Integral Calculus

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Main Contents

1. The Natural Logarithmic Function
2. The Natural Exponential Function
3. General Exponential and Logarithmic Function

Remember: Rule 1:

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c$$

Now evaluate the integrals:

(1) $\int \frac{1}{x^2} \, dx$

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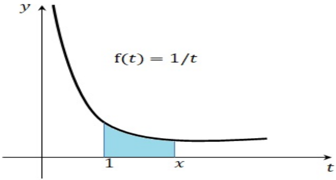
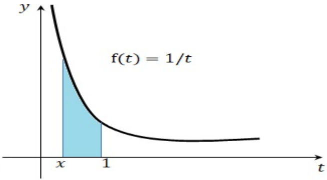
We want to find a function $F(x)$ such that $\int \frac{1}{x} \, dx = F(x) + c$.

Consider the function $f(t) = \frac{1}{t}$. It is continuous on the interval $(0, +\infty)$ and this implies that the function is integrable on the interval $[1, x]$. The area of the region under the graph can be expressed as

$$F(x) = \int_1^x \frac{1}{t} \, dt$$





Section 1: The Natural Logarithmic Function

Definition

The natural logarithmic function, denoted by ln, *is defined as follows:*

$$\ln : (0, \infty) \to \mathbb{R} ,$$

$$\ln x = \int_1^x \frac{1}{t}\, dt \quad \forall x > 0$$

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(1) The **domain** of the natural logarithmic function is the positive real numbers.
(2) The **range** of the natural logarithmic function is $\mathbb{R}$.

- Let $x = 1$, then $\ln x = \displaystyle\int_1^1 \frac{1}{t} \ dt = 0$.
- For $x > 1$, $\ln x = \displaystyle\int_1^x \frac{1}{t} \ dt > 0$ because $\frac{1}{t} > 0$ for each $t \in [1, x]$.
- For $0 < x < 1$, $\displaystyle\int_x^1 \frac{1}{t} \ dt = -\int_1^x \frac{1}{t} \ dt < 0$

$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

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$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

(3) The natural logarithmic function is differentiable and continuous on the domain. From the fundamental theorem of calculus, we have

$$\frac{d}{dx}(\ln x) = \frac{d}{dx} \int_1^x \frac{1}{t} \, dt = \frac{1}{x} > 0, \quad \forall x > 0.$$

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f(h(x)) \, h'(x) - f(g(x)) \, g'(x) \quad \forall x \in J.$$

The logarithm is an **increasing** function in the interval $(0, \infty)$.

(4) The second derivative

$$\frac{d^2}{dx^2}(\ln x) = \frac{d}{dx}(\frac{1}{x}) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2} < 0 \quad \forall x \in (0, \infty)$$

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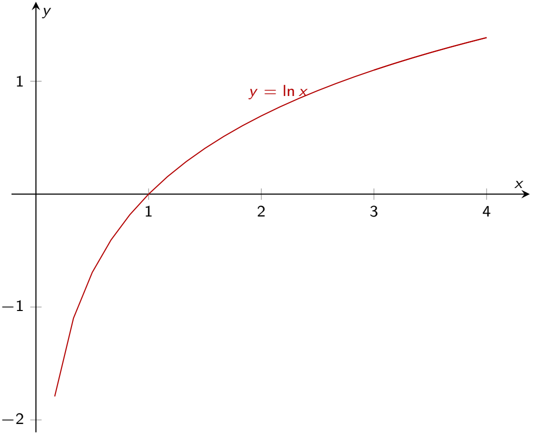
From the previous properties, we have the graph of the natural logarithmic function.



$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

(5) $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.

(6) Rules of the natural logarithmic function:

Theorem

If $a, b > 0$ *and* $r \in \mathbb{Q}$*, then*

1. $\ln ab = \ln a + \ln b$.
2. $\ln \frac{a}{b} = \ln a - \ln b$.
3. $\ln a^r = r \ln a$.

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Differentiation of Natural Logarithmic Function

From our discussion above, we found that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Hence,

$$\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx} \ln(|\ x\ |) = \frac{1}{x}\ \forall x \neq 0.$$

Theorem

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Example

Find the derivative of the function.

1. $f(x) = \ln(x + 1)$
2. $g(x) = \ln(x^3 + 2x - 1)$
3. $h(x) = \ln \sqrt{x^2 + 1}$
4. $f(x) = \ln \cos\ x$
5. $g(x) = \sqrt{x}\ \ln x$
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Solution:

(1) $f'(x) = \dfrac{1}{x+1}$.

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Solution:

(1) $f'(x) = \dfrac{1}{x+1}$.

(2) $g'(x) = \dfrac{3x^2+2}{x^3+2x-1}$.

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$$\frac{d}{dx}\sqrt{u(x)} = \frac{u'(x)}{2\sqrt{u(x)}}$$

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OR

$h(x) = \ln \sqrt{x^2+1} = \ln(x^2+1)^{\frac{1}{2}} = \frac{1}{2}\ \ln(x^2+1)$

$\Rightarrow h'(x) = \dfrac{1}{2} \dfrac{2x}{x^2+1} = \dfrac{x}{x^2+1}$

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(5) $g'(x) = \dfrac{1}{2\sqrt{x}}\ln x + \sqrt{x}\dfrac{1}{x} = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x} = \dfrac{\ln x + 2}{2\sqrt{x}}.$

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(6) $h'(x) = \cos\ (\ln x)(\dfrac{1}{x}) = \dfrac{\cos\ (\ln x)}{x}.$

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Find the derivative of the function $y = \sqrt[5]{\frac{x-1}{x+1}}$.

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Solution: By taking the logarithm of each side, we have

$$\ln \mid y \mid = \ln \left| \sqrt[5]{\frac{x-1}{x+1}} \right| = \ln \left| \left(\frac{x-1}{x+1}\right)^{\frac{1}{5}} \right| = \frac{1}{5} \ln \left| \left(\frac{x-1}{x+1}\right) \right| = \frac{1}{5} \Big(\ln |x-1| - \ln |x+1| \Big).$$

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By differentiating both sides with respect to x, we have

$$\frac{y'}{y} = \frac{1}{5}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) \qquad \left(\frac{d}{dx} \ln y = \frac{y'}{y}\right)$$

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By multiplying both sides by y, we obtain

$$y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big) y$$

$$\Rightarrow y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big) \sqrt[5]{\frac{x-1}{x+1}}.$$

Section 1: The Natural Logarithmic Function

■ **Recall:** $\frac{d}{dx} \ln \mid u \mid = \frac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we obtain

$$\int \frac{u'}{u}\, dx = \ln \mid u \mid + c$$

If $u = x$, we have the following special case

$$\int \frac{1}{x}\, dx = \ln \mid x \mid + c$$

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Example

Evaluate the integral.

1. $\int \frac{2x}{x^2+1}\, dx$
2. $\int \frac{6x^2+1}{4x^3+2x+1}\, dx$

Recall: $\frac{d}{dx}\ln|u| = \frac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we obtain

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1. $\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + c.$

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1. $\int \frac{2x}{x^2+1}\, dx = \ln(x^2+1) + c.$
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Recall: $\dfrac{d}{dx}\ln\mid u\mid = \dfrac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we obtain

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2. $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\,dx = \frac{1}{2}\int \frac{2(6x^2+1)}{4x^3+2x+1}\,dx = \tfrac{1}{2}\ln\mid 4x^3+2x+1\mid + c.$

Example

Evaluate the integral.

1. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$
2. $\displaystyle\int \tan\ x\ dx$
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Solution:

1. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\ dx \Rightarrow 2\sqrt{x}\ du = dx$. By substitution, we have

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \int \frac{2\sqrt{x}du}{\sqrt{x}\ u} = 2\int \frac{1}{u}\ du = 2\ln|\ u\ | = 2\ln|\ 1+\sqrt{x}\ | + c$$

$$\Rightarrow \int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln|\ 1+\sqrt{x}\ |\Big]_1^4 = 2(\ln 3 - \ln 2).$$

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$$\Rightarrow \int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln \mid 1+\sqrt{x} \mid \Big]_1^4 = 2(\ln 3 - \ln 2).$$

2. We know that $\tan\ x = \dfrac{\sin\ x}{\cos\ x}$. Therefore,

$$\int \tan\ x\ dx = \int \frac{\sin\ x}{\cos\ x}\ dx = -\int \frac{-\sin\ x}{\cos\ x}\ dx$$
$$= -\ln \mid \cos\ x \mid +c$$

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2. We know that $\tan\ x = \dfrac{\sin\ x}{\cos\ x}$. Therefore,

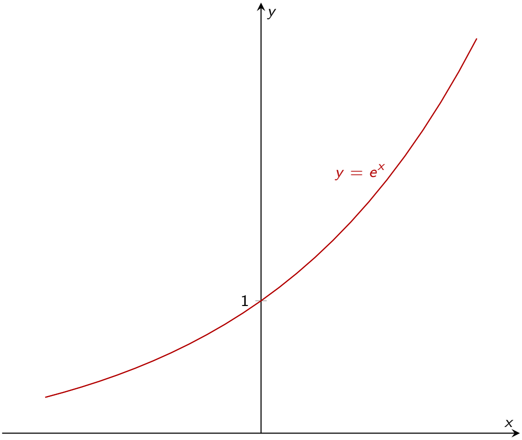
$$\int \tan\ x\ dx = \int \frac{\sin\ x}{\cos\ x}\ dx = -\int \frac{-\sin\ x}{\cos\ x}\ dx$$
$$= -\ln \mid \cos\ x \mid +c$$

3. $\int \sec\ x\ dx = \int \frac{\sec\ x\ (\sec\ x + \tan\ x)}{(\sec\ x + \tan\ x)}\ dx = \int \frac{\sec^2\ x + \sec\ x \tan x}{\sec\ x + \tan\ x}\ dx = \ln \mid \sec\ x + \tan\ x \mid +c.$

Definition

The natural exponential function, denoted by exp, is defined as follows: $exp : \mathbb{R} \longrightarrow (0, \infty)$,

$$y = exp(x) = e^x \Leftrightarrow \ln y = x$$



(1) The domain of $exp(x)$ is $\mathbb{R}$.

(2) The range of $exp(x)$ is $(0, \infty)$ as follows:

$$y = \begin{cases} exp\ x > 1 & : x > 0 \\ exp\ x = 1 & : x = 0 \\ exp\ x < 1 & : x < 0 \end{cases}$$

(3) $exp\ (1) = e \approx 2.71828$.
Also, $\ln e = 1$ and $\ln e^r = r \ln e = r$.

(4) The function $exp(x)$ is increasing on the domain:

$$y = e^x \Rightarrow \ln y = x.$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{y'}{y} = 1 \Rightarrow y' = y.$$

$$\Rightarrow \frac{d}{dx} e^x = e^x \ \forall x \in \mathbb{R}.$$

(5) The second derivative $\frac{d^2}{dx^2} e^x = e^x > 0$ for all $x \in \mathbb{R}$. Hence, the function $exp(x)$ is concave upward on the domain.

(6) $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$.

(7) Since e^x and $\ln x$ are inverse functions, then

- $\ln e^x = x, \ \forall x \in \mathbb{R}$
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Theorem

If $a, b > 0$ *and* $r \in \mathbb{Q}$, *then*

1. $e^a e^b = e^{a+b}$
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Example

Solve for x.

1. $\ln x = 2$
2. $\ln(\ln x) = 0$

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   $\Rightarrow \ln x = 1$

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Differentiation of Natural Exponential Function

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If $u = g(x)$ is differentiable, then

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Find the derivative of the function.

1. $y = e^{-5x^2}$
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Recall that $\frac{d}{dx}e^u = e^u u'$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

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$$= \int e^{\tan\ x}\ \sec^2\ x\ dx \qquad \text{Use the identity } \sec\ x = \frac{1}{\cos\ x} \Rightarrow \sec^2\ x = \frac{1}{\cos^2\ x}$$
$$= e^{\tan\ x} + c$$

(1) General Exponential Function

Definition

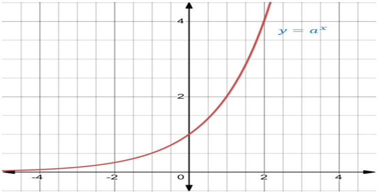
The general exponential function is defined as follows: $a^x : \mathbb{R} \to (0, \infty)$,

$$\ln a^x = x \ln a \Rightarrow e^{\ln a^x} = e^{x \ln a} \Rightarrow a^x = e^{x \ln a} \quad \textit{for every} \quad a > 0.$$

(1) The function a^x is called the general exponential function with base a.

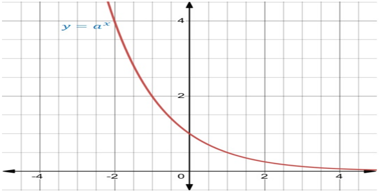
(2) If $a > 1$, $\ln a > 0$ and this implies that $x \ln a$ and $f(x)$ are increasing functions:

$$y = \begin{cases} a^x > 1 & : x > 0 \\ a^x = 1 & : x = 0 \\ a^x < 1 & : x < 0 \end{cases}$$



(3) If $a < 1$, $\ln a < 0$ and this implies that $x \ln a$ and $f(x)$ are decreasing functions.

$$y = \begin{cases} a^x < 1 & : x > 0 \\ a^x = 1 & : x = 0 \\ a^x > 1 & : x < 0 \end{cases}$$



Rules of the general exponential function:

Theorem

If $a, b > 0$ *and* $x, y \in \mathbb{R}$, *then*

1. $a^x a^y = a^{x+y}$
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Section 3: General Exponential and Logarithmic Functions

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1. $y = 2^{\sqrt{x}}$
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2. $y' = 3^{x^2 \sin x} (2x \sin x + x^2 \cos x) \ln 3$.

Example

Find the derivative of the function $y = (\sin\ x)^x$.

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Solution:
By taking the natural logarithm of both sides, we have $\ln y = x\ \ln|\sin\ x|$. By differentiating both sides, we obtain

$$\begin{aligned} \frac{y'}{y} &= (1)\ln|\sin\ x| + x\,\frac{\cos\ x}{\sin\ x} \\ \Rightarrow y' &= \big(\ln|\sin\ x| + x\cot\ x\big)y \\ \Rightarrow y' &= \big(\ln|\sin\ x| + x\cot\ x\big)(\sin\ x)^x. \end{aligned}$$

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Section 3: General Exponential and Logarithmic Functions

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   OR Let $u = 5^x + 1 \Rightarrow du = 5^x \ln 5 \, dx \Rightarrow \frac{du}{5^x \ln 5} = dx$. By substitution, we obtain

$$\int 5^x \sqrt{u} \, \frac{du}{5^x \ln 5}$$

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Solution:

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2. $\int 5^x \; \sqrt{5^x + 1} \; dx = \int 5^x \; (5^x + 1)^{\frac{1}{2}} \; dx = \frac{1}{\ln 5} \int 5^x \; (1) \; \ln 5 \; (5^x + 1)^{\frac{1}{2}} \; dx = \frac{1}{\ln 5} \; \frac{2}{3} (5^x + 1)^{\frac{3}{2}} + c$

OR Let $u = 5^x + 1 \Rightarrow du = 5^x \ln 5 \; dx \Rightarrow \frac{du}{5^x \ln 5} = dx$. By substitution, we obtain

$$\int 5^x \; \sqrt{u} \; \frac{du}{5^x \ln 5} = \frac{1}{\ln 5} \int u^{\frac{1}{2}} \; du$$

- If $u = g(x)$ is differentiable, then

$$\frac{d}{dx} a^u = a^u \; u' \; \ln a.$$

$$(1) \int a^u \; u' \; \ln a \; dx = a^u + c$$
$$(2) \int a^u \; u' \; dx = \frac{1}{\ln a} a^u + c$$

Example

Evaluate the integral.

1. $\int x \; 3^{-x^2} \; dx$
2. $\int 5^x \; \sqrt{5^x + 1} \; dx$

Solution:

1. $\int x \; 3^{-x^2} \; dx = \frac{1}{-2 \; \ln 3} \int -2x \; 3^{-x^2} \; \ln 3 \; dx = \frac{-1}{2 \; \ln 3} \; 3^{-x^2} + c$
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Example

Evaluate the integral.

1. $\int x \, 3^{-x^2} \, dx$
2. $\int 5^x \sqrt{5^x + 1} \, dx$

Solution:

1. $\int x \, 3^{-x^2} \, dx = \frac{1}{-2 \; \ln 3} \int -2x \, 3^{-x^2} \ln 3 \; dx = \frac{-1}{2 \; \ln 3} 3^{-x^2} + c$

2. $\int 5^x \sqrt{5^x + 1} \, dx = \int 5^x \, (5^x + 1)^{\frac{1}{2}} \, dx = \frac{1}{\ln 5} \int 5^x \; (1) \; \ln 5 \; (5^x + 1)^{\frac{1}{2}} \, dx = \frac{1}{\ln 5} \; \frac{2}{3} (5^x + 1)^{\frac{3}{2}} + c$

   OR Let $u = 5^x + 1 \Rightarrow du = 5^x \ln 5 \; dx \Rightarrow \frac{du}{5^x \ln 5} = dx$. By substitution, we obtain

$$\int 5^x \sqrt{u} \, \frac{du}{5^x \ln 5} = \frac{1}{\ln 5} \int u^{\frac{1}{2}} \, du = \frac{1}{\ln 5} \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2(5^x + 1)^{\frac{3}{2}}}{3 \ln 5} + c.$$

(2) General Logarithmic Function

Definition

The general logarithmic function is defined as follows:

$$\log_a : (0, \infty) \to \mathbb{R} \ ,$$

$$x = a^y \Leftrightarrow y = \log_a x.$$

$$y = \begin{cases} logx > 0 & : x > 1 \\ logx = 0 & : x = 1 \\ logx < 0 & : 0 < x < 1 \end{cases}$$

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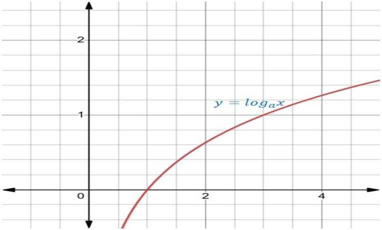
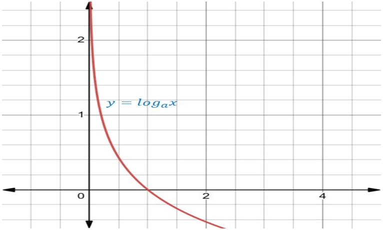
$a > 1$



$a < 1$



Section 3: General Exponential and Logarithmic Functions

Properties of General Logarithmic Function

1. The general logarithmic function $\log_a x = \dfrac{\ln x}{\ln a}$.
   Proof. From the definition, we have $y = \log_a x \Rightarrow x = a^y$.
   By taking the natural logarithm of both sides, we have

$$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a}$$

2. If $a > 1$, the function $\log_a x$ is increasing while if $0 < a < 1$, the function $\log_a x$ is decreasing.
3. The natural logarithmic function $\ln x = \log_e x$.
4. The general logarithmic function $\log_{10} x = \log x$.
5. The general logarithm $\log_a a = 1$.
6. Rules of the general logarithmic function:

Theorem

If $x, y > 0$ and $r \in \mathbb{R}$, then

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^r = r \log_a x$

Differentiation of General Logarithmic Function

Since $\log_a x = \frac{\ln x}{\ln a}$, then

$$\frac{d}{dx}\left(\log_a x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right) = \frac{1}{\ln a}\frac{1}{x} = \frac{1}{x \ln a}.$$

By integrating both sides, we have

$$\int \frac{1}{x \ln a}\, dx = \frac{1}{\ln a}\int \frac{1}{x}\, dx = \frac{\ln |x|}{\ln a} = \log_a |x| + c.$$

Theorem

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{\ln a}\, \frac{u'}{u} = \frac{1}{u\, \ln a}\, u'$$

From the previous theorem, we have

$$\int \frac{1}{u \ln a} u'\, dx = \frac{1}{\ln a}\int \frac{u'}{u}\, dx = \frac{\ln |u|}{\ln a} = \log_a |u| + c$$

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

Example

Find the derivative of the function.

1. $y = \log_3 \sin\ x$
2. $y = \log \sqrt{x}$

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

Example

Find the derivative of the function.

1. $y = \log_3 \sin\ x$
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Solution:

1. $y' = \dfrac{1}{\ln 3} \dfrac{\cos\ x}{\sin\ x} = \dfrac{\cot\ x}{\ln 3}.$

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

Example

Find the derivative of the function.

1. $y = \log_3 \sin\ x$
2. $y = \log \sqrt{x}$

Solution:

1. $y' = \dfrac{1}{\ln 3}\dfrac{\cos\ x}{\sin\ x} = \dfrac{\cot\ x}{\ln 3}.$
2. $y = \log(x)^{\frac{1}{2}} = \frac{1}{2}\log x \Rightarrow y' = \dfrac{1}{(2\ln 10)\ x}.$

Exercise: Choose the correct answer. If $\log_2 \frac{x}{x-1} = 1$, then x is equal to ...

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) -1

Exercise: **Choose the correct answer.** If $\log_2 \frac{x}{x-1} = 1$, then x is equal to ...

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) -1

Solution:

$$\log_2 \frac{x}{x-1} = 1 \Rightarrow 2^1 = \frac{x}{x-1} \qquad \textit{Remember} : x = a^y \Leftrightarrow log_a x = y$$

$$2x - 2 = x \Rightarrow 2x - x = 2 \Rightarrow x = 2$$