Integral Calculus

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Exercise. Evaluate the integral.

(1) $\int xe^{x^2}\ dx$

Section 1: Integration by Parts

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Solution: $\int x^2 \cos\ x^3\ dx = \frac{1}{3}\int 3\ x^2 \cos(x^3)\ dx = \frac{1}{3}\ \sin\ (x^3) + c$

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(4) $\int x \cos(x)\ dx$

Section 1: Integration by Parts

Let $u = f(x)$ and $v = g(x)$, we know that

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x)\ .$$

Thus,

$$f(x)g'(x) = \frac{d}{dx}\left(f(x)g(x)\right) - f'(x)g(x)\ .$$

By integrating both sides, we have

$$\begin{aligned}\int f(x)g'(x)\ dx &= \int \frac{d}{dx}\left(f(x)g(x)\right)\ dx - \int f'(x)g(x)\ dx \\ &= f(x)g(x) - \int f'(x)g(x)\ dx.\end{aligned}$$

Since $u = f(x) \Rightarrow du = f'(x)\ dx$ and $v = g(x) \Rightarrow dv = g'(x)\ dx$. Therefore,

$$\int u\ dv = uv - \int v\ du\ .$$

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Since $u = f(x) \Rightarrow du = f'(x)\ dx$ and $v = g(x) \Rightarrow dv = g'(x)\ dx$. Therefore,

$$\int u\ dv = uv - \int v\ du \ .$$

Theorem

If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u\ dv = uv - \int v\ du.$$

$$u \xrightarrow{\text{Derivation}} du$$

$$dv \xrightarrow{\text{Integration}} v = \int dv$$

Example

Evaluate the integral $\int x\, e^{x}\, dx$.

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Solution: The integrand $x\, e^x$ is a product of two functions x and e^x.

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Choose $u = x$, and $dv = e^x\, dx$. Then,

$$u = x \Rightarrow du = dx\ ,$$
$$dv = e^x\, dx \Rightarrow v = \int e^x\, dx = e^x .$$

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Choose $u = x$, and $dv = e^x\, dx$. Then,

$$u = x \Rightarrow du = dx \ ,$$
$$dv = e^x\, dx \Rightarrow v = \int e^x\, dx = e^x.$$

From the theorem

$$\int u\, dv = uv - \int v\, du$$
$$\int x\, e^x\, dx = x\, e^x - \int e^x\, dx$$
$$= x\, e^x - e^x + c \ .$$

Example

Evaluate the integral $\int x\ e^x\ dx$.

Solution: The integrand $x\ e^x$ is a product of two functions x and e^x.

Choose $u = x$, and $dv = e^x\ dx$. Then,

$$u = x \Rightarrow du = dx\ ,$$
$$dv = e^x\ dx \Rightarrow v = \int e^x\ dx = e^x.$$

From the theorem

$$\int u\ dv = uv - \int v\ du$$
$$\int x\ e^x\ dx = x\ e^x - \int e^x\ dx$$
$$= x\ e^x - e^x + c\ .$$

Notes.

- We choose $u = x$ because it can be differentiated to a constant. Thus the new product integral will not involve a product anymore.
- Try to choose

$$u = e^x \text{ and } dv = x\ dx$$

You will obtain

$$I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x\ dx.$$

However, the integral $\int \frac{x^2}{2} e^x\ dx$ is more difficult than the original one $\int xe^x\ dx$.

Example

Evaluate the integral $\int x \ \cos \ x \, dx$.

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution: In the same manner as in the preceding example, set $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx \,,$$
$$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.$$

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution: In the same manner as in the preceding example, set $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx \,,$$
$$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.$$

From the theorem,

$$\int u \, dv = uv - \int v \, du$$
$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + c \,.$$

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution: In the same manner as in the preceding example, set $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx \,,$$
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From the theorem,

$$\int u \, dv = uv - \int v \, du$$
$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + c \,.$$

Notes.

- We choose $u = x$ because it can be differentiated to a constant. Thus the new product integral will not involve a product anymore.
- Try to choose

  $u = \cos x$ and $dv = x \, dx$

  Do you have the same result?

Example

Evaluate the integral $\int \ln x \; dx$.

Example

Evaluate the integral $\int \ln x \, dx$.

Solution: Choose $u = \ln x$, and $dv = dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} \, dx \ ,$$

$$dv = dx \Rightarrow v = \int 1 \, dx = x.$$

Remember.
If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}$$

Example

Evaluate the integral $\int \ln x \; dx$.

Solution: Choose $u = \ln x$, and $dv = dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} \; dx \; ,$$
$$dv = dx \Rightarrow v = \int 1 \; dx = x.$$

Remember.
If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}$$

Apply the theorem

$$\int u \; dv = uv - \int v \; du$$
$$\int \ln x \; dx = x \ln x - \int x \frac{1}{x} \; dx$$
$$= x \ln x - \int 1 \; dx$$
$$= x \ln x - x + c$$

Example

Evaluate the integral $\int x^3 \ln x \, dx$.

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Solution: Choose $u = \ln x$, and $dv = x^3 \, dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} dx \ ,$$
$$dv = x^3 \, dx \Rightarrow v = \int x^3 \, dx = \frac{x^4}{4}.$$

Example

Evaluate the integral $\int x^3 \ln x \, dx$.

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$$dv = x^3 \, dx \Rightarrow v = \int x^3 \, dx = \frac{x^4}{4}.$$

From the theorem,

$$\int u \, dv = uv - \int v \, du$$
$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} \, dx$$
$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$
$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + c \ .$$

Example

Evaluate the integral $\int x^3 \ \ln x \ dx$.

Solution: Choose $u = \ln x$, and $dv = x^3 \ dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} dx \ ,$$

$$dv = x^3 \ dx \Rightarrow v = \int x^3 \ dx = \frac{x^4}{4}.$$

From the theorem,

$$\int u \ dv = uv - \int v \ du$$

$$\int x^3 \ \ln x \ dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \ \frac{1}{x} \ dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \ dx$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + c \ .$$

Rule.

To evaluate $\int x^n \ \ln x \ dx$, let

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$dv = x^n \ dx \Rightarrow v = \int x^n \ dx = \frac{x^{n+1}}{n+1}$$

Hence,

$$\int x^n \ \ln x \ dx = \frac{x^{n+1}}{n+1} \ \ln x + \frac{x^{n+1}}{(n+1)^2} + c$$

Example

Evaluate the integral $\int \sin\ x\ \ln(\cos\ x)\ dx$.

Example

Evaluate the integral $\int \sin x \ln(\cos x)\, dx$.

Solution: Let $u = \ln(\cos x)$ for $\cos x > 0$, and $dv = \sin x\, dx$. Then,

$$u = \ln(\cos x) \Rightarrow du = \frac{-\sin x}{\cos x}\, dx,$$

$$dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x.$$

Example

Evaluate the integral $\int \sin\ x\ \ln(\cos\ x)\ dx$.

Solution: Let $u = \ln(\cos\ x)$ for $\cos\ x > 0$, and $dv = \sin\ x\ dx$. Then,

$$u = \ln(\cos\ x) \Rightarrow du = \frac{-\sin\ x}{\cos\ x}\ dx,$$
$$dv = \sin\ x\ dx \Rightarrow v = \int \sin\ x\ dx = -\cos\ x.$$

Hence,

$$\int u\ dv = uv - \int v\ du$$
$$\int \sin\ x\ \ln(\cos\ x)\ dx = -\cos\ x\ \ln(\cos\ x) - \int \cos\ x\ \frac{\sin\ x}{\cos\ x}\ dx\ dx$$
$$= -\cos\ x\ \ln(\cos\ x) - \int \sin\ x\ dx$$
$$= -\cos\ x\ \ln(\cos\ x) + \cos\ x + c\ .$$

Note. Sometimes we need to use the integration by parts twice.

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Example

Evaluate the integral $\int x^2 e^x \, dx$.

Section 1: Integration by Parts

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Example

Evaluate the integral $\int x^2 e^x \, dx$.

Solution: Let $I = \int x^2 e^x \, dx$ and choose $u = x^2$, and $dv = e^x \, dx$. Then,

$$u = x^2 \Rightarrow du = 2x \, dx \,,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

This implies $I = x^2 e^x - 2\int x e^x \, dx$.

Note

In successive application of the integration by parts, do not switch choices for u and dv.

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$$u = x^2 \Rightarrow du = 2x \, dx \,,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

This implies $I = x^2 e^x - 2 \int x e^x \, dx$.

We use the integration by parts again for the integral $\int x \, e^x \, dx$.

Let $J = \int x \, e^x \, dx$.
Choose $u = x$ and $dv = e^x \, dx$, then

$$u = x \Rightarrow du = \, dx \,,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

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In successive application of the integration by parts, do not switch choices for u and dv.

Section 1: Integration by Parts

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Evaluate the integral $\int x^2 e^x \, dx$.

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Let $J = \int x \, e^x \, dx$.

Choose $u = x$ and $dv = e^x \, dx$, then

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Note

In successive application of the integration by parts, do not switch choices for u and dv.

Therefore, $J = xe^x - \int e^x \, dx = xe^x - e^x + c$.

By substituting the result of J into I, we have

Note. Sometimes we need to use the integration by parts twice.

Example

Evaluate the integral $\int x^2 e^x \, dx$.

Solution: Let $I = \int x^2 e^x \, dx$ and choose $u = x^2$, and $dv = e^x \, dx$. Then,

$$u = x^2 \Rightarrow du = 2x \, dx \, ,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x .$$

Note

In successive application of the integration by parts, do not switch choices for u and dv.

This implies $I = x^2 e^x - 2\int xe^x \, dx$.

We use the integration by parts again for the integral $\int x \, e^x \, dx$.

Let $J = \int x \, e^x \, dx$.
Choose $u = x$ and $dv = e^x \, dx$, then

$$u = x \Rightarrow du = dx \, ,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x .$$

Therefore, $J = xe^x - \int e^x \, dx = xe^x - e^x + c$.
By substituting the result of J into I, we have

$$I = x^2 e^x - 2(xe^x - e^x) + c = e^x(x^2 - 2x + 2) + c.$$

Section 2.1: Integration of Powers of Trigonometric Functions

In this section, we evaluate integrals of forms

- $\int \sin^n x \ \cos^m x \, dx$,
- $\int \tan^n x \ \sec^m x \, dx$ and
- $\int \cot^n x \ \csc^m x \, dx$.

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- $\int \sin^n x \cos^m x \, dx$,
- $\int \tan^n x \sec^m x \, dx$ and
- $\int \cot^n x \csc^m x \, dx$.

Form 1. $\int \sin^n x \cos^m x \, dx$.

This form is treated as follows:

1. If n is an odd integer, write
$$\sin^n x \cos^m x = \sin^{n-1} x \cos^m x \sin x$$
Then, use the identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.
2. If m is an odd integer, write
$$\sin^n x \cos^m x = \sin^n x \cos^{m-1} x \cos x$$
Then, use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$.
3. If m and n are even, use the identities $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

Example

Evaluate the integral $\int \sin^5 x \cos^4 x \, dx$.

Example

Evaluate the integral $\int \sin^5 x \cos^4 x \, dx$.

Solution:

$$\begin{aligned} \sin^5 x \cos^4 x &= \sin^4 x \cos^4 x \sin x \\ &= (\sin^2 x)^2 \cos^4 x \sin x \\ &= (1 - \cos^2 x)^2 \cos^4 x \sin x \qquad : \cos^2 x + \sin^2 x = 1 \end{aligned}$$

The integral becomes

$$\int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx$$

Example

Evaluate the integral $\int \sin^5 x \cos^4 x \, dx$.

Solution:

$$\begin{aligned}\sin^5 x \cos^4 x &= \sin^4 x \cos^4 x \sin x \\ &= (\sin^2 x)^2 \cos^4 x \sin x \\ &= (1-\cos^2 x)^2 \cos^4 x \sin x \qquad : \cos^2 x + \sin^2 x = 1\end{aligned}$$

The integral becomes

$$\int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$.

Example

Evaluate the integral $\int \sin^5 x \cos^4 x \, dx$.

Solution:

$$\begin{aligned} \sin^5 x \cos^4 x &= \sin^4 x \cos^4 x \sin x \\ &= (\sin^2 x)^2 \cos^4 x \sin x \\ &= (1 - \cos^2 x)^2 \cos^4 x \sin x \qquad : \cos^2 x + \sin^2 x = 1 \end{aligned}$$

The integral becomes

$$\int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$. Thus, $-\int (1 - \cos^2 x)^2 \cos^4 x \, (-\sin x) \, dx$.

Example

Evaluate the integral $\int \sin^5 x \cos^4 x \, dx$.

Solution:

$$\begin{aligned}\sin^5 x \cos^4 x &= \sin^4 x \cos^4 x \sin x \\ &= (\sin^2 x)^2 \cos^4 x \sin x \\ &= (1-\cos^2 x)^2 \cos^4 x \sin x \qquad : \cos^2 x + \sin^2 x = 1\end{aligned}$$

The integral becomes

$$\int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$. Thus, $-\int (1-\cos^2 x)^2 \cos^4 x (-\sin x) \, dx$. By substituting, we have

$$\begin{aligned}-\int (1-u^2)^2 u^4 \, du = \int (1-2u^2+u^4)u^4 \, du &= -\int (u^4 - 2u^6 + u^8) \, du \\ &= -\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right) + c \\ &= -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c. \qquad \text{Return to the original variable } x\end{aligned}$$

Example

Evaluate the integral $\int \cos^3 x \, dx$.

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$$\begin{aligned} \cos^3 x &= \cos^2 x \cos x \\ &= (1 - \sin^2 x) \cos x \qquad : \cos^2 x + \sin^2 x = 1 \end{aligned}$$

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Evaluate the integral $\int \cos^3 x \, dx$.

Solution:

$$\begin{aligned} \cos^3 x &= \cos^2 x \cos x \\ &= (1 - \sin^2 x) \cos x \qquad : \cos^2 x + \sin^2 x = 1 \end{aligned}$$

Thus, $\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx$.

Example

Evaluate the integral $\int \cos^3 x \, dx$.

Solution:

$$\begin{aligned} \cos^3 x &= \cos^2 x \cos x \\ &= (1 - \sin^2 x) \cos x \qquad : \cos^2 x + \sin^2 x = 1 \end{aligned}$$

Thus, $\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx$.
Let $u = \sin x \Rightarrow du = \cos x \, dx$.

Example

Evaluate the integral $\int \cos^3 x \, dx$.

Solution:

$$\begin{aligned} \cos^3 x &= \cos^2 x \cos x \\ &= (1 - \sin^2 x) \cos x \qquad : \cos^2 x + \sin^2 x = 1 \end{aligned}$$

Thus, $\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx$.
Let $u = \sin x \Rightarrow du = \cos x \, dx$. By substitution, we have

$$\begin{aligned} \int (1 - u^2) \, du &= u - \frac{u^3}{3} + c \\ &= \sin x - \frac{1}{3} \sin^3 x + c \; . \qquad \text{Return to the original variable } x \end{aligned}$$

Form 2. $\int \tan^n x \sec^m x \, dx$. This form is treated as follows:

1. If $n = 0$, write
$$\sec^m x = \sec^{m-2} x \sec^2 x$$
   - If $m > 1$ is odd, use the integration by parts.
   - If m is even, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$.
2. If $m = 0$ and n is odd or even, write
$$\tan^n x = \tan^{n-2} x \tan^2 x$$
Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \tan x$.
3. If n is even and m is odd, use the identity $\tan^2 x = \sec^2 x - 1$ to reduce the power m and then use the integration by parts.
4. If $m \geq 2$ is even, write
$$\tan^n x \sec^m x = \tan^n x \sec^{m-2} x \sec^2 x$$
Then, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$.
5. If n is odd and $m \geq 1$, write
$$\tan^n x \sec^m x = \tan^{n-1} x \sec^{m-1} x \tan x \sec x$$
Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$.

Example

Evaluate the integral $\int \tan^5 x \sec^4 x \, dx$.

Example

Evaluate the integral $\int \tan^5 x \sec^4 x \, dx$.

Solution: Express the integrand $\tan^5 x \sec^4 x$ as follows

$$\begin{aligned} \tan^5 x \sec^4 x &= \tan^5 x \sec^2 x \sec^2 x \\ &= \tan^5 x (\tan^2 x + 1) \sec^2 x \end{aligned}$$

Example

Evaluate the integral $\int \tan^5 x \sec^4 x \, dx$.

Solution: Express the integrand $\tan^5 x \sec^4 x$ as follows

$$\begin{aligned} \tan^5 x \sec^4 x &= \tan^5 x \sec^2 x \sec^2 x \\ &= \tan^5 x (\tan^2 x + 1) \sec^2 x \end{aligned}$$

This implies

$$\int \tan^5 x \sec^4 x \, dx = \int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx = \int (\tan^7 x + \tan^5 x) \sec^2 x \, dx$$

Example

Evaluate the integral $\int \tan^5 x \sec^4 x \, dx$.

Solution: Express the integrand $\tan^5 x \sec^4 x$ as follows

$$\begin{aligned} \tan^5 x \sec^4 x &= \tan^5 x \sec^2 x \sec^2 x \\ &= \tan^5 x (\tan^2 x + 1) \sec^2 x \end{aligned}$$

This implies

$$\int \tan^5 x \sec^4 x \, dx = \int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx = \int (\tan^7 x + \tan^5 x) \sec^2 x \, dx$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$ and by substituting, we have

$$\begin{aligned} \int (u^7 + u^5) \, du &= \frac{u^8}{8} + \frac{u^6}{6} + c \\ &= \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6} + c . \end{aligned}$$

Return to the original variable x

Example

Evaluate the integral $\int \sec^3 x \, dx$.

Example

Evaluate the integral $\int \sec^3 x \, dx$.

Solution: Write $\sec^3 x = \sec x \sec^2 x$ and let $I = \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$. We use the integration by parts to evaluate the integral as follows:

Example

Evaluate the integral $\int \sec^3 x \, dx$.

Solution: Write $\sec^3 x = \sec x \sec^2 x$ and let $I = \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$. We use the integration by parts to evaluate the integral as follows:

$u = \sec x \Rightarrow du = \sec x \tan x \, dx$,

$dv = \sec^2 x \, dx \Rightarrow v = \int \sec^2 x \, dx = \tan x$.

Example

Evaluate the integral $\int \sec^3 x \, dx$.

Solution: Write $\sec^3 x = \sec x \sec^2 x$ and let $I = \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$. We use the integration by parts to evaluate the integral as follows:

$$u = \sec x \Rightarrow du = \sec x \tan x \, dx ,$$

$$dv = \sec^2 x \, dx \Rightarrow v = \int \sec^2 x \, dx = \tan x.$$

Hence,

$$I = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$I = \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$$

$$I = \sec x \tan x - I + \ln | \sec x + \tan x |$$

$$2I = \sec x \tan x + \ln | \sec x + \tan x |$$

$$I = \frac{1}{2}(\sec x \tan x + \ln | \sec x + \tan x |) + c.$$

- $\tan^2 x = \sec^2 x - 1$
- $-\int (\sec^3 x - \sec x) \, dx = -\int \sec^3 x \, dx + \int \sec x \, dx$
- $\int \sec x \, dx = \ln | \sec x + \tan x | + c$

Remember. Example 3.4 No. 7 in Chapter 3

Form 3. $\int \cot^n x \csc^m x \, dx$.

The treatment of this form is similar to the integral $\int \tan^n x \sec^m x \, dx$, except we use the identity

$$\cot^2 x + 1 = \csc^2 x.$$

Example

Evaluate the integral $\int \cot^5 x \csc^4 x \, dx$.

Form 3. $\int \cot^n x \csc^m x \, dx$.

The treatment of this form is similar to the integral $\int \tan^n x \sec^m x \, dx$, except we use the identity

$$\cot^2 x + 1 = \csc^2 x.$$

Example

Evaluate the integral $\int \cot^5 x \csc^4 x \, dx$.

Solution:
Write $\cot^5 x \csc^4 x = \csc^3 x \cot^4 x \csc x \cot x$. This implies

$$\begin{aligned} \int \cot^5 x \csc^4 x \, dx &= \int \csc^3 x \cot^4 x \csc x \cot x \, dx \\ &= \int \csc^3 x (\csc^2 x - 1)^2 \csc x \cot x \, dx \\ &= \int (\csc^7 x - 2\csc^5 x + \csc^3 x) \csc x \cot x \, dx \\ &= -\frac{\csc^8 x}{8} + \frac{\csc^6 x}{3} - \frac{\csc^4 x}{4} + c. \end{aligned}$$

Section 2.2: Integration of Forms $sinux\ cosvx$, $sinux\ sinvx$ and $cosux\ cosvx$

We deal with the integrals $\int \sin\ ux\ \cos\ vx\ dx$, $\int \sin\ ux\ \sin\ vx\ dx$ and $\int \cos\ ux\ \cos\ vx\ dx$ by using the following formulas:

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$$\begin{aligned}\sin\ ux\ \cos\ vx &= \frac{1}{2}\left(\sin\ (u-v)\ x + \sin\ (u+v)\ x\right)\\ \sin\ ux\ \sin\ vx &= \frac{1}{2}\left(\cos\ (u-v)\ x - \cos\ (u+v)\ x\right)\\ \cos\ ux\ \cos\ vx &= \frac{1}{2}\left(\cos\ (u-v)\ x + \cos\ (u+v)\ x\right)\end{aligned}$$

Example

Evaluate the integral $\int \sin\ 5x\ \sin\ 3x\ dx$.

Solution: From the previous formulas, we have $\sin\ 5x\ \sin\ 3x = \frac{1}{2}\left(\cos\ 2x - \cos\ 8x\right)$. Hence,

Section 2.2: Integration of Forms *sinux cosvx*, *sinux sinvx* and *cosux cosvx*

We deal with the integrals $\int \sin\ ux\ \cos\ vx\ dx$, $\int \sin\ ux\ \sin\ vx\ dx$ and $\int \cos\ ux\ \cos\ vx\ dx$ by using the following formulas:

$$\begin{aligned}
\sin\ ux\ \cos\ vx &= \frac{1}{2}\left(\sin\ (u-v)\ x + \sin\ (u+v)\ x\right)\\
\sin\ ux\ \sin\ vx &= \frac{1}{2}\left(\cos\ (u-v)\ x - \cos\ (u+v)\ x\right)\\
\cos\ ux\ \cos\ vx &= \frac{1}{2}\left(\cos\ (u-v)\ x + \cos\ (u+v)\ x\right)
\end{aligned}$$

Example

Evaluate the integral $\int \sin\ 5x\ \sin\ 3x\ dx$.

Solution: From the previous formulas, we have $\sin\ 5x\ \sin\ 3x = \frac{1}{2}\left(\cos\ 2x - \cos\ 8x\right)$. Hence,

$$\begin{aligned}
\int \sin\ 5x\ \sin\ 3x\ dx &= \frac{1}{2}\int (\cos\ 2x - \cos\ 8x)\ dx\\
&= \frac{1}{2(2)}\int (2)\cos\ 2x\ dx - \frac{1}{2(8)}\int (8)\cos\ 8x\ dx\\
&= \frac{1}{4}\sin\ 2x - \frac{1}{16}\sin\ 8x + c.
\end{aligned}$$

Section 3: Trigonometric Substitutions

We are going to study integrals containing the following expressions for $a > 0$:

- $\sqrt{a^2 - x^2}$
- $\sqrt{a^2 + x^2}$
- $\sqrt{x^2 - a^2}$

Section 3: Trigonometric Substitutions

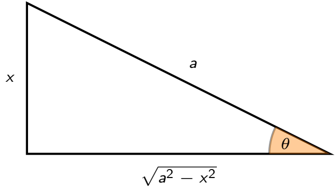
We are going to study integrals containing the following expressions for $a > 0$:

- $\sqrt{a^2 - x^2}$
- $\sqrt{a^2 + x^2}$
- $\sqrt{x^2 - a^2}$

■ $\sqrt{a^2 - x^2} = a \ \cos \ \theta$ if $x = a \ \sin \ \theta$.

If $x = a \ \sin \ \theta$ where $\theta \in [-\pi/2, \pi/2]$, then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2(1 - \sin^2 \ \theta)} \\ &= \sqrt{a^2 \cos^2 \ \theta} \\ &= a \ \cos \ \theta.\end{aligned}$$



$x = a \ \sin \ \theta \Rightarrow \sin \ \theta = \frac{x}{a}$

Notes.

- If the expression $\sqrt{a^2 - x^2}$ is in a denominator, then we assume $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- We can also use the previous substitution for $\sqrt[n]{(a^2 - x^2)^m} = (a^2 - x^2)^{\frac{m}{n}}$.

Example

Evaluate the integral $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$.

Example

Evaluate the integral $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$.

Remember. $\sqrt{a^2 - x^2} = a \ \cos \ \theta$ if $x = a \ \sin \ \theta$.

Example

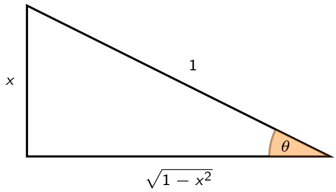
Evaluate the integral $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$.

Remember. $\sqrt{a^2 - x^2} = a \cos\theta$ if $x = a \sin\theta$.

Solution: Let $x = \sin\theta$ where $\theta \in (-\pi/2, \pi/2)$, thus $dx = \cos\theta\, d\theta$. By substitution, we have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\, dx &= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta \\
&= \int \frac{\sin^2\theta}{\cos\theta} \cos\theta\, d\theta \\
&= \int \sin^2\theta\, d\theta \qquad \text{where } \sin^2\theta = \frac{1-\cos 2\theta}{2} \\
&= \frac{1}{2}\int (1-\cos 2\theta)\, d\theta \qquad : \frac{1}{2}\int 2\cos 2\theta\, d\theta \\
&= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c \quad : \quad \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{1}{2}\left(\theta - \sin\theta\cos\theta\right) + c \\
&= \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right) + c
\end{aligned}$$

Return to the original variable x



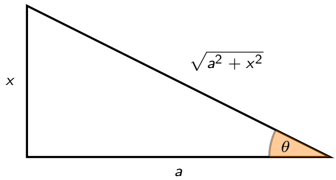
$\sin\theta = x \Rightarrow \theta = \sin^{-1} x$

$\cos\theta = \sqrt{1-x^2}$

- $\sqrt{a^2+x^2} = a \ \sec \ \theta$ if $x = a \ \tan \ \theta$.

If $x = a \ \tan \ \theta$ where $\theta \in (-\pi/2, \pi/2)$, then

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2\theta} \\ &= \sqrt{a^2(1+\tan^2 \ \theta)} \\ &= \sqrt{a^2\sec^2 \ \theta} \\ &= a \ \sec \ \theta.\end{aligned}$$



$x = a \ \tan \ \theta \Rightarrow \tan \ \theta = \frac{x}{a}$

Note. We can also use the previous substitution for $\sqrt[n]{(a^2+x^2)^m} = (a^2+x^2)^{\frac{m}{n}}$.

Example

Evaluate the integral $\int \sqrt{x^2 + 9}\, dx$

Example

Evaluate the integral $\int \sqrt{x^2+9}\, dx$

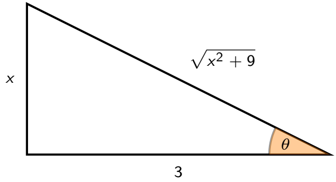
Remember. $\sqrt{a^2+x^2} = a\ \sec\ \theta$ if $x = a\ \tan\ \theta$.

Example

Evaluate the integral $\int \sqrt{x^2+9}\,dx$

Remember. $\sqrt{a^2+x^2} = a \sec\theta$ if $x = a\tan\theta$.

Solution: Let $x = 3\tan\theta$ where $\theta \in (-\pi/2, \pi/2)$. This implies $dx = 3\sec^2\theta\, d\theta$. By substitution, we have



$$\begin{aligned}\int \sqrt{x^2+9}\,dx &= \int \sqrt{9\tan^2\theta+9}\,(3\sec^2\theta)\,d\theta \\ &= 9\int \sec^3\theta\,d\theta \qquad \text{(see Example 5.8 item 3)} \\ &= \frac{9}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right)+c \\ &= \frac{9}{2}\left(\frac{x\sqrt{x^2+9}}{9} + \ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|\right)+c\end{aligned}$$

Return to the original variable x

$\tan\theta = \frac{x}{3}$

$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

Remember. Lecture 12

Example

Evaluate the integral $\int \sec^3 x \, dx$.

Solution: Write $\sec^3 x = \sec x \sec^2 x$ and let $I = \int \sec x \sec^2 x \, dx$. We use the integration by parts to evaluate the integral as follows:

$$u = \sec x \Rightarrow du = \sec x \tan x \, dx \, ,$$

$$dv = \sec^2 x \, dx \Rightarrow v = \int \sec^2 x \, dx = \tan x.$$

Hence,

$$I = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$I = \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$$

$$I = \sec x \tan x - I + \ln |\sec x + \tan x|$$

$$2I = \sec x \tan x + \ln |\sec x + \tan x|$$

$$I = \frac{1}{2}(\sec x \tan x + \ln |\sec x + \tan x|) + c.$$

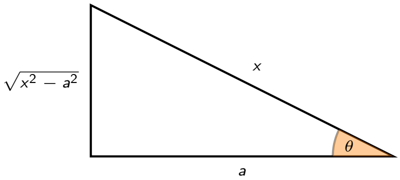
- $\tan^2 x = \sec^2 x - 1$
- $-\int (\sec^3 x - \sec x) \, dx = -\int \sec^3 x \, dx + \int \sec x \, dx$
- $\int \sec x \, dx = \ln |\sec x + \tan x| + c$

Remember: Chapter 3

(Example 3.4 No. 7)

■ $\sqrt{x^2 - a^2} = a \tan \theta$ if $x = a \sec \theta$.
If $x = a \sec \theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, then

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2 \theta - a^2} \\ &= \sqrt{a^2(\sec^2 \theta - 1)} \\ &= \sqrt{a^2 \tan^2 \theta} \\ &= a \tan \theta.\end{aligned}$$



$x = a \sec \theta \Rightarrow \sec \theta = \frac{x}{a}$

Note. We can also use the previous substitution for $\sqrt[n]{(x^2 - a^2)^m} = (x^2 - a^2)^{\frac{m}{n}}$.

Example

Evaluate the integral $\int_5^6 \frac{\sqrt{x^2 - 25}}{x^4}\, dx$.

Example

Evaluate the integral $\int_5^6 \frac{\sqrt{x^2 - 25}}{x^4}\, dx$.

Remember. $\sqrt{x^2 - a^2} = a \tan \theta$ if $x = a \sec \theta$.

Example

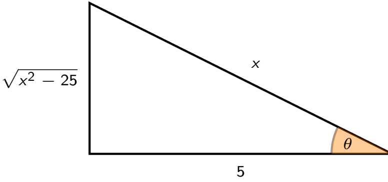
Evaluate the integral $\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\, dx$.

Remember. $\sqrt{x^2-a^2} = a\ \tan\ \theta$ if $x = a\ \sec\ \theta$.

Solution: Let $x = 5\ \sec\ \theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, thus $dx = 5\ \sec\ \theta\ \tan\ \theta\ d\theta$. After substitution, the integral becomes

$$\begin{aligned}
\int \frac{5\ \tan\ \theta}{625\ \sec^4\ \theta}\, 5\sec\ \theta\ \tan\ \theta\ d\theta &= \frac{1}{25}\int \frac{\tan^2\ \theta}{\sec^3\ \theta}\, d\theta \\
&= \frac{1}{25}\int \sin^2\ \theta\ \cos\ \theta\ d\theta \\
&= \frac{1}{75}\sin^3\ \theta + c \\
&= \frac{\left(\sqrt{x^2-25}\right)^3}{75x^3} + c \\
&= \frac{(x^2-25)^{3/2}}{75x^3} + c
\end{aligned}$$

Return to the original variable x



$\sin\ \theta = \dfrac{\sqrt{x^2-25}}{x}$

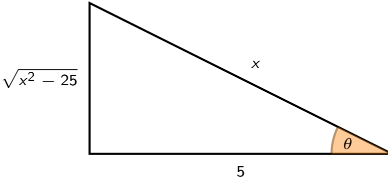
Example

Evaluate the integral $\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\, dx$.

Remember. $\sqrt{x^2-a^2} = a \tan\theta$ if $x = a \sec\theta$.

Solution: Let $x = 5 \sec\theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, thus $dx = 5 \sec\theta \tan\theta \, d\theta$. After substitution, the integral becomes

$$\int \frac{5 \tan\theta}{625 \sec^4\theta} 5 \sec\theta \tan\theta \, d\theta = \frac{1}{25}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$
$$= \frac{1}{25}\int \sin^2\theta \cos\theta \, d\theta$$
$$= \frac{1}{75}\sin^3\theta + c$$
$$= \frac{\left(\sqrt{x^2-25}\right)^3}{75x^3} + c$$
$$= \frac{(x^2-25)^{3/2}}{75x^3} + c$$



$\sin\theta = \frac{\sqrt{x^2-25}}{x}$

Return to the original variable x

Thus,

$$\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\, dx = \frac{1}{75}\left[\frac{(x^2-25)^{3/2}}{x^3}\right]_5^6 = \frac{1}{600}.$$

Exercise. Evaluate the integral.

1. $\displaystyle\int \frac{x}{x^2+1}\,dx$

Exercise. Evaluate the integral.

1. $\displaystyle\int \frac{x}{x^2+1}\,dx$

Solution: $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

Exercise. Evaluate the integral.

1. $\displaystyle\int \frac{x}{x^2+1}\,dx$

   Solution: $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

2. $\displaystyle\int \frac{x+1}{x^2+2x-8}\,dx$

Exercise. Evaluate the integral.

1. $\int \frac{x}{x^2+1}\,dx$

   Solution: $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

2. $\int \frac{x+1}{x^2+2x-8}\,dx$

   Solution: $\int \frac{x+1}{x^2+2x-8}\,dx = \frac{1}{2}\int \frac{2\,(x+1)}{x^2+2x-8}\,dx = \frac{1}{2}\ln|x^2+2x-8|+c$

Exercise. Evaluate the integral.

1. $\int \frac{x}{x^2+1}\,dx$

   Solution: $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

2. $\int \frac{x+1}{x^2+2x-8}\,dx$

   Solution: $\int \frac{x+1}{x^2+2x-8}\,dx = \frac{1}{2}\int \frac{2\,(x+1)}{x^2+2x-8}\,dx = \frac{1}{2}\ln|x^2+2x-8|+c$

3. $\int \frac{x+1}{x^2-2x-8}\,dx$

Section 4: Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

1. A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x(x^3 - 1)$.
2. The degree of a polynomial $f(x)$ is the highest power occurring in the polynomial, for example the degree of $f(x)$ is 3 and the degree of $g(x)$ is 4.

Steps of integrals of rational functions:

Step 1: If the degree of $f(x)$ is equal or greater than the degree of $g(x)$, we do polynomial long-division; otherwise we move to step 2.

By doing the long-division, we reduce the fraction to a mixed quantity.

$$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)} \ ,$$

```
            h(x)
       ____________
g(x)  )    f(x)
 -          ...
            ...
       ____________
            r(x)
```

Note: The degree of the numerator of the new fraction should be less than the degree of the denominator.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials.

Step 3: Find the partial fractions. This step depends on the result of step 2 where the fraction $\frac{f(x)}{g(x)}$ or $\frac{r(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x) = P_1(x) + P_2(x) + P_3(x) + \ldots + P_n(x) \ ,$$

where

$$P_k(x) = \frac{A_k}{(ax+b)^n} \text{ or } P_k(x) = \frac{A_k x + B_k}{(ax^2+bx+c)^n} \text{ such that } b^2 - 4ac < 0$$

Step 4: Integrate the result of step 3:

$$\int q(x)\,dx = \int P_1(x)\,dx + \int P_2(x)\,dx + \int P_3(x)\,dx + \ldots + \int P_n(x)\,dx \ .$$

Section 4: Integrals of Rational Functions

Review:

Factoring Polynomial.

(1) Common Factor Example: $6x^2 - 2x = 2x(3x - 1)$

(2) A Method For Simple Cases $ax^2 + bx + c$

Example 1:

$$x^2 + 3x + 2$$

$$x^2 + 3x + 2$$

$$1 + 2 = 3 \text{ and } 1 \times 2 = 2$$

$$(x + 1)(x + 2)$$

Example 2:

$$x^2 + x - 12$$

$$x^2 + 1x - 12$$

$$-3 + 4 = 1 \text{ and } -3 \times 4 = -12$$

$$(x - 3)(x + 4)$$

Example 3:

$$x^3 + x^2 - 12x \Rightarrow x(x^2 + x - 12) \qquad \text{common factor}$$

$$\Rightarrow x(x^2 + 1x - 12) = x(x - 3)(x + 4)$$

(3) Difference Of Two Squares $a^2 - b^2 = (a - b)(a + b)$

Example :

$$x^2 - 16 = (x - 4)(x + 4)$$

(4) Quadratic Formula Solutions $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4\ a\ c}}{2\ a} \quad \text{where } a, b, c \text{ are constants and } a \neq 0$$

1. $b^2 - 4ac > 0 \Rightarrow$ two distinct real solutions.
2. $b^2 - 4ac = 0 \Rightarrow$ one real solution.
3. $b^2 - 4ac < 0 \Rightarrow$ no real solutions.

Example :

$$x^2 - 2x - 8$$

$$a = 1, \; b = -2, \; c = -8$$

$$ax^2 + bx + c$$

$$x = \frac{-b \pm \sqrt{b^2 - 4\ .a\ .c}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\ (1)\ (-8)}}{2(1)} \Rightarrow x = \frac{2 \pm \sqrt{4 + 32}}{2} = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2}$$

$$\Rightarrow x = \frac{2+6}{2} = \frac{8}{2} = 4 \Rightarrow (x - 4) = 0 \text{ OR } x = \frac{2-6}{2} = \frac{-4}{2} = -2 \Rightarrow (x + 2) = 0$$

$$\Rightarrow x^2 - 2x - 8 = (x - 4)(x + 2) = 0$$

Algebraic Expressions

Let a and b be real numbers. Then,

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$
2. $(a + b)(a - b) = a^2 - b^2$
3. $(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$
4. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
5. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$

Addition of Fractional Functions.

Example 1:

$$\frac{1}{x+1} + \frac{3}{x-5} =$$

■ **Addition of Fractional Functions.**

Example 1:

$$\frac{1}{x+1}+\frac{3}{x-5}=\frac{1(x-5)+3(x+1)}{(x+1)(x-5)}=\frac{4x-2}{(x+1)(x-5)}$$

Addition of Fractional Functions.

Example 1:

$$\frac{1}{x+1} + \frac{3}{x-5} = \frac{1(x-5)+3(x+1)}{(x+1)(x-5)} = \frac{4x-2}{(x+1)(x-5)}$$

Example 2:

$$\frac{1}{x+1} + \frac{3}{x-5} + \frac{4}{(x-5)^2} = \frac{1(x-5)^2}{(x+1)(x-5)^2} + \frac{3(x-5)(x+1)}{(x-5)(x-5)(x+1)} + \frac{4(x+1)}{(x-5)^2(x+1)}$$

$$= \frac{x^2-10+25}{(x+1)(x-5)^2} + \frac{3(x^2-4x-5)}{(x-5)(x-5)(x+1)} + \frac{4x+4}{(x-5)^2(x+1)} = \frac{4x^2-18x+14}{(x+1)(x-5)^2}$$

Example

Evaluate the integral $\displaystyle\int \frac{x+1}{x^2-2x-8}\,dx$.

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:

Step 1: This step can be skipped since the degree of the function $f(x) = x + 1$ is less than the degree of the function $g(x) = x^2 - 2x - 8$.

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:

Step 1: This step can be skipped since the degree of the function $f(x) = x + 1$ is less than the degree of the function $g(x) = x^2 - 2x - 8$.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 - 2x - 8 = (x + 2)(x - 4)$$

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:

Step 1: This step can be skipped since the degree of the function $f(x) = x + 1$ is less than the degree of the function $g(x) = x^2 - 2x - 8$.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 - 2x - 8 = (x+2)(x-4)$$

Step 3: Find the partial fractions

$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax - 4A + Bx + 2B}{(x+2)(x-4)}$$

We need to find the constants A and B by equating the coefficients of like powers of x in the two sides of the equation:

$$x + 1 = (A+B)x - 4A + 2B$$

Coefficients of the numerators:

coefficients of x: $A + B = 1 \rightarrow 1$

constants: $-4A + 2B = 1 \rightarrow 2$

By doing some calculation, we obtain $A = \frac{1}{6}$ and $B = \frac{5}{6}$. Thus,

$4\times$ equation $1+$ equation 2

$$4A + 4B = 4$$
$$-4A + 2B = 1$$

$- - - - - - - - - -$

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$$\frac{x+1}{x^2-2x-8} = \frac{1/6}{x+2} + \frac{5/6}{x-4} \ .$$

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Evaluate the integral $\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2}\, dx.$

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Section 4: Integrals of Rational Functions

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Hence, we have

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$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}.$$

We need to find the constants A and B.
Coefficients of the numerators:

$$\begin{aligned} \text{coefficients of } x: &\quad A + B = 11 \to 1 \\ \text{constants:} &\quad 2A + B = 25 \to 2 \end{aligned}$$

By doing some calculation, we have $A = 14$ and $B = -3$. Hence,

$$q(x) = (2x-10) + \frac{14}{x+1} + \frac{-3}{x+2}.$$

$-2\times$ equation $1+$ equation 2

$$\begin{aligned} -2A - 2B &= -22 \\ 2A + B &= 25 \\ \hline -B &= 3 \end{aligned}$$

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Step 4: Integrate the result of step 3.

$$\int q(x)\,dx = \int (2x-10)\,dx + \int \frac{14}{x+1}\,dx + \int \frac{-3}{x+2}\,dx$$
$$= x^2 - 10x + 14\ln|x+1| - 3\ln|x+2| + c.$$

Example

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Since the denominator $g(x)$ has repeated factors, then

$$\begin{aligned}
\frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} &= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-5)} \\
&= \frac{A(x+1)(x-5)}{(x+1)(x+1)(x-5)} + \frac{B(x-5)}{(x+1)^2(x-5)} + \frac{C(x+1)^2}{(x-5)(x+1)^2} \\
&= \frac{A(x^2-4x-5) + B(x-5) + C(x^2+2x+1)}{(x+1)^2(x-5)}
\end{aligned}$$

Coefficients of the numerators:

$$\begin{aligned}
\text{coefficients of } x^2: &\quad A + C = 2 \to 1 \\
\text{coefficients of } x: &\quad -4A + B + 2C = -25 \to 2 \\
\text{constants:} &\quad -5A - 5B + C = -33 \to 3
\end{aligned}$$

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$.

$5\times$ equation $2+$ equation 3

$-25A + 11C = -158 \to 4$

$25\times$ equation $1+$ equation 4

$36C = -108 \Rightarrow C = -3$

Step 4: Integrate the result of step 3.

$$\begin{aligned}\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx &= \int \frac{5}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{-3}{x-5}\,dx \\ &= 5\ln\mid x+1\mid + \int (x+1)^{-2}\,dx - 3\ln\mid x-5\mid \\ &= 5\ln\mid x+1\mid - \frac{1}{(x+1)} - 3\ln\mid x-5\mid + c.\end{aligned}$$

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$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}.$$

Coefficients of the numerators:

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5.5 Integrals Involving Quadratic Forms

(1) We provide a new technique for integrals that contain irreducible quadratic expressions $ax^2 + bx + c$ where $b \neq 0$.

(2) This technique depends on completing square method: $u^2 \pm 2uv + v^2 = (u \pm v)^2$.

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Notes:
Assume we have a quadratic polynomial $ax^2 + bx + c$.

- If a quadratic polynomial has real roots, it is called reducible; otherwise it is called irreducible i.e., $b^2 - 4ac < 0$.

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Example: For the quadratic expression $x^2 - 6x + 13$, we have $a = 1$, $b = -6$ and $c = 13$.

$$b^2 - 4ac = -16 < 0 \Rightarrow \text{the quadratic expression is irreducible}$$

.
To complete the square, we find $\left(\frac{b}{2\sqrt{a}}\right)^2 = 9$, then we add and substrate it as follows:

$$x^2 - 6x + 13 = \underbrace{x^2 - 6x + 9}_{=(x-3)^2} - \underbrace{9 + 13}_{=4}$$

Hence, $x^2 - 6x + 13 = (x - 3)^2 + 4$.

Example

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From the previous example, we have $x^2 - 6x + 13 = (x-3)^2 + 4$. So,

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$$\int \frac{1}{u^2 + 4}\, du = \frac{1}{2}\tan^{-1}\frac{u}{2} + c = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c.$$

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Hence

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Let $u = x - 1$, then $du = dx$. By substitution, the integral becomes

$$\int \frac{1}{\sqrt{1 - u^2}}\, du = \sin^{-1} u + c = \sin^{-1}(x-1) + c.$$

Fractional Functions in *sin x* and *cos x*

Exercise: Evaluate the integral.

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$$\Rightarrow du = \frac{u^2+1}{2}\, dx \qquad \sec^2 x = \tan^2 x + 1$$

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Fractional Functions in *sin x* and *cos x*

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$$\cos x = \cos 2\left(\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

We can find that

$$\cos\frac{x}{2} = \frac{1}{\sqrt{u^2+1}} \text{ and } \sin\frac{x}{2} = \frac{u}{\sqrt{u^2+1}} \qquad \text{use the identity } \cos^2\frac{x}{2} + \sin^2\frac{x}{2} = 1$$

This implies

$$\cos x = \frac{1-u^2}{1+u^2}.$$

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For an integral that contains a rational expression in $\sin x$ *and* $\cos x$, *we take* $u = \tan\,(x/2)$, *then*

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$$= 2\int (u+1)^{-2}\ du$$

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$$\begin{aligned}
\int \frac{1}{u^2 + u} 4u^3\, du &= 4 \int \frac{u^3}{u(u+1)}\, du \\
&= 4 \int \frac{u^2}{u+1}\, du \\
&= 4 \int (u-1)\, du + 4 \int \frac{1}{u+1}\, du \\
&= 2u^2 - 4u + 4 \ln \mid u+1 \mid + c \\
&= 2\sqrt{x} - 4\sqrt[4]{x} + 4 \ln \mid \sqrt[4]{x} + 1 \mid + c.
\end{aligned}$$

$$\begin{array}{r|l}
 & u \quad - \quad 1 \\
u+1) & u^2 \\
 & -(u^2 \quad +u \quad) \\
 & \quad -u \\
 & \quad -(-u \quad -1 \quad) \\
 & \quad 1
\end{array}$$

$\frac{u^2}{u+1} = (u-1) + \frac{1}{u+1}$

Integrals of Form $\sqrt[n]{f(x)}$

If the integrand is of from $\sqrt[n]{f(x)}$, it is useful to assume $u = \sqrt[n]{f(x)}$.

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By substitution, we have

$$\begin{aligned}
\int u\, \frac{2u}{u^2 - 1}\, du &= \int \frac{2u^2}{u^2 - 1}\, du \\
&= \int 2\, du + 2 \int \frac{1}{u^2 - 1}\, du \\
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Note: This case differs from that given in the substitution method in Chapter 1 i.e., $\sqrt[n]{f(x)}\ f'(x)$.

$$\int \sqrt[n]{g(x)}\, dx$$

Let $u = \sqrt[n]{g(x)}$

$$\int \sqrt[n]{g(x)}\ g'(x)\, dx = \int (g(x))^{\frac{1}{n}}\ g'(x)\, dx$$

Let $u = g(x) \Rightarrow du = g'(x)\, dx$