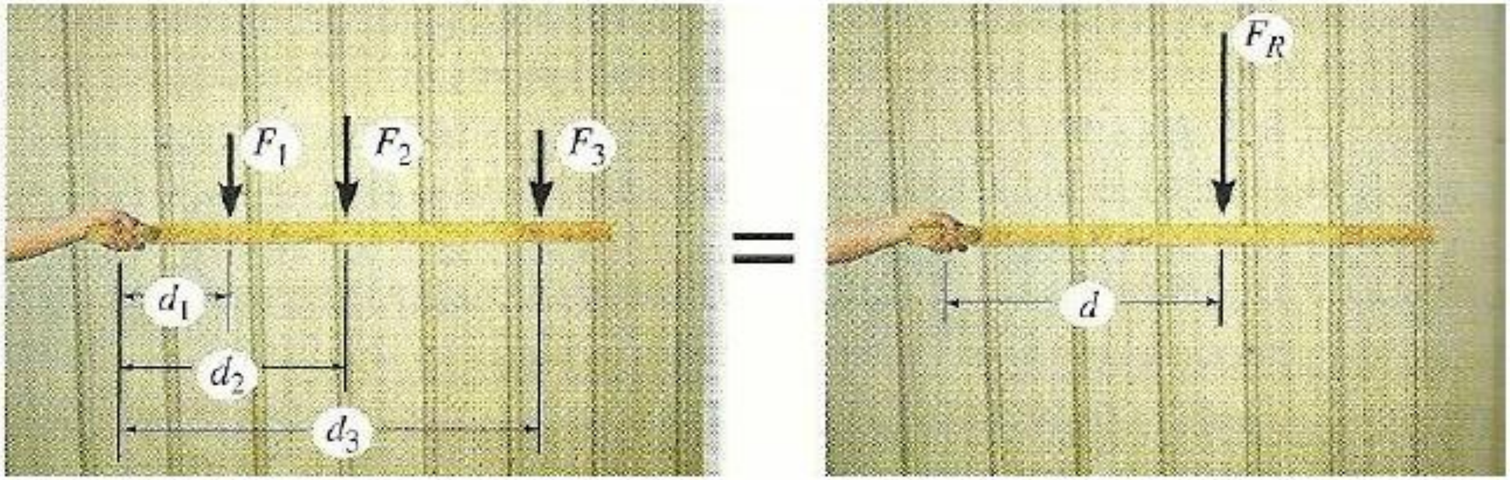


At support O:

$W_R = W_1 + W_2$

$(M_R)_O = W_1 d_1 + W_2 d_2$

Equivalent Systems: Resultants



$F_R = F_1 + F_2 + F_3$

How to find d?

Moment of the Resultant force about the grip must be equal to the moment of the forces about the grip

$F_R d = F_1 d_1 + F_2 d_2 + F_3 d_3$

→ Equilibrium Conditions

Equivalent Systems: Resultants

Equilibrium

Equilibrium of a body is a condition in which the resultants of all forces acting on the body is zero.

→Condition studied in Statics

When the resultant of all forces on a body is not zero, acceleration of the body is obtained by equating the force resultant to the product of the mass and acceleration of the body.

→ Condition studied in Dynamics

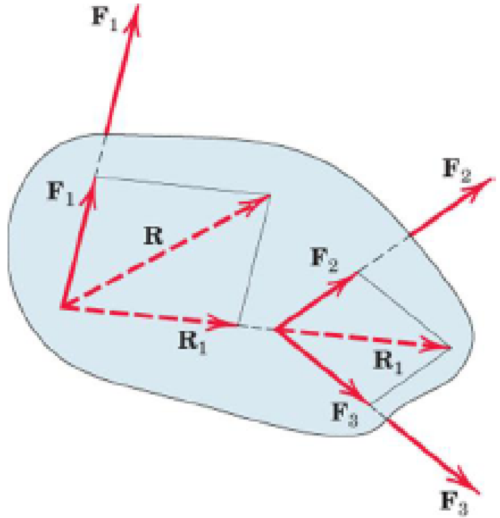
4- Resultants

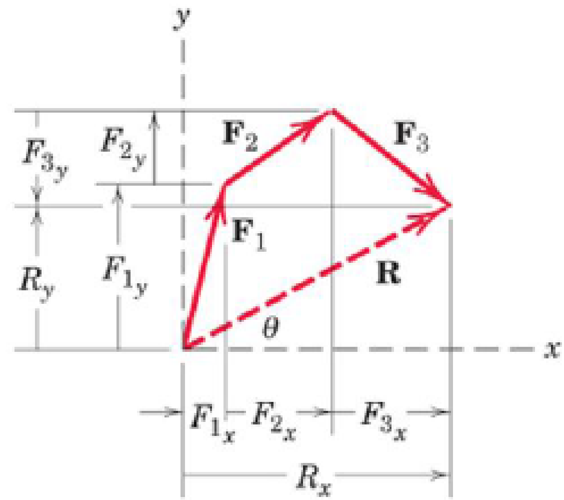
The resultant of a system of forces: is the simplest force combination which can replace the original forces without altering the external effect on the rigid body to which the forces are applied.

Equilibrium of a body: is the condition in which the resultant of all forces acting on the body is ***zero.***

Equivalent Systems: Resultants

Vector Approach: Principle of Transmissibility can be used





Magnitude and direction of the resultant force R is obtained by forming the force polygon where the forces are added head to tail in any sequence

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma \mathbf{F}$$

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

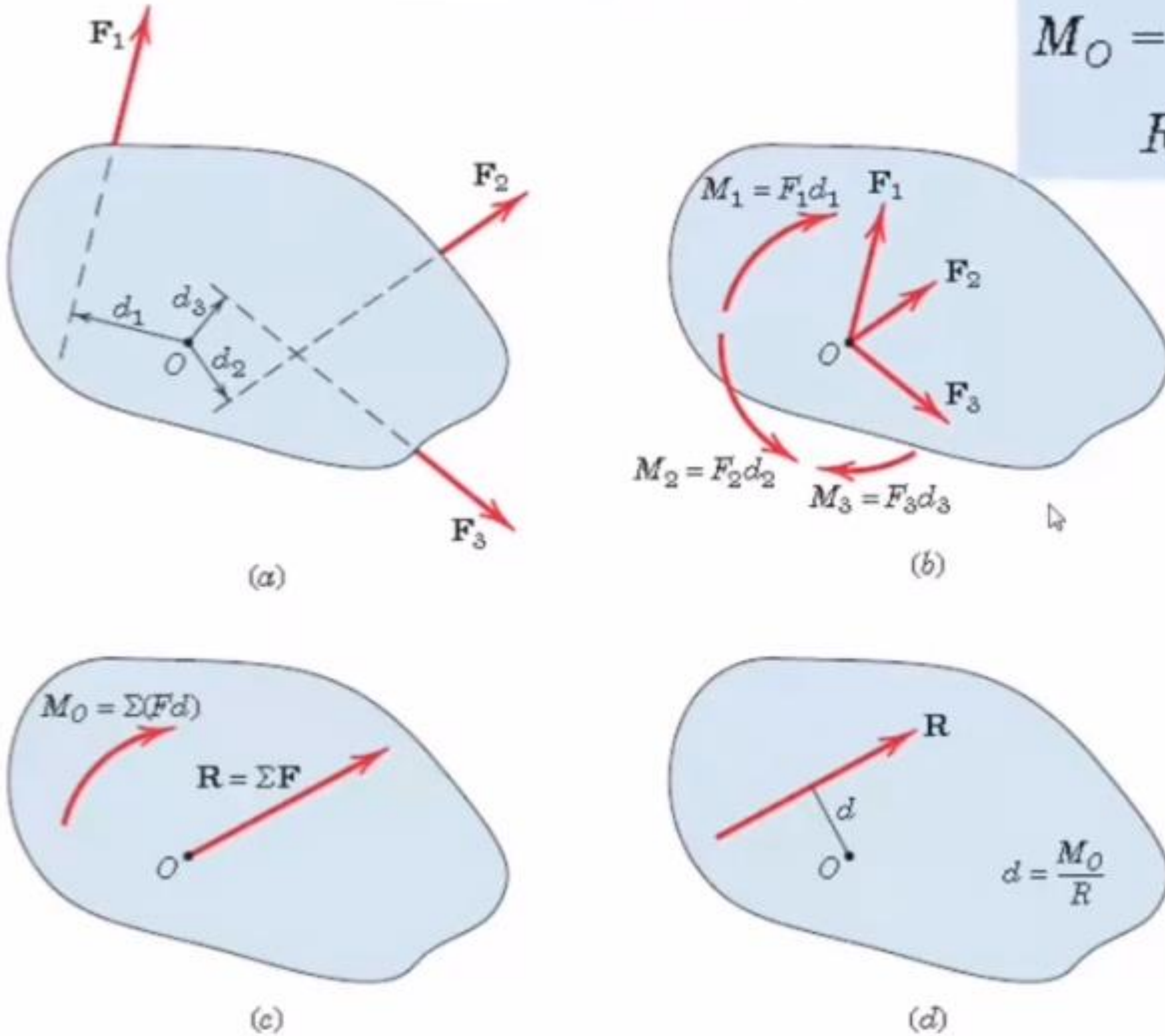
$$\theta = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}$$

Algebraic Method

$$R = \Sigma F$$

$$M_O = \Sigma M = \Sigma(Fd)$$

$$Rd = M_O$$

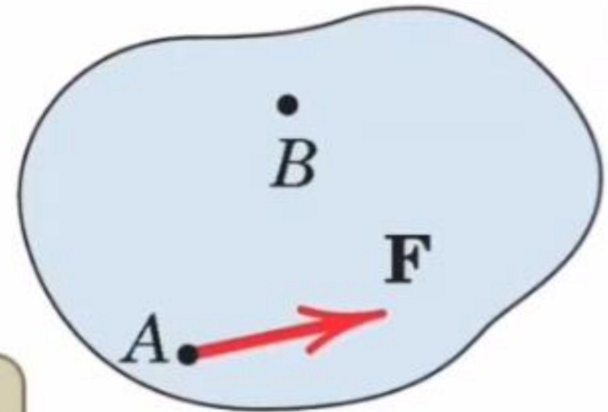


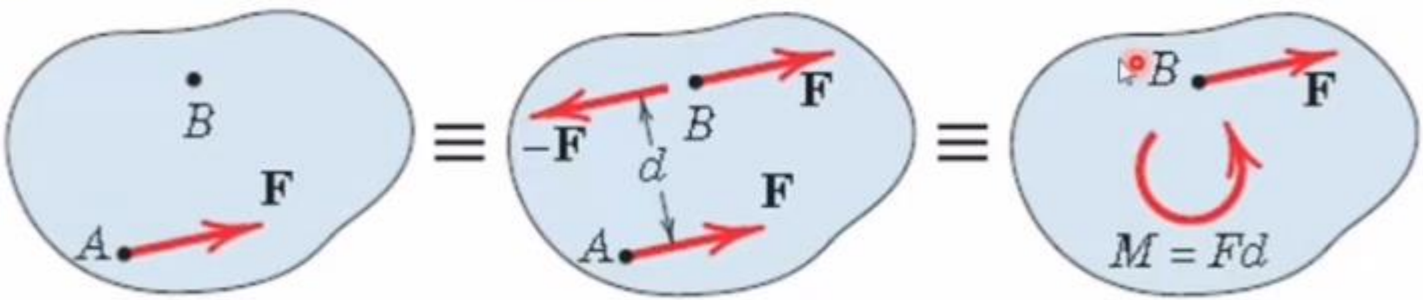
Force–Couple Systems

A force acting on a body

+ any fixed axis which does not intersect the line of the force.

= The same force at the fixed point + couple moment ($M=Fd$)





The combination of the force and couple in the right-hand is referred to as *a force–couple system.*

➢ *We may reverse this process.*

The principle of moments.

For the case of ***non-concurrent*** force systems; the moment of the resultant force about any point O **equals** the sum of the moments of the original forces of the system about the same point.

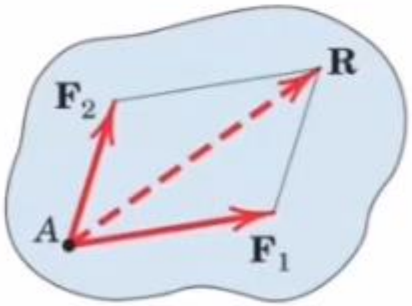
$$\mathbf{R} = \Sigma \mathbf{F}$$

$$M_O = \Sigma M = \Sigma(Fd)$$

$$Rd = M_O$$

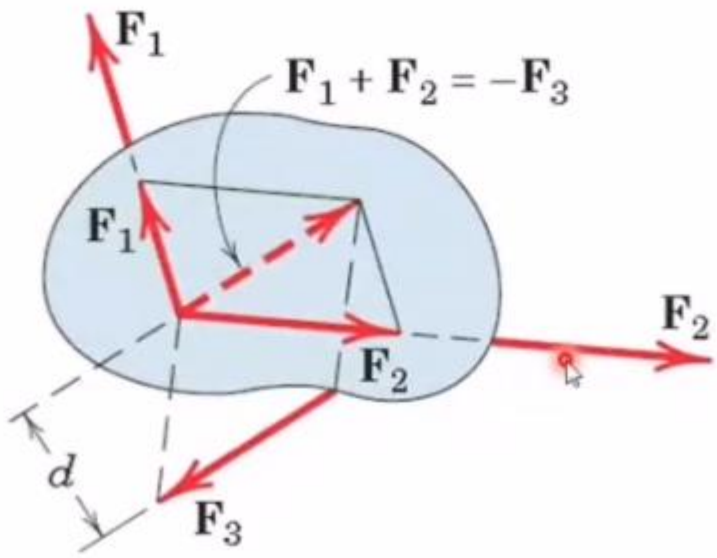
NOTES:

- For a concurrent system of forces:
  $\Sigma M_A = \text{zero}$.
  The resultant $\mathbf{R} = \Sigma \mathbf{F}$, passes through point A.



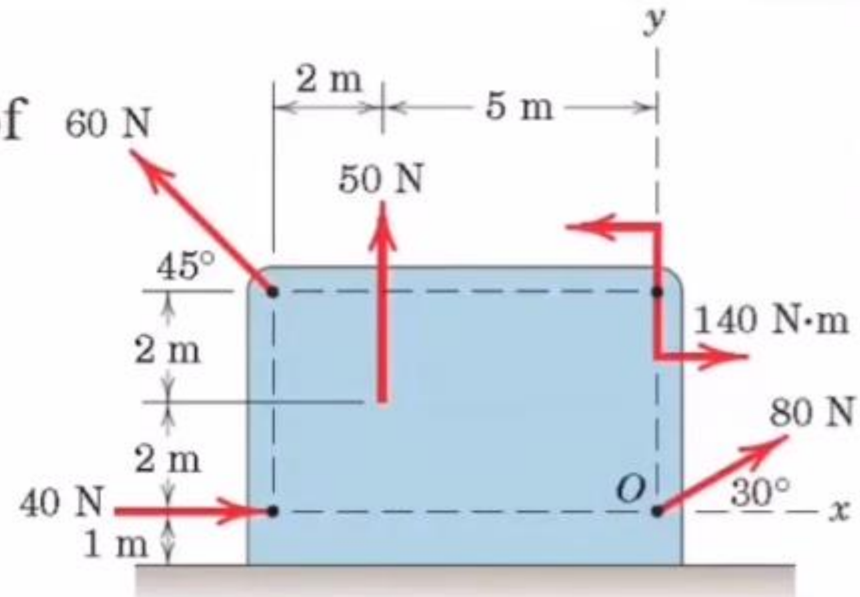
- For a parallel force system, select a coordinate axis in the direction of the forces.
- If the resultant force R for a given force system is zero, the resultant of the system need not be zero because the resultant may be a couple.

EX: The three forces in Figure, for instance, have a zero resultant force but have a resultant clockwise couple $M = F_3 d$.



SAMPLE PROBLEM

Determine the resultant of the four forces and one couple which act on the plate shown.



Solution.

$[R_x = \Sigma F_x]$ $\quad R_x = 40 + 80 \cos 30° - 60 \cos 45° = 66.9 \text{ N}$

$[R_y = \Sigma F_y]$ $\quad R_y = 50 + 80 \sin 30° + 60 \cos 45° = 132.4 \text{ N}$

$[R = \sqrt{R_x^2 + R_y^2}]$ $\quad R = \sqrt{(66.9)^2 + (132.4)^2} = 148.3 \text{ N}$

$\left[\theta = \tan^{-1} \frac{R_y}{R_x}\right]$ $\quad \theta = \tan^{-1} \frac{132.4}{66.9} = 63.2°$

$$[M_O = \Sigma(Fd)]$$

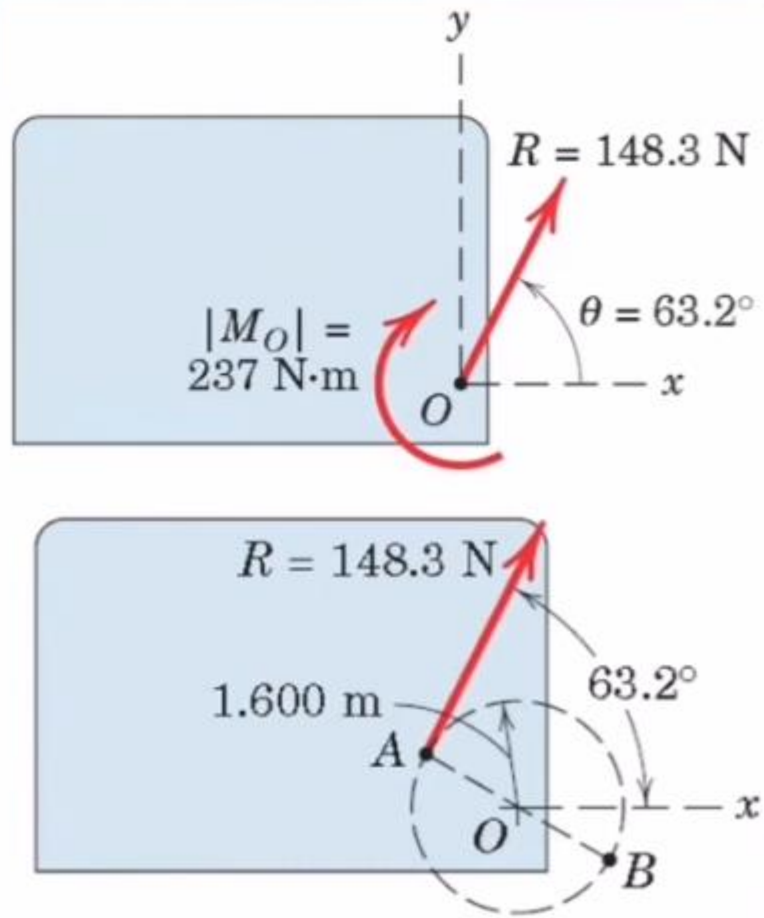
$$M_O = 140 - 50(5) + 60 \cos 45°(4) - 60 \sin 45°(7)$$

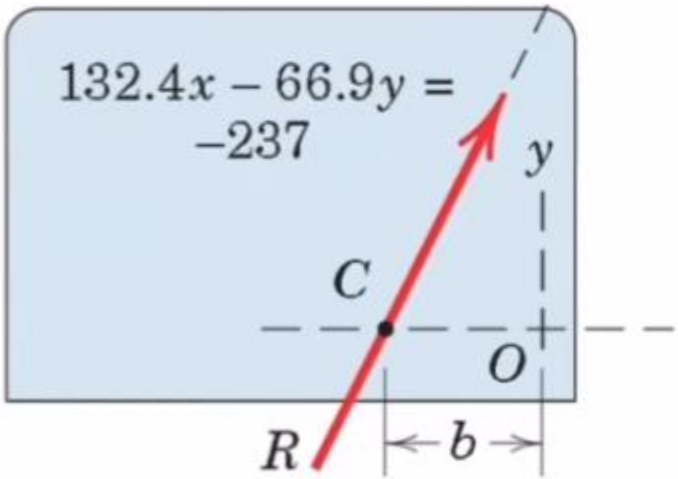
$$= -237 \text{ N} \cdot \text{m}$$

$$[Rd = |M_O|]$$

$$148.3d = 237$$

$$d = 1.600 \text{ m}$$





The resultant **R** may also be located

$$R_y b = |M_O|$$

$$b = \frac{237}{132.4} = 1.792 \text{ m}$$

Another approach : Use the vector expression

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$$

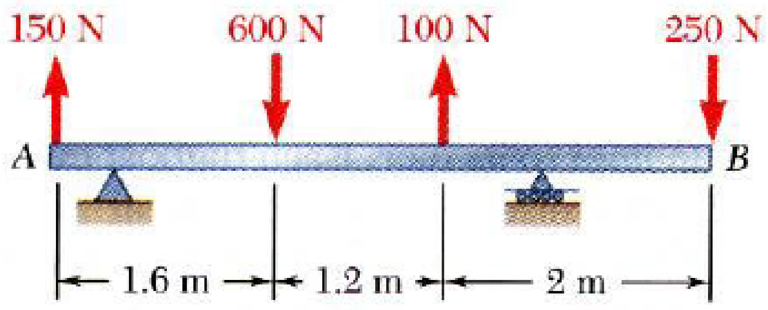
where **r**= *x* **i** +*y* **j** is a position vector running from point ***O*** to any point on the line of action of **R**.

$$(x\mathbf{i} + y\mathbf{j}) \times (66.9\mathbf{i} + 132.4\mathbf{j}) = -237\mathbf{k}$$

$$(132.4x - 66.9y)\mathbf{k} = -237\mathbf{k}$$

The desired line of action, $132.4x - 66.9y = -237$

Example on Equivalent System



For the beam, reduce the system of forces shown to

(a) an **equivalent force-couple** system at ***A***,

(b) an **equivalent force couple** system at ***B***, and

(c) a **single force** or **resultant**

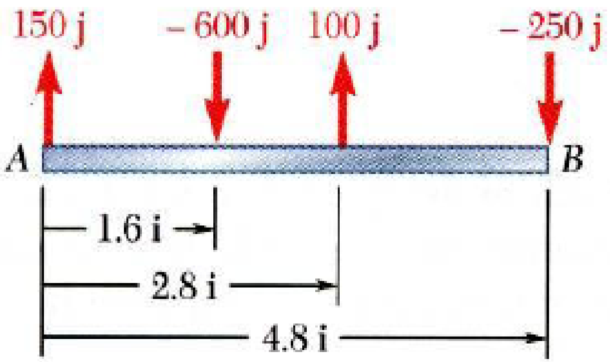
Solution:

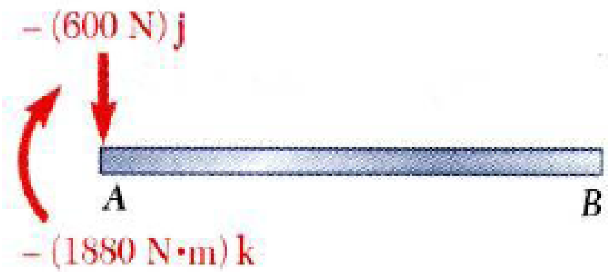
a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about *A*.

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

c) Determine the point of application for the resultant force such that its moment about *A* is equal to the resultant couple at *A*.

Example on Equivalent System





SOLUTION:

a) Compute the resultant force and the resultant couple at *A*.

$$\vec{R} = \sum \vec{F}$$

$$= (150\text{ N})\vec{j} - (600\text{ N})\vec{j} + (100\text{ N})\vec{j} - (250\text{ N})\vec{j}$$

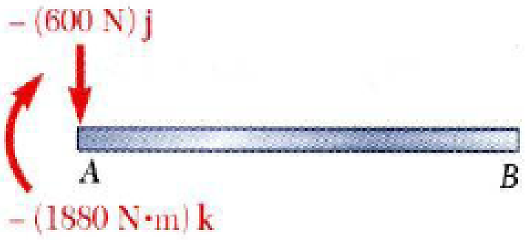
$$\vec{R} = -(600\text{ N})\vec{j}$$

$$\vec{M}_A^R = \sum\left(\vec{r} \times \vec{F}\right)$$

$$= \left(1.6\vec{i}\right) \times \left(-600\vec{j}\right) + \left(2.8\vec{i}\right) \times \left(100\vec{j}\right) + \left(4.8\vec{i}\right) \times \left(-250\vec{j}\right)$$

$$\vec{M}_A^R = -(1880\text{ N}\cdot\text{m})\vec{k}$$

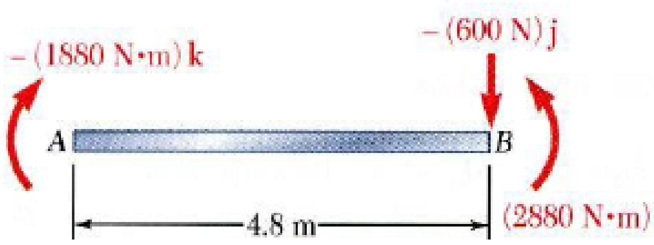
Example on Equivalent System



b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

The force is unchanged by the movement of the force-couple system from *A* to *B*.

$$\boxed{\vec{R} = -(600\,\mathrm{N})\vec{j}}$$

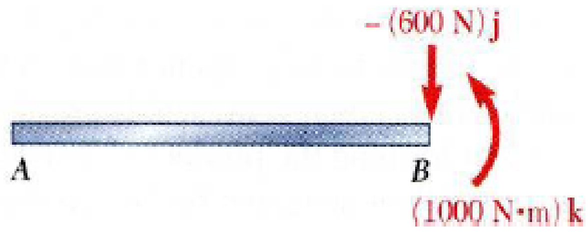


The couple at *B* is equal to the moment about *B* of the force-couple system found at *A*.

$$\vec{M}_B^R = \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R}$$

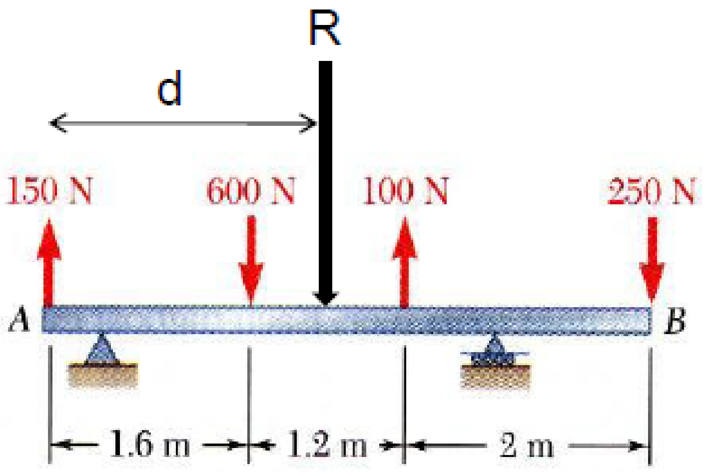
$$= -(1880\,\mathrm{N\cdot m})\vec{k} + (-4.8\,\mathrm{m})\vec{i} \times (-600\,\mathrm{N})\vec{j}$$

$$= -(1880\,\mathrm{N\cdot m})\vec{k} + (2880\,\mathrm{N\cdot m})\vec{k}$$



$$\boxed{\vec{M}_B^R = +(1000\,\mathrm{N\cdot m})\vec{k}}$$

Example on Equivalent System



c)

$F_R = F_1 + F_2 + F_3 + F_4$

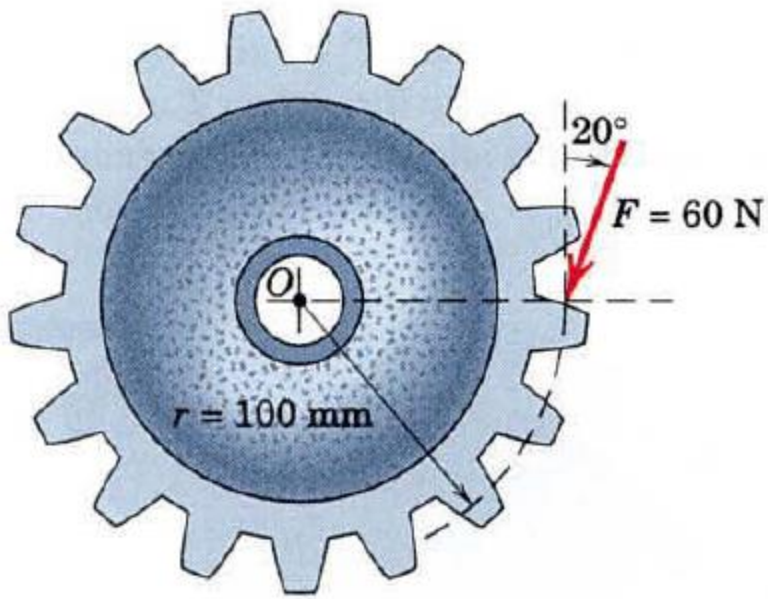
$R = 150 - 600 + 100 - 250 = -650\ N$

$F_R d = F_1 d_1 + F_2 d_2 + F_3 d_3 + F_4 d_4$

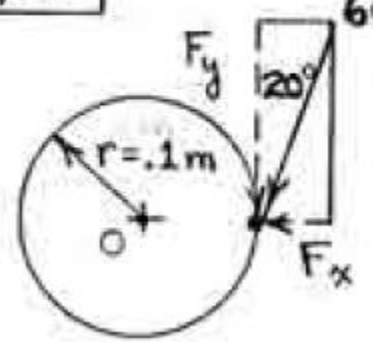
$d = 3.13\ m$

2/31 A force **F** of magnitude 40 N is applied to the gear. Determine the moment of **F** about point O.

Ans. $M_O = 5.64\ \text{N}\cdot\text{m CW}$



2/31

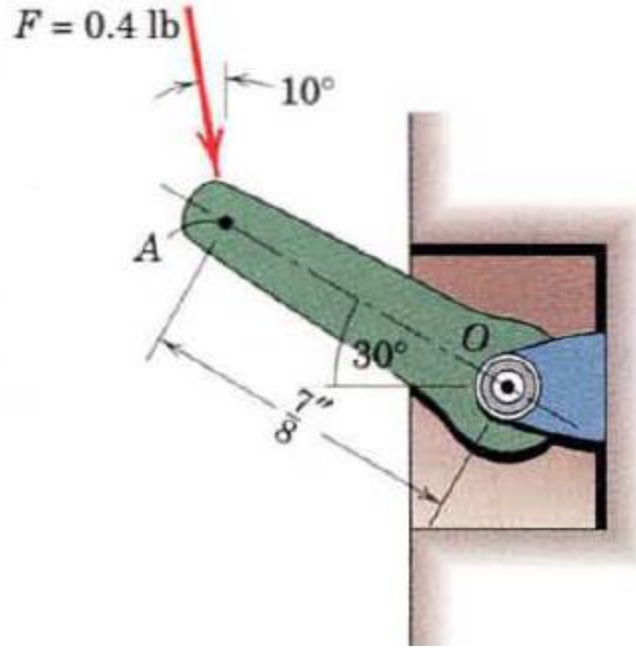


$$+\circlearrowright M_O = r F_y$$

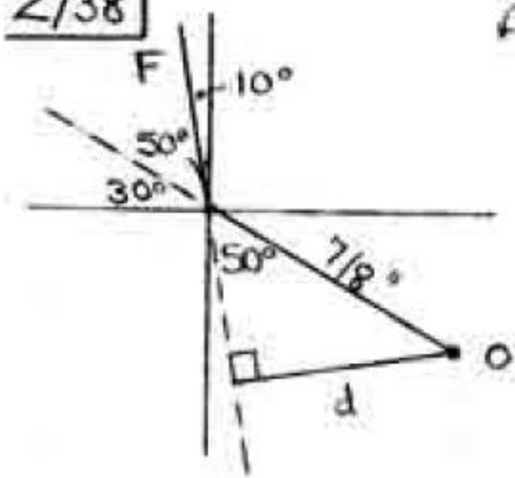
$$= (0.1)(60 \cos 20°)$$

$$= \underline{5.64\ \text{N}\cdot\text{m}}$$

2/38 Compute the moment of the 0.4-lb force about the pivot O of the wall-switch toggle.



2/38



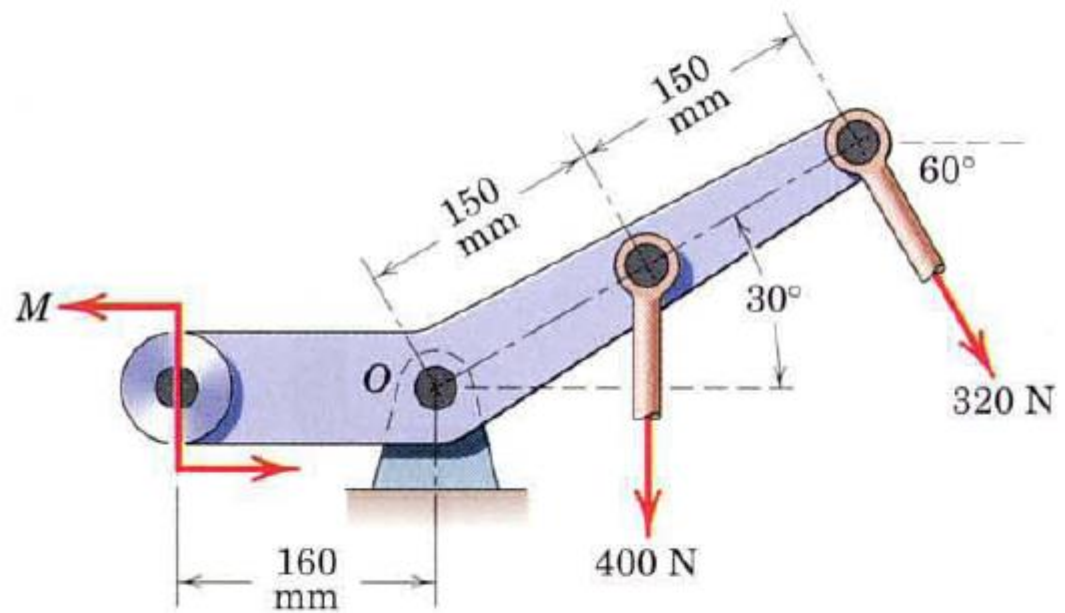
$$\curvearrowleft + \; M_O = Fd$$

$$= 0.4\left(\tfrac{7}{8}\sin 50^\circ\right)$$

$$= \underline{0.268 \text{ lb-in.}}$$

2/77 If the resultant of the two forces and couple M passes through point O, determine M.

Ans. $M = 148.0\ \text{N}\cdot\text{m CCW}$

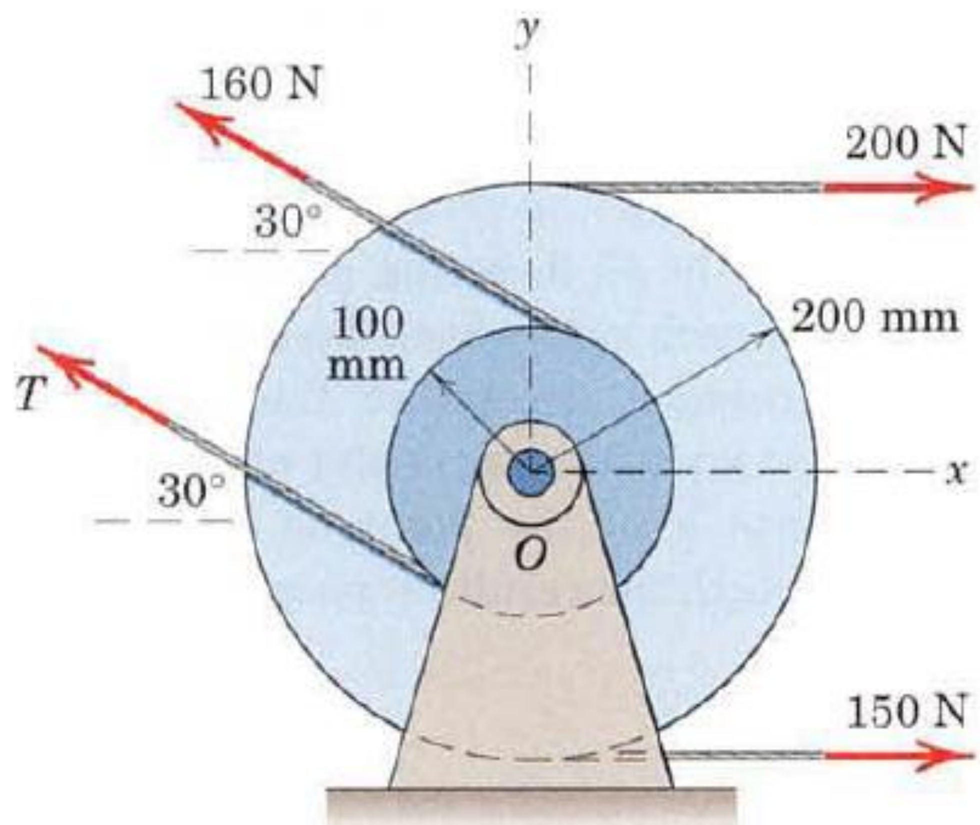


2/77 $M_O = 0$, so

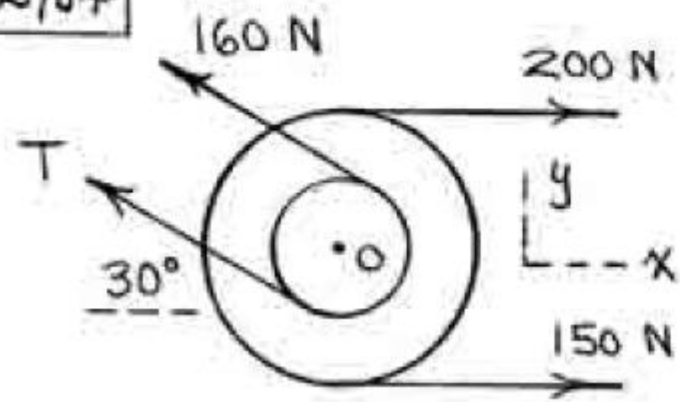
$$\circlearrowleft^{+}\ M - 400(0.150 \cos 30°) - 320(0.300) = 0$$

$M = 148.0\ \text{N}\cdot\text{m}$

2/84 Two integral pulleys are subjected to the belt tensions shown. If the resultant **R** of these forces passes through the center O, determine T and the magnitude of **R** and the counterclockwise angle θ it makes with the x-axis.



2/84



$$\begin{cases} r_1 = 0.1\text{ m} \\ r_2 = 0.2\text{ m} \end{cases}$$

$$\overset{+}{\circlearrowleft} M_O = 0: \; 200(0.2) - 150(0.2) - 160(0.1) + (0.1)T = 0$$

$$\underline{T = 60\text{ N}}$$

$$R_x = \Sigma F_x = 200 + 150 - (160 + 60)\cos 30° = 159.5\text{ N}$$

$$R_y = \Sigma F_y = (160 + 30)\sin 30° = 110\text{ N}$$

$$R = \sqrt{R_x^2 + R_y^2} = \underline{193.7\text{ N}}$$

$$\theta = \tan^{-1}(R_y/R_x) = \underline{34.6°}$$