

# Integral Calculus

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Department of Mathematics

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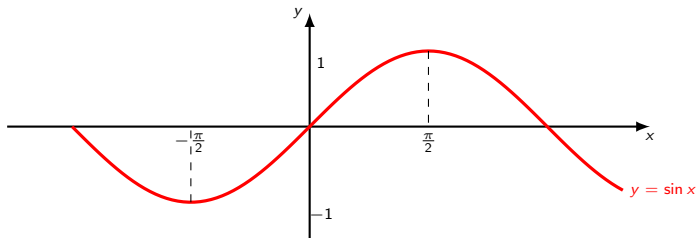
# Chapter 4: Inverse Trigonometric and Hyperbolic Functions

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- 3 Inverse Hyperbolic Functions

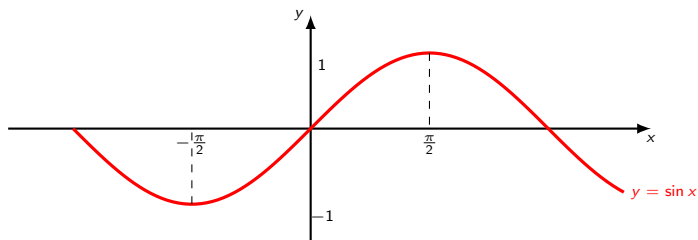
# Section 1: Inverse Trigonometric Functions

$$\sin x : \mathbb{R} \rightarrow [-1, 1]$$

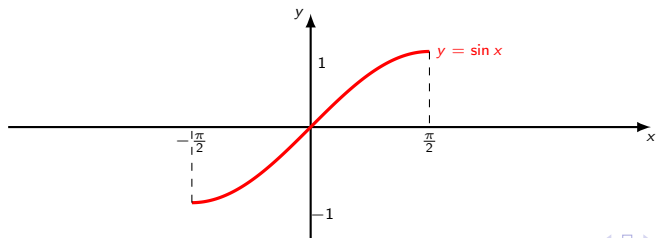


# Section 1: Inverse Trigonometric Functions

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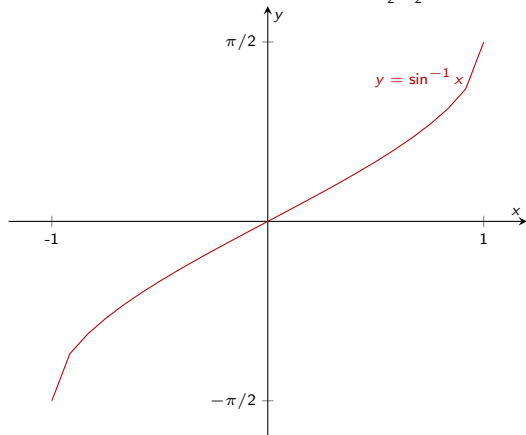


$$\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$



# Section 1: Inverse Trigonometric Functions

The inverse sine function  $\sin^{-1} x : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



■ **Notes:**

■ For other inverse trigonometric functions, see your book.

■ The most common notations to name the inverse trigonometric functions are *arcsinx*, *arccosx*, *arctanx*, etc. However, the notations  $\sin^{-1} x$ ,  $\cos^{-1} x$ ,  $\tan^{-1} x$ , etc. are often used as well.

■ **Common mistake:** some students write  $\sin^{-1} x = (\sin x)^{-1} = \frac{1}{\sin x}$  and **this is not true**.

# Section 1: Inverse Trigonometric Functions

## ■ Differentiation of Inverse Trigonometric Functions

### Theorem

If  $u = g(x)$  is a differentiable function, then

$$1 \quad \frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} u' = \frac{u'}{\sqrt{1-u^2}}$$

$$2 \quad \frac{d}{dx} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} u' = \frac{-u'}{\sqrt{1-u^2}}$$

$$3 \quad \frac{d}{dx} \tan^{-1} u = \frac{1}{u^2+1} u' = \frac{u'}{u^2+1}$$

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$$5 \quad \frac{d}{dx} \sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} u' = \frac{u'}{|u|\sqrt{u^2-1}}$$

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### Example

Find the derivative of the function.

$$① \quad y = \sin^{-1} 5x$$

$$② \quad y = \tan^{-1} e^x$$

$$③ \quad y = \sec^{-1} 2x$$

$$④ \quad y = \sin^{-1} (x-1)$$

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Solution:

$$① \quad y' = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25x^2}}$$



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$$\textcircled{4} \quad y' = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}}$$

Remember:  $(u \pm v)^2 = u^2 \pm 2uv + v^2$

# Section 1: Inverse Trigonometric Functions

## Theorem

For  $a > 0$ ,

$$\textcircled{1} \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$$

$$\textcircled{2} \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

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## Example

Evaluate the integral  $\int \frac{1}{\sqrt{4 - 25x^2}} dx$ .

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## Example

Evaluate the integral  $\int \frac{1}{\sqrt{4 - 25x^2}} dx$ .

**Solution:**

$$\int \frac{1}{\sqrt{4 - 25x^2}} dx = \int \frac{1}{\sqrt{4 - (5x)^2}} dx.$$

Let  $u = 5x \Rightarrow du = 5dx \Rightarrow dx = \frac{du}{5}$ . By substitution, we have

$$\int \frac{1}{\sqrt{2^2 - u^2}} \frac{du}{5} = \frac{1}{5} \int \frac{1}{\sqrt{2^2 - u^2}} du = \frac{1}{5} \sin^{-1} \frac{u}{2} + c = \frac{1}{5} \sin^{-1} \frac{5x}{2} + c.$$

# Section 1: Inverse Trigonometric Functions

## Example

Evaluate the integral.

$$① \int \frac{1}{x\sqrt{x^6 - 4}} dx$$

$$② \int \frac{1}{9x^2 + 5} dx$$

$$③ \int \frac{1}{\sqrt{e^{2x} - 1}} dx$$

# Section 1: Inverse Trigonometric Functions

## Example

Evaluate the integral.

$$\textcircled{1} \int \frac{1}{x\sqrt{x^6-4}} dx$$

$$\textcircled{2} \int \frac{1}{9x^2+5} dx$$

$$\textcircled{3} \int \frac{1}{\sqrt{e^{2x}-1}} dx$$

Solution:

$$\textcircled{1} \int \frac{1}{x\sqrt{x^6-4}} dx = \int \frac{1}{x\sqrt{(x^3)^2-4}} dx.$$

Let  $u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$ . By substitution, we obtain

$$\int \frac{1}{x\sqrt{u^2-2^2}} \frac{du}{3x^2} = \frac{1}{3} \int \frac{1}{u\sqrt{u^2-2^2}} du = \frac{1}{3} \frac{1}{2} \sec^{-1} \frac{|u|}{2} + c = \frac{1}{6} \sec^{-1} \frac{|x^3|}{2} + c.$$



# Section 1: Inverse Trigonometric Functions

## Example

Evaluate the integral.

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Solution:

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$$\textcircled{2} \int \frac{1}{9x^2+5} dx = \int \frac{1}{(3x)^2+5} dx.$$

Let  $u = 3x \Rightarrow du = 3dx \Rightarrow dx = \frac{du}{3}$ . By substitution, we have

$$\int \frac{1}{u^2+(\sqrt{5})^2} \frac{du}{3} = \frac{1}{3} \int \frac{1}{u^2+(\sqrt{5})^2} du = \frac{1}{3} \frac{1}{\sqrt{5}} \tan^{-1} \frac{u}{\sqrt{5}} + c = \frac{1}{3\sqrt{5}} \tan^{-1} \frac{3x}{\sqrt{5}} + c.$$

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Evaluate the integral.

$$\textcircled{1} \int \frac{1}{x\sqrt{x^6-4}} dx$$

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$$\textcircled{3} \int \frac{1}{\sqrt{e^{2x}-1}} dx = \int \frac{1}{\sqrt{(e^x)^2-1}} dx.$$

Let  $u = e^x \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$ . By substituting that into the integral, we have

$$\int \frac{1}{\sqrt{u^2-1}} \frac{du}{e^x} = \int \frac{1}{u\sqrt{u^2-1}} du = \sec^{-1} |u| + c = \sec^{-1} e^x + c.$$

## Section 2: Hyperbolic Functions

### Definition

The hyperbolic sine function ( $\sinh$ ) and the hyperbolic cosine function ( $\cosh$ ) are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \forall x \in \mathbb{R},$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \forall x \in \mathbb{R}.$$

Other hyperbolic functions can be defined from the hyperbolic sine and the hyperbolic cosine as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \forall x \in \mathbb{R}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \forall x \in \mathbb{R}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

## Section 2: Hyperbolic Functions

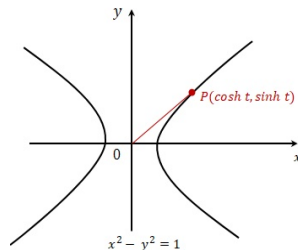
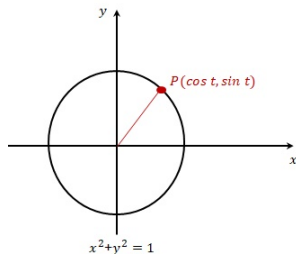
■ The hyperbolic functions  $\cosh x$  and  $\sinh x$  satisfy the following identity:

$$\cosh^2 x - \sinh^2 x = 1, \quad \forall x \in \mathbb{R} \quad (1)$$

*Proof.* From the definition, we have  $\cosh x - \sinh x = e^{-x}$  and  $\cosh x + \sinh x = e^x$ . Thus,

$$\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x) = e^{-x}e^x = e^0 = 1.$$

Since  $\cos^2 t + \sin^2 t = 1$  for any  $t \in \mathbb{R}$ , then the point  $P(\cos t, \sin t)$  is located on the unit circle  $x^2 + y^2 = 1$ .



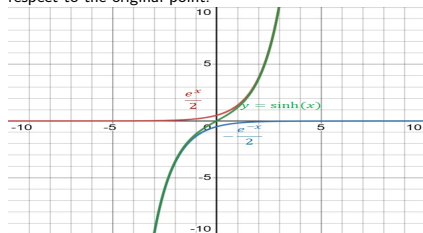
# Section 2: Hyperbolic Functions

- The function  $\sinh x$  is odd

$$\sinh(-x) = -\sinh x$$

- The functions  $\tanh$ ,  $\coth$  and  $\operatorname{csch}$  are odd functions.

- The graphs of these functions are symmetric with respect to the original point.

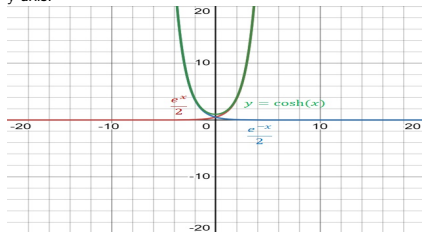


- The function  $\cosh x$  is even

$$\cosh(-x) = \cosh x$$

- The function  $\operatorname{sech}$  is an even function.

- The graphs of these functions are symmetric around the y-axis.



## Section 2: Hyperbolic Functions

### Theorem

$$① \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$② \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$③ \sinh 2x = 2 \sinh x \cosh x$$

$$④ \cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1 = \cosh^2 x + \sinh^2 x$$

$$⑤ 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$⑥ \coth^2 x - 1 = \operatorname{csch}^2 x$$

$$⑦ \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$⑧ \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

# Section 2: Hyperbolic Functions

## ■ Differentiation of Hyperbolic Functions

### Theorem

If  $u = g(x)$  is differentiable function, then

$$\textcircled{1} \quad \frac{d}{dx} \sinh u = \cosh u \, u'$$

$$\textcircled{2} \quad \frac{d}{dx} \cosh u = \sinh u \, u'$$

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### Example

Find the derivative of the functions.

$$\textcircled{1} \quad y = e^{\sinh x}$$

$$\textcircled{2} \quad y = (x + 1) \tanh^2(x^3)$$

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Solution:

$$\textcircled{1} \quad y' = e^{\sinh x} \cosh x. \quad \frac{d}{dx}(e^u) = e^u \cdot u'$$



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$$\textcircled{3} \quad \frac{d}{dx} \tanh u = \operatorname{sech}^2 u \, u'$$

$$\textcircled{4} \quad \frac{d}{dx} \coth u = -\operatorname{csch}^2 u \, u'$$

$$\textcircled{5} \quad \frac{d}{dx} \operatorname{sech} u = -\operatorname{sech} u \tanh u \, u'$$

$$\textcircled{6} \quad \frac{d}{dx} \operatorname{csch} u = -\operatorname{csch} u \coth u \, u'$$

### Example

Find the derivative of the functions.

$$\textcircled{1} \quad y = e^{\sinh x}$$

$$\textcircled{2} \quad y = (x + 1) \tanh^2 (x^3)$$

Solution:

$$\textcircled{1} \quad y' = e^{\sinh x} \cosh x. \quad \frac{d}{dx}(e^u) = e^u \cdot u'$$

$$\textcircled{2} \quad y' = (1) \tanh^2 (x^3) + (x + 1) \left( 2 \tanh (x^3) \operatorname{sech}^2 (x^3) (3x^2) \right) \Rightarrow y' = \tanh^2 (x^3) + 6x^2(x + 1) \tanh (x^3) \operatorname{sech}^2 (x^3)$$

$$\text{Remember: } \frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

## Section 2: Hyperbolic Functions

- $\int \sinh x \, dx = \cosh x + c$

- $\int \cosh x \, dx = \sinh x + c$

- $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

- $\int \operatorname{csch}^2 x \, dx = -\operatorname{coth} x + c$

- $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

- $\int \operatorname{csch} x \operatorname{coth} x \, dx = -\operatorname{csch} x + c$

### Example

Evaluate the integral  $\int \sinh^2 x \cosh x \, dx$ .

## Section 2: Hyperbolic Functions

$$\bullet \int \sinh x \, dx = \cosh x + c$$

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$$\bullet \int \operatorname{csch} x \operatorname{coth} x \, dx = -\operatorname{csch} x + c$$

### Example

Evaluate the integral  $\int \sinh^2 x \cosh x \, dx$ .

**Solution:**

$$\text{Let } u = \sinh x \Rightarrow du = \cosh x \, dx$$

$$\Rightarrow dx = \frac{du}{\cosh x}$$

By substitution, we have

$$\int u^2 \cosh x \times \frac{du}{\cosh x} = \int u^2 \, du = \frac{u^3}{3} + c$$

$$\int \sinh^2 x \cosh x \, dx = \frac{\sinh^3 x}{3} + c.$$

## Section 2: Hyperbolic Functions

$$\bullet \int \sinh x \, dx = \cosh x + c$$

$$\bullet \int \cosh x \, dx = \sinh x + c$$

$$\bullet \int \operatorname{sech}^2 x \, dx = \tanh x + c$$

$$\bullet \int \operatorname{csch}^2 x \, dx = -\operatorname{coth} x + c$$

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### Example

Evaluate the integral  $\int \sinh^2 x \cosh x \, dx$ .

**Solution:**

$$\text{Let } u = \sinh x \Rightarrow du = \cosh x \, dx$$

$$\Rightarrow dx = \frac{du}{\cosh x}$$

By substitution, we have

$$\int u^2 \cosh x \cdot \frac{du}{\cosh x} = \int u^2 \, du = \frac{u^3}{3} + c$$

$$\int \sinh^2 x \cosh x \, dx = \frac{\sinh^3 x}{3} + c.$$

**OR:**

$$\begin{aligned} & \int \sinh^2 x \cosh x \, dx \\ &= \int (\sinh x)^2 \cosh x \, dx \\ &= \frac{\sinh^3 x}{3} + c \end{aligned}$$

## Section 2: Hyperbolic Functions

### Example

Evaluate the integral.

1  $\int e^{\cosh x} \sinh x \, dx$

2  $\int \tanh x \, dx$

3  $\int e^x \operatorname{sech} x \, dx$

## Section 2: Hyperbolic Functions

### Example

Evaluate the integral.

$$\textcircled{1} \int e^{\cosh x} \sinh x \, dx$$

$$\textcircled{2} \int \tanh x \, dx$$

$$\textcircled{3} \int e^x \operatorname{sech} x \, dx$$

$$\textcircled{1} \int e^{\cosh x} \sinh x \, dx = e^{\cosh x} + c.$$

**Remember:**  $\int e^u u' \, dx = e^u + c.$

## Section 2: Hyperbolic Functions

### Example

Evaluate the integral.

$$\textcircled{1} \int e^{\cosh x} \sinh x \, dx$$

$$\textcircled{2} \int \tanh x \, dx$$

$$\textcircled{3} \int e^x \operatorname{sech} x \, dx$$

$$\textcircled{1} \int e^{\cosh x} \sinh x \, dx = e^{\cosh x} + c. \quad \text{Remember: } \int e^u u' \, dx = e^u + c.$$

$$\textcircled{2} \int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \ln(\cosh x) + c. \quad \text{Remember: } \int \frac{u'}{u} \, dx = \ln |u| + c.$$

## Section 2: Hyperbolic Functions

### Example

Evaluate the integral.

$$\textcircled{1} \int e^{\cosh x} \sinh x \, dx$$

$$\textcircled{2} \int \tanh x \, dx$$

$$\textcircled{3} \int e^x \operatorname{sech} x \, dx$$

$$\textcircled{1} \int e^{\cosh x} \sinh x \, dx = e^{\cosh x} + c. \quad \text{Remember: } \int e^u u' \, dx = e^u + c.$$

$$\textcircled{2} \int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \ln(\cosh x) + c. \quad \text{Remember: } \int \frac{u'}{u} \, dx = \ln |u| + c.$$

$$\textcircled{3} \int e^x \operatorname{sech} x \, dx = \int \frac{2e^x}{e^x + e^{-x}} \, dx = \int \frac{2e^x}{e^x + \frac{1}{e^x}} \, dx = \int \frac{2e^x}{\frac{e^{2x}+1}{e^x}} \, dx = \int \frac{2e^{2x}}{e^{2x}+1} \, dx = \ln(e^{2x} + 1) + c.$$



# Section 3: Inverse Hyperbolic Functions

## ■ Inverse Hyperbolic Functions

The function  $\sinh : \mathbb{R} \rightarrow \mathbb{R}$  is bijective, so it has an inverse function

$$\sinh^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

$$\sinh y = x \Leftrightarrow y = \sinh^{-1} x$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow e^y - 2x - e^{-y} = 0$$

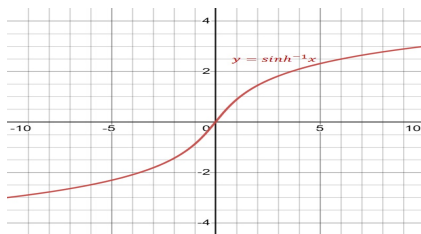
$$\Rightarrow e^{2y} - 2xe^y - 1 = 0 \quad \text{Multiply both sides by } e^y : (e^y)^2 - 2x(e^y) - e^{-y}e^y = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \quad \text{By using the discriminant method } ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2 + 1} \quad \frac{2x}{2} \pm \frac{\sqrt{4x^2 + 4}}{2} = \frac{2x}{2} \pm \frac{2\sqrt{x^2 + 1}}{2}$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \quad \text{Since } \sqrt{x^2 + 1} > x \text{ and } e^y > 0$$

$$\Rightarrow y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad \text{By taking the natural logarithm of both sides}$$



## Section 3: Inverse Hyperbolic Functions

### Theorem

- 1  $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \forall x \in \mathbb{R}$
- 2  $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \forall x \in [1, \infty)$
- 3  $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \forall x \in (-1, 1)$
- 4  $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \forall x \in \mathbb{R} \setminus [-1, 1]$
- 5  $\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \forall x \in (0, 1]$
- 6  $\operatorname{csch}^{-1} x = \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right), \forall x \in \mathbb{R} \setminus \{0\}$

**Example:**  $\tanh^{-1}(0) = \frac{1}{2} \ln\left(\frac{1+0}{1-0}\right) = \frac{1}{2} \ln(1) = 0$

# Section 3: Inverse Hyperbolic Functions

## ■ Differentiation of Inverse Hyperbolic Functions

### Theorem

If  $u = g(x)$  is differentiable function, then

$$\textcircled{1} \quad \frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{u^2 + 1}} u'$$

$$\textcircled{2} \quad \frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} u', \quad \forall u \in (1, \infty)$$

$$\textcircled{3} \quad \frac{d}{dx} \tanh^{-1} u = \frac{1}{1 - u^2} u', \quad \forall u \in (-1, 1)$$

$$\textcircled{4} \quad \frac{d}{dx} \coth^{-1} u = \frac{1}{1 - u^2} u', \quad \forall u \in \mathbb{R} \setminus [-1, 1]$$

$$\textcircled{5} \quad \frac{d}{dx} \operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1 - u^2}} u', \quad \forall u \in (0, 1)$$

$$\textcircled{6} \quad \frac{d}{dx} \operatorname{csch}^{-1} u = \frac{-1}{|u| \sqrt{u^2 + 1}} u', \quad \forall u \in \mathbb{R} \setminus \{0\}$$

### Example

Find the derivative of the functions.

$$\textcircled{1} \quad y = \sinh^{-1} \sqrt{x}$$

$$\textcircled{2} \quad y = e^x \operatorname{sech}^{-1} x$$

# Section 3: Inverse Hyperbolic Functions

## ■ Differentiation of Inverse Hyperbolic Functions

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### Example

Find the derivative of the functions.

$$\textcircled{1} \quad y = \sinh^{-1} \sqrt{x}$$

$$\textcircled{2} \quad y = e^x \operatorname{sech}^{-1} x$$

Solution:

$$\textcircled{1} \quad y' = \frac{1}{\sqrt{(\sqrt{x})^2 + 1}} \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}.$$

# Section 3: Inverse Hyperbolic Functions

## Differentiation of Inverse Hyperbolic Functions

### Theorem

If  $u = g(x)$  is differentiable function, then

$$\textcircled{1} \frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{u^2 + 1}} u'$$

$$\textcircled{2} \frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} u', \quad \forall u \in (1, \infty)$$

$$\textcircled{3} \frac{d}{dx} \tanh^{-1} u = \frac{1}{1 - u^2} u', \quad \forall u \in (-1, 1)$$

$$\textcircled{4} \frac{d}{dx} \coth^{-1} u = \frac{1}{1 - u^2} u', \quad \forall u \in \mathbb{R} \setminus [-1, 1]$$

$$\textcircled{5} \frac{d}{dx} \operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1 - u^2}} u', \quad \forall u \in (0, 1)$$

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### Example

Find the derivative of the functions.

$$\textcircled{1} y = \sinh^{-1} \sqrt{x}$$

$$\textcircled{2} y = e^x \operatorname{sech}^{-1} x$$

Solution:

$$\textcircled{1} y' = \frac{1}{\sqrt{(\sqrt{x})^2 + 1}} \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}.$$

$$\textcircled{2} y' = e^x \operatorname{sech}^{-1} x + e^x \frac{-1}{x\sqrt{1-x^2}} \Rightarrow y' = e^x \operatorname{sech}^{-1} x - \frac{e^x}{x\sqrt{1-x^2}}$$

## Section 3: Inverse Hyperbolic Functions

### Theorem

$$\textcircled{1} \int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \frac{x}{a} + c$$

$$\textcircled{2} \int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \frac{x}{a} + c, x > a$$

$$\textcircled{3} \int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \frac{x}{a} + c, |x| < a$$

$$|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$$

$$\textcircled{4} \int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \coth^{-1} \frac{x}{a} + c, |x| > a$$

$$|x| > a \Rightarrow x > a \text{ or } x < -a$$

$$\textcircled{5} \int \frac{1}{x\sqrt{a^2 - x^2}} dx = -\frac{1}{a} \operatorname{sech}^{-1} \frac{|x|}{a} + c, |x| < a$$

$$\textcircled{6} \int \frac{1}{x\sqrt{x^2 + a^2}} dx = -\frac{1}{a} \operatorname{csch}^{-1} \frac{|x|}{a} + c, |x| > a$$

# Section 3: Inverse Hyperbolic Functions

## Theorem

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$$\textcircled{6} \int \frac{1}{x\sqrt{x^2 + a^2}} dx = -\frac{1}{a} \operatorname{csch}^{-1} \frac{|x|}{a} + c, |x| > a$$

## Example

Evaluate the integral  $\int \frac{1}{\sqrt{e^{2x} + 9}} dx$ .

# Section 3: Inverse Hyperbolic Functions

## Theorem

$$\textcircled{1} \int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \frac{x}{a} + c$$

$$\textcircled{2} \int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \frac{x}{a} + c, x > a$$

$$\textcircled{3} \int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \frac{x}{a} + c, |x| < a$$

$$|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$$

$$\textcircled{4} \int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \coth^{-1} \frac{x}{a} + c, |x| > a$$

$$|x| > a \Rightarrow x > a \text{ or } x < -a$$

$$\textcircled{5} \int \frac{1}{x\sqrt{a^2 - x^2}} dx = -\frac{1}{a} \operatorname{sech}^{-1} \frac{|x|}{a} + c, |x| < a$$

$$\textcircled{6} \int \frac{1}{x\sqrt{x^2 + a^2}} dx = -\frac{1}{a} \operatorname{csch}^{-1} \frac{|x|}{a} + c, |x| > a$$

## Example

Evaluate the integral  $\int \frac{1}{\sqrt{e^{2x} + 9}} dx$ .

**Solution:**

$$\int \frac{1}{\sqrt{e^{2x} + 9}} dx = \int \frac{1}{\sqrt{(e^x)^2 + 3^2}} dx.$$

Let  $u = e^x \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$ . By substituting that into the integral, we have

$$\int \frac{1}{\sqrt{u^2 + 3^2}} \frac{du}{e^x} = \int \frac{1}{u\sqrt{u^2 + 3^2}} du = -\frac{1}{3} \operatorname{csch}^{-1} \frac{|u|}{3} + c = -\frac{1}{3} \operatorname{csch}^{-1} \frac{e^x}{3} + c$$



## Section 3: Inverse Hyperbolic Functions

### Example

Evaluate the integral.

$$1 \int_0^1 \frac{1}{16 - x^2} dx$$

$$2 \int_5^7 \frac{1}{16 - x^2} dx$$

# Section 3: Inverse Hyperbolic Functions

## Example

Evaluate the integral.

$$\textcircled{1} \int_0^1 \frac{1}{16 - x^2} dx$$

$$\textcircled{2} \int_5^7 \frac{1}{16 - x^2} dx$$

Solution:

$$\textcircled{1} \int_0^1 \frac{1}{16 - x^2} dx = \int_0^1 \frac{1}{4^2 - x^2} dx$$

Since the interval of the integral  $[0, 1]$  is subinterval of  $(-4, 4)$ , then the value of the integral is  $\tanh^{-1}$ . Hence,

$$\int_0^1 \frac{1}{16 - x^2} dx = \frac{1}{4} \left[ \tanh^{-1} \frac{x}{4} \right]_0^1 = \frac{1}{4} \left[ \tanh^{-1} \left( \frac{1}{4} \right) - \tanh^{-1}(0) \right] = \frac{1}{4} \left[ \frac{1}{2} \ln \left( \frac{5}{3} \right) - \frac{1}{2} \ln(1) \right] = \frac{1}{8} \ln \left( \frac{5}{3} \right).$$

# Section 3: Inverse Hyperbolic Functions

## Example

Evaluate the integral.

$$\textcircled{1} \int_0^1 \frac{1}{16 - x^2} dx$$

$$\textcircled{2} \int_5^7 \frac{1}{16 - x^2} dx$$

Solution:

$$\textcircled{1} \int_0^1 \frac{1}{16 - x^2} dx = \int_0^1 \frac{1}{4^2 - x^2} dx$$

Since the interval of the integral  $[0, 1]$  is subinterval of  $(-4, 4)$ , then the value of the integral is  $\tanh^{-1}$ . Hence,

$$\int_0^1 \frac{1}{16 - x^2} dx = \frac{1}{4} \left[ \tanh^{-1} \frac{x}{4} \right]_0^1 = \frac{1}{4} \left[ \tanh^{-1} \left( \frac{1}{4} \right) - \tanh^{-1}(0) \right] = \frac{1}{4} \left[ \frac{1}{2} \ln \left( \frac{5}{3} \right) - \frac{1}{2} \ln(1) \right] = \frac{1}{8} \ln \left( \frac{5}{3} \right).$$

$$\textcircled{2} \int_5^7 \frac{1}{16 - x^2} dx = \int_5^7 \frac{1}{4^2 - x^2} dx$$

The interval of the integral is not subinterval of  $(-4, 4)$ , so the value of the integral is  $\coth^{-1}$ . This implies

$$\int_5^7 \frac{1}{16 - x^2} dx = \frac{1}{4} \left[ \coth^{-1} \frac{x}{4} \right]_5^7 = \frac{1}{4} \left[ \coth^{-1} \frac{7}{4} - \coth^{-1} \frac{5}{4} \right] = \frac{1}{8} \left[ \ln(11) - 3 \ln(3) \right].$$