GENERAL MATHEMATICS 2

Dr. M. Alghamdi

Department of Mathematics

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1. Linear Systems
2. Solution of Linear Equations Systems
   (1) Cramer's method,
   (2) Gauss elimination method, and
   (3) Gauss-Jordan method

Definition

A linear system of m equations in n variables $x_1, x_2, \ldots, x_n$ *is a set of equations of the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \cdots \quad + \cdots \quad + \cdots \quad + \cdots \quad + \cdots &= \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + +a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where $a_{ij}, b_j \in \mathbb{R}$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*.*

The above system of linear equations can be written as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

A is called the coefficients matrix,
X is called the column vector of the variables (or column vector of the unknowns),
B is called the column vector of constants (or column vector of the resultants).

Section 1: Linear Systems

A special case of the linear system of equations is a system of **two different variables** x_1 and x_2:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

The above system of linear equations can be written as $AX = B$ where A $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, X $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and B $= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

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Solving Systems of Linear Equations

The following theory shows the basic condition for a system of linear equations in order to have a solution.

Theorem

Let $AX = B$ be a linear system with n equations in n variables. The system has a solution if $det(A) \neq 0$.

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Theorem

Let $AX = B$ be a linear system with n equations in n variables. The system has a solution if $det(A) \neq 0$.

In this chapter, we present three methods to solve the systems of linear equations $Ax = B$:

(1) Cramer's method,
(2) Gauss elimination method, and
(3) Gauss-Jordan method.

Section 2: Solution of Linear Equations Systems

(1) Cramer's Method

Theorem

Let $AX = B$ be a linear system with n equations in n variables. If $det(A) \neq 0$, then the unique solution to this system is

$$x_i = \frac{det(A_i)}{det(A)} \quad \text{for every} \quad i = 1, 2, \ldots, n \ ,$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector of constants B.

The matrix A_1 is formed by replacing the first column of A by the column vector of constants B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \Rightarrow A_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The matrix A_2 is formed by replacing the second column of A by the column vector of constants B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \Rightarrow A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}.$$

By continuing doing so, the matrix A_n is formed by replacing the last column of A by the column vector of constants B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \Rightarrow A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{bmatrix}.$$

By continuing doing so, the matrix A_n is formed by replacing the last column of A by the column vector of constants B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \Rightarrow A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{bmatrix}.$$

Remember: The determinant of 3×3 Matrices

Let A be a square matrix of order 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

To calculate the determinant, choose the first row of A and multiply each of its elements by the corresponding cofactor:

$$det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$det(A) = a_{11}(a_{22}\ a_{33} - a_{23}\ a_{32}) - a_{12}(a_{21}\ a_{33} - a_{23}\ a_{31}) + a_{13}(a_{21}\ a_{32} - a_{22}\ a_{31})$$

Example

Solve the linear system of equations using Cramer's rule.

$$\begin{aligned} 2x + y + z &= 3 \\ 4x + y - z &= -2 \\ 2x - 2y + z &= 6 \end{aligned}$$

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$$\begin{aligned} 2x + y + z &= 3 \\ 4x + y - z &= -2 \\ 2x - 2y + z &= 6 \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & -1 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow det(A) = -18.$$

$$A_1 = \begin{bmatrix} 3 & 1 & 1 \\ -2 & 1 & -1 \\ 6 & -2 & 1 \end{bmatrix} \Rightarrow det(A_1) = -9.$$

$$A_2 = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -2 & -1 \\ 2 & 6 & 1 \end{bmatrix} \Rightarrow det(A_2) = 18.$$

$$A_3 = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \\ 2 & -2 & 6 \end{bmatrix} \Rightarrow det(A_3) = -54.$$

Hence,

$$x = \frac{det(A_1)}{det(A)} = \frac{-9}{-18} = \frac{1}{2}$$

$$y = \frac{det(A_2)}{det(A)} = \frac{18}{-18} = -1$$

$$z = \frac{det(A_3)}{det(A)} = \frac{-54}{-18} = 3$$

The column vector of the variables is $X = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 3 \end{bmatrix}$.

Example

Solve the linear system of equations using Cramer's rule.

$$
\begin{aligned}
x + y + z &= 12 \\
x - y &= 2 \\
x - z &= 4
\end{aligned}
$$

Example

Solve the linear system of equations using Cramer's rule.

$$x + y + z = 12$$
$$x - y = 2$$
$$x - z = 4$$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow det(A) = 3.$$

$$A_1 = \begin{bmatrix} 12 & 1 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \Rightarrow det(A_1) = 18.$$

$$A_2 = \begin{bmatrix} 1 & 12 & 1 \\ 1 & 2 & 0 \\ 1 & 4 & -1 \end{bmatrix} \Rightarrow det(A_2) = 12.$$

$$A_3 = \begin{bmatrix} 1 & 1 & 12 \\ 1 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix} \Rightarrow det(A_3) = 6.$$

Therefore,

$$x = \frac{det(A_1)}{det(A)} = \frac{18}{3} = 6$$

$$y = \frac{det(A_2)}{det(A)} = \frac{12}{3} = 4$$

$$z = \frac{det(A_3)}{det(A)} = \frac{6}{3} = 2$$

The column vector of the variables is $X = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$.

(2) Gauss Elimination Method
Linear Equations Systems:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \cdots \quad + \cdots \quad + \cdots \quad + \cdots \quad + \cdots &= \cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + +a_{nn}x_n &= b_n \end{aligned} \tag{2}$$

where $a_{ij}, b_j \in \mathbb{R}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.

The above system of linear equations can be written as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

A is called the coefficients matrix,
X is called the column vector of the variables (or column vector of the unknowns),
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Definition

Gaussian elimination is a method for solving a linear system $AX = B$ by constructing the augmented matrix $[A|B]$ and transforming the matrix A to an upper triangular matrix $[C|D]$.

The Method:

1. Construct the augmented matrix $[A|B]$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right],$$

where A is the coefficients matrix and B is the column vector of constants.

2. Use the elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with a leading coefficient of each row equals 1:

$$\left[\begin{array}{cccccc|c} 1 & c_{12} & c_{13} & c_{14} & \cdots & c_{1n} & d_1 \\ 0 & 1 & c_{23} & c_{24} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & c_{(n-1)n} & d_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & d_n \end{array}\right].$$

3. From the last augmented matrix, we have $x_n = d_n$ and the rest of the unknowns can be calculated by backward substitutions.

$$CX = D$$

$$\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1,n-1} & c_{1n} \\ 0 & 1 & c_{23} & \cdots & c_{2,n-1} & c_{2n} \\ 0 & 0 & 1 & \cdots & c_{3,n-1} & c_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & c_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{bmatrix}$$

$$\begin{aligned} 1x_1 + c_{12}x_2 + c_{13}x_3 + \ldots + c_{1n}x_n &= d_1 \\ 0 + 1x_2 + c_{23}x_3 + \ldots + c_{2n}x_n &= d_2 \\ 0 + 0 + 1x_3 + \ldots + c_{3n}x_n &= d_3 \\ \cdots \quad + \cdots \quad + \cdots \quad + \cdots \quad + \cdots &= \cdots \\ 0 + 0 + 0 + \ldots + 1x_{n-1} + c_{n-1,n}x_n &= d_{n-1} \\ 0 + 0 + 0 + \ldots + 1x_n &= d_n \end{aligned} \tag{3}$$

Elementary Row Operations

(1) Replace i^{th} row (R_i) by j^{th} row (R_j): $R_i \leftrightarrow R_j$

$$\begin{bmatrix} 2 & -3 & 1 \\ -1 & 5 & 2 \\ 1 & -2 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 5 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix}$$

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(2) Multiply i^{th} row (R_i) by λ: $\xrightarrow{\lambda R_i}$

$$\begin{bmatrix} -1 & 5 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} -2 & 10 & 4 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix}.$$

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(3) Multiply i^{th} row (R_i) by λ and add the result to j^{th} row (R_j): $\xrightarrow{\lambda R_i + R_j}$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 11 \\ 2 & 1 & 25 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & -1 & 3 \end{array}\right]$$

Example

Solve the linear system by Gauss elimination method.

$$\begin{aligned} x - 2y + z &= 4 \\ -x + 2y + z &= -2 \\ 4x - 3y - z &= -4 \end{aligned}$$

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Solution: Construct the augmented matrix $[A|B]$. Then, use elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with a leading coefficient of each row equals 1.

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$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -20 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

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$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -20 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Example

Solve the linear system by Gauss elimination method.

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Solution: Construct the augmented matrix $[A|B]$. Then, use elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with a leading coefficient of each row equals 1.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow{1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow{-4R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 5 & -5 & -20 \end{array}\right]$$

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$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$$

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Solve the linear system by Gauss elimination method.

$$\begin{aligned} x - 2y + z &= 4 \\ -x + 2y + z &= -2 \\ 4x - 3y - z &= -4 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with a leading coefficient of each row equals 1.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow{1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow{-4R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 5 & -5 & -20 \end{array}\right]$$

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$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{aligned} z &= 1 \\ y - z &= -4 \Rightarrow y - 1 = -4 \Rightarrow y = -4 + 1 = -3 \\ x - 2y + z &= 4 \Rightarrow x - 2(-3) + 1 = 4 \Rightarrow x = -3 \end{aligned}$$

Section 2: Solution of Linear Equations Systems

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Solution: Construct the augmented matrix $[A|B]$. Then, use elementary row operations on the augmented matrix to transform the matrix A to an upper triangular matrix with a leading coefficient of each row equals 1.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow{1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 4 & -3 & -1 & -4 \end{array}\right] \xrightarrow{-4R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 5 & -5 & -20 \end{array}\right]$$

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$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{aligned} z &= 1 \\ y - z &= -4 \Rightarrow y - 1 = -4 \Rightarrow y = -4 + 1 = -3 \\ x - 2y + z &= 4 \Rightarrow x - 2(-3) + 1 = 4 \Rightarrow x = -3 \end{aligned}$$

The column vector of variables is $X = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$

Example

Solve the linear system by Gauss elimination method.

$$
\begin{aligned}
x + y + z &= 2\\
x - y + 2z &= 0\\
2x + z &= 2
\end{aligned}
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$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right]$$

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-2R_1 + R_3]{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

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Example

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[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-2R_1+R_3]{-1R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right]
$$

$$
\xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].
$$

Example

Solve the linear system by Gauss elimination method.

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

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$$\xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} z &= 0 \\ y - \frac{1}{2}z &= 1 \Rightarrow y - 0 = 1 \Rightarrow y = 1 \\ x + y + z &= 2 \Rightarrow x + 1 + 0 = 2 \Rightarrow x = 1 \end{aligned}$$

The column vector of variables is $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(3) Gauss-Jordan Method

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3\\
\cdots \quad + \cdots \quad + \cdots \quad + \cdots \quad + \cdots &= \cdots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + +a_{nn}x_n &= b_n
\end{aligned} \tag{4}
$$

where $a_{ij}, b_j \in \mathbb{R}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.

The above system of linear equations can be written as $AX = B$ where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \; X = \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_n \end{bmatrix}.
$$

A is called the coefficients matrix,
X is called the column vector of the variables (or column vector of the unknowns),
B is called the column vector of constants (or column vector of the resultants).

Definition

Gauss-Jordan elimination is a method for solving a linear system $AX = B$ by constructing the augmented matrix $[A|B]$ and transforming the matrix A to an identity matrix $[I_n|D]$.

The Method:

1. Construct the augmented matrix $[A|B]$.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right],$$

   where A is the coefficients matrix and B is the column vector of constants.

2. Use the elementary row operations on the augmented matrix $[A|B]$ to transform the matrix A to the identity matrix I_n.

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & d_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & d_n \end{array}\right].$$

3. From the last augmented matrix, $x_k = d_k$ for every $k = 1, 2, \ldots, n$.

$$CX = D$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{bmatrix}$$

$$\begin{aligned} x_1 &= d_1 \\ x_2 &= d_2 \\ x_3 &= d_3 \\ \cdots &= \cdots \\ x_{n-1} &= d_{n-1} \\ x_n &= d_n \end{aligned} \tag{5}$$

Elementary Row Operations

(1) Replace i^{th} row (R_i) by j^{th} row (R_j): $R_i \leftrightarrow R_j$

$$\begin{bmatrix} 2 & -3 & 1 \\ -1 & 5 & 2 \\ 1 & -2 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 5 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix}$$

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(2) Multiply i^{th} row (R_i) by λ: $\xrightarrow{\lambda R_i}$

$$\begin{bmatrix} -1 & 5 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} -2 & 10 & 4 \\ 2 & -3 & 1 \\ 1 & -2 & -7 \end{bmatrix}.$$

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(3) Multiply i^{th} row (R_i) by λ and add the result to j^{th} row (R_j): $\xrightarrow{\lambda R_i + R_j}$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 11 \\ 2 & 1 & 25 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 11 \\ 0 & -1 & 3 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

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Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right]$$

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-1R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$x + y + z = 2$$
$$x - y + 2z = 0$$
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Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-1R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Section 2: Solution of Linear Equations Systems

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-1R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x+y+z&=2\\ x-y+2z&=0\\ 2x+z&=2 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 1 & -1 & 2 & 0\\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & -2 & 1 & -2\\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & -2 & 1 & -2\\ 0 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & -2 & 1 & -2\\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 1 & -\frac{1}{2} & 1\\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x+y+z&=2\\ x-y+2z&=0\\ 2x+z&=2 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B]=\left[\begin{array}{ccc|c} 1&1&1&2\\ 1&-1&2&0\\ 2&0&1&2 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|c} 1&1&1&2\\ 0&-2&1&-2\\ 2&0&1&2 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 1&1&1&2\\ 0&-2&1&-2\\ 0&-2&-1&-2 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1&1&1&2\\ 0&-2&1&-2\\ 0&0&-2&0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1&1&1&2\\ 0&1&-\frac{1}{2}&1\\ 0&0&1&0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3+R_2} \left[\begin{array}{ccc|c} 1&1&1&2\\ 0&1&0&1\\ 0&0&1&0 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|c} 1&0&1&1\\ 0&1&0&1\\ 0&0&1&0 \end{array}\right] \xrightarrow{-1R_3+R_1} \left[\begin{array}{ccc|c} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&0 \end{array}\right].$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 0 \\ 2x + z &= 2 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-1R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-1R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-1R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-1R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence, $x = 1$, $y = 1$ and $z = 0$. The column vector of variables is $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x - 2y + 2z &= 5 \\ 5x + 3y + 6z &= 57 \\ x + 2y + 2z &= 21 \end{aligned}$$

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Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 5 & 3 & 6 & 57 \\ 1 & 2 & 2 & 21 \end{array}\right]$$

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 5 & 3 & 6 & 57 \\ 1 & 2 & 2 & 21 \end{array}\right] \xrightarrow[-1R_1 + R_3]{-5R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 4 & 0 & 16 \end{array}\right]$$

Example

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 5 & 3 & 6 & 57 \\ 1 & 2 & 2 & 21 \end{array}\right] \xrightarrow[-1R_1 + R_3]{-5R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 4 & 0 & 16 \end{array}\right] \xrightarrow{\frac{1}{4} R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

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$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 13 & -4 & 32 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

$$\begin{aligned} x - 2y + 2z &= 5 \\ 5x + 3y + 6z &= 57 \\ x + 2y + 2z &= 21 \end{aligned}$$

Solution: Construct the augmented matrix $[A|B]$. Then, use the elementary row operations on the augmented matrix to transform the matrix A to the identity matrix I_n.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 5 & 3 & 6 & 57 \\ 1 & 2 & 2 & 21 \end{array}\right] \xrightarrow[-1R_1 + R_3]{-5R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 4 & 0 & 16 \end{array}\right] \xrightarrow{\frac{1}{4} R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 13 & -4 & 32 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 13 & -4 & 32 \end{array}\right] \xrightarrow{-13R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

Example

Solve the linear system by Gauss-Jordan elimination method.

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Example

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$$x - 2y + 2z = 5$$
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Example

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Hence, $x = 3$, $y = 4$ and $z = 5$. The column vector of variables is $X = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$