Integral Calculus

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Areas

Example

Sketch the region bounded by the graphs of $y = x^2$ *and* $y = x + 6$ *over the interval* $[-2, 3]$*, then find its area.*

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Solution: The intersection points:

$$x^2 = x + 6 \Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x+2)(x-3) = 0 \Rightarrow x = -2 \text{ and } x = 3$$

$$x = -2 \Rightarrow y = 4 \Rightarrow (-2, 4)$$

$$x = 3 \Rightarrow y = 9 \Rightarrow (3, 9)$$

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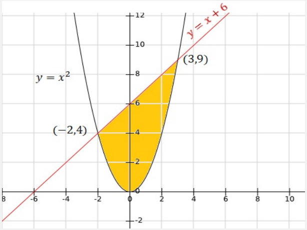

The area of the region is

$$A = \int_{-2}^{3} (x + 6 - x^2)\, dx$$

$$= \left[\ \frac{x^2}{2} + 6x - \frac{x^3}{3}\ \right]_{-2}^{3}$$

$$= \left(\frac{3^2}{2} + 6(3) - \frac{3^3}{3}\right) - \left(\frac{(-2)^2}{2} + 6(-2) - \frac{(-2)^3}{3}\right)$$

$$= \frac{27}{2} + \frac{22}{3} = \frac{125}{6}\ .$$

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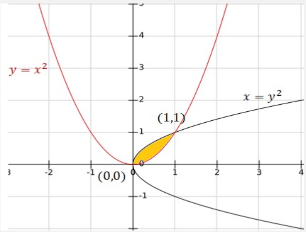
Solution: We write the two functions in terms of x, so the upper graph: $x = y^2 \Rightarrow y = \sqrt{x}$.

The intersection points:
$x^2 = \sqrt{x} \Rightarrow x^4 = x$
$x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$
$x = 0 \Rightarrow (0, 0)$
$x = 1 \Rightarrow (1, 1)$

$$\begin{aligned} A &= \int_0^1 (\sqrt{x} - x^2)\, dx \\ &= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 \\ &= \left[\frac{2}{3} - \frac{1}{3} \right] \\ &= \frac{1}{3} . \end{aligned}$$



Example

Sketch the region bounded by the graphs of $x = 2y$ *and* $x = \frac{y}{2} + 3$ *and* $x-$*axis, then find its area.*

Example

Sketch the region bounded by the graphs of $x = 2y$ and $x = \frac{y}{2} + 3$ and $x-$axis, then find its area.

Solution:
The intersection points:

$$\begin{aligned} \frac{y}{2} + 3 &= 2y \\ \Rightarrow y + 6 &= 4y \\ \Rightarrow y &= 2. \end{aligned}$$

Substitute $y = 2$ in both functions to have $x = 4$.
Thus, the two curves intersect at $(4, 2)$.

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Sketch the region bounded by the graphs of $x = 2y$ and $x = \frac{y}{2} + 3$ and $x-$axis, then find its area.

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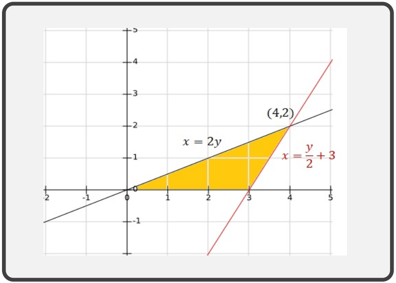
Thus, the two curves intersect at $(4, 2)$.

$$A = \int_0^2 (\frac{y}{2} + 3 - 2y)\ dy$$
$$= \int_0^2 (-\frac{3}{2}y + 3)\ dy$$
$$= \Big[-\frac{3}{4}y^2 + 3y\Big]_0^2$$
$$= -3 + 6 = 3.$$



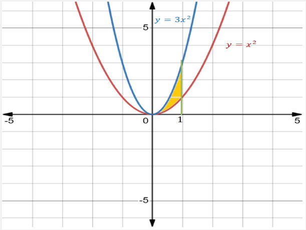
Example

Sketch the region bounded by the graphs of $y = x^2$, $y = 3x^2$, $x = 0$ and $x = 1$, then find its area.

Solution:

The two curves intersect at one point $(0, 0)$.

$$\begin{aligned} A &= \int_0^1 (3x^2 - x^2)\, dx \\ &= \int_0^1 (2x^2)\, dx \\ &= \frac{2}{3}\left[x^3\right]_0^1 \\ &= \frac{2}{3}. \end{aligned}$$



Example

Sketch the region bounded by the graphs of $y = 2x^2 + 1, \ y = 4 - x^2, \ x = 0$ *and* $x = 1$*, then find its area.*

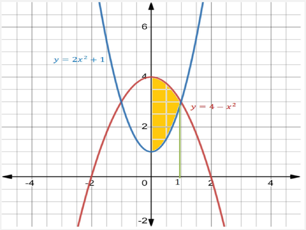
Solution:

The intersection points:

$$2x^2 + 1 = 4 - x^2$$
$$\Rightarrow 3x^2 = 3$$
$$\Rightarrow x = \pm 1.$$

Substitute $x = \pm 1$ in both functions to have $y = 3$. Thus, the two curves intersect at $(-1, 3)$ and $(1, 3)$. However, the region for which the area is to be calculated lies in the interval $[0, 1]$. So the area is:

$$\begin{aligned} A &= \int_0^1 \left((4 - x^2) - (2x^2 + 1)\right) \, dx \\ &= \int_0^1 (3 - 3x^2) \, dx \\ &= \left[3x - x^3\right]_0^1 \\ &= 2. \end{aligned}$$

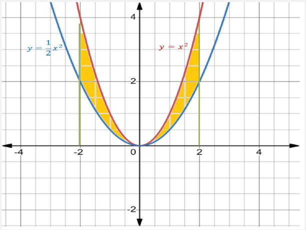


Example

Sketch the region bounded by the graphs of $y = \frac{1}{2}x^2$, $y = x^2$, $x = -2$ *and* $x = 2$*, then find its area.*

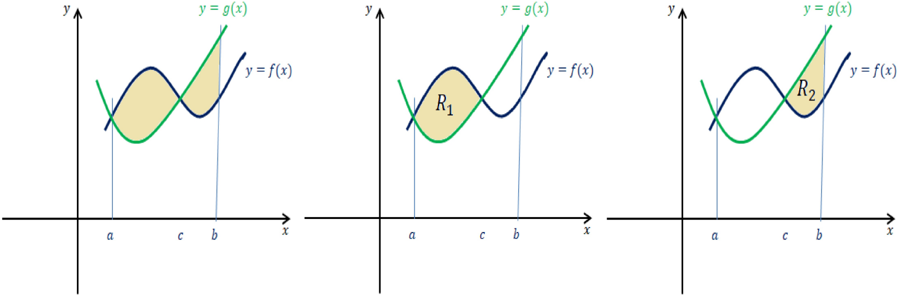
Solution:
The two curves intersect at one point $(0, 0)$.

$$\begin{aligned} A &= \int_{-2}^{2} (x^2 - \frac{1}{2}x^2))\ dx \\ &= \frac{1}{2}\int_{-2}^{2} x^2\ dx \\ &= \frac{1}{6}\left[x^3\right]_{-2}^{2} \\ &= \frac{8}{3}. \end{aligned}$$



Notes.

(1) If the functions f and g are continuous and $f(x) \geq g(x)\ \forall x \in [a, c]$, $g(x) \geq f(x)\ \forall x \in [c, b]$ as shown the figure. Then, we have two regions: R_1 and R_2.



For The region R_1 in $[a, c]$: the upper boundary is the graph of the function f and the lower boundary is the graph of the function g. Then, the area is

$$A_1 = \int_a^c \Big(f(x) - g(x)\Big)\ dx$$

For The region R_2 in $[c, b]$: the upper boundary is the graph of the function g and the lower boundary is the graph of the function f. Then, the area is

$$A_2 = \int_c^b \Big(g(x) - f(x)\Big)\ dx$$

The area of region is the sum of the two areas: $A = A_1 + A_2$.

Areas

(2) When calculating the area between a curve and the x-axis, you should carry out separate calculations for the parts of the curve above the axis, and the parts of the curve below the axis. The integral for a part of the curve below the axis gives minus the area for that part.

Example

Sketch the region bounded by the graph of $y = x(x-1)(x-2)$ and $x-$axis from $x = 0$ to $x = 2$, then find its area.

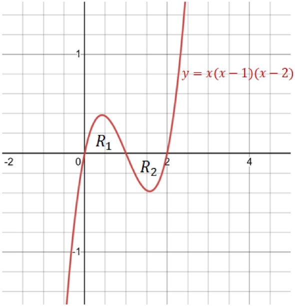
Solution: To simplify the calculation, we write $y = x(x-1)(x-2) = x^3 - 3x^2 + 2x$.

- For The region R_1 in $[0, 1]$: the area is

$$A_1 = \int_0^1 \left(x^3 - 3x^2 + 2x\right)\, dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{1}{4}$$

- For The region R_2 in $[1, 2]$: the area is

$$A_2 = \int_1^2 \left(x^3 - 3x^2 + 2x\right)\, dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = -\frac{1}{4}$$



The actual value of the area of the region R_2 is $+\frac{1}{4}$.
The area of region is the sum of the two areas: $A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Note. If we had decided to work this out by finding the value of the integral between $x = 0$ and $x = 2$, without separating calculations for the parts of the curve above the axis, and the parts of the curve below the axis, we have

$$A = \int_0^2 \left(x^3 - 3x^2 + 2x\right)\, dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^2 = 0$$ But we know that this is not true.