Integral Calculus

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Main Contents.

- Review
- Indeterminate Forms
- L'Hôpital's Rule
- Improper Integrals

Review

In the beginning of this section, we remind the reader with definition of limits and list some rules of the limits. Let f be a defined function on an open interval I and $c \in I$ where f may not be defined at c. Then,

$$\lim_{x \to c} f(x) = L, \ L \in \mathbb{R}$$

means for every $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Some Rules of the Limits: If $\lim_{x \longrightarrow c} f(x)$ and $\lim_{x \longrightarrow c} g(x)$ both exist, then

1. **Sum Rule:** $\lim_{x \longrightarrow c} \big(f(x) + g(x)\big) = \lim_{x \longrightarrow c} f(x) + \lim_{x \longrightarrow c} g(x)$.
2. **Difference Rule:** $\lim_{x \longrightarrow c} \big(f(x) - g(x)\big) = \lim_{x \longrightarrow c} f(x) - \lim_{x \longrightarrow c} g(x)$.
3. **Product Rule:** $\lim_{x \longrightarrow c} \big(f(x).g(x)\big) = \lim_{x \longrightarrow c} f(x) \times \lim_{x \longrightarrow c} g(x)$.
4. **Constant Multiple Rule:** $\lim_{x \longrightarrow c} \big(k\ f(x)\big) = k \lim_{x \longrightarrow c} f(x)$.
5. **Quotient Rule:** $\lim_{x \longrightarrow c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \longrightarrow c} f(x)}{\lim_{x \longrightarrow c} g(x)}$.
6. **Power Rule:** $\lim_{x \longrightarrow c} \big(f(x)\big)^{m/n} = \big(\lim_{x \longrightarrow c} f(x)\big)^{m/n}$.

Notes.

- $\frac{0}{a} = 0$ where $a \neq 0$
- $\frac{a}{\pm\infty} = 0$ where a is a number.
- $\frac{\pm\infty}{a} = \pm\infty$ where a is a positive number.

Review

Example

Find each limit if it exists.

1. $\lim_{x \to 1} x$
2. $\lim_{x \to 8} \sqrt{x}$
3. $\lim_{x \to 0} (x^2 - 2x + 1)$
4. $\lim_{x \to \pi} \sin\ x\ \cos\ x$
5. $\lim_{x \to 1} \frac{x}{(x^2 + 1)}$
6. $\lim_{x \to 3^+} \frac{1}{(x - 3)}$

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1. $\lim_{x \to 1} x = 1$
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4. $\lim_{x \to \pi} \sin\ x\ \cos\ x = \lim_{x \to \pi} \sin\ x \lim_{x \to \pi} \cos\ x = 0$
5. $\lim_{x \to 1} \dfrac{x}{(x^2 + 1)} = \dfrac{\lim_{x \to 1} x}{\lim_{x \to 1}(x^2 + 1)} = \dfrac{1}{2}$

Example

Find each limit if it exists.

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2. $\lim\limits_{x \to 8} \sqrt{x}$
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Solution:

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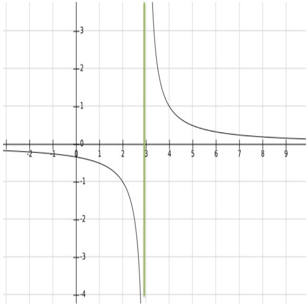
2. $\lim\limits_{x \to 8} \sqrt{x} = \sqrt{8} = 2\sqrt{2}$
3. $\lim\limits_{x \to 0} (x^2 - 2x + 1) = \lim\limits_{x \to 0} x^2 - 2 \lim\limits_{x \to 0} x + \lim\limits_{x \to 0} 1 = 1.$
4. $\lim\limits_{x \to \pi} \sin x \cos x = \lim\limits_{x \to \pi} \sin x \lim\limits_{x \to \pi} \cos x = 0$
5. $\lim\limits_{x \to 1} \dfrac{x}{(x^2 + 1)} = \dfrac{\lim_{x \to 1} x}{\lim_{x \to 1} (x^2 + 1)} = \dfrac{1}{2}$
6. $\lim\limits_{x \to 3^+} \dfrac{1}{(x - 3)} = \infty$



Indeterminate Forms.

Example

1. $\lim_{x \to 0} \frac{\sin\ x}{x} = \frac{0}{0}$
2. $\lim_{x \to \infty} \frac{e^x}{x} = \frac{\infty}{\infty}$
3. $\lim_{x \to 0^+} x^2 \ln x = 0.\infty$
4. $\lim_{x \to 1^+} (\frac{1}{x-1} - \frac{1}{\ln x}) = \infty - \infty$

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In the following table, we categorize the indeterminate forms:

List of the indeterminate forms.

Case	Indeterminate Form
Quotient	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
Product	$0.\infty$ or $0.(-\infty)$
Sum & Difference	$(-\infty) + \infty$ or $\infty - \infty$
Exponent	0^0, 1^∞, $1^{-\infty}$ or ∞^0

Indeterminate Forms & L'Hôpital's Rule

L'Hôpital's Rule

The following theorem examines the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Theorem

Suppose f and g are differentiable on an interval I and $c \in I$ where f and g may not be differentiable at c. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$ and $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or equals to ∞.

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if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or equals to ∞.

Notes.

1. We can apply L'Hôpital's rule for $c = \pm\infty$ and when $x \to c^+$ or $x \to c^-$.
2. When applying L'Hôpital's rule, we should calculate the derivatives of $f(x)$ and $g(x)$ separately.
3. Sometimes, we need to apply L'Hôpital's rule twice.

Example

Use L'Hôpital's rule to find each limit if it exists.

1. $\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x^2-25}$
2. $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$
3. $\lim_{x \to \infty} \frac{e^x}{x}$

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Solution:

1. Since $\lim\limits_{x \to 5} \sqrt{x-1} - 2 = 0$ and $\lim\limits_{x \to 5} x^2 - 2 = 0$, we have the indeterminate form $\dfrac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x \to 5} \frac{\frac{1}{2\sqrt{x-1}}}{2x} = \lim_{x \to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$

Example

Use L'Hôpital's rule to find each limit if it exists.

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Solution:

1. Since $\lim_{x\to 5} \sqrt{x-1}-2=0$ and $\lim_{x\to 5} x^2-2=0$, we have the indeterminate form $\frac{0}{0}$. By applying L'Hôpital's rule, we have

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$$\frac{\frac{1}{2\sqrt{x-1}}}{2x} = \frac{1}{2\sqrt{x-1}} \div \frac{2x}{1} = \frac{1}{2\sqrt{x-1}} \times \frac{1}{2x} = \frac{1}{4x\sqrt{x-1}}$$

Indeterminate Forms & L'Hôpital's Rule

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1. Since $\lim_{x \to 5} \sqrt{x-1} - 2 = 0$ and $\lim_{x \to 5} x^2 - 2 = 0$, we have the indeterminate form $\frac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x \to 5} \frac{\frac{1}{2\sqrt{x-1}}}{2x} = \lim_{x \to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$

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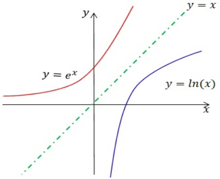
2. The indeterminate form is $\frac{\infty}{\infty}$. Apply L'Hôpital's rule to obtain

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

$$\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{1}{x} \div \frac{1}{2\sqrt{x}} = \frac{1}{x} \times \frac{2\sqrt{x}}{1} = \frac{2\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{2}{\sqrt{x}}$$



Example

Use L'Hôpital's rule to find each limit if it exists.

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2. $\lim\limits_{x \to \infty} \dfrac{\ln x}{\sqrt{x}}$
3. $\lim\limits_{x \to \infty} \dfrac{e^x}{x}$

Solution:

1. Since $\lim\limits_{x \to 5} \sqrt{x-1}-2 = 0$ and $\lim\limits_{x \to 5} x^2-2 = 0$, we have the indeterminate form $\frac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x \to 5} \frac{\frac{1}{2\sqrt{x-1}}}{2x} = \lim_{x \to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$

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2. The indeterminate form is $\frac{\infty}{\infty}$. Apply L'Hôpital's rule to obtain

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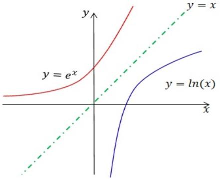
$$\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{1}{x} \div \frac{1}{2\sqrt{x}} = \frac{1}{x} \times \frac{2\sqrt{x}}{1} = \frac{2\sqrt{x}}{\sqrt{x}\,\sqrt{x}} = \frac{2}{\sqrt{x}}$$



3. The indeterminate form is $\frac{\infty}{\infty}$. By applying L'Hôpital's rule, we have $\lim\limits_{x \to \infty} \frac{e^x}{x} = \lim\limits_{x \to \infty} \frac{e^x}{1} = \infty$.

Techniques for other indeterminate forms.

Indeterminate form $0.\infty$.

1. Write $f(x)\,g(x)$ as $\dfrac{f(x)}{1/g(x)}$ or $\dfrac{g(x)}{1/f(x)}$.
2. Apply L'Hôpital's rule to the resulting indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Indeterminate Forms & L'Hôpital's Rule

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Find the limit if it exists $\lim_{x \to 0^+} x^2 \ln x$

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Example

Find the limit if it exists $\lim_{x\to 0^+} x^2 \ln x$

Solution: The indeterminate form is $0.(-\infty)$, so we cannot apply L'Hôpital's rule. We need to rearrange the expression in a way that enables us to apply L'Hôpital's rule. By using the previous techniques, we have

$$x^2 \ln x = \frac{\ln x}{\frac{1}{x^2}}.$$

The indeterminate form of the new expression is $\dfrac{\infty}{\infty}$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Note: $y = \frac{1}{x^2} = x^{-2} \Rightarrow y' = -2\,x^{-3} = \frac{-2}{x^3}$

Hence, $\lim_{x\to 0^+} x^2 \ln x = 0.$

Indeterminate Forms & L'Hôpital's Rule

- Indeterminate form $(-\infty) + \infty$ or $\infty - \infty$.
  1. Write the form as a quotient or product.
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Example

Find the limit if it exists $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

Solution: The indeterminate form is $\infty - \infty$.

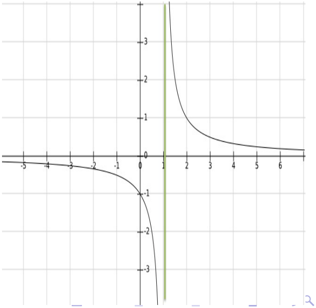
$$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x}.$$

We have the indeterminate form $\frac{0}{0}$. From L'Hôpital's rule,

$$\lim_{x \to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x \to 1^+} \frac{1-x}{x \ \ln x + x - 1}.$$

We have the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule again to have

$$\lim_{x \to 1^+} \frac{1-x}{x \ \ln x + x - 1} = \lim_{x \to 1^+} \frac{-1}{\ln x + 2} = \frac{-1}{2}.$$



Indeterminate Forms & L'Hôpital's Rule

- Indeterminate forms 0^0, 1^{∞}, $1^{-\infty}$ or ∞^0.
  1. Let $y = f(x)^{g(x)}$
  2. Take the natural logarithm $\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$.
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Find the limit if it exists $\lim\limits_{x \to 0^+} (1 + x)^{\frac{1}{x}}$.

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Solution: The indeterminate form is 1^{∞}. To treat this form, let $y = (1+x)^{\frac{1}{x}}$. By taking the natural logarithm of both sides, we have

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$$\ln y = \frac{1}{x} \ln(1+x) \Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

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$$\ln y = \frac{1}{x} \ln(1+x) \Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

The indeterminate form is $\frac{0}{0}$.

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$$\ln y = \frac{1}{x} \ln(1+x) \Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$

The indeterminate form is $\frac{0}{0}$. By applying L'Hôpital's rule, we obtain

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1 \ .$$

- Indeterminate forms 0^0, 1^∞, $1^{-\infty}$ or ∞^0.
  1. Let $y = f(x)^{g(x)}$
  2. Take the natural logarithm $\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$.
  3. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

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Hence,

$$\lim_{x \to 0} \ln y = 1 \Rightarrow e^{\lim\limits_{x \to 0} \ln y} = e^1 \qquad \text{(take the natural exponent of both sides)}$$

$$\Rightarrow \lim_{x \to 0} e^{(\ln y)} = e$$

$$\Rightarrow \lim_{x \to 0} y = e \Rightarrow \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e.$$

Improper Integrals

Remember in Chapter 2.

For any function f bounded and defined on a closed bounded interval $[a, b]$, the definite integral of f from a to b is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_k f(\omega_k)\Delta x_k, (\parallel P \parallel \to 0)$$

if the limit exists. The numbers a and b are called the limits of the integration.

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The proper integral is the Riemann integral (the function f must be bounded and the interval must be closed and bounded). If one of these conditions is not satisfied, we define a new sense of the integral called **the improper integral**.

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From this, there are two cases of the improper integrals:

- **The first case: Infinite Intervals**

For continuous function f, we study integrals of forms:

- $\displaystyle\int_a^{\infty} f(x)\,dx$,
- $\displaystyle\int_{-\infty}^{b} f(x)\,dx$
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- **The second case: Discontinuous Integrands** $\int_a^b f(x)\, dx$

Over the interval $[a, b]$:

- If f is continuous on $[a, b)$ and has an infinite discontinuity at b i.e., $\lim_{x\to b^-} f(x) = \pm\infty$.
- If f is continuous on $(a, b]$ and has an infinite discontinuity at a i.e., $\lim_{x\to a^+} f(x) = \pm\infty$,
- If f is continuous on $[a, b]$ except at $c \in (a, b)$ such that $\lim_{x\to c^\pm} f(x) = \pm\infty$.

Improper Integrals

The first case: Infinite Intervals

Definition

1. *Let f be a continuous function on $[a, \infty)$. The improper integral $\int_a^\infty f(x)\,dx$ is defined as follows:*

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \text{if the limit exists.}$$

2. *Let f be a continuous function on $(-\infty, b]$. The improper integral $\int_{-\infty}^b f(x)\,dx$ is defined as follows:*

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx \quad \text{if the limit exists.}$$

The previous integrals are convergent (or to converge) if the limit exists as a finite number. However, if the limit does not exist or equals $\pm\infty$, the integral is called divergent (or to diverge).

3. *Let f be a continuous function on $\mathbb{R}$ and $a \in \mathbb{R}$. The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is defined as follows:*

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Improper Integrals

Notes.

1. If an improper integral is convergent, the value of the improper integral is the value of the limit.
2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

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Example

Determine whether the integral $\int_0^\infty \frac{1}{(x+2)^2}\,dx$ *converges or diverges.*

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Example

Determine whether the integral $\int_0^{\infty} \frac{1}{(x+2)^2}\, dx$ *converges or diverges.*

Solution:

$$\int_0^{\infty} \frac{1}{(x+2)^2}\, dx = \lim_{t\to\infty} \int_0^{t} \frac{1}{(x+2)^2}\, dx$$

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1. If an improper integral is convergent, the value of the improper integral is the value of the limit.
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Example

Determine whether the integral $\int_0^\infty \frac{1}{(x+2)^2}\, dx$ *converges or diverges.*

Solution:

$$\int_0^\infty \frac{1}{(x+2)^2}\, dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x+2)^2}\, dx$$

The integral

$$\int_0^t \frac{1}{(x+2)^2}\, dx = \int_0^t (x+2)^{-2}\, dx = \Big[\frac{-1}{x+2}\Big]_0^t = -\Big(\frac{1}{t+2} - \frac{1}{2}\Big).$$

Notes.

1. If an improper integral is convergent, the value of the improper integral is the value of the limit.
2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

Example

Determine whether the integral $\int_0^\infty \frac{1}{(x+2)^2}\,dx$ *converges or diverges.*

Solution:

$$\int_0^\infty \frac{1}{(x+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx$$

The integral

$$\int_0^t \frac{1}{(x+2)^2}\,dx = \int_0^t (x+2)^{-2}\,dx = \Big[\frac{-1}{x+2}\Big]_0^t = -\Big(\frac{1}{t+2} - \frac{1}{2}\Big).$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx = -\lim_{t\to\infty}\Big(\frac{1}{t+2} - \frac{1}{2}\Big) = -(0 - \frac{1}{2}) = \frac{1}{2}.$$

This implies that the integral converges and has the value $\frac{1}{2}$.

Improper Integrals

Example

Determine whether the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ *converges or diverges.*

Improper Integrals

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Determine whether the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ *converges or diverges.*

Solution:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2}\, dx + \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2}\, dx$$

Improper Integrals

Example

Determine whether the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ *converges or diverges.*

Solution:

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We know that $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$, so

$$\lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}\, dx + \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}$$
$$= \lim_{t\to-\infty}\left[0-\tan^{-1}(t)\right] + \lim_{t\to\infty}\left[\tan^{-1} t - 0\right]$$
$$= -\lim_{t\to-\infty}\tan^{-1} t + \lim_{t\to\infty}\tan^{-1} t$$
$$= -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi.$$

The integral is convergent and has the value π.

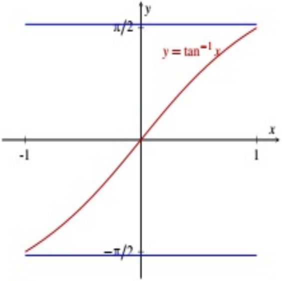

Figure 4.3

Improper Integrals

The second case: Discontinuous Integrands

Definition

1. *If f is continuous on $[a, b)$ and has an infinite discontinuity at b i.e., $\lim_{x \to b^-} f(x) = \pm\infty$, then*

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx \text{ if the limit exists.}$$

2. *If f is continuous on $(a, b]$ and has an infinite discontinuity at a i.e., $\lim_{x \to a^+} f(x) = \pm\infty$, then*

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^a f(x)\, dx \text{ if the limit exists.}$$

In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.

3. *If f is continuous on $[a, b]$ except at $c \in (a, b)$ such that $\lim_{x \to c^\pm} f(x) = \pm\infty$, the improper integral $\int_a^b f(x)\, dx$ is defined as follows:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Example

Determine whether the integral $\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\, dx$ *converges or diverges.*

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Solution: Since $\lim_{x\to 4^-} \frac{1}{(4-x)^{\frac{3}{2}}} = \infty$ and the integrand is continuous on $[0, 4)$, then from Definition .2,

Example

Determine whether the integral $\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx$ *converges or diverges.*

Solution: Since $\lim_{x\to 4^-} \frac{1}{(4-x)^{\frac{3}{2}}} = \infty$ and the integrand is continuous on $[0,4)$, then from Definition .2,

$$\begin{aligned}\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx &= \lim_{t\to 4^-} \int_0^t (4-x)^{-\frac{3}{2}}\,dx \\ &= \lim_{t\to 4^-} \left[\frac{2}{\sqrt{4-x}}\right]_0^t \\ &= \lim_{t\to 4^-} \left(\frac{2}{\sqrt{4-t}} - 1\right) \\ &= \infty.\end{aligned}$$

$$\begin{aligned}\int (4-x)^{-3/2}\,dx &= -\int -(4-x)^{-3/2}\,dx \\ &= 2(4-x)^{-1/2} + c \\ &= \frac{2}{\sqrt{4-x}} + c\end{aligned}$$

Thus, the improper integral diverges.

Improper Integrals

Example

Determine whether the integral $\int_{-3}^{1} \frac{1}{x^2}\, dx$ *converges or diverges.*

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Solution:

Since $\lim_{x \to 0^-} \frac{1}{x^2} = \lim_{x \to 0^+} \frac{1}{x^2} = \infty$ and the integrand is continuous on $[-3, 0) \cup (0, 1]$, then

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Determine whether the integral $\int_{-3}^{1} \frac{1}{x^2}\, dx$ *converges or diverges.*

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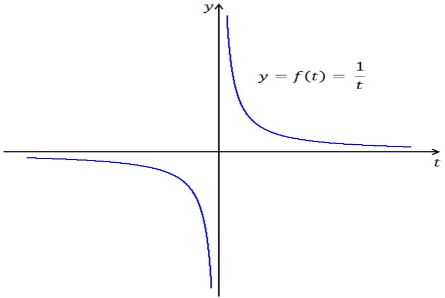

$$\begin{aligned}\int_{-3}^{1} \frac{1}{x^2}\, dx &= \int_{-3}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx \\ &= \lim_{t\to 0^-} \int_{-3}^{t} \frac{1}{x^2} + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2} \\ &= -\lim_{t\to 0^-} \left[\frac{1}{x}\right]_{-3}^{t} - \lim_{t\to 0^+} \left[\frac{1}{x}\right]_{t}^{1} \\ &= -\lim_{t\to 0^-} \left[\frac{1}{t} + \frac{1}{3}\right] - \lim_{t\to 0^+} \left[1 - \frac{1}{t}\right] \\ &= \infty.\end{aligned}$$

$$\int \frac{1}{x^2}\, dx = \int x^{-2}\, dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$