Integral Calculus

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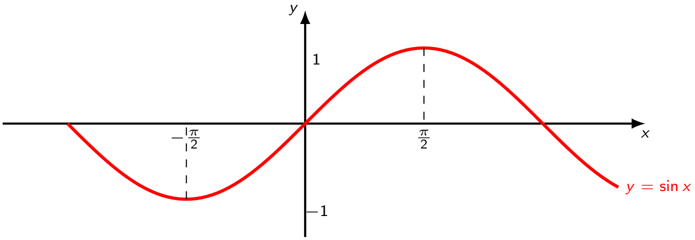
Main Contents.

- Inverse Trigonometric Functions
- Hyperbolic Functions
- Inverse Hyperbolic Functions

$$\sin\ x : \mathbb{R} \longrightarrow [-1, 1]$$



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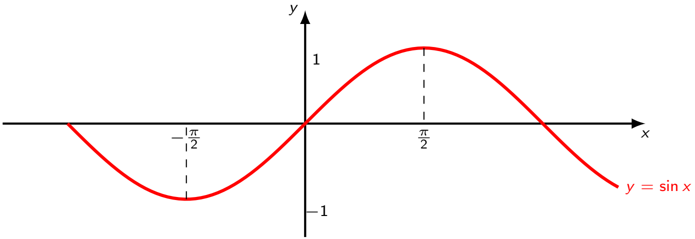
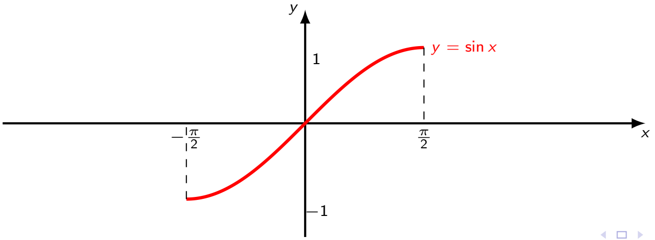
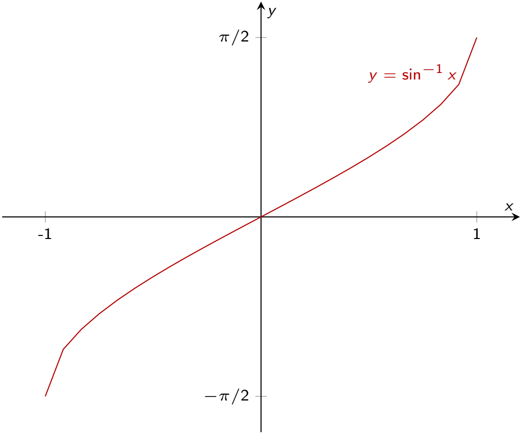


$$\sin\ x : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$$



Section 1: Inverse Trigonometric Functions

The inverse sine function $\sin^{-1} x : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



Notes.

- For other inverse trigonometric functions, see your book.

- The most common notations to name the inverse trigonometric functions are *arcsinx*, *arccosx*, *arctanx*, etc. However, the notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc. are often used as well.

- Common mistake: some students write $\sin^{-1} x = (\sin x)^{-1} = \frac{1}{\sin x}$ and **this is not true** .

Differentiation of Inverse Trigonometric Functions

Theorem

If $u = g(x)$ is a differentiable function, then

1. $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}u' = \dfrac{u'}{\sqrt{1-u^2}}$
2. $\dfrac{d}{dx}\cos^{-1} u = \dfrac{-1}{\sqrt{1-u^2}}u' = \dfrac{-u'}{\sqrt{1-u^2}}$
3. $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{u^2+1}u' = \dfrac{u'}{u^2+1}$
4. $\dfrac{d}{dx}\cot^{-1} u = \dfrac{-1}{u^2+1}u' = \dfrac{-u'}{u^2+1}$
5. $\dfrac{d}{dx}\sec^{-1} u = \dfrac{1}{|u|\sqrt{u^2-1}}u' = \dfrac{u'}{|u|\sqrt{u^2-1}}$
6. $\dfrac{d}{dx}\csc^{-1} u = \dfrac{-1}{|u|\sqrt{u^2-1}}u' = \dfrac{-u'}{|u|\sqrt{u^2-1}}$

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Example

Find the derivative of the function.

1. $y = \sin^{-1} 5x$
2. $y = \tan^{-1} e^x$
3. $y = \sec^{-1} 2x$
4. $y = \sin^{-1}(x-1)$

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Solution:

1. $y' = \frac{1}{\sqrt{1-(5x)^2}}\, 5 = \frac{5}{\sqrt{1-25x^2}}$.

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1. $y' = \frac{1}{\sqrt{1-(5x)^2}}\, 5 = \frac{5}{\sqrt{1-25x^2}}.$
2. $y' = \frac{1}{(e^x)^2+1}\, e^x = \frac{e^x}{e^{2x}+1}.$

Differentiation of Inverse Trigonometric Functions

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2. $y' = \frac{1}{(e^x)^2+1}\,e^x = \frac{e^x}{e^{2x}+1}.$
3. $y' = \frac{2}{|2x|\sqrt{4x^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}.$

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4. $y' = \dfrac{1}{\sqrt{1-(x-1)^2}} = \dfrac{1}{\sqrt{2x-x^2}}.$

Remember. $(u \pm v)^2 = u^2 \pm 2uv + v^2$

Theorem

For $a > 0$,

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c$
2. $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a} \sec^{-1} \frac{|x|}{a} + c$

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Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{4 - 25x^2}}\, dx.$

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Example

Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{4 - 25x^2}}\, dx$.

Solution:

$\displaystyle\int \frac{1}{\sqrt{4 - 25x^2}}\, dx = \int \frac{1}{\sqrt{4 - (5x)^2}}\, dx.$

Let $u = 5x \Rightarrow du = 5dx \Rightarrow dx = \frac{du}{5}$. By substitution, we have

$$\int \frac{1}{\sqrt{2^2 - u^2}}\,\frac{du}{5} = \frac{1}{5}\int \frac{1}{\sqrt{2^2 - u^2}}\, du = \frac{1}{5}\sin^{-1}\frac{u}{2} + c = \frac{1}{5}\sin^{-1}\frac{5x}{2} + c.$$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{x\sqrt{x^6-4}}\,dx$
2. $\displaystyle\int \frac{1}{9x^2+5}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

Example

Evaluate the integral.

1. $\int \frac{1}{x\sqrt{x^6-4}}\, dx$
2. $\int \frac{1}{9x^2+5}\, dx$
3. $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx$

Solution:

1. $\int \frac{1}{x\sqrt{x^6-4}}\, dx = \int \frac{1}{x\sqrt{(x^3)^2-4}}\, dx.$

   Let $u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$. By substitution, we obtain

$$\int \frac{1}{x\sqrt{u^2-2^2}}\, \frac{du}{3x^2} = \frac{1}{3}\int \frac{1}{u\sqrt{u^2-2^2}}\, du = \frac{1}{3}\frac{1}{2}\sec^{-1}\frac{|u|}{2} + c = \frac{1}{6}\sec^{-1}\frac{|x^3|}{2} + c.$$

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2. $\int \frac{1}{9x^2+5}\,dx = \int \frac{1}{(3x)^2+5}\,dx.$

   Let $u = 3x \Rightarrow du = 3dx \Rightarrow dx = \frac{du}{3}$. By substitution, we have

$$\int \frac{1}{u^2+(\sqrt{5})^2}\,\frac{du}{3} = \frac{1}{3}\int \frac{1}{u^2+(\sqrt{5})^2}\,du = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + c = \frac{1}{3\sqrt{5}}\tan^{-1}\frac{3x}{\sqrt{5}} + c.$$

Example

Evaluate the integral.

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3. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{\sqrt{(e^x)^2-1}}\,dx.$

   Let $u = e^x \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. By substituting that into the integral, we have

$$\int \frac{1}{\sqrt{u^2-1}}\,\frac{du}{e^x} = \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + c = \sec^{-1} e^x + c.$$

Section 2: Hyperbolic Functions

Definition

The hyperbolic sine function (sinh) *and the hyperbolic cosine function* (cosh) *are defined as follows:*

$$\sinh\ x = \frac{e^x - e^{-x}}{2},\ \forall x \in \mathbb{R}\ ,$$

$$\cosh\ x = \frac{e^x + e^{-x}}{2},\ \forall x \in \mathbb{R}.$$

■ Other hyperbolic functions can be defined from the hyperbolic sine and the hyperbolic cosine as follows:

$$\tanh\ x = \frac{\sinh\ x}{\cosh\ x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},\quad \forall x \in \mathbb{R}$$

$$\coth\ x = \frac{\cosh\ x}{\sinh\ x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},\quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\text{sech}\ x = \frac{1}{\cosh\ x} = \frac{2}{e^x + e^{-x}},\quad \forall x \in \mathbb{R}$$

$$\text{csch}\ x = \frac{1}{\sinh\ x} = \frac{2}{e^x - e^{-x}},\quad \forall x \in \mathbb{R} \setminus \{0\}$$

- The hyperbolic functions $\cosh x$ and $\sinh x$ satisfy the following identity:

$$\cosh^2 x - \sinh^2 x = 1, \ \forall x \in \mathbb{R} \tag{1}$$

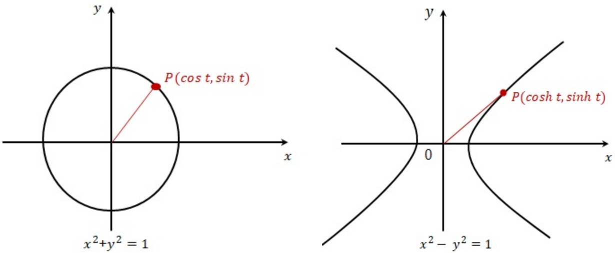
Proof. From the definition, we have

$$\cosh x - \sinh x = e^{-x} \quad \text{and} \quad \cosh x + \sinh x = e^{x}$$

Thus,

$$\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x) = e^{-x}e^{x} = e^{0} = 1 \ .$$

Since $\cos^2 t + \sin^2 t = 1$ for any $t \in \mathbb{R}$, then the point $P(\cos t, \sin t)$ is located on the unit circle $x^2 + y^2 = 1$.

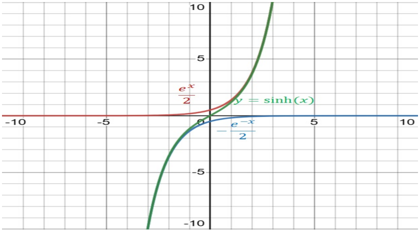


■ The function $\sinh x$ is odd

$$\sinh(-x) = -\sinh x$$

■ The functions tanh, coth and csch are odd functions.

■ The graphs of these functions are symmetric with respect to the original point.

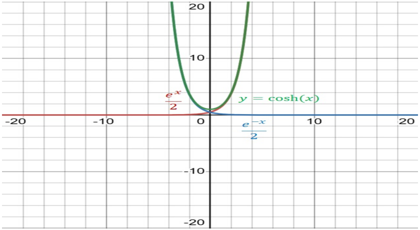


■ The function $\cosh x$ is even

$$\cosh(-x) = \cosh x$$

■ The function sech is an even function.

■ The graphs of these functions are symmetric around the y-axis.



Theorem

1. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
2. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
3. $\sinh 2x = 2\sinh x \cosh x$
4. $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 = \cosh^2 x + \sinh^2 x$
5. $1 - \tanh^2 x = sech^2 x$
6. $\coth^2 x - 1 = csch^2 x$
7. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
8. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Differentiation of Hyperbolic Functions

Theorem

If $u = g(x)$ is differentiable function, then

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4. $\frac{d}{dx}\coth\ u = -csch^2\ u\ u'$
5. $\frac{d}{dx}sech\ u = -sech\ u\ \tanh\ u\ u'$
6. $\frac{d}{dx}csch\ u = -csch\ u\ \coth\ u\ u'$

Example

Find the derivative of the functions.

1. $y = e^{\sinh\ x}$
2. $y = (x+1)\ \tanh^2\ (x^3)$

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Solution:

1. $y' = e^{\sinh\ x}\ \cosh\ x.$ $\qquad \frac{d}{dx}(e^u) = e^u.u'$

Section 2: Hyperbolic Functions

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Theorem

If $u = g(x)$ is differentiable function, then

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Example

Find the derivative of the functions.

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Solution:

1. $y' = e^{\sinh\ x}\ \cosh\ x.$ $\quad \frac{d}{dx}(e^u) = e^u.u'$
2. $y' = (1)\tanh^2\ (x^3) + (x+1)\Big(2\tanh\ (x^3)\ \text{sech}^2\ (x^3)\ (3x^2)\Big) \Rightarrow y' = \tanh^2\ (x^3) + 6x^2(x+1)\ \tanh\ (x^3)\ \text{sech}^2\ (x^3)$

Remember. $\frac{d}{dx}(f(x).g(x)) = f'(x).g(x) + f(x).g'(x)$

- $\int \sinh\ x\, dx = \cosh\ x + c$
- $\int \cosh\ x\, dx = \sinh\ x + c$
- $\int \text{sech}^2\ x\, dx = \tanh\ x + c$
- $\int \text{csch}^2\ x\, dx = -\coth\ x + c$
- $\int \text{sech}\ x\ \tanh\ x\, dx = -\text{sech}\ x + c$
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Example

Evaluate the integral $\int \sinh^2\ x\ \cosh\ x\, dx$.

- $\int \sinh\ x\ dx = \cosh\ x + c$
- $\int \cosh\ x\ dx = \sinh\ x + c$
- $\int \operatorname{sech}^2\ x\ dx = \tanh\ x + c$
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- $\int \operatorname{csch}\ x\ \coth\ x\ dx = -\operatorname{csch}\ x + c$

Example

Evaluate the integral $\int \sinh^2\ x\ \cosh\ x\ dx$.

Solution:

Let $u = \sinh\ x \Rightarrow du = \cosh\ x\ dx$

$$\Rightarrow dx = \frac{du}{\cosh\ x}$$

By substitution, we have

$$\int u^2 \cosh\ x \frac{du}{\cosh\ x} = \int u^2\ du = \frac{u^3}{3} + c$$

$$\int \sinh^2\ x\ \cosh\ x\ dx = \frac{\sinh^3\ x}{3} + c.$$

- $\int \sinh x \, dx = \cosh x + c$
- $\int \cosh x \, dx = \sinh x + c$
- $\int \operatorname{sech}^2 x \, dx = \tanh x + c$
- $\int \operatorname{csch}^2 x \, dx = -\coth x + c$
- $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$
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Example

Evaluate the integral $\int \sinh^2 x \cosh x \, dx$.

Solution:

Let $u = \sinh x \Rightarrow du = \cosh x \, dx$

$$\Rightarrow dx = \frac{du}{\cosh x}$$

By substitution, we have

$$\int u^2 \cosh x \frac{du}{\cosh x} = \int u^2 \, du = \frac{u^3}{3} + c$$

$$\int \sinh^2 x \cosh x \, dx = \frac{\sinh^3 x}{3} + c.$$

OR:

$$\int \sinh^2 x \cosh x \, dx$$

$$= \int (\sinh x)^2 \cosh x \, dx$$

$$= \frac{\sinh^3 x}{3} + c$$

Example

Evaluate the integral.

1. $\int e^{\cosh x} \sinh x \, dx$
2. $\int \tanh x \, dx$
3. $\int e^{x} \operatorname{sech} x \, dx$

Example

Evaluate the integral.

1. $\int e^{\cosh\ x} \sinh\ x\,dx$
2. $\int \tanh\ x\,dx$
3. $\int e^{x}\ sech\ x\,dx$

1. $\int e^{\cosh\ x} \sinh\ x\,dx = e^{\cosh\ x} + c.$

Remember. $\int e^{u}\ u'\ dx = e^{u} + c.$

Example

Evaluate the integral.

1. $\int e^{\cosh\ x}\ \sinh\ x\, dx$
2. $\int \tanh\ x\, dx$
3. $\int e^{x}\ sech\ x\, dx$

1. $\int e^{\cosh\ x} \sinh\ x\, dx = e^{\cosh\ x} + c.$ **Remember.** $\int e^{u}\ u'\ dx = e^{u} + c.$

2. $\int \tanh\ x\, dx = \int \frac{\sinh\ x}{\cosh\ x}\, dx = \ln(\cosh\ x) + c.$ **Remember.** $\int \frac{u'}{u}\, dx = \ln|u| + c.$

Example

Evaluate the integral.

1. $\int e^{\cosh\ x} \sinh\ x\ dx$
2. $\int \tanh\ x\ dx$
3. $\int e^{x}\ sech\ x\ dx$

1. $\int e^{\cosh\ x} \sinh\ x\ dx = e^{\cosh\ x} + c.$ **Remember.** $\int e^{u}\ u'\ dx = e^{u} + c.$

2. $\int \tanh\ x\ dx = \int \frac{\sinh\ x}{\cosh\ x}\ dx = \ln(\cosh\ x) + c.$ **Remember.** $\int \frac{u'}{u}\ dx = \ln|u| + c.$

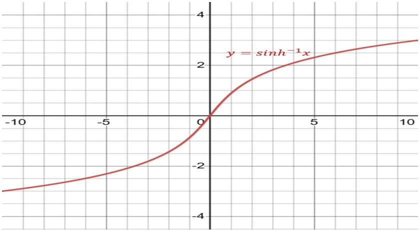
3. $\int e^{x}\ sech\ x\ dx = \int \frac{2e^{x}}{e^{x} + e^{-x}}\ dx = \int \frac{2e^{x}}{e^{x} + \frac{1}{e^{x}}}\ dx = \int \frac{2e^{x}}{\frac{e^{2x}+1}{e^{x}}}\ dx = \int \frac{2e^{2x}}{e^{2x} + 1}\ dx = \ln\left(e^{2x} + 1\right) + c.$

Inverse Hyperbolic Functions

The function $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective, so it has an inverse function

$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$

$$\sinh\ y = x \Leftrightarrow y = \sinh^{-1}\ x$$



$$x = \sinh\ y = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow e^y - 2x - e^{-y} = 0$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$ Multiply both sides by e^y : $(e^y)^2 - 2x(e^y) - e^{-y}e^y = 0$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$ By using the discriminant method $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow e^y = x \pm \sqrt{x^2 + 1}$$ $\dfrac{2x}{2} \pm \dfrac{\sqrt{4x^2 + 4}}{2} = \dfrac{2x}{2} \pm \dfrac{2\sqrt{x^2 + 1}}{2}$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1}$$ Since $\sqrt{x^2 + 1} > x$ and $e^y > 0$

$$\Rightarrow y = \sinh^{-1}\ x = \ln(x + \sqrt{x^2 + 1})$$ By taking the natural logarithm of both sides

Theorem

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), \ \forall x \in \mathbb{R}$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), \ \forall x \in [1, \infty)$
3. $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x}\right), \ \forall x \in (-1, 1)$
4. $\coth^{-1} x = \frac{1}{2} \ln \left(\frac{x+1}{x-1}\right), \ \forall x \in \mathbb{R} \setminus [-1, 1]$
5. $sech^{-1} x = \ln \left(\frac{1+\sqrt{1-x^2}}{x}\right), \ \forall x \in (0, 1]$
6. $csch^{-1} x = \ln \left(\frac{1+\sqrt{x^2+1}}{x}\right), \ \forall x \in \mathbb{R} \setminus \{0\}$

Example. $\tanh^{-1}(0) = \frac{1}{2} \ln \left(\frac{1+0}{1-0}\right) = \frac{1}{2} \ln(1) = 0$

Differentiation of Inverse Hyperbolic Functions

Theorem

If $u = g(x)$ is differentiable function, then

1. $\dfrac{d}{dx}\sinh^{-1} u = \dfrac{1}{\sqrt{u^2+1}}u'$
2. $\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2-1}}u', \ \forall u \in (1, \infty)$
3. $\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1-u^2}u', \ \forall u \in (-1, 1)$
4. $\dfrac{d}{dx}\coth^{-1} u = \dfrac{1}{1-u^2}u', \ \forall u \in \mathbb{R} \setminus [-1, 1]$
5. $\dfrac{d}{dx}sech^{-1} u = \dfrac{-1}{u\sqrt{1-u^2}}u', \ \forall u \in (0, 1)$
6. $\dfrac{d}{dx}csch^{-1} u = \dfrac{-1}{|\ u\ |\ \sqrt{u^2+1}}u', \ \forall u \in \mathbb{R} \setminus \{0\}$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = e^x \ sech^{-1} \ x$

Differentiation of Inverse Hyperbolic Functions

Theorem

If $u = g(x)$ is differentiable function, then

1. $\dfrac{d}{dx}\sinh^{-1} u = \dfrac{1}{\sqrt{u^2+1}}u'$
2. $\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2-1}}u', \ \forall u \in (1, \infty)$
3. $\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1-u^2}u', \ \forall u \in (-1, 1)$
4. $\dfrac{d}{dx}\coth^{-1} u = \dfrac{1}{1-u^2}u', \ \forall u \in \mathbb{R} \setminus [-1, 1]$
5. $\dfrac{d}{dx}sech^{-1} u = \dfrac{-1}{u\sqrt{1-u^2}}u', \ \forall u \in (0, 1)$
6. $\dfrac{d}{dx}csch^{-1} u = \dfrac{-1}{|\,u\,|\sqrt{u^2+1}}u', \ \forall u \in \mathbb{R} \setminus \{0\}$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = e^x \, sech^{-1} \, x$

Solution:

1. $y' = \dfrac{1}{\sqrt{(\sqrt{x})^2+1}} \, \dfrac{1}{2\sqrt{x}} = \dfrac{1}{\sqrt{x+1}} \, \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x(x+1)}}.$

Differentiation of Inverse Hyperbolic Functions

Theorem

If $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}}u'$
2. $\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}u', \ \forall u \in (1, \infty)$
3. $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}u', \ \forall u \in (-1, 1)$
4. $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}u', \ \forall u \in \mathbb{R} \setminus [-1, 1]$
5. $\frac{d}{dx} sech^{-1} u = \frac{-1}{u\sqrt{1-u^2}}u', \ \forall u \in (0, 1)$
6. $\frac{d}{dx} csch^{-1} u = \frac{-1}{\mid u \mid \sqrt{u^2+1}}u', \ \forall u \in \mathbb{R} \setminus \{0\}$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = e^x \ sech^{-1} \ x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}.$
2. $y' = e^x \ \text{sech}^{-1} x + e^x \frac{-1}{x\sqrt{1-x^2}} \Rightarrow y' = e^x \ \text{sech}^{-1} x - \frac{e^x}{x\sqrt{1-x^2}}$

Section 3: Inverse Hyperbolic Functions

Theorem

1. $\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \sinh^{-1} \frac{x}{a} + c$
2. $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \cosh^{-1} \frac{x}{a} + c,\ x > a$
3. $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{a} \tanh^{-1} \frac{x}{a} + c,\ | x | < a$

   $| x | < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$
4. $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{a} \coth^{-1} \frac{x}{a} + c,\ | x | > a$

   $| x | > a \Rightarrow x > a \text{ or } x < -a$
5. $\int \frac{1}{x\sqrt{a^2 - x^2}}\, dx = -\frac{1}{a} sech^{-1} \frac{|x|}{a} + c,\ | x | < a$
6. $\int \frac{1}{x\sqrt{x^2 + a^2}}\, dx = -\frac{1}{a} csch^{-1} \frac{|x|}{a} + c,\ | x | > a$

Section 3: Inverse Hyperbolic Functions

Theorem

1. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a} + c,\ x > a$
3. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c,\ |x| < a$

   $|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$

4. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\coth^{-1}\frac{x}{a} + c,\ |x| > a$

   $|x| > a \Rightarrow x > a \text{ or } x < -a$

5. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}sech^{-1}\frac{|x|}{a} + c,\ |x| < a$
6. $\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}csch^{-1}\frac{|x|}{a} + c,\ |x| > a$

Example

Evaluate the integral $\int \frac{1}{\sqrt{e^{2x}+9}}\,dx$.

Section 3: Inverse Hyperbolic Functions

Theorem

1. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \sinh^{-1} \frac{x}{a} + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \cosh^{-1} \frac{x}{a} + c,\ x > a$
3. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\tanh^{-1} \frac{x}{a} + c,\ |x| < a$

   $|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$
4. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\coth^{-1} \frac{x}{a} + c,\ |x| > a$

   $|x| > a \Rightarrow x > a \text{ or } x < -a$
5. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = -\frac{1}{a} sech^{-1} \frac{|x|}{a} + c,\ |x| < a$
6. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = -\frac{1}{a} csch^{-1} \frac{|x|}{a} + c,\ |x| > a$

Example

Evaluate the integral $\int \frac{1}{\sqrt{e^{2x}+9}}\, dx.$

Solution:

$\int \frac{1}{\sqrt{e^{2x}+9}}\, dx = \int \frac{1}{\sqrt{(e^x)^2+3^2}}\, dx.$

Let $u = e^x \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$. By substituting that into the integral, we have

$$\int \frac{1}{\sqrt{u^2+3^2}} \frac{du}{e^x} = \int \frac{1}{u\sqrt{u^2+3^2}}\, du = -\frac{1}{3}\text{csch}^{-1} \frac{|u|}{3} + c = -\frac{1}{3}\text{csch}^{-1} \frac{e^x}{3} + c$$

Example

Evaluate the integral.

1. $\int_0^1 \frac{1}{16 - x^2}\, dx$
2. $\int_5^7 \frac{1}{16 - x^2}\, dx$

Example

Evaluate the integral.

1. $\int_0^1 \frac{1}{16-x^2}\,dx$
2. $\int_5^7 \frac{1}{16-x^2}\,dx$

Solution:

1. $\int_0^1 \frac{1}{16-x^2}\,dx = \int_0^1 \frac{1}{4^2-x^2}\,dx$
   Since the interval of the integral $[0,1]$ is subinterval of $(-4,4)$, then the value of the integral is $\tanh^{-1}$. Hence,

$$\int_0^1 \frac{1}{16-x^2}\,dx = \frac{1}{4}\left[\tanh^{-1}\frac{x}{4}\right]_0^1 = \frac{1}{4}\left[\tanh^{-1}(\frac{1}{4}) - \tanh^{-1}(0)\right] = \frac{1}{4}\left[\frac{1}{2}\ln(\frac{5}{3}) - \frac{1}{2}\ln(1)\right] = \frac{1}{8}\ln(\frac{5}{3}).$$

Example

Evaluate the integral.

1. $\int_0^1 \frac{1}{16 - x^2}\, dx$
2. $\int_5^7 \frac{1}{16 - x^2}\, dx$

Solution:

1. $\int_0^1 \frac{1}{16 - x^2}\, dx = \int_0^1 \frac{1}{4^2 - x^2}\, dx$
   Since the interval of the integral $[0, 1]$ is subinterval of $(-4, 4)$, then the value of the integral is $\tanh^{-1}$. Hence,
   $$\int_0^1 \frac{1}{16 - x^2}\, dx = \frac{1}{4}\left[\tanh^{-1} \frac{x}{4}\right]_0^1 = \frac{1}{4}\left[\tanh^{-1}(\frac{1}{4}) - \tanh^{-1}(0)\right] = \frac{1}{4}\left[\frac{1}{2}\ln(\frac{5}{3}) - \frac{1}{2}\ln(1)\right] = \frac{1}{8}\ln(\frac{5}{3}).$$
2. $\int_5^7 \frac{1}{16 - x^2}\, dx = \int_5^7 \frac{1}{4^2 - x^2}\, dx$
   The interval of the integral is not subinterval of $(-4, 4)$, so the value of the integral is $\coth^{-1}$. This implies
   $$\int_5^7 \frac{1}{16 - x^2}\, dx = \frac{1}{4}\left[\coth^{-1} \frac{x}{4}\right]_5^7 = \frac{1}{4}\left[\coth^{-1} \frac{7}{4} - \coth^{-1} \frac{5}{4}\right] = \frac{1}{8}\left[\ln(11) - 3\ln(3)\right].$$