Integral Calculus

Prof. Mohamad Alghamdi

Department of Mathematics

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Main Contents.

- The Natural Logarithmic Function
- The Natural Exponential Function
- General Exponential and Logarithmic Function

Section 1: The Natural Logarithmic Function

Remember. Rule 1

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c$$

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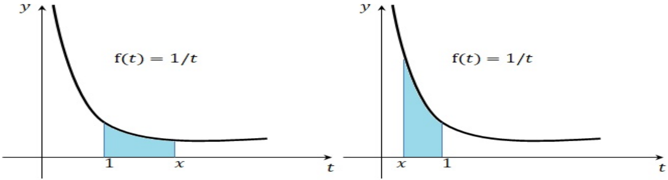
- Consider the function $f(t) = \frac{1}{t}$. It is continuous on the interval $(0, +\infty)$ and this implies that the function is integrable on the interval $[1, x]$. The area of the region under the graph can be expressed as

$$F(x) = \int_1^x \frac{1}{t} \, dt$$



Section 1: The Natural Logarithmic Function

Definition

The natural logarithmic function, denoted by ln, *is defined as follows:*

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(1) The **domain** of the natural logarithmic function is the positive real numbers.
(2) The **range** of the natural logarithmic function is $\mathbb{R}$.

- Let $x = 1$, then $\ln x = \int_1^1 \frac{1}{t} \, dt = 0$.
- For $x > 1$, $\ln x = \int_1^x \frac{1}{t} \, dt > 0$ because $\frac{1}{t} > 0$ for each $t \in [1, x]$.
- For $0 < x < 1$, $\int_x^1 \frac{1}{t} \, dt = -\int_1^x \frac{1}{t} \, dt < 0$

$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

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$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

(3) The natural logarithmic function is differentiable and continuous on the domain. From the fundamental theorem of calculus, we have

$$\frac{d}{dx}(\ln x) = \frac{d}{dx} \int_1^x \frac{1}{t} \, dt = \frac{1}{x} > 0, \quad \forall x > 0.$$

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f(h(x)) \, h'(x) - f(g(x)) \, g'(x) \quad \forall x \in J.$$

The logarithm is an **increasing** function in the interval $(0, \infty)$.

(4) The second derivative

$$\frac{d^2}{dx^2}(\ln x) = \frac{d}{dx}(\frac{1}{x}) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2} < 0 \quad \forall x \in (0, \infty)$$

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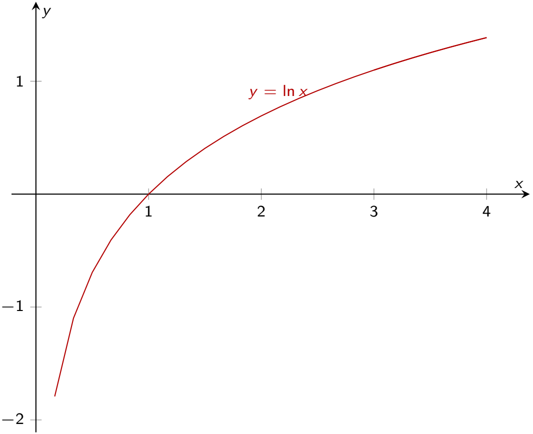
From the previous properties, we have the graph of the natural logarithmic function.



$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

(5) $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$.

(6) Rules of the natural logarithmic function:

Theorem

If $a, b > 0$ and $r \in \mathbb{Q}$, then

1. $\ln ab = \ln a + \ln b$.
2. $\ln \frac{a}{b} = \ln a - \ln b$.
3. $\ln a^r = r \ln a$.

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Differentiation of Natural Logarithmic Function

From our discussion above, we found that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Hence,

$$\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx} \ln(|\ x\ |) = \frac{1}{x}\ \forall x \neq 0.$$

Theorem

If $u = g(x)$ *is differentiable, then*

$$\frac{d}{dx} \ln\ u = \frac{1}{u}\ .\ u' = \frac{u'}{u}$$

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Example

Find the derivative of the function.

1. $f(x) = \ln(x + 1)$
2. $g(x) = \ln(x^3 + 2x - 1)$
3. $h(x) = \ln \sqrt{x^2 + 1}$
4. $f(x) = \ln \cos \ x$
5. $g(x) = \sqrt{x} \ \ln x$
6. $h(x) = \sin \ (\ln x)$

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$$\frac{d}{dx}\ln\ u\ =\ \frac{1}{u}\ .\ u'\ =\ \frac{u'}{u}$$

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Solution:

(1) $f'(x) = \dfrac{1}{x+1}$.

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(1) $f'(x) = \dfrac{1}{x+1}$.

(2) $g'(x) = \dfrac{3x^2+2}{x^3+2x-1}$.

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(3) $h'(x) = \dfrac{1}{\sqrt{x^2 + 1}} \dfrac{2x}{2\sqrt{x^2 + 1}} = \dfrac{x}{x^2 + 1}$.

$$\frac{d}{dx} \sqrt{u(x)} = \frac{u'(x)}{2\sqrt{u(x)}}$$

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OR

$h(x) = \ln \sqrt{x^2+1} = \ln(x^2+1)^{\frac{1}{2}} = \frac{1}{2}\ \ln(x^2+1)$

$\Rightarrow h'(x) = \dfrac{1}{2} \dfrac{2x}{x^2+1} = \dfrac{x}{x^2+1}$

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(5) $g'(x) = \dfrac{1}{2\sqrt{x}} \ln x + \sqrt{x}\dfrac{1}{x} = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x} = \dfrac{\ln x + 2}{2\sqrt{x}}.$

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(6) $h'(x) = \cos \ (\ln x)(\dfrac{1}{x}) = \dfrac{\cos \ (\ln x)}{x}.$

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Solution: By taking the logarithm of each side, we have

$$\ln \mid y \mid = \ln \left| \sqrt[5]{\frac{x-1}{x+1}} \right| = \ln \left| \left(\frac{x-1}{x+1}\right)^{\frac{1}{5}} \right| = \frac{1}{5} \ln \left| \left(\frac{x-1}{x+1}\right) \right| = \frac{1}{5} \Big(\ln |x-1| - \ln |x+1| \Big).$$

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By differentiating both sides with respect to x, we have

$$\frac{y'}{y} = \frac{1}{5} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) \qquad \left(\frac{d}{dx} \ln y = \frac{y'}{y} \right)$$

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By multiplying both sides by y, we obtain

$$y' = \frac{1}{5} \Big(\frac{1}{x-1} - \frac{1}{x+1} \Big) y$$

$$\Rightarrow y' = \frac{1}{5} \Big(\frac{1}{x-1} - \frac{1}{x+1} \Big) \sqrt[5]{\frac{x-1}{x+1}}.$$

Section 1: The Natural Logarithmic Function

■ **Recall.** $\frac{d}{dx}\ln|u| = \frac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we obtain

$$\int \frac{u'}{u}\,dx = \ln|u| + c$$

If $u = x$, we have the following special case

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

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Example

Evaluate the integral.

1. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$
2. $\displaystyle\int \tan\ x\ dx$
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Solution:

1. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\ dx \Rightarrow 2\sqrt{x}\ du = dx$. By substitution, we have

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \int \frac{2\sqrt{x}du}{\sqrt{x}\ u} = 2\int \frac{1}{u}\ du = 2\ln|\ u\ | = 2\ln|\ 1+\sqrt{x}\ | + c$$

$$\Rightarrow \int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln|\ 1+\sqrt{x}\ |\ \Big]_1^4 = 2(\ln 3 - \ln 2).$$

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$$\Rightarrow \int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln|\ 1+\sqrt{x}\ |\Big]_1^4 = 2(\ln 3 - \ln 2).$$

2. We know that $\tan\ x = \frac{\sin\ x}{\cos\ x}$. Therefore,

$$\int \tan\ x\, dx = \int \frac{\sin\ x}{\cos\ x}\, dx = -\int \frac{-\sin\ x}{\cos\ x}\, dx$$
$$= -\ln|\cos\ x| + c$$

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$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \int \frac{2\sqrt{x}du}{\sqrt{x}\ u} = 2\int \frac{1}{u}\ du = 2\ln \mid u \mid = 2\ln \mid 1+\sqrt{x} \mid + c$$

$$\Rightarrow \int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln \mid 1+\sqrt{x} \mid \Big]_1^4 = 2(\ln 3 - \ln 2).$$

2. We know that $\tan\ x = \dfrac{\sin\ x}{\cos\ x}$. Therefore,

$$\int \tan\ x\ dx = \int \frac{\sin\ x}{\cos\ x}\ dx = -\int \frac{-\sin\ x}{\cos\ x}\ dx$$
$$= -\ln \mid \cos\ x \mid + c$$

3. $$\int \sec\ x\ dx = \int \frac{\sec\ x\ (\sec\ x + \tan\ x)}{(\sec\ x + \tan\ x)}\ dx = \int \frac{\sec^2\ x + \sec\ x \tan x}{\sec\ x + \tan\ x}\ dx = \ln \mid \sec\ x + \tan\ x \mid + c.$$

Definition

The natural exponential function, denoted by exp, is defined as follows: $exp : \mathbb{R} \longrightarrow (0, \infty)$,

$$y = exp(x) = e^x \Leftrightarrow \ln y = x$$

(1) The domain of $exp(x)$ is $\mathbb{R}$.

(2) The range of $exp(x)$ is $(0, \infty)$ as follows:

$$y = \begin{cases} exp\ x > 1 & : x > 0 \\ exp\ x = 1 & : x = 0 \\ exp\ x < 1 & : x < 0 \end{cases}$$

(3) $exp\ (1) = e \approx 2.71828$.

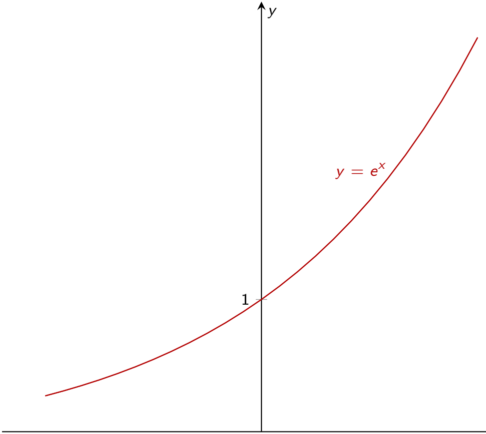
Also, $\ln e = 1$ and $\ln e^r = r \ln e = r$.

(4) The function $exp(x)$ is **increasing** on the domain:

$$y = e^x \Rightarrow \ln y = x.$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{y'}{y} = 1 \Rightarrow y' = y.$$

$$\Rightarrow \frac{d}{dx} e^x = e^x \ \forall x \in \mathbb{R}.$$



(5) The second derivative $\frac{d^2}{dx^2}e^x = e^x > 0$ for all $x \in \mathbb{R}$. Hence, the function $exp(x)$ is **concave upward** on the domain.

(6) $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

(7) Since e^x and $\ln x$ are inverse functions, then **(a)** $\ln e^x = x, \ \forall x \in \mathbb{R}$ **(b)** $e^{\ln x} = x, \ \forall x \in (0, \infty)$

(8) Rules of the natural exponential function:

Theorem

If $a, b > 0$ and $r \in \mathbb{Q}$, then

1. $e^a e^b = e^{a+b}$
2. $\frac{e^a}{e^b} = e^{a-b}$
3. $(e^a)^r = e^{ar}$

Section 2: The Natural Exponential Function

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Solve for x.

1. $\ln x = 2$
2. $\ln(\ln x) = 0$

Section 2: The Natural Exponential Function

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$\Rightarrow x = e^2$

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   $\Rightarrow \ln x = 1$
   $\Rightarrow e^{\ln x} = e^1$ (take *exp* of both sides)
   $\Rightarrow x = e$.

Differentiation of Natural Exponential Function

Theorem

If $u = g(x)$ is differentiable, then

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Example

Find the derivative of the function.

1. $y = e^{-5x^2}$
2. $y = e^{\sin(x^2)}$
3. $y = e^{3\cos(x) - 4x^2}$

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1. $y' = e^{-5x^2}(-10x).$

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Solution:

1. $y' = e^{-5x^2}(-10x).$
2. $y' = e^{\sin(x^2)} \left(\cos(x^2)\,(2x)\right) \Rightarrow y' = 2x\, \cos(x^2)\, e^{\sin(x^2)}$

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3. $y' = e^{3\cos(x) - 4x^2}(-3\sin(x) - 8x).$

Section 2: The Natural Exponential Function

■ **Recall.** $\frac{d}{dx} e^u = e^u u'$ where $u = g(x)$ is a differentiable function.

By integrating both sides, we have

$$(1) \int e^u \; u' \; dx = e^u + c$$

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Section 2: The Natural Exponential Function

Recall. $\frac{d}{dx}e^u = e^u u'$ where $u = g(x)$ is a differentiable function.

By integrating both sides, we have

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1. $\int x \, e^{-x^2} \, dx$
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2. $\int \frac{e^{\tan x}}{\cos^2 x} \, dx = \int e^{\tan x} \frac{1}{\cos^2 x} \, dx$

$= \int e^{\tan x} \sec^2 x \, dx$ Use the identity $\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x}$

$= e^{\tan x} + c$

(1) General Exponential Function

Definition

The general exponential function is defined as follows: $a^x : \mathbb{R} \to (0, \infty)$,

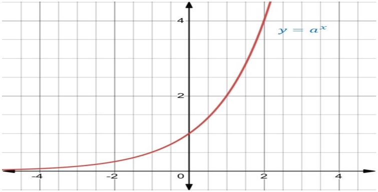
$$a^x = e^{x \ln a} \quad \textit{for every} \quad a > 0.$$

Explanation: $\ln a^x = x \ln a \Rightarrow e^{\ln a^x} = e^{x \ln a} \Rightarrow a^x = e^{x \ln a}$ *for every* $a > 0$.

(1) The function a^x is called the general exponential function with base a.

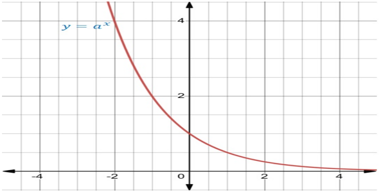
(2) If $a > 1$, $\ln a > 0$ and this implies that $x \ln a$ and $f(x)$ are increasing functions:

$$y = \begin{cases} a^x > 1 & : x > 0 \\ a^x = 1 & : x = 0 \\ a^x < 1 & : x < 0 \end{cases}$$



(3) If $a < 1$, $\ln a < 0$ and this implies that $x \ln a$ and $f(x)$ are decreasing functions.

$$y = \begin{cases} a^x < 1 & : x > 0 \\ a^x = 1 & : x = 0 \\ a^x > 1 & : x < 0 \end{cases}$$



Rules of the general exponential function

Theorem

If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^x a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
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Differentiation of General Exponential Function

Theorem

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx} a^u = a^u \; u' \; \ln a$$

Note. Using the previous derivation rule, we have $\frac{d}{dx} e^u = e^u \; u' \; \ln e = e^u \; u'$

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Example

Find the derivative of the function.

1. $y = 2^{\sqrt{x}}$
2. $y = 3^{x^2 \sin x}$

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Solution:

1. $y' = 2^{\sqrt{x}} \, \frac{1}{2\,\sqrt{x}} \, \ln 2 = \frac{2^{\sqrt{x}} \ln 2}{2\,\sqrt{x}}.$

Section 3: General Exponential and Logarithmic Functions

Rules of the general exponential function

Theorem

If $a, b > 0$ and $x, y \in \mathbb{R}$, then

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Example

Find the derivative of the function.

1. $y = 2^{\sqrt{x}}$
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Solution:

1. $y' = 2^{\sqrt{x}} \dfrac{1}{2\sqrt{x}} \ln 2 = \dfrac{2^{\sqrt{x}} \ln 2}{2\sqrt{x}}$.
2. $y' = 3^{x^2 \sin\, x} (2x \sin\, x + x^2 \cos\, x) \ln 3$.

Example

Find the derivative of the function $y = (\sin\ x)^x$.

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Solution: By taking the natural logarithm of both sides, we have $\ln y = x\ \ln|\sin\ x|$. By differentiating both sides, we obtain

$$\begin{aligned} \frac{y'}{y} &= (1)\ln|\sin\ x| + x\,\frac{\cos\ x}{\sin\ x} \\ \Rightarrow y' &= \big(\ln|\sin\ x| + x\cot\ x\big)y \\ \Rightarrow y' &= \big(\ln|\sin\ x| + x\cot\ x\big)(\sin\ x)^x. \end{aligned}$$

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Exercise. Find the derivative of the function: (1) $y = x^2$ (2) $y = 2^x$ (3) $y = x^x$

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(2) $y = 2^x \Rightarrow y' = 2^x(1)\ln 2$

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$$\frac{y'}{y} = \ln|x| + 1$$
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Section 3: General Exponential and Logarithmic Functions

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Evaluate the integral.

1. $\int x \, 3^{-x^2} \, dx$
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- If $u = g(x)$ is differentiable, then

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2. $\int 5^x\ \sqrt{5^x + 1}\ dx = \int 5^x\ (5^x + 1)^{\frac{1}{2}}\ dx$

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2. $\int 5^x \; \sqrt{5^x + 1} \; dx = \int 5^x \; (5^x + 1)^{\frac{1}{2}} \; dx = \frac{1}{\ln 5} \int 5^x \; (1) \; \ln 5 \; (5^x + 1)^{\frac{1}{2}} \; dx$

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2. $\int 5^x \sqrt{5^x + 1} \; dx = \int 5^x \; (5^x + 1)^{\frac{1}{2}} \; dx = \frac{1}{\ln 5} \int 5^x \; (1) \; \ln 5 \; (5^x + 1)^{\frac{1}{2}} \; dx = \frac{1}{\ln 5} \; \frac{2}{3} (5^x + 1)^{\frac{3}{2}} + c$

- If $u = g(x)$ is differentiable, then

$$\frac{d}{dx} a^u = a^u \; u' \; \ln a.$$

$$(1) \int a^u \; u' \; \ln a \; dx = a^u + c$$

$$(2) \int a^u \; u' \; dx = \frac{1}{\ln a} a^u + c$$

Example

Evaluate the integral.

1. $\int x \, 3^{-x^2} \, dx$
2. $\int 5^x \sqrt{5^x + 1} \, dx$

Solution:

1. $$\int x \, 3^{-x^2} \, dx = \frac{1}{-2 \; \ln 3} \int -2x \, 3^{-x^2} \ln 3 \; dx = \frac{-1}{2 \; \ln 3} \, 3^{-x^2} + c$$
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   OR Let $u = 5^x + 1 \Rightarrow du = 5^x \ln 5 \; dx \Rightarrow \frac{du}{5^x \ln 5} = dx$. By substitution, we obtain

$$\int 5^x \; \sqrt{u} \; \frac{du}{5^x \ln 5}$$

- If $u = g(x)$ is differentiable, then

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$$\int 5^x \; \sqrt{u} \; \frac{du}{5^x \ln 5} = \frac{1}{\ln 5} \int u^{\frac{1}{2}} \; du = \frac{1}{\ln 5} \; \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2(5^x + 1)^{\frac{3}{2}}}{3 \ln 5} + c.$$

(2) General Logarithmic Function

Definition

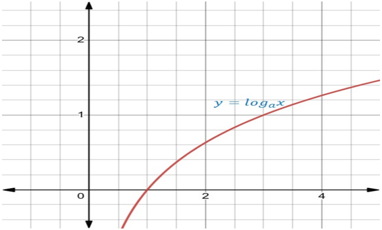
The general logarithmic function is defined as follows:

$$\log_a : (0, \infty) \to \mathbb{R} ,$$

$$x = a^y \Leftrightarrow y = \log_a x.$$

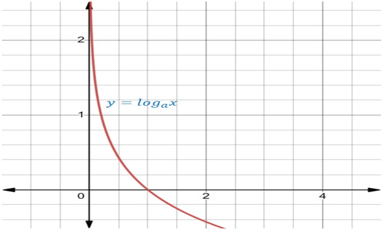
$$y = \begin{cases} logx > 0 & : x > 1 \\ logx = 0 & : x = 1 \\ logx < 0 & : 0 < x < 1 \end{cases}$$

$a > 1$



$$y = \begin{cases} logx < 0 & : x > 1 \\ logx = 0 & : x = 1 \\ logx > 0 & : 0 < x < 1 \end{cases}$$

$a < 1$



Section 3: General Exponential and Logarithmic Functions

Properties of General Logarithmic Function

1. The general logarithmic function $\log_a x = \dfrac{\ln x}{\ln a}$.
   Proof. From the definition, we have $y = \log_a x \Rightarrow x = a^y$.
   By taking the natural logarithm of both sides, we have

$$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a}$$

2. If $a > 1$, the function $\log_a x$ is increasing while if $0 < a < 1$, the function $\log_a x$ is decreasing.
3. The natural logarithmic function $\ln x = \log_e x$.
4. The general logarithmic function $\log_{10} x = \log x$.
5. The general logarithm $\log_a a = 1$.

Rules of the general logarithmic function

Theorem

If $x, y > 0$ and $r \in \mathbb{R}$, then

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^r = r \log_a x$

Differentiation of General Logarithmic Function

Since $\log_a x = \frac{\ln x}{\ln a}$, then

$$\frac{d}{dx}\left(\log_a x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right) = \frac{1}{\ln a}\frac{1}{x} = \frac{1}{x\ln a}.$$

By integrating both sides, we have

$$\int \frac{1}{x\ln a}\,dx = \frac{1}{\ln a}\int \frac{1}{x}\,dx = \frac{\ln|x|}{\ln a} = \log_a|x| + c.$$

Theorem

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a|u|\right) = \frac{d}{dx}\left(\frac{\ln|u|}{\ln a}\right) = \frac{1}{\ln a}\,\frac{u'}{u} = \frac{1}{u\,\ln a}\,u'$$

From the previous theorem, we have

$$\int \frac{1}{u\ln a}u'\,dx = \frac{1}{\ln a}\int \frac{u'}{u}\,dx = \frac{\ln|u|}{\ln a} = \log_a|u| + c$$

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

Example

Find the derivative of the function.

1. $y = \log_3 \sin\ x$
2. $y = \log \sqrt{x}$

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

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Find the derivative of the function.

1. $y = \log_3 \sin\ x$
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Solution:

1. $y' = \dfrac{1}{\ln 3} \dfrac{\cos\ x}{\sin\ x} = \dfrac{\cot\ x}{\ln 3}.$

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

Example

Find the derivative of the function.

1. $y = \log_3 \sin\ x$
2. $y = \log \sqrt{x}$

Solution:

1. $y' = \dfrac{1}{\ln 3} \dfrac{\cos\ x}{\sin\ x} = \dfrac{\cot\ x}{\ln 3}$.
2. $y = \log(x)^{\frac{1}{2}} = \frac{1}{2} \log x \Rightarrow y' = \dfrac{1}{(2 \ln 10)\ x}$.

Exercise: **Choose the correct answer.** If $\log_2 \frac{x}{x-1} = 1$, then x is equal to ...

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) -1

Exercise: Choose the correct answer. If $\log_2 \frac{x}{x-1} = 1$, then x is equal to ...

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) -1

Solution:

$$\log_2 \frac{x}{x-1} = 1 \Rightarrow 2^1 = \frac{x}{x-1} \qquad \textit{Remember}: x = a^y \Leftrightarrow \log_a x = y$$

$$2x - 2 = x \Rightarrow 2x - x = 2 \Rightarrow x = 2$$