Integral Calculus

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Main Content

Review

In the beginning of this section, we remind the reader with definition of limits and list some rules of the limits. Let f be a defined function on an open interval I and $c \in I$ where f may not be defined at c. Then,

$$\lim_{x \to c} f(x) = L, \ \ L \in \mathbb{R}$$

means for every $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Some Rules of the Limits: If $\lim_{x \longrightarrow c} f(x)$ and $\lim_{x \longrightarrow c} g(x)$ both exist, then

1. **Sum Rule:** $\lim_{x \longrightarrow c} \big(f(x) + g(x)\big) = \lim_{x \longrightarrow c} f(x) + \lim_{x \longrightarrow c} g(x)$.
2. **Difference Rule:** $\lim_{x \longrightarrow c} \big(f(x) - g(x)\big) = \lim_{x \longrightarrow c} f(x) - \lim_{x \longrightarrow c} g(x)$.
3. **Product Rule:** $\lim_{x \longrightarrow c} \big(f(x).g(x)\big) = \lim_{x \longrightarrow c} f(x) \times \lim_{x \longrightarrow c} g(x)$.
4. **Constant Multiple Rule:** $\lim_{x \longrightarrow c} \big(k\ f(x)\big) = k \lim_{x \longrightarrow c} f(x)$.
5. **Quotient Rule:** $\lim_{x \longrightarrow c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \longrightarrow c} f(x)}{\lim_{x \longrightarrow c} g(x)}$.
6. **Power Rule:** $\lim_{x \longrightarrow c} \big(f(x)\big)^{m/n} = \big(\lim_{x \longrightarrow c} f(x)\big)^{m/n}$.

Notes:

- $\frac{0}{a} = 0$ where $a \neq 0$
- $\frac{a}{\pm\infty} = 0$ where a is a number.
- $\frac{\pm\infty}{a} = \pm\infty$ where a is a positive number.

Example

Find each limit if it exists.

1. $\lim_{x \to 1} x$
2. $\lim_{x \to 8} \sqrt{x}$
3. $\lim_{x \to 0} (x^2 - 2x + 1)$
4. $\lim_{x \to \pi} \sin\ x\ \cos\ x$
5. $\lim_{x \to 1} \frac{x}{(x^2 + 1)}$
6. $\lim_{x \to 3^+} \frac{1}{(x - 3)}$

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Solution:

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Solution:

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2. $\lim_{x \to 8} \sqrt{x} = \sqrt{8} = 2\sqrt{2}$

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Solution:

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3. $\lim_{x \to 0} (x^2 - 2x + 1) = \lim_{x \to 0} x^2 - 2 \lim_{x \to 0} x + \lim_{x \to 0} 1 = 1.$

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5. $\lim_{x \to 1} \frac{x}{(x^2 + 1)} = \frac{\lim_{x \to 1} x}{\lim_{x \to 1} (x^2 + 1)} = \frac{1}{2}$

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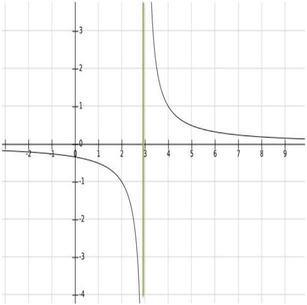
6. $\lim_{x \to 3^+} \frac{1}{(x - 3)} = \infty$



■ **Indeterminate Forms.**

Example

1. $\lim_{x\to 0} \frac{\sin\ x}{x} = \frac{0}{0}$
2. $\lim_{x\to \infty} \frac{e^x}{x} = \frac{\infty}{\infty}$
3. $\lim_{x\to 0^+} x^2 \ln x = 0.\infty$
4. $\lim_{x\to 1^+} (\frac{1}{x-1} - \frac{1}{\ln x}) = \infty - \infty$

Indeterminate Forms & L'Hôpital's Rule

Indeterminate Forms.

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In the following table, we categorize the indeterminate forms:

List of the indeterminate forms.

Case	Indeterminate Form
Quotient	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
Product	$0.\infty$ or $0.(-\infty)$
Sum & Difference	$(-\infty) + \infty$ or $\infty - \infty$
Exponent	0^0, 1^∞, $1^{-\infty}$ or ∞^0

L'Hôpital's Rule

The following theorem examines the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Theorem

Suppose f and g are differentiable on an interval I and $c \in I$ where f and g may not be differentiable at c. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$ and $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or equals to ∞.

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if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or equals to ∞.

Notes:

1. We can apply L'Hôpital's rule for $c = \pm\infty$ and when $x \to c^+$ or $x \to c^-$.
2. When applying L'Hôpital's rule, we should calculate the derivatives of $f(x)$ and $g(x)$ separately.
3. Sometimes, we need to apply L'Hôpital's rule twice.

Example

Use L'Hôpital's rule to find each limit if it exists.

1. $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25}$
2. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$
3. $\lim_{x\to\infty} \frac{e^x}{x}$

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Solution:

1. Since $\lim\limits_{x\to 5} \sqrt{x-1}-2=0$ and $\lim\limits_{x\to 5} x^2-2=0$, we have the indeterminate form $\dfrac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x\to 5} \frac{\frac{1}{2\sqrt{x-1}}}{2x} = \lim_{x\to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$

Example

Use L'Hôpital's rule to find each limit if it exists.

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$$\frac{\frac{1}{2\sqrt{x-1}}}{2x} = \frac{1}{2\sqrt{x-1}} \div \frac{2x}{1} = \frac{1}{2\sqrt{x-1}} \times \frac{1}{2x} = \frac{1}{4x\sqrt{x-1}}$$

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Use L'Hôpital's rule to find each limit if it exists.

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Solution:

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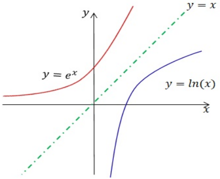
2. The indeterminate form is $\frac{\infty}{\infty}$. Apply L'Hôpital's rule to obtain

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$

$$\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\frac{1}{x}\div\frac{1}{2\sqrt{x}}=\frac{1}{x}\times\frac{2\sqrt{x}}{1}=\frac{2\sqrt{x}}{\sqrt{x}\,\sqrt{x}}=\frac{2}{\sqrt{x}}$$



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Use L'Hôpital's rule to find each limit if it exists.

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Solution:

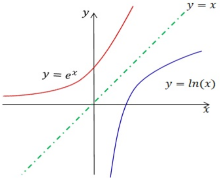
1. Since $\lim\limits_{x\to 5} \sqrt{x-1}-2 = 0$ and $\lim\limits_{x\to 5} x^2-2=0$, we have the indeterminate form $\dfrac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x\to 5} \frac{\frac{1}{2\sqrt{x-1}}}{2x} = \lim_{x\to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$

$$\frac{\frac{1}{2\sqrt{x-1}}}{2x} = \frac{1}{2\sqrt{x-1}} \div \frac{2x}{1} = \frac{1}{2\sqrt{x-1}} \times \frac{1}{2x} = \frac{1}{4x\sqrt{x-1}}$$

2. The indeterminate form is $\dfrac{\infty}{\infty}$. Apply L'Hôpital's rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$



$$\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{1}{x} \div \frac{1}{2\sqrt{x}} = \frac{1}{x} \times \frac{2\sqrt{x}}{1} = \frac{2\sqrt{x}}{\sqrt{x}\,\sqrt{x}} = \frac{2}{\sqrt{x}}$$

3. The indeterminate form is $\dfrac{\infty}{\infty}$. By applying L'Hôpital's rule, we have $\lim\limits_{x\to\infty} \dfrac{e^x}{x} = \lim\limits_{x\to\infty} \dfrac{e^x}{1} = \infty$.

Indeterminate Forms & L'Hôpital's Rule

Techniques for other indeterminate forms.

Indeterminate form $0.\infty$.

1. Write $f(x)\ g(x)$ as $\dfrac{f(x)}{1/g(x)}$ or $\dfrac{g(x)}{1/f(x)}$.
2. Apply L'Hôpital's rule to the resulting indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

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Example

Find the limit if it exists $\lim\limits_{x \to 0^+} x^2 \ln x$

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Example

Find the limit if it exists $\lim\limits_{x \to 0^+} x^2 \ln x$

Solution: The indeterminate form is $0.(-\infty)$, so we cannot apply L'Hôpital's rule. We need to rearrange the expression in a way that enables us to apply L'Hôpital's rule. By using the previous techniques, we have

$$x^2 \ln x = \frac{\ln x}{\frac{1}{x^2}}.$$

The indeterminate form of the new expression is $\dfrac{\infty}{\infty}$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \to 0^+} \frac{x^2}{-2} = 0.$$

Note: $y = \frac{1}{x^2} = x^{-2} \Rightarrow y' = -2\, x^{-3} = \frac{-2}{x^3}$

Hence, $\lim\limits_{x \to 0^+} x^2 \ln x = 0$.

Indeterminate Forms & L'Hôpital's Rule

- Indeterminate form $(-\infty) + \infty$ or $\infty - \infty$.
  1. Write the form as a quotient or product.
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Find the limit if it exists $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

Solution: The indeterminate form is $\infty - \infty$.

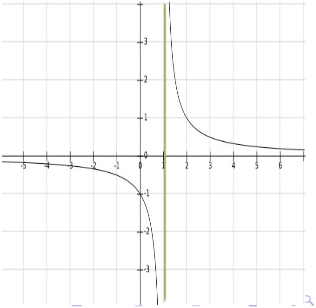
$$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x}.$$

We have the indeterminate form $\frac{0}{0}$. From L'Hôpital's rule,

$$\lim_{x \to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x \to 1^+} \frac{1-x}{x\ \ln x + x - 1}.$$

We have the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule again to have

$$\lim_{x \to 1^+} \frac{1-x}{x\ \ln x + x - 1} = \lim_{x \to 1^+} \frac{-1}{\ln x + 2} = \frac{-1}{2}.$$



Indeterminate Forms & L'Hôpital's Rule

- Indeterminate forms 0^0, 1^∞, $1^{-\infty}$ or ∞^0.
  1. Let $y = f(x)^{g(x)}$
  2. Take the natural logarithm $\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$.
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Find the limit if it exists $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}}$.

Solution: The indeterminate form is 1^∞. To treat this form, let $y = (1+x)^{\frac{1}{x}}$. By taking the natural logarithm of both sides, we have

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$$\ln y = \frac{1}{x} \ln(1+x) \Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

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$$\ln y = \frac{1}{x} \ln(1+x) \Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

The indeterminate form is $\frac{0}{0}$.

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The indeterminate form is $\frac{0}{0}$. By applying L'Hôpital's rule, we obtain

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Indeterminate Forms & L'Hôpital's Rule

- Indeterminate forms 0^0, 1^{∞}, $1^{-\infty}$ or ∞^0.
  1. Let $y = f(x)^{g(x)}$
  2. Take the natural logarithm $\ln y = \ln f(x)^{g(x)} = g(x)\ln f(x)$.
  3. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

Find the limit if it exists $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$.

Solution: The indeterminate form is 1^{∞}. To treat this form, let $y=(1+x)^{\frac{1}{x}}$. By taking the natural logarithm of both sides, we have

$$\ln y = \frac{1}{x}\ln(1+x) \Rightarrow \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\frac{\ln(1+x)}{x}$$

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Hence,

$$\lim_{x\to 0}\ln y = 1 \Rightarrow e^{\lim_{x\to 0}\ln y} = e^1 \qquad \text{(take the natural exponent of both sides)}$$

$$\Rightarrow \lim_{x\to 0} e^{(\ln y)} = e$$

$$\Rightarrow \lim_{x\to 0} y = e \Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$$

Improper Integrals

■ **Remember: In Chapter 2,**
For any function f bounded and defined on a closed bounded interval $[a, b]$, the definite integral of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_k f(\omega_k)\Delta x_k, (\| P \| \to 0)$$

if the limit exists. The numbers a and b are called the limits of the integration.

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The proper integral is the Riemann integral (the function f must be bounded and the interval must be closed and bounded). If one of these conditions is not satisfied, we define a new sense of the integral called **the improper integral**.

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From this, there are two cases of the improper integrals:

The first case: Infinite Intervals

For continuous function f, we study integrals of forms:

- $\int_a^{\infty} f(x)\ dx$,
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The second case: **Discontinuous Integrands** $\int_a^b f(x)\ dx$

Over the interval $[a, b]$:
- If f is continuous on $[a, b)$ and has an infinite discontinuity at b i.e., $\lim_{x \to b^-} f(x) = \pm\infty$.
- If f is continuous on $(a, b]$ and has an infinite discontinuity at a i.e., $\lim_{x \to a^+} f(x) = \pm\infty$,
- If f is continuous on $[a, b]$ except at $c \in (a, b)$ such that $\lim_{x \to c^{\pm}} f(x) = \pm\infty$.

Improper Integrals

The first case: Infinite Intervals

Definition

1. *Let f be a continuous function on $[a, \infty)$. The improper integral $\int_a^\infty f(x)\,dx$ is defined as follows:*

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx \quad \text{if the limit exists.}$$

2. *Let f be a continuous function on $(-\infty, b]$. The improper integral $\int_{-\infty}^b f(x)\,dx$ is defined as follows:*

$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx \quad \text{if the limit exists.}$$

The previous integrals are convergent (or to converge) if the limit exists as a finite number. However, if the limit does not exist or equals $\pm\infty$, the integral is called divergent (or to diverge).

3. *Let f be a continuous function on $\mathbb{R}$ and $a \in \mathbb{R}$. The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is defined as follows:*

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Improper Integrals

Notes:

1. If an improper integral is convergent, the value of the improper integral is the value of the limit.
2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

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Example

Determine whether the integral $\int_0^{\infty} \frac{1}{(x+2)^2}\, dx$ *converges or diverges.*

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Example

Determine whether the integral $\int_0^\infty \frac{1}{(x+2)^2}\,dx$ *converges or diverges.*

Solution:

$$\int_0^\infty \frac{1}{(x+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx$$

Improper Integrals

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Determine whether the integral $\int_0^\infty \frac{1}{(x+2)^2}\ dx$ *converges or diverges.*

Solution:

$$\int_0^\infty \frac{1}{(x+2)^2}\ dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x+2)^2}\ dx$$

The integral

$$\int_0^t \frac{1}{(x+2)^2}\ dx = \int_0^t (x+2)^{-2}\ dx = \Big[\frac{-1}{x+2}\Big]_0^t = -(\frac{1}{t+2} - \frac{1}{2}).$$

Improper Integrals

Notes:

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Example

Determine whether the integral $\int_0^\infty \frac{1}{(x+2)^2}\, dx$ *converges or diverges.*

Solution:

$$\int_0^\infty \frac{1}{(x+2)^2}\, dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x+2)^2}\, dx$$

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$$\int_0^t \frac{1}{(x+2)^2}\, dx = \int_0^t (x+2)^{-2}\, dx = \Big[\frac{-1}{x+2}\Big]_0^t = -\big(\frac{1}{t+2} - \frac{1}{2}\big).$$

Thus,

$$\lim_{t\to\infty} \int_0^t \frac{1}{(x+2)^2}\, dx = -\lim_{t\to\infty} \big(\frac{1}{t+2} - \frac{1}{2}\big) = -(0 - \frac{1}{2}) = \frac{1}{2}.$$

This implies that the integral converges and has the value $\frac{1}{2}$.

Example

Determine whether the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ *converges or diverges.*

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Solution:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

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We know that $\int \frac{1}{1+x^2}\, dx = \tan^{-1}\ x + c$, so

$$\begin{aligned}
&\lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\, dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} \\
&= \lim_{t\to-\infty} \left[0 - \tan^{-1}(t)\right] + \lim_{t\to\infty} \left[\tan^{-1}\ t - 0\right] \\
&= -\lim_{t\to-\infty} \tan^{-1}\ t + \lim_{t\to\infty} \tan^{-1}\ t \\
&= -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi.
\end{aligned}$$

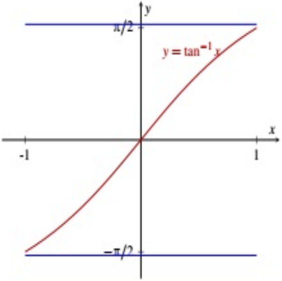

Figure 4.3

The integral is convergent and has the value π.

Improper Integrals

The second case: Discontinuous Integrands

Definition

1. *If f is continuous on $[a, b)$ and has an infinite discontinuity at b i.e.,* $\lim_{x \to b^-} f(x) = \pm\infty$, *then*

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx \quad \text{if the limit exists.}$$

2. *If f is continuous on $(a, b]$ and has an infinite discontinuity at a i.e.,* $\lim_{x \to a^+} f(x) = \pm\infty$, *then*

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^a f(x)\, dx \quad \text{if the limit exists.}$$

In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.

3. *If f is continuous on $[a, b]$ except at $c \in (a, b)$ such that* $\lim_{x \to c^\pm} f(x) = \pm\infty$, *the improper integral* $\int_a^b f(x)\, dx$ *is defined as follows:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Example

Determine whether the integral $\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx$ *converges or diverges.*

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Example

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Solution: Since $\lim_{x\to 4^-} \frac{1}{(4-x)^{\frac{3}{2}}} = \infty$ and the integrand is continuous on $[0,4)$, then from Definition .2,

$$\begin{aligned}\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx &= \lim_{t\to 4^-} \int_0^t (4-x)^{-\frac{3}{2}}\,dx \\ &= \lim_{t\to 4^-} \left[\frac{2}{\sqrt{4-x}}\right]_0^t \\ &= \lim_{t\to 4^-} \left(\frac{2}{\sqrt{4-t}} - 1\right) \\ &= \infty.\end{aligned}$$

$$\begin{aligned}\int (4-x)^{-3/2}\,dx &= -\int -(4-x)^{-3/2}\,dx \\ &= 2(4-x)^{-1/2} + c \\ &= \frac{2}{\sqrt{4-x}} + c\end{aligned}$$

Thus, the improper integral diverges.

Example

Determine whether the integral $\int_{-3}^{1} \frac{1}{x^2}\, dx$ *converges or diverges.*

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Solution:
Since $\lim_{x \to 0^-} \frac{1}{x^2} = \lim_{x \to 0^+} \frac{1}{x^2} = \infty$ and the integrand is continuous on $[-3, 0) \cup (0, 1]$, then

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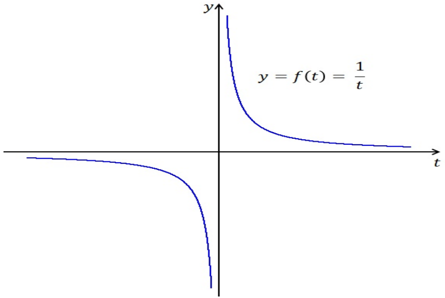

$$\begin{aligned}\int_{-3}^{1} \frac{1}{x^2}\, dx &= \int_{-3}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx \\ &= \lim_{t\to 0^-} \int_{-3}^{t} \frac{1}{x^2} + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2} \\ &= -\lim_{t\to 0^-} \left[\frac{1}{x}\right]_{-3}^{t} - \lim_{t\to 0^+} \left[\frac{1}{x}\right]_{t}^{1} \\ &= -\lim_{t\to 0^-} \left[\frac{1}{t} + \frac{1}{3}\right] - \lim_{t\to 0^+} \left[1 - \frac{1}{t}\right] \\ &= \infty.\end{aligned}$$

$$\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$