Integral Calculus

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April 26, 2024

Main Contents.

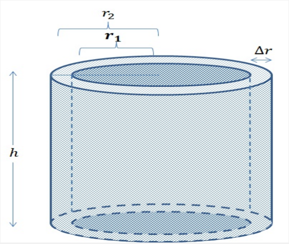
Volumes of Revolution Solids (Cylindrical Shells Method)

We study a new method to evaluate the volume of revolution solid called **cylindrical shells method** .

- In the washer method, we assume that the rectangle from each subinterval is **perpendicular** to the revolution axis.
- In the cylindrical shells method, the rectangle will be **parallel** to the revolution axis.

The figure shows a cylindrical shell. Let

- r_1 be the inner radius of the shell,
- r_2 be the outer radius of the shell,
- h be high of the shell,
- $\Delta r = r_2 - r_1$ be the thickness of the shell,
- $r = \frac{r_1+r_2}{2}$ be the average radius of the shell.



The volume of the cylindrical shell:
$$\begin{aligned} V = V_2 - V_1 &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi\left(\frac{r_2 + r_1}{2}\right)h(r_2 - r_1) \\ &= 2\pi rh\Delta r. \end{aligned}$$

Volumes of Revolution Solids (Cylindrical Shells Method)

- Let R is a region bounded by the graph of $y = f(x)$ and x-axis on the interval $[a, b]$.
- Let S be a solid generated by revolving the region about y-axis.
- Let P be a partition of the interval $[a, b]$ and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a mark on P where ω_k is the midpoint of $[x_{k-1}, x_k]$.
- The revolution of the rectangle about the y-axis generates a cylindrical shell where
- the high $= f(\omega_k)$,
- the average radius $= \omega_k$,
- the thickness $= \Delta x_k$.
- The volume of the cylindrical shell is

$$V_k = 2\pi \omega_k f(\omega_k) \Delta x_k$$

To evaluate the volume of the whole solid, we sum the volumes of all cylindrical shells. This implies

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} \omega_k f(\omega_k) \Delta x_k .$$

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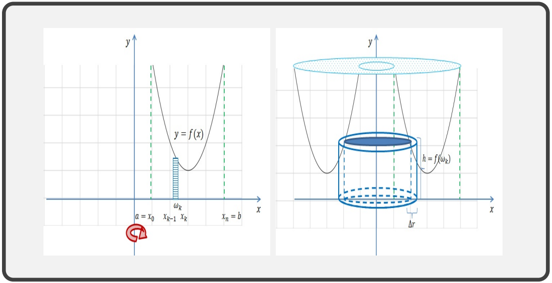

From the Riemann sum

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} \omega_k f(\omega_k) \Delta x_k = \int_a^b x \, f(x) \, dx \, .$$

This implies

$$V = 2\pi \int_a^b x \, f(x) \, dx .$$

Similarly, we can find that if the revolution of the region is about x-axis.

- Let R is a region bounded by the graph of $x = f(y)$ and y-axis on the interval $[c, d]$.
- Let S be a solid generated by revolving the region about x-axis.
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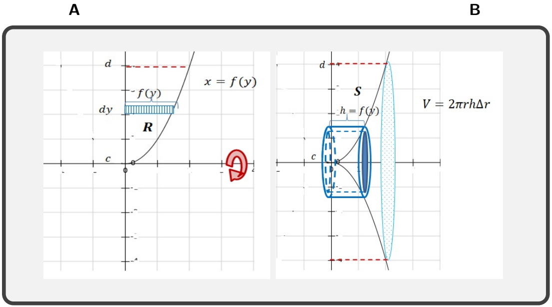


From the Riemann sum $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta y_k = \int_c^d y\ f(y)\ dy$.

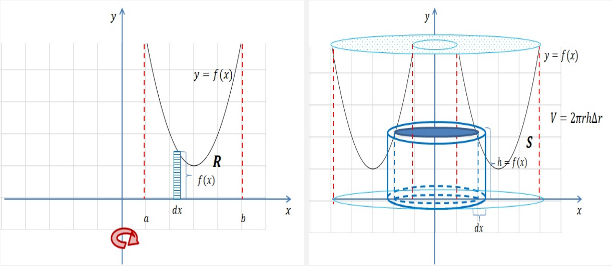
This implies

$$V = 2\pi \int_c^d y\ f(y)\ dy.$$

Volumes of Revolution Solids (Cylindrical Shells Method)

Theorem.

(1) If R is a region bounded by the graph of $y = f(x)$ and x-axis on the interval $[a, b]$, the volume of the revolution solid generated by revolving R about y-axis is



1. The two points (area boundaries) on the $x-$axis.

2. Rotation about the $y-$axis.

3. The rectangle is parallel to the axis of rotation ($y-$axis).

$$V = 2\pi \int_a^b x\, f(x)\, dx.$$

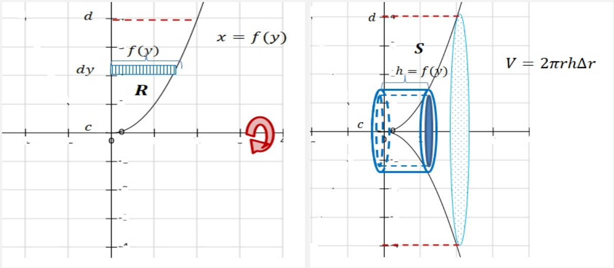
١. حدود المنطقة على محور س

٢. الدوران حول محور ص

٣. المستطيل يوازي محور الدوران (محور ص)

Volumes of Revolution Solids (Cylindrical Shells Method)

(2) If R is a region bounded by the graph of $x = f(y)$ and y-axis on the interval $[c, d]$, the volume of the revolution solid generated by revolving R about x-axis is



1. The two points (area boundaries) on the $y-$axis.

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Note. The importance of the cylindrical shells method appears when solving equations for one variable in terms of another (i.e., solving x in terms of y). For example, let S be a solid generated by revolving the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about y-axis. By the washer method, we have to solve the cubic equation for x in terms of y, but this is not simple.

Volumes of Revolution Solids (Cylindrical Shells Method)

Example

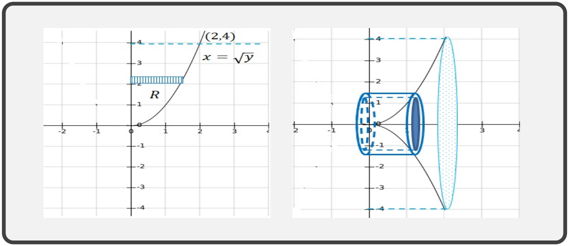
Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $y = 4$, and y-axis. Then, find the volume of the solid generated by revolving R about x-axis.

Volumes of Revolution Solids (Cylindrical Shells Method)

Example

Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $y = 4$, and y-axis. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: First, we find the intersection point: $x = \sqrt{y} \Rightarrow y = x^2$ and $y = 4$, so $x^2 = 4 \Rightarrow x = \pm 2$
We ignore $x = -2$ since $x = \sqrt{y}$. Now the graphs of the two functions intersect in one point $(2, 4)$.



Since the revolution is about the x-axis, the rectangle is horizontal and by revolving it, we have a cylindrical shell where

- the high: $x = \sqrt{y}$,
- the average radius: y,
- the thickness: dy.

The volume of the cylindrical shell is $dV = 2\pi\, y\, \sqrt{y}\, dy$.

Thus, the volume of the solid over the interval $[0, 4]$ is

$$V = 2\pi \int_0^4 y\sqrt{y}\, dy = 2\pi \int_0^4 y^{\frac{3}{2}}\, dy = \frac{4\pi}{5}\left[\, y^{\frac{5}{2}}\,\right]_0^4 = \frac{4\pi}{5}\Big[32 - 0\Big] = \frac{128\pi}{5}.$$

Example

Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $y = 4$, and y-axis. Then, find the volume of the solid generated by revolving R about x-axis.

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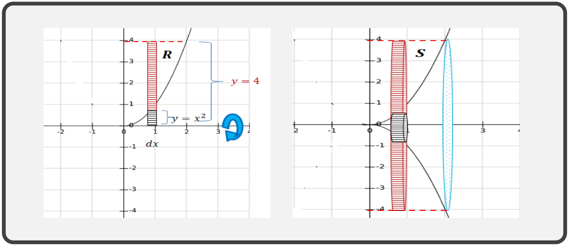
Solution: We want to find the volume using washer method.



The figure shows the region R and the solid S generated by revolving R about the x-axis. The vertical rectangles generate a washer:

- the outer radius: $y_1 = 4$,
- the inner radius: $y_2 = x^2$ and
- the thickness: dx.

The volume of the washer is $dV = \pi\left[(4)^2 - (x^2)^2\right]\, dx$.

Hence, the volume of the solid over the interval $[0, 2]$ is

$$V = \pi \int_0^2 \left((4)^2 - (x^2)^2\right)\, dx = \pi \int_0^2 (16 - x^4)\, dx = \pi \left[16x - \frac{x^5}{5}\right]_0^2 = \frac{128}{5}\pi.$$

Example

Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$ from $x = 0$ to $x = 2$. Evaluate the volume of the solid generated by revolving R about y-axis.

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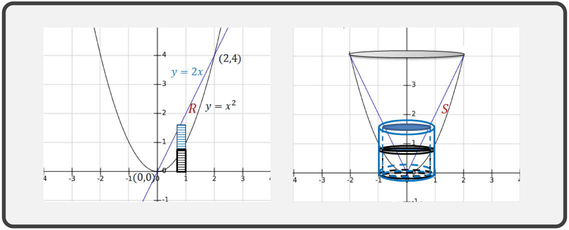
Solution: Intersection points: $f(x) = g(x) \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$
By substitution, we have that the two curves intersect in two points $(0, 0)$ and $(2, 4)$.



Since the revolution is about the y-axis, the rectangles are horizontal and by revolving them, we have a cylindrical shells where

- the high: $y = 2x - x^2$,
- the average radius: x,
- the thickness: dx.

The volume of the cylindrical shell is $dV = 2\pi\ x\ (2x - x^2)\ dx$.

Thus, the volume of the solid over the interval $[0, 2]$ is

$$V = 2\pi \int_0^2 x(2x - x^2)\ dx = 2\pi \int_0^2 (2x^2 - x^3)\ dy = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = \frac{8\pi}{3}.$$

Example

Sketch the region R bounded by the graph of $y = 2x - x^2$ and x-axis. Then, by the cylindrical shells method, find the volume of the solid generated by revolving R about y-axis.

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Solution:

- Intersection with y-axis:

$$\Rightarrow x = 0 \Rightarrow y = 2(0) - 0^2 = 0 \Rightarrow (0, 0)$$

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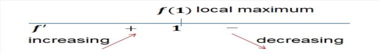
- First derivative test.

$$y' = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1$$



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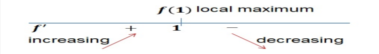

- Second derivative test.

$$y'' = 0 \Rightarrow y'' = -2 < 0$$

The graph of $f(x)$ is concave downward.

Example

Sketch the region R bounded by the graph of $y = 2x - x^2$ and x-axis. Then, by the cylindrical shells method, find the volume of the solid generated by revolving R about y-axis.

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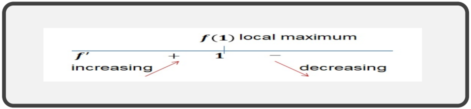

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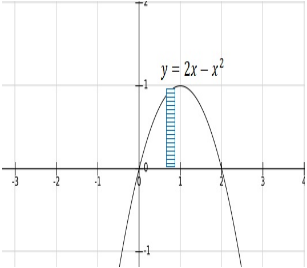
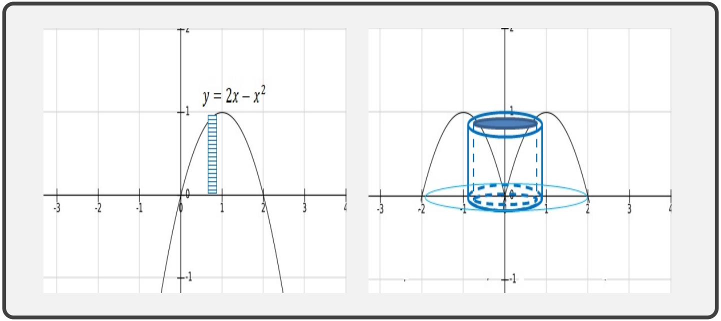


Volumes of Revolution Solids (Cylindrical Shells Method)



Since the revolution is about the y-axis, the rectangle is vertical and by revolving it, we obtain a cylindrical shell where

- the high: $y = 2x - x^2$,
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The difference between disk method and method of cylindrical shells

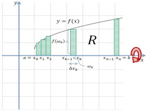
Disk Method

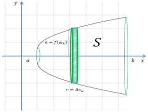
Region R bounded by :
◇ $y = f(x)$ ◇ x-axis ◇ $[a, b]$ on x-axis

Revolution about x-axis

Rectangle vertical on the x-axis

$V = \pi \int_a^b (f(x))^2 \, dx$





Region R bounded by :
◇ $x = f(y)$ ◇ y-axis ◇ $[c, d]$ on y-axis

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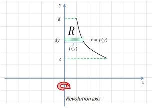
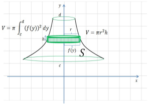




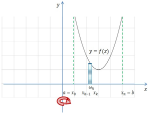
Cylindrical Shells Method

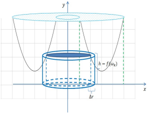
Region R bounded by :
◇ $y = f(x)$ ◇ x-axis ◇ $[a, b]$ on x-axis

Revolution about y-axis

Rectangle parallel to the y-axis

$V = 2\pi \int_a^b x \, f(x) \, dx$





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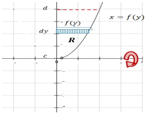
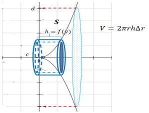




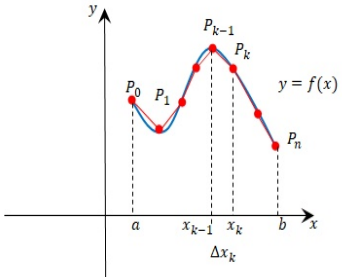
Arc Length.

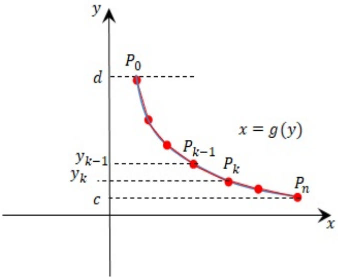
Let $y = f(x)$ be a smooth function on $[a, b]$. The length of the arc of f from $(a, f(a))$ to $(b, f(b))$ is

$$L(f) = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

Let $x = g(y)$ be a smooth function on $[c, d]$. The length of the arc of g from $(g(c), c)$ to $(g(d), d)$ is

$$L(g) = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy.$$





Example

Find the arc length of the given function $y = 5 - \sqrt{x^3}$ *from* $A(0, 5)$ *to* $B(4, -3)$.

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$$\begin{aligned} y = f(x) = 5 - \sqrt{x^3} &\Rightarrow f'(x) = -\frac{3}{2}x^{\frac{1}{2}} \\ &\Rightarrow (f'(x))^2 = \frac{9}{4}x \\ &\Rightarrow 1 + (f'(x))^2 = \frac{4 + 9x}{4} \\ &\Rightarrow \sqrt{1 + (f'(x))^2} = \frac{\sqrt{4 + 9x}}{2}. \end{aligned}$$

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Find the arc length of the given function $y = 5 - \sqrt{x^3}$ from $A(0,5)$ to $B(4,-3)$.

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The arc length of the function is

$$\begin{aligned} L(f) = \frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx = \frac{1}{2}\int_0^4 (4+9x)^{\frac{1}{2}}\, dx &= \frac{1}{27}\left[(4+9x)^{\frac{3}{2}}\right]_0^4 \\ &= \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] \\ &= \frac{8}{27}\left[10\sqrt{10} - 1\right]. \end{aligned}$$

Note:

$$40^{\frac{3}{2}} = (\sqrt{4 \times 10})^3 = (2\sqrt{10})^3$$

Example

Find the length of the curve $y = \cosh\ x$ *over the interval* $0 \leq x \leq 2$.

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$$\begin{aligned} y = f(x) = \cosh x &\Rightarrow f'(x) = \sinh\ x \\ &\Rightarrow (f'(x))^2 = \sinh^2\ x \\ &\Rightarrow 1 + (f'(x))^2 = 1 + \sinh^2\ x = \cosh^2\ x \\ &\Rightarrow \sqrt{1 + (f'(x))^2} = \cosh\ x. \end{aligned}$$

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The length of the curve is

$$\begin{aligned} L(f) = \int_0^2 \cosh\ x\, dx &= \left[\sinh\ x\right]_0^2 \\ &= \sinh 2 - \sinh 0 \\ &= \sinh 2 \end{aligned}$$

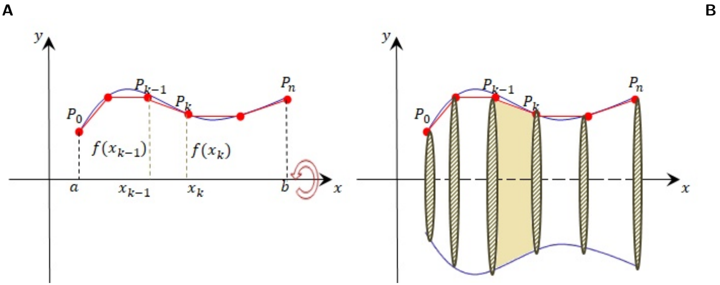
Note:

$$(\sinh\ 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0)$$

Surfaces of Revolution.

Definition

Let f be a continuous function on $[a, b]$. The surface of revolution is generated by revolving the graph of the function f about an axis.



Theorem

1. *Let $y = f(x)$ be a smooth function on $[a, b]$.*
   - *If the revolution is about x-axis, the area of the revolution surface is*

$$S.A = 2\pi \int_a^b \mid y \mid \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

   - *If the revolution is about y-axis, the area of the revolution surface is*

$$S.A = 2\pi \int_a^b \mid x \mid \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

2. *Let $x = g(y)$ be a smooth function on $[c, d]$.*
   - *If the revolution is about y-axis, the area of the revolution surface is*

$$S.A = 2\pi \int_c^d \mid x \mid \sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

   - *If the revolution is about x-axis, the area of the revolution surface is*

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Example

Find the surface area generated by revolving the graph of the function $\sqrt{4 - x^2}$, $-2 \leq x \leq 2$ *about x-axis.*

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Solution:

We apply the formula $S.A = 2\pi \int_a^b \mid y \mid \sqrt{1+(f'(x))^2}\, dx$.

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$$y = \sqrt{4-x^2} \Rightarrow f'(x) = \frac{-2x}{2\sqrt{4-x^2}}$$

$$\Rightarrow (f'(x))^2 = \frac{x^2}{4-x^2}$$

$$\Rightarrow 1 + (f'(x))^2 = \frac{4}{4-x^2}$$

$$\Rightarrow \sqrt{1+(f'(x))^2} = \frac{2}{\sqrt{4-x^2}}.$$

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$$\begin{aligned} y = \sqrt{4-x^2} \Rightarrow f'(x) &= \frac{-2x}{2\sqrt{4-x^2}} \\ \Rightarrow (f'(x))^2 &= \frac{x^2}{4-x^2} \\ \Rightarrow 1+(f'(x))^2 &= \frac{4}{4-x^2} \\ \Rightarrow \sqrt{1+(f'(x))^2} &= \frac{2}{\sqrt{4-x^2}}. \end{aligned}$$

The area of the revolution surface is

$$\begin{aligned} S.A &= 2\pi \int_{-2}^{2} \sqrt{4-x^2}\, \frac{2}{\sqrt{4-x^2}}\, dx \\ &= 2\pi \int_{-2}^{2} 2\, dx \\ &= 4\pi \left[2+2\right] = 16\pi \end{aligned}$$

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Solution:

We apply the formula $S.A = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2}\, dx$.

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Find the surface area generated by revolving the graph of the function $y = 2x$, $0 \leq x \leq 3$ about y-axis.

Solution:

We apply the formula $S.A = 2\pi \int_a^b \mid x \mid \sqrt{1 + (f'(x))^2}\, dx$.

$$
\begin{aligned}
y = 2x &\Rightarrow f'(x) = 2 \\
&\Rightarrow (f'(x))^2 = 4 \\
&\Rightarrow 1 + (f'(x))^2 = 5 \\
&\Rightarrow \sqrt{1 + (f'(x))^2} = \sqrt{5}.
\end{aligned}
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Example

Find the surface area generated by revolving the graph of the function $y = 2x$, $0 \leq x \leq 3$ about y-axis.

Solution:

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The area of the revolution surface is

$$\begin{aligned} S.A &= 2\pi \int_0^3 \mid x \mid \ \sqrt{5}\ dx \\ &= 2\pi\ \sqrt{5}\ \int_0^3 x\ dx \\ &= \sqrt{5}\pi \Big[\ x^2\ \Big]_0^3 = 9\sqrt{5}\pi \end{aligned}$$