Integral Calculus

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May 1, 2024

Main Content

Parametric Equations of Plane Curves

Definition

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Let C be a curve consists of all ordered pairs $\big(f(t), g(t)\big)$, where f and g are continuous on an interval I. The equations

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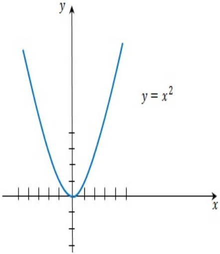
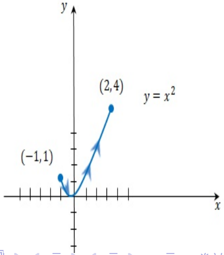
Example: Consider the plane curve C given by $y = x^2$.



Now, let $x = t$ and $y = t^2$ for $-1 \leq t \leq 2$.

We have the same graph where $x(t)$ and $y(t)$ are called parametric equations for the curve C with parameter t.



Parametric Equations of Plane Curves

Notes.

1. The parametric equations give the same graph of $y = f(x)$.

2. The parametric equations give the orientation of the curve C indicated by arrows and determined by increasing values of the parameter as shown in the previous figure.

3. To find the parametric equations, we introduce a third variable t. Then, we rewrite x and y as functions of t.

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Example

Write the curve given by $x(t) = 2t + 1$ *and* $y(t) = 4t^2 - 9$ *as* $y = f(x)$.

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Example

Write the curve given by $x(t) = 2t + 1$ *and* $y(t) = 4t^2 - 9$ *as* $y = f(x)$.

Solution:

Since $x = 2t + 1$, then $t = (x - 1)/2$.

This implies

$$\begin{aligned} y &= 4t^2 - 9 \\ &= 4\left(\frac{x-1}{2}\right)^2 - 9 \\ \Rightarrow y &= x^2 - 2x - 8 \end{aligned}$$

Example

Sketch and identify the curve defined by the parametric equations

$$x = 5\cos\ t, \quad y = 4\sin\ t, \quad 0 \le t \le 2\pi.$$

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Solution:

$$x = 5\ \cos\ t \Rightarrow \cos\ t = \frac{x}{5}$$

$$y = 4\ \sin\ t \Rightarrow \sin\ t = \frac{y}{4}$$

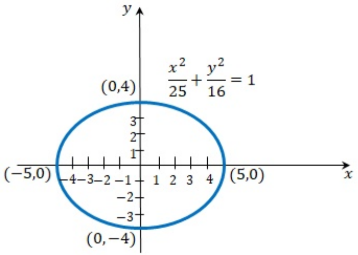
By using the identity

$$\cos^2\ t + \sin^2\ t = 1$$

we have

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Thus, the curve is an ellipse.



Parametric Equations of Plane Curves

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The curve C is given parametrically $x = \sin\ t,\ \ y = \cos\ t,\ \ 0 \leq t \leq 2\pi$. *Find an equation in* x *and* y, *then sketch the graph and indicate the orientation.*

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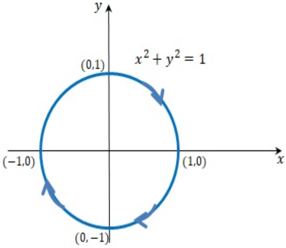
Solution: By using the identity $\cos^2\ t + \sin^2\ t = 1$, we obtain

$$x^2 + y^2 = 1.$$

Therefore, the curve is a circle. The orientation can be indicated as follows:

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	1	0	-1	0
y	1	0	-1	0	1
(x, y)	$(0, 1)$	$(1, 0)$	$(0, -1)$	$(-1, 0)$	$(0, 1)$



Note. The arrows on the curve indicate the direction in which the point (x, y) traces the curve as t increases from 0 to 2π.

Tangent Lines

Suppose that f and g are differentiable functions. We want to find the slope of the tangent line to a smooth curve C given by the parametric equations $x = f(t)$ and $y = g(t)$ where y is a differentiable function of x.

Chain Rule

$$y = h(x)$$

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if} \quad \frac{dx}{dt} \neq 0 \qquad (1)$$

Remark

- *If $dy/dt = 0$ such that $dx/dt \neq 0$, the curve has a horizontal tangent line.*
- *If $dx/dt = 0$ such that $dy/dt \neq 0$, the curve has a vertical tangent line.*

Tangent Lines

Example

Find the slope of the tangent line to the curve at the indicated value.

1. $x = t + 1$, $y = t^2 + 3t$; at $t = -1$
2. $x = t^3 - 3t$, $y = t^2 - 5t - 1$; at $t = 2$
3. $x = \sin\ t$, $y = \cos\ t$; at $t = \frac{\pi}{4}$

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Solution:

1. The slope of the tangent line at $P(x, y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3}{1} = 2t + 3.$$

   The slope of the tangent line at $t = -1$ is $m = 2(-1) + 3 = 1$.

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2. The slope of the tangent line at $P(x, y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-5}{3t^2-3}.$$

The slope of the tangent line at $t = 2$ is $m = \dfrac{2(2)-5}{3(2)^2-3} = \dfrac{-1}{9}$.

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3. The slope of the tangent line at $P(x, y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin\ t}{\cos\ t} = -\tan\ t.$$

The slope of the tangent line at $t = \frac{\pi}{4}$ is $m = -\tan \dfrac{\pi}{4} = -1$.

Tangent Lines

Example

Find the equations of the tangent line and the vertical tangent line at $t = 2$ *to the curve* C *given parametrically by* $x = 2t, \quad y = t^2 - 1.$

Tangent Lines

Example

Find the equations of the tangent line and the vertical tangent line at $t = 2$ to the curve C given parametrically by $x = 2t, \quad y = t^2 - 1$.

Solution: The slope of the tangent line at $P(x, y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

The slope of the tangent line at $t = 2$ is $m = 2$.

The point corresponding to $t = 2$ is $(x_0, y_0) = (4, 3)$. Thus, the equations of the tangent line is

$$y - 3 = 2(x - 4)$$

Point-Slope form: $y - y_0 = m(x - x_0)$

Since the slope of the tangent line at $t = 2$ is $m = 2$, then the slope of the vertical tangent line is $\frac{-1}{m} = \frac{-1}{2}$.

Thus, the equation of the vertical tangent line is

$$y - 3 = -\frac{1}{2}(x - 4).$$

Tangent Lines

Let the curve C has the parametric equations $x = f(t)$ and $y = g(t)$ where f and g are differentiable functions. To find the second derivative $\frac{d^2y}{dx^2}$, we apply the formula:

$$\frac{d^2y}{dx^2} = \frac{d(y^{'})}{dx} = \frac{dy^{'}/dt}{dx/dt} \qquad (2)$$

Note that $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$.

Example

If $x = t,\ y = t^2 - 1$, *find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* $t = 1$.

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Let the curve C has the parametric equations $x = f(t)$ and $y = g(t)$ where f and g are differentiable functions. To find the second derivative $\frac{d^2y}{dx^2}$, we apply the formula:

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Example

If $x = t$, $y = t^2 - 1$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 1$.

Solution: $\frac{dy}{dt} = 2t, \quad \frac{dx}{dt} = 1$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1} = 2t \Rightarrow \text{ at } t = 1, \text{we have} \frac{dy}{dx} = 2(1) = 2$$

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Example

If $x = \sin\ t$, $y = \cos\ t$, *find* $\dfrac{dy}{dx}$ *and* $\dfrac{d^2y}{dx^2}$ *at* $t = \frac{\pi}{3}$.

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Solution: $\frac{dy}{dt} = -\ \sin\ t, \quad \frac{dx}{dt} = \cos\ t$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\ \sin\ t}{\cos\ t} = -\ \tan\ t \Rightarrow \text{ at } t = \frac{\pi}{3}, \text{we have} \frac{dy}{dx} = -\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\sec^2\ t}{\cos\ t} = -\sec^3\ t \Rightarrow \text{ at } t = \frac{\pi}{3}, \text{we have} \frac{d^2y}{dx^2} = -8$$

Theorem

Let C be a smooth curve has the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$, and f' and g' are continuous. Assume that the curve C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$.

1. *The arc length of the curve is*

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

2. *If $y \geq 0$ over $[a, b]$, the surface area generated by revolving C about x-axis is*

$$S.A = 2\pi \int_a^b y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt\ .$$

3. *If $x \geq 0$ over $[a, b]$, the surface area generated by revolving C about y-axis is*

$$S.A = 2\pi \int_a^b x \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

Example

Find the length of the curve $x = e^t \cos\ t,\quad y = e^t \sin\ t,\quad 0 \leq t \leq \frac{\pi}{2}$.

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Solution: We apply $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$. First, we find $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{dx}{dt} = e^t \cos\ t - e^t \sin\ t \Rightarrow (\frac{dx}{dt})^2 = (e^t \cos\ t - e^t \sin\ t)^2\ ,$$

$$\frac{dy}{dt} = e^t \sin\ t + e^t \cos\ t \Rightarrow (\frac{dy}{dt})^2 = (e^t \sin\ t + e^t \cos\ t)^2.$$

Thus,

$$\begin{aligned}(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= e^{2t}\cos^2\ t - 2e^{2t}\cos\ t\ \sin\ t + e^{2t}\sin^2\ t + e^{2t}\sin^2\ t + 2e^{2t}\sin\ t\ \cos\ t + e^{2t}\cos^2\ t \\ &= e^{2t}(\cos^2\ t + \sin^2\ t) + e^{2t}(\cos^2\ t + \sin^2\ t) \qquad \text{Remember: } \cos^2\ t + \sin^2\ t = 1 \\ &= e^{2t} + e^{2t} = 2e^{2t}\end{aligned}$$

$$\Rightarrow \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{2}\ e^t$$

When applying $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$, we have

$$L = \sqrt{2}\int_0^{\frac{\pi}{2}} e^t\ dt = \sqrt{2}\ \left[\ e^t\ \right]_0^{\frac{\pi}{2}} = \sqrt{2}\ (e^{\frac{\pi}{2}} - 1)\ .$$

Example

Find the surface area generated by revolving the curve $x = 3\cos\ t,\ y = 3\sin\ t,\ 0 \leq t \leq \frac{\pi}{3}$ *about* x*-axis.*

Example

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Solution: Since the revolution is about x-axis, we apply the formula

$$S.A = 2\pi \int_a^b y\ \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt.$$

We find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as follows:

$$\frac{dx}{dt} = -3\sin\ t \Rightarrow (\frac{dx}{dt})^2 = 9\sin^2\ t$$

$$\frac{dy}{dt} = 3\cos\ t \Rightarrow (\frac{dx}{dt})^2 = 9\cos^2\ t$$

Thus,

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 9(\sin^2\ t + \cos^2\ t) = 9$$

Remember: $\cos^2\ t + \sin^2\ t = 1$

This implies

$$S.A = 2\pi \int_0^{\frac{\pi}{3}} 3\ \sin\ t\ \sqrt{9}\ dt = 18\pi \int_0^{\frac{\pi}{3}} \sin\ t\ dt = -18\pi\ \Big[\cos\ t\Big]_0^{\frac{\pi}{3}}$$

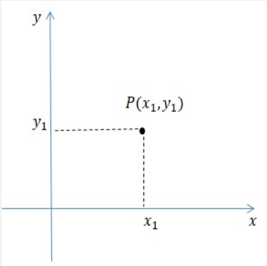
$$= -18\pi\ \Big[\frac{1}{2} - 1\Big] = 9\pi\ .$$

Polar Coordinates

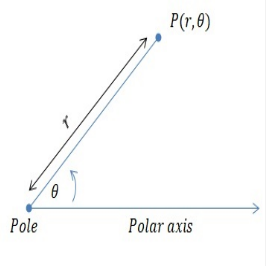
Definition

The polar coordinate system is a two-dimensional system consisted of a pole and a polar axis (half line). Each point P on a plane is determined by a distance r from a fixed point O called the pole (or origin) and an angle θ from a fixed direction.

Cartesian Coordinates (Rectangular Coordinates)



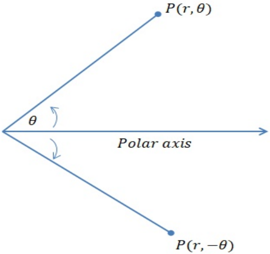
Polar Coordinate



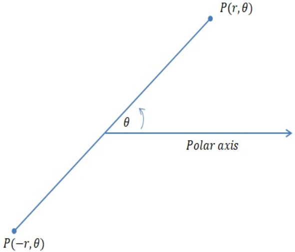
■ **Notes.**

(1) From the definition, the point P in the polar coordinate system is represented by the ordered pair (r, θ) where r, θ are called polar coordinates.

(2) The angle θ is positive if it is measured counterclockwise from the axis, but if it is measured clockwise the angle is negative.



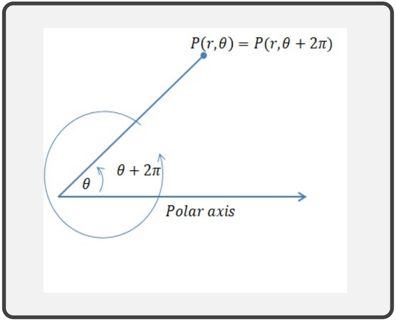
(3) In the polar coordinates, if $r > 0$, the point $P(r, \theta)$ will be in the same quadrant as θ; if $r < 0$, it will be in the quadrant on the opposite side of the pole with the half line. That is, the points $P(r, \theta)$ and $P(-r, \theta)$ lie in the same line through the pole O, but on opposite sides of O.

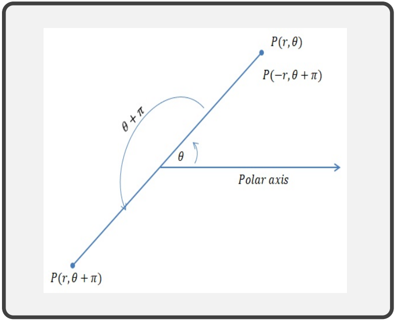


Polar Coordinates

(4) In the Cartesian coordinate system, every point has only one representation while in a polar coordinate system each point has many representations. The following formula gives all representations of a point $P(r, \theta)$ in the polar coordinate system

$$P(r, \theta + 2n\pi) = P(r, \theta) = P(-r, \theta + (2n+1)\pi), \quad n \in \mathbb{Z}\,. \tag{4}$$





Polar Coordinates

Example

Plot the points whose polar coordinates are given.

1. $(1, 5\pi/4)$
2. $(1, -3\pi/4)$
3. $(1, 13\pi/4)$
4. $(-1, \pi/4)$

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Solution:

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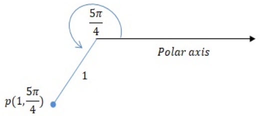

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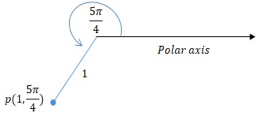
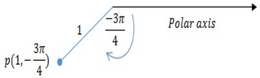


(2)



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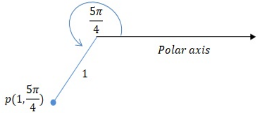

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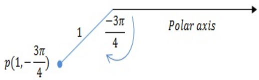

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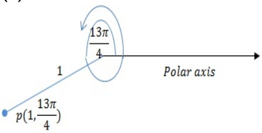

$\frac{13\pi}{4} = \frac{5\pi}{4} + 2\pi$

$P(r, \theta) = P(r, \theta + 2n\pi)$ choosing $n = 1$

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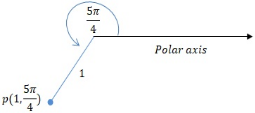

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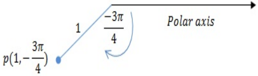

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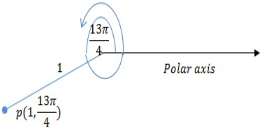

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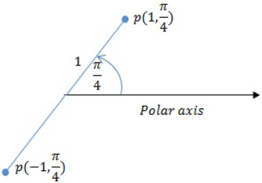


$\frac{\pi}{4} = \frac{5\pi}{4} - \pi$

$P(r, \theta) = P(-r, \theta + (2n+1)\pi)$ choosing $n = -1$

The Relationship between Cartesian and Polar Coordinates

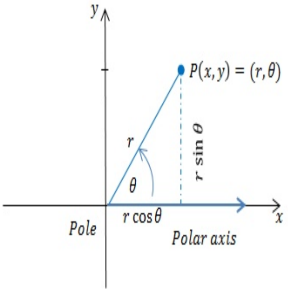
- Let (x, y) be the Cartesian (rectangular) coordinates and (r, θ) be the polar coordinates of the same point P.

- Let the pole be at the origin of the Cartesian coordinates system, and let the polar axis be the positive x-axis and the line $\theta = \frac{\pi}{2}$ be the positive y-axis as shown in the figure.

In the triangle, we have

$$\cos\ \theta = \frac{x}{r} \Rightarrow x = r\cos\ \theta\ ,$$

$$\sin\ \theta = \frac{y}{r} \Rightarrow y = r\sin\ \theta .$$



Hence,

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2, \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2 \qquad (\cos^2\theta + \sin^2\theta = 1) \end{aligned}$$

The previous relationships can be summarized as follows:

$$x = r\cos\ \theta, \quad y = r\sin\ \theta,$$

$$\tan\ \theta = \frac{y}{x} \ \text{ for } \ x \neq 0,$$

$$x^2 + y^2 = r^2$$

Example

Convert from polar coordinates to rectangular coordinates.

1. $(1, \pi/4)$
2. $(2, \pi)$

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Solution:

1. We have $r = 1$ and $\theta = \frac{\pi}{4}$. From the formulas given above,

$$x = r\cos\ \theta = (1)\cos\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}\ ,$$

$$y = r\sin\ \theta = (1)\sin\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Therefore, in the Cartesian coordinates, the point $(1, \pi/4)$ is represented by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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Therefore, in the Cartesian coordinates, the point $(1, \pi/4)$ is represented by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

2. We have $r = 2$ and $\theta = \pi$, hence

$$x = r\cos\ \theta = 2\cos\ \pi = -2\ ,$$

$$y = r\sin\ \theta = 2\sin\ \pi = 0.$$

The point $(2, \pi)$ is represented in the Cartesian coordinates by $(-2, 0)$.

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For the given Cartesian point, find one representation in the polar coordinates.

1. $(1, -1)$
2. $(2\sqrt{3}, -2)$

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Solution:

1. We have $x = 1$ and $y = -1$. By using the formulas given above,

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{2},$$

$$\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = -\frac{\pi}{4}\ .$$

In the polar coordinates, the point $(1, -1)$ can be represented by $(\sqrt{2}, -\frac{\pi}{4})$.

As mentioned in Note 4, in a polar coordinate system, each point has several representations:

$$P(r, \theta) = P(r, \theta + 2n\pi), \quad n \in \mathbb{Z}\ .$$

$$P(r, \theta) = P(-r, \theta + (2n+1)\pi)$$

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2. We have $x = 2\sqrt{3}$ and $y = -2$. Hence,

$$x^2 + y^2 = r^2 \Rightarrow r = 4,$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}} \Rightarrow \theta = \frac{5\pi}{6}.$$

Therefore, the point $(4, \frac{5\pi}{6})$ in the polar coordinate system is one representation of the point $(2\sqrt{3}, -2)$.

Polar Equations

- In Cartesian coordinates (x, y), the equations are either of the form $y = f(x)$ or $x = f(y)$.

- In polar coordinates (r, θ), the equations take one form, which is $r = f(\theta)$.

Note: a **solution** of the polar equation is an ordered pair (r_0, θ_0) satisfies the equation i.e., $r_0 = f(\theta_0)$.

Example The polar points $(1, \frac{\pi}{3})$ and $(\sqrt{2}, \frac{\pi}{4})$ are solutions of the polar equation $r = 2\cos\ \theta$:
Put $\theta = \frac{\pi}{3} \Rightarrow r = 2\cos(\frac{\pi}{3}) = 2(\frac{1}{2}) = 1$. This means $(1, \frac{\pi}{3})$ is a solution of the polar equation $r = 2\cos\ \theta$.

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Put $\theta = \frac{\pi}{4} \Rightarrow r = 2\cos(\frac{\pi}{4}) = 2(\frac{1}{\sqrt{2}}) = \sqrt{2}$. This means $(\sqrt{2}, \frac{\pi}{4})$ is a solution of the polar equation $r = 2\cos\ \theta$.

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Example

Find a polar equation that has the same graph as the equation in x and y.

1. $x = 7$
2. $y = -3$
3. $x^2 + y^2 = 4$
4. $y^2 = 9x$

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Solution:

1. $x = 7 \Rightarrow r\cos\ \theta = 7$

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Solution:

1. $x = 7 \Rightarrow r\cos\ \theta = 7 \Rightarrow r = 7.\frac{1}{\cos\theta}$

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Solution:

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Remember $x = r\cos\ \theta$

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3. $x^2 + y^2 = 4 \Rightarrow r^2 = 4$ Remember $x^2 + y^2 = r^2$
4. $y^2 = 9x \Rightarrow r^2\sin^2\ \theta = 9r\cos\ \theta$

$$\Rightarrow r\sin^2\ \theta = 9\cos\ \theta$$

$$\Rightarrow r = 9\frac{\cos\ \theta}{\sin\ \theta}.\frac{1}{\sin\ \theta} \Rightarrow r = 9\cot\ \theta\csc\ \theta.$$

Example

Find an equation in x and y that has the same graph as the polar equation.

1. $r = 3$
2. $r = \sin \theta$
3. $r = 6 \cos \theta$
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2. $r = \sin\theta \Rightarrow r^2 = r\sin\theta \Rightarrow r^2 = y \Rightarrow x^2 + y^2 = y \Rightarrow x^2 + y^2 - y = 0.$

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4. $r = \sec\ \theta \Rightarrow r = \dfrac{1}{\cos\ \theta} \Rightarrow r\cos\ \theta = 1 \Rightarrow x = 1.$

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Tangent Line to Polar Curves

Since $r = f(\theta)$ is a polar equation, then $x = r\cos\ \theta \Rightarrow x = f(\theta)\cos\ \theta$ and $y = r\sin\ \theta \Rightarrow y = f(\theta)\sin\ \theta$.
From the chain rule, we have

$$\frac{dx}{d\theta} = -f(\theta)\sin\ \theta + f'(\theta)\cos\ \theta = -r\sin\ \theta + \frac{dr}{d\theta}\cos\ \theta\ ,$$

$$\frac{dy}{d\theta} = f(\theta)\cos\ \theta + f'(\theta)\sin\ \theta = r\cos\ \theta + \frac{dr}{d\theta}\sin\ \theta.$$

If $\dfrac{dx}{d\theta} \neq 0$, the slope of the tangent line to the graph of $r = f(\theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\ \theta + \sin\ \theta(dr/d\theta)}{-r\sin\ \theta + \cos\ \theta(dr/d\theta)}.$$

Tangent Line to Polar Curves

Theorem

Let $r = f(\theta)$ be a polar equation where f' is continuous. The slope of the tangent line to the graph of $r = f(\theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\ \theta + \sin\ \theta(dr/d\theta)}{-r\sin\ \theta + \cos\ \theta(dr/d\theta)}.$$

- **Remark**
- If $\frac{dy}{d\theta} = 0$ such that $\frac{dx}{d\theta} \neq 0$, the curve has a horizontal tangent line.
- If $\frac{dx}{d\theta} = 0$ such that $\frac{dy}{d\theta} \neq 0$, the curve has a vertical tangent line.
- If $\frac{dx}{d\theta} \neq 0$ at $\theta = \theta_0$, the slope of the tangent line to the graph of $r = f(\theta)$ is

$$\frac{r_0\cos\ \theta_0 + \sin\ \theta_0(dr/d\theta)_{\theta=\theta_0}}{-r_0\sin\ \theta_0 + \cos\ \theta_0(dr/d\theta)_{\theta=\theta_0}}, \quad \text{where} \quad r_0 = f(\theta_0)$$

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Find the slope of the tangent line to the graph of $r = \sin\ \theta$ at $\theta = \frac{\pi}{4}$.

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Example

Find the slope of the tangent line to the graph of $r = \sin\theta$ at $\theta = \frac{\pi}{4}$.

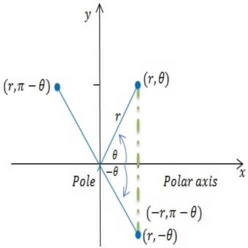
Solution: $x = r\cos\theta \Rightarrow x = \sin\theta\cos\theta \Rightarrow \frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta$, and
$y = r\sin\theta \Rightarrow y = \sin^2\theta \Rightarrow \frac{dy}{d\theta} = 2\sin\theta\cos\theta$.
From the theorem, we have $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$.
At $\theta = \frac{\pi}{4}$, we have $\frac{dy}{d\theta} = 1$ and $\frac{dx}{d\theta} = 0$. Thus, the slope is undefined and in this case, the curve has a vertical tangent line.

Symmetry in Polar Coordinates

Theorem

1. **Symmetry about the polar axis.**
   The graph of $r = f(\theta)$ *is symmetric with respect to the polar axis if replacing* (r, θ) *with* $(r, -\theta)$ *or with* $(-r, \pi - \theta)$ *does not change the equation.*
2. **Symmetry about the vertical line** $\theta = \frac{\pi}{2}$**.**
   The graph of $r = f(\theta)$ *is symmetric with respect to the vertical line if replacing* (r, θ) *with* $(r, \pi - \theta)$ *or with* $(-r, -\theta)$ *does not change the equation.*
3. **Symmetry about the pole** $\theta = 0$**.**
   The graph of $r = f(\theta)$ *is symmetric with respect to the pole if replacing* (r, θ) *with* $(-r, \theta)$ *or with* $(r, \theta + \pi)$ *does not change the equation.*

A



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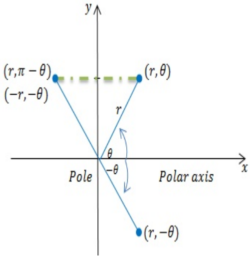

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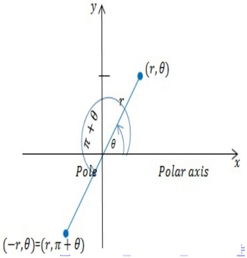


Graphs in Polar Coordinates

Some Special Polar Graphs

Lines in polar coordinates

1. The polar equation of a straight line $ax + by = c$ is $r = \dfrac{c}{a\cos\ \theta + b\sin\ \theta}$.

   Since $x = r\cos\ \theta$ and $y = r\sin\ \theta$, then

   $$ax + by = c \Rightarrow r(a\cos\ \theta + b\sin\ \theta) = c \Rightarrow r = \frac{c}{(a\cos\ \theta + b\sin\ \theta)}$$

2. The polar equation of a vertical line $x = k$ is $r = k\sec\ \theta$.

   Let $x = k$, then $r\cos\ \theta = k$. This implies $r = \dfrac{k}{\cos\ \theta} = k\sec\ \theta$.

3. The polar equation of a horizontal line $y = k$ is $r = k\csc\ \theta$.

   Let $y = k$, then $r\sin\ \theta = k$. This implies $r = \dfrac{k}{\sin\ \theta} = r\csc\ \theta$.

4. The polar equation of a line that passes the origin point and makes an angle θ_0 with the positive x-axis is $\theta = \theta_0$.

Circles in polar coordinates

1. The circle equation with center at the pole O and radius $|a|$ is $r = a$.
2. The circle equation with center at $(a, 0)$ and radius $|a|$ is

$$r = 2a\cos\ \theta$$

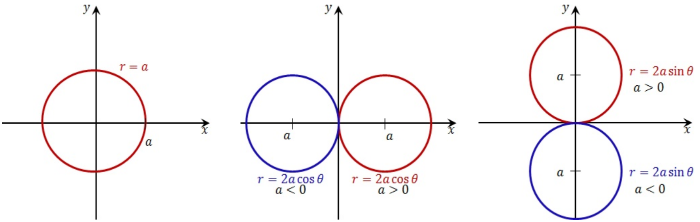
3. The circle equation with center at $(0, a)$ and radius $|a|$ is

$$r = 2a\sin\ \theta$$

For item 2,

$$r = 2a\ \cos\ \theta$$
$$r^2 = 2a\ r\cos\ \theta$$
$$x^2 + y^2 = 2\ a\ x$$
$$(x - a)^2 + y^2 = a^2$$

Equation of a circle with center $(a, 0)$.



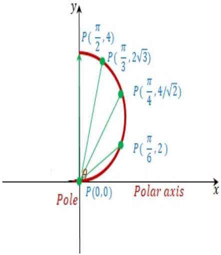
Example

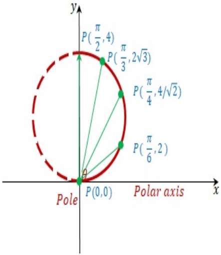
Sketch the graph of $r = 4 \sin \ \theta$.

Solution:
Note that the graph of $r = 4 \sin \ \theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since $4 \sin \ (\pi - \theta) = 4 \sin \ \theta$.

We restrict our attention to the interval $[0, \pi/2]$ and by the symmetry, we complete the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	2	$4/\sqrt{2}$	$2\sqrt{3}$	4

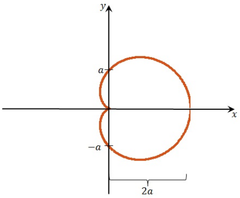




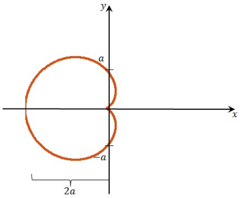
Cardioid curves

1. $r = a(1 \pm \cos\ \theta)$

$r = a(1 + \cos\ \theta)$

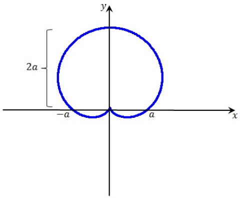


$r = a(1 - \cos\ \theta)$

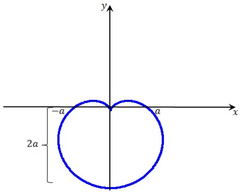


2. $r = a(1 \pm \sin\ \theta)$

$r = a(1 + \sin\ \theta)$



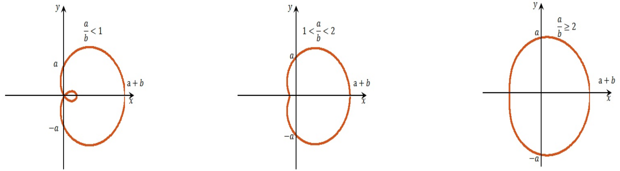
$r = a(1 - \sin\ \theta)$



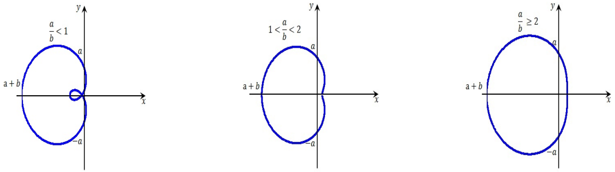
Limaçons curves

1. $r = a \pm b\cos\ \theta$

1. $r = a + b\cos\ \theta$



2. $r = a - b\cos\ \theta$



2. $r = a \pm b\sin\ \theta$

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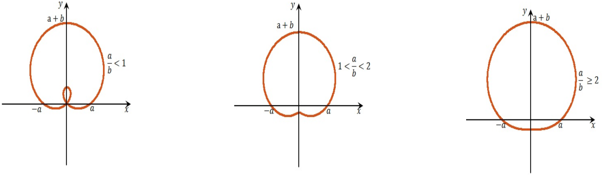
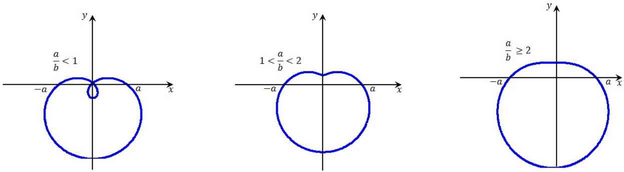


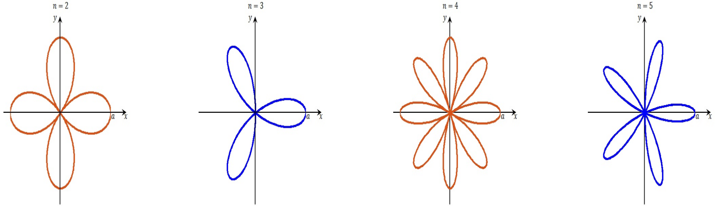
2. $r = a - b\sin\ \theta$



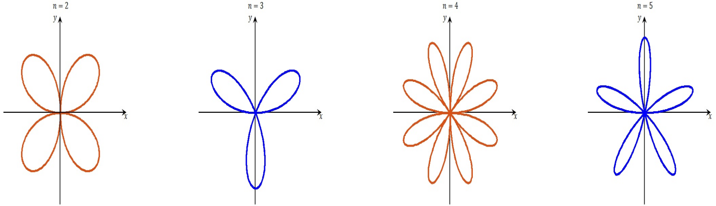
Graphs in Polar Coordinates

Roses (Note that if n is odd, there are n petals; however, if n is even, there are $2n$ petals.)

1. $r = a\ \cos(n\theta)$ where $n \in \mathbb{N}$

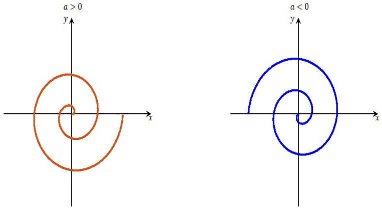


2. $r = a\ \sin(n\theta)$ where $n \in \mathbb{N}$.



Graphs in Polar Coordinates

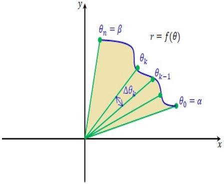
Spiral of Archimedes
$r = a\ \theta$



Area in Polar Coordinates

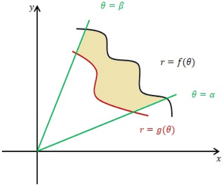
Case 1

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$$

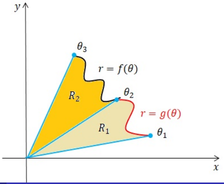


Case 2

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] d\theta$$



Case 3



$$A_1 = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left(g(\theta)\right)^2 \, d\theta$$

$$A_2 = \frac{1}{2}\int_{\theta_2}^{\theta_3} \left(f(\theta)\right)^2 \, d\theta$$

Total: $A = A_1 + A_2$

Area in Polar Coordinates

Example

Find the area of the region bounded by the graph of the polar equation.

1. $r = 3$
2. $r = 2\cos\ \theta$

Example

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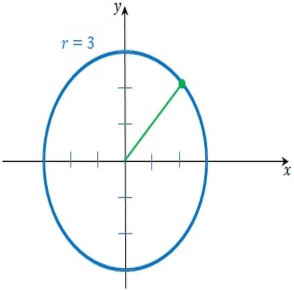
Solution:



(1) The graph of $r = 3$ is obtained by letting θ takes values from 0 to 2π. Thus, the area is

$$A = \frac{1}{2}\int_0^{2\pi} 3^2\ d\theta = \frac{9}{2}\int_0^{2\pi}\ d\theta = \frac{9}{2}\Big[\ \theta\ \Big]_0^{2\pi} = 9\pi.$$

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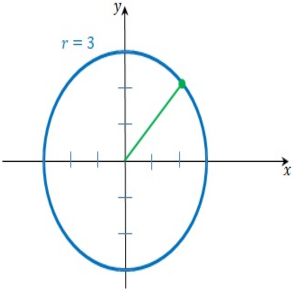

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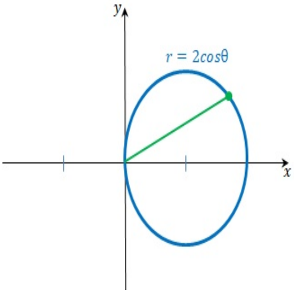
Note that one can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.,

$$A = 4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} 3^2\ d\theta\Big) = 2\int_0^{\frac{\pi}{2}} 9\ d\theta = 18\Big[\ \theta\ \Big]_0^{\frac{\pi}{2}} = 9\pi.$$

(2) We find the area of the upper half circle and multiply the result by 2 as follows:

$$
\begin{aligned}
A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos\ \theta)^2\ d\theta\right)\\
&= \int_0^{\frac{\pi}{2}} 4\cos^2\ \theta\ d\theta \qquad : \cos^2\ \theta = \frac{1+\cos\ 2\theta}{2}\\
&= 2\int_0^{\frac{\pi}{2}} (1+\cos\ 2\theta)\ d\theta\\
&= 2\left[\theta + \frac{\sin\ 2\theta}{2}\right]_0^{\frac{\pi}{2}}\\
&= 2\left[\frac{\pi}{2} - 0\right]\\
&= \pi.
\end{aligned}
$$



Area in Polar Coordinates

Example

Find the area of the region that is outside the graph of $r = 3$ *and inside the graph of* $r = 2 + 2\cos\theta$.

Example

Find the area of the region that is outside the graph of $r = 3$ and inside the graph of $r = 2 + 2\cos\theta$.

Solution: The intersection point of the two curves in the first quadrant is

$$2 + 2\cos\ \theta = 3 \Rightarrow \cos\ \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

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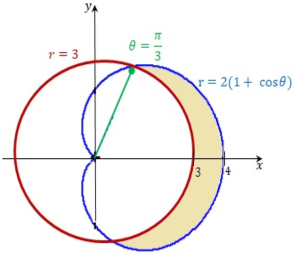
Solution: The intersection point of the two curves in the first quadrant is

$$2 + 2\cos\ \theta = 3 \Rightarrow \cos\ \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

As shown in the figure, we find the area in the first quadrant and then we double the result to find the area of the whole region.

$$\begin{aligned} A &= 2\Big(\frac{1}{2}\int_0^{\frac{\pi}{3}} \big(4(1+\cos\ \theta)^2 - 9\big)\ d\theta\Big) \\ &= \int_0^{\frac{\pi}{3}} \big(4(1 + 2\cos\ \theta + \cos^2\ \theta) - 9\big)\ d\theta \\ &= \int_0^{\frac{\pi}{3}} (8\cos\ \theta + 4\cos^2\ \theta - 5)\ d\theta \\ &= \Big[8\sin\ \theta + \sin\ 2\theta - 3\theta\Big]_0^{\frac{\pi}{3}} \\ &= \frac{9}{2}\sqrt{3} - \pi. \end{aligned}$$



Example

Find the area of the region that is inside the graphs of the equations $r = \sin\theta$ *and* $r = \sqrt{3}\cos\theta$.

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Solution:

First, we find the intersection points of the two curves

$$\sin\ \theta = \sqrt{3}\cos\ \theta \Rightarrow \tan\ \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}.$$

Area in Polar Coordinates

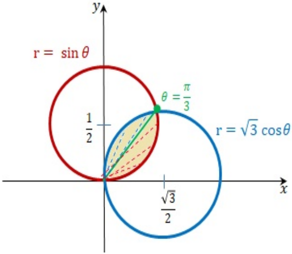
Example

Find the area of the region that is inside the graphs of the equations $r = \sin\theta$ *and* $r = \sqrt{3}\cos\theta$*.*

Solution:

First, we find the intersection points of the two curves

$$\sin\ \theta = \sqrt{3}\cos\ \theta \Rightarrow \tan\ \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}.$$

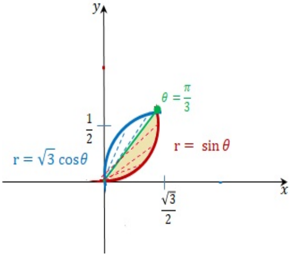


The origin O is in each circle, but it cannot be found by solving the equations. Therefore, when looking for the intersection points of the polar graphs, we sometimes take under consideration the graphs.

The region is divided into two small regions: below and above the line $\frac{\pi}{3}$.

Region(1): below the line $\frac{\pi}{3}$.

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\ \theta\ d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1 - \cos\ 2\theta)\ d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{\sin\ 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sin\ \frac{2\pi}{3}}{2}\right]$$
$$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right].$$



Area in Polar Coordinates

Region(1): below the line $\frac{\pi}{3}$.

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\ \theta\ d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos\ 2\theta)\ d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{\sin\ 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sin\ \frac{2\pi}{3}}{2}\right]$$
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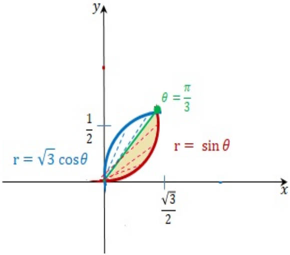
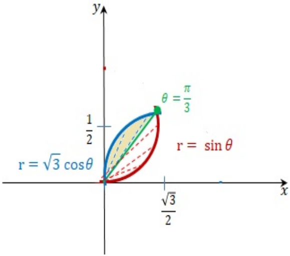


Region(2): above the line $\frac{\pi}{3}$.

$$A_2 = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\ \theta)^2\ d\theta = \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos\ 2\theta)\ d\theta$$
$$= \frac{3}{4}\left[\theta + \frac{\sin\ 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= \frac{3}{4}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)\right]$$
$$= \frac{3}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right].$$



Area in Polar Coordinates

Region(1): below the line $\frac{\pi}{3}$.

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\ \theta\ d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1 - \cos\ 2\theta)\ d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{\sin\ 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sin\ \frac{2\pi}{3}}{2}\right]$$
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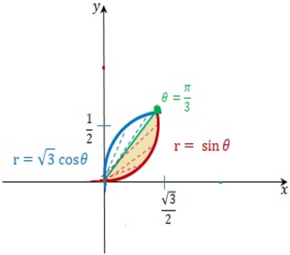

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$$A_2 = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\ \theta)^2\ d\theta = \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + \cos\ 2\theta)\ d\theta$$
$$= \frac{3}{4}\left[\theta + \frac{\sin\ 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= \frac{3}{4}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)\right]$$
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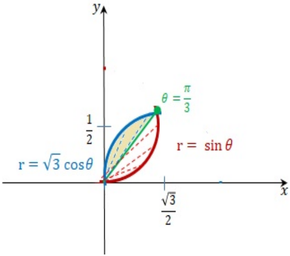

Total area $A = A_1 + A_2 = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$.

Arc Length and Surface of Revolution in Polar Coordinates

Arc Length in Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} \; d\theta \tag{9}$$

Arc Length in Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} \ d\theta \tag{9}$$

Example

Find the length of the curve.

1. $r = 2$
2. $r = 2\sin\ \theta$

Arc Length in Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} \; d\theta \qquad (9)$$

Example

Find the length of the curve.

1. $r = 2$
2. $r = 2\sin\ \theta$

Solution:

1. $r^2 + (\frac{dr}{d\theta})^2 = 4$. Hence,

$$L = \int_0^{2\pi} \sqrt{4}\, d\theta = 2\Big[\ \theta\ \Big]_0^{2\pi} = 4\pi.$$

Arc Length in Polar Coordinates

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} \; d\theta \qquad (9)$$

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Solution:

1. $r^2 + (\frac{dr}{d\theta})^2 = 4$. Hence,

$$L = \int_0^{2\pi} \sqrt{4}\, d\theta = 2\Big[\ \theta\ \Big]_0^{2\pi} = 4\pi.$$

2. $r^2 + (\frac{dr}{d\theta})^2 = 4\sin^2\ \theta + 4\cos^2\ \theta = 4(\sin^2\ \theta + \cos^2\ \theta) = 4.$ Remember: $\cos^2\ \theta + \sin^2\ \theta = 1$

This implies

$$L = \int_0^{\pi} \sqrt{4}\, d\theta = 2\Big[\ \theta\ \Big]_0^{\pi} = 2\pi.$$

Surface of Revolution in Polar Coordinates

- The surface area generated by revolving the curve $r = f(\theta)$ about the polar axis (x-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} \mid r \sin \ \theta \mid \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta \qquad (10)$$

Surface of Revolution in Polar Coordinates

- The surface area generated by revolving the curve $r = f(\theta)$ about the polar axis (x-axis) is

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- The surface area generated by revolving the curve $r = f(\theta)$ about the line $\theta = \frac{\pi}{2}$ (y-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} \mid r \cos \ \theta \mid \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta \qquad (11)$$

Surface of Revolution in Polar Coordinates

- The surface area generated by revolving the curve $r = f(\theta)$ about the polar axis (x-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} | \, r \sin \theta \, | \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta \qquad (10)$$

- The surface area generated by revolving the curve $r = f(\theta)$ about the line $\theta = \frac{\pi}{2}$ (y-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} | \, r \cos \theta \, | \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta \qquad (11)$$

Example

Find the area of the surface generated by revolving the curve $r = 2\sin \theta$ about the polar axis.

Surface of Revolution in Polar Coordinates

The surface area generated by revolving the curve $r = f(\theta)$ about the polar axis (x-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} |\ r \sin\ \theta\ | \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta \qquad (10)$$

The surface area generated by revolving the curve $r = f(\theta)$ about the line $\theta = \frac{\pi}{2}$ (y-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} |\ r \cos\ \theta\ | \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta \qquad (11)$$

Example

Find the area of the surface generated by revolving the curve $r = 2\sin\ \theta$ about the polar axis.

Solution: We apply the formula $S.A = 2\pi \int_{\alpha}^{\beta} |\ r \sin\theta\ | \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$.

$$r^2 + (\frac{dr}{d\theta})^2 = 4\sin^2\ \theta + 4\cos^2\ \theta = 4(\sin^2\ \theta + \cos^2\ \theta) = 4.$$

$$\Rightarrow S.A = 2\pi \int_0^{\pi} |\ (2\ \sin\ \theta)\ \sin\theta\ | \sqrt{4}\ d\theta = 8\pi \int_0^{\pi} \sin^2\ \theta\ d\theta = 4\pi \int_0^{\pi} (1 - \cos\ 2\theta)\ d\theta = 4\pi \left[\theta - \frac{\sin\ 2\theta}{2}\right]_0^{\pi} = 4\pi^2.$$