Integral Calculus

Prof. Mohamad Alghamdi

Department of Mathematics

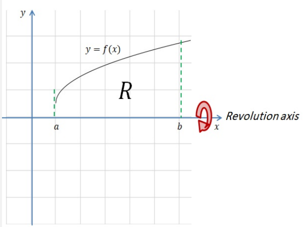
April 25, 2024

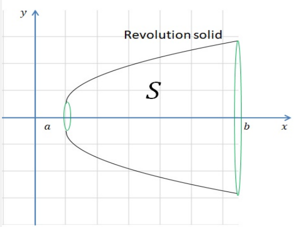
Main Contents.

Definition

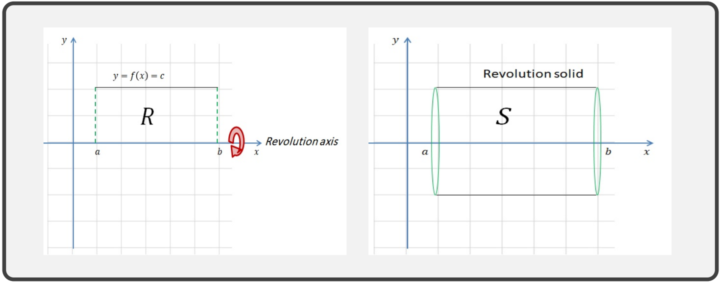
If R is a plane region, the solid of revolution S is a solid generated from revolving R about a line in the same plane where the line is called the axis of revolution.

Example: Let $y = f(x) \geq 0$ be a continuous function for every $x \in [a, b]$. Let R be a region bounded by the graph of f and the x-axis from $x = a$ to $x = b$. The region revolution about x-axis generates a solid given in Figure 3 (right).

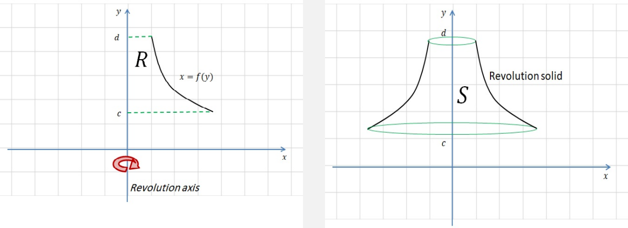




Example: Let $y = f(x)$ be a constant function from $x = a$ to $x = b$, as in Figure 4. The region R is a rectangle and by revolving it about x-axis, we obtain a circular cylinder.



Example: Consider a region R bounded by the graph of $x = f(y)$ from $y = c$ to $y = d$. Revolution of R about y-axis generates a solid given in the figure.



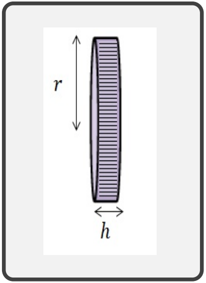
(1) Disk Method

- Let f be continuous on $[a, b]$ and let R be a region bounded by the graph of f and x-axis form $x = a$ to $x = b$.
- Let S be a solid generated by revolving R about x-axis.
- Assume that P is a partition of $[a, b]$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a mark where $\omega_k \in [x_{k-1}, x_k]$.
- From each subinterval $[x_{k-1}, x_k]$, we form a vertical rectangle, its high and width are $f(\omega_k)$ and Δx_k, respectively.

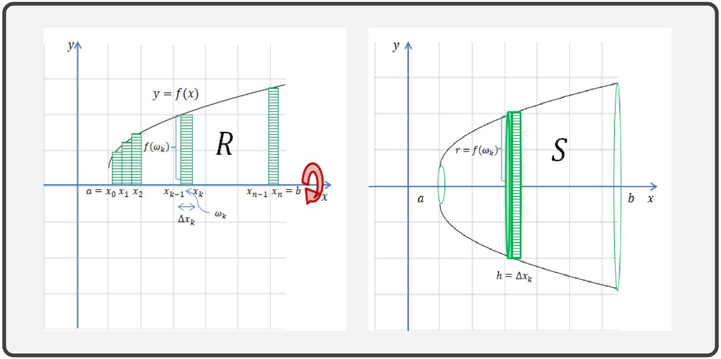
The revolution of the vertical rectangle about x-axis generates a circular disk as shown in the figure.

The volume of the circular disk with radius r and high h is

$$V = \pi r^2 h \ .$$



Volumes of Revolution Solids (Disk Method)



From the figure, the volume of each circular disk is

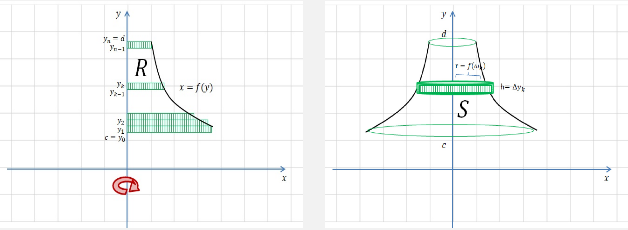
$$V_k = \pi(f(\omega_k))^2 \Delta x_k, \quad k = 1, 2, \ldots, n \; .$$

The sum of the volumes of the circular disks approximates the volume of the revolution solid:

$$V = \sum_{k=1}^{n} V_k = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi \; (f(\omega_k))^2 \; \Delta x_k = \pi \; \int_a^b \left[\; f(x) \; \right]^2 \, dx.$$

Volumes of Revolution Solids (Disk Method)

Similarly, we can find the volume of the revolution generated by revolving a region about y-axis. Let f be continuous on $[c, d]$ and let R be a region bounded by the graph of f and y-axis from $y = c$ to $y = d$.



From the figure, the volume of each circular disk is

$$V_k = \pi (f(\omega_k))^2 \Delta y_k, \quad k = 1, 2, \ldots, n\ .$$

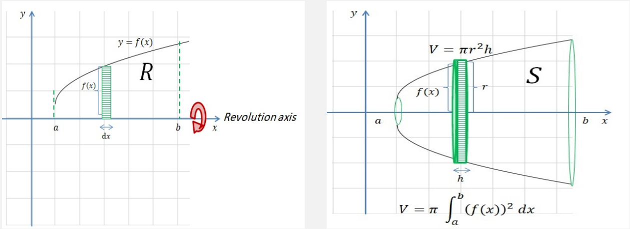
The sum of the volumes of the circular disks approximates the volume of the revolution solid:

$$V = \sum_{k=1}^{n} V_k = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi (f(\omega_k))^2 \Delta y_k$$
$$= \pi \int_c^d \Big[f(y) \Big]^2 \, dy.$$

Volumes of Revolution Solids (Disk Method)

Theorem:

(1) If R is a region bounded by the graph of f on the interval $[a, b]$, the volume of the revolution solid generated by revolving R about x-axis is



1. The two points (area boundaries) on the $x-$axis.

2. Rotation about the $x-$axis.

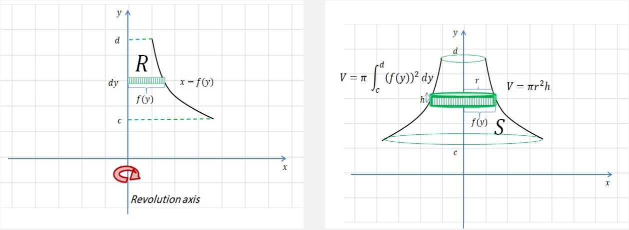
3. The rectangle is perpendicular to the axis of rotation ($x-$axis).

$$V = \pi \int_a^b \left[f(x) \right]^2 dx.$$

١. حدود المنطقة على محور س
٢. الدوران حول محور س
٣. المستطيل عمودي على محور الدوران (محور س)

Volumes of Revolution Solids (Disk Method)

(2) If R is a region bounded by the graph of f on the interval $[c, d]$, the volume of the revolution solid generated by revolving R about y-axis is



1. The two points (area boundaries) on the $y-$axis.
2. Rotation about the $y-$axis.
3. The rectangle is perpendicular to the axis of rotation ($y-$axis).

$$V = \pi \int_c^d \left[f(y) \right]^2 dy.$$

١. حدود المنطقة على محور ص

٢. الدوران حول محور ص

٣. المستطيل عمودي على محور الدوران (محور ص)

Example

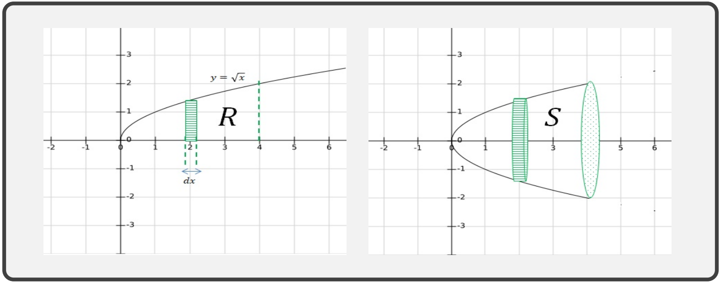
Sketch the region R bounded by the graph of $y = \sqrt{x}$ *from* $x = 0$ *to* $x = 4$. *Then, find the volume of the solid generated by revolving R about* x*-axis.*

Volumes of Revolution Solids (Disk Method)

Example

Sketch the region R bounded by the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 4$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: The figure shows the region R and the solid S generated by revolving the region about x-axis.

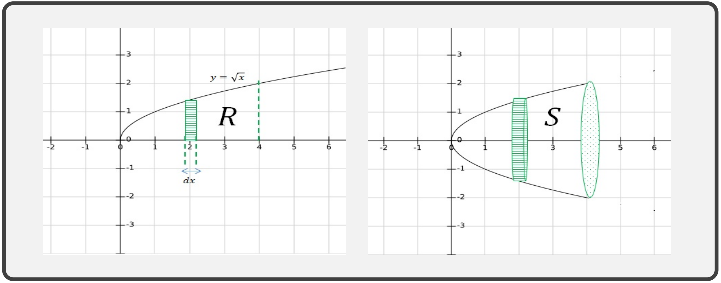


Volumes of Revolution Solids (Disk Method)

Example

Sketch the region R bounded by the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 4$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: The figure shows the region R and the solid S generated by revolving the region about x-axis.



Since the revolution is about x-axis, we have a vertical disk with radius $y = \sqrt{x}$ and thickness dx.
Thus, the volume of the solid S is

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \frac{\pi}{2} \left[x^2 \right]_0^4 = \frac{\pi}{2} \left[16 - 0 \right] = 8\pi.$$

Volumes of Revolution Solids (Disk Method)

Example

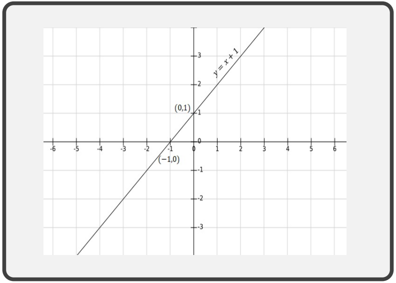
Sketch the region R bounded by the graph of $y = x + 1$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about x-axis.

Example

Sketch the region R bounded by the graph of $y = x + 1$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: First, we sketch the graph of the function $y = x + 1$:

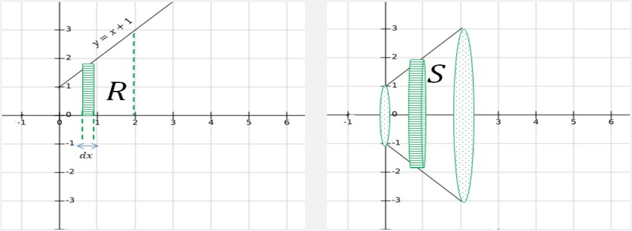
x	0	-1
y	1	0



Example

Sketch the region R bounded by the graph of $y = x + 1$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: First, we sketch the graph of the function $y = x + 1$ and determine the region R in the interval $[0, 2]$. Then, we sketch the solid generated by revolving R about the x-axis.

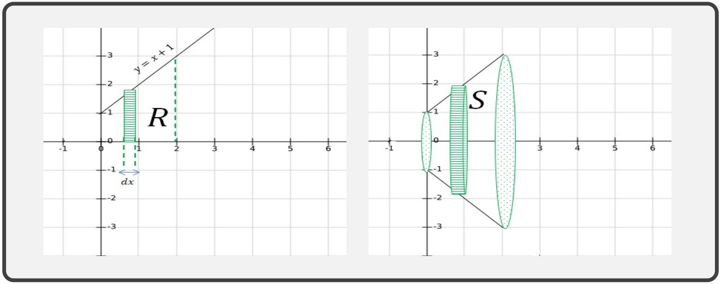


Volumes of Revolution Solids (Disk Method)

Example

Sketch the region R bounded by the graph of $y = x + 1$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: First, we sketch the graph of the function $y = x + 1$ and determine the region R in the interval $[0, 2]$. Then, we sketch the solid generated by revolving R about the x-axis.



From the figure, we have a vertical disk with radius $y = x + 1$ and thickness dx. Thus, the volume of the solid S is as follows:

$$V = \pi \int_0^2 (x+1)^2 \, dx = \frac{\pi}{3}\left[(x+1)^3\right]_0^2 = \frac{\pi}{3}(27-1) = \frac{26\pi}{3}.$$

Example

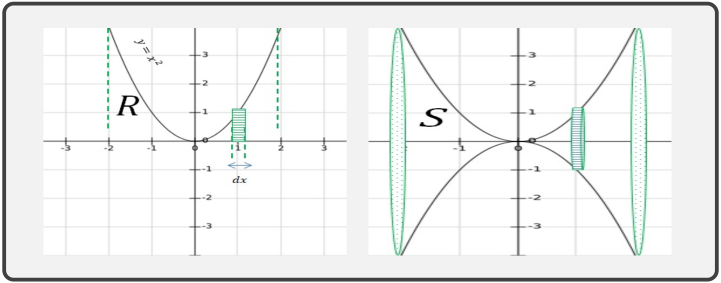
Sketch the region R bounded by the graph of the function $y = x^2$ *and* x*-axis from* $x = -2$ *to* $x = 2$*. Then, find the volume of the solid generated by revolving R about* x*-axis.*

Volumes of Revolution Solids (Disk Method)

Example

Sketch the region R bounded by the graph of the function $y = x^2$ and x-axis from $x = -2$ to $x = 2$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: The figure on the left shows the region R bounded by the graph of $y = x^2$ in the interval $[-2, 2]$. The figure to the right shows the solid S generated by revolving the region about the x-axis.

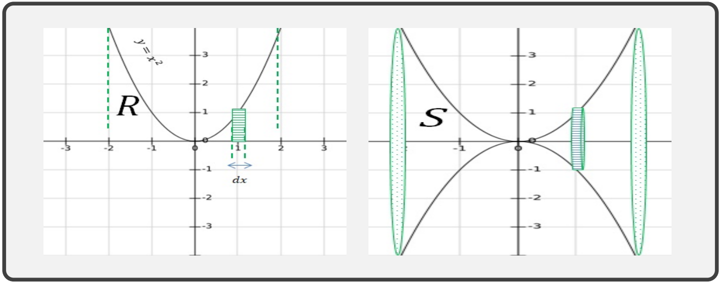


Volumes of Revolution Solids (Disk Method)

Example

Sketch the region R bounded by the graph of the function $y = x^2$ and x-axis from $x = -2$ to $x = 2$. Then, find the volume of the solid generated by revolving R about x-axis.

Solution: The figure on the left shows the region R bounded by the graph of $y = x^2$ in the interval $[-2, 2]$. The figure to the right shows the solid S generated by revolving the region about the x-axis.



From the figure, we have a vertical disk with radius $y = x^2$ and thickness dx. Thus, the volume of the solid S is as follows:

$$V = \pi \int_{-2}^{2} (x^2)^2 \, dx = \pi \int_{-2}^{2} x^4 \, dx = \frac{\pi}{5} \left[x^5 \right]_{-2}^{2} = \frac{64\pi}{5}.$$

Volumes of Revolution Solids (Disk Method)

Example

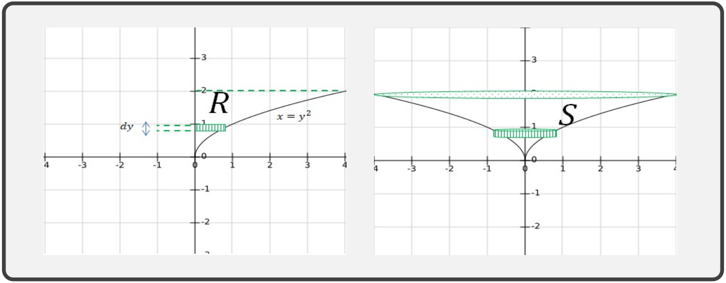
Sketch the region R bounded by the graph of the equation $x = y^2$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about y-axis.

Volumes of Revolution Solids (Disk Method)

Example

Sketch the region R bounded by the graph of the equation $x = y^2$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about y-axis.

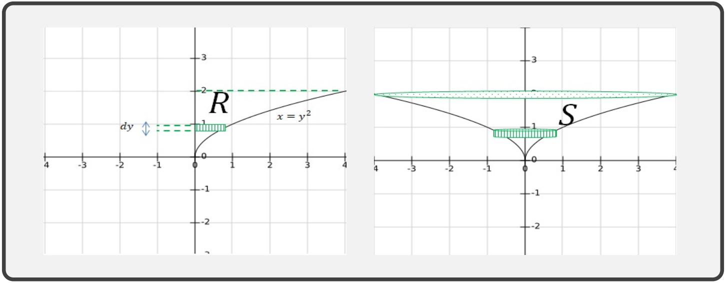
Solution: The figure on the left shows the region R bounded by the graph of $x = y^2$ in the interval $[0, 2]$. The figure to the right shows the solid S generated by revolving the region about the y-axis.



Example

Sketch the region R bounded by the graph of the equation $x = y^2$ on the interval $[0, 2]$. Then, find the volume of the solid generated by revolving R about y-axis.

Solution: The figure on the left shows the region R bounded by the graph of $x = y^2$ in the interval $[0, 2]$. The figure to the right shows the solid S generated by revolving the region about the y-axis.



Since the revolution of R is about the y-axis, we have a horizontal disk with radius $x = y^2$ and thickness dy. Thus, the volume of the solid S is as follows:

$$V = \pi \int_0^2 (y^2)^2 \, dy = \frac{\pi}{5} \left[y^5 \right]_0^2 = \frac{32\pi}{5}.$$