Integral Calculus

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Main Contents.

Review

Graph of Some Functions

(1) Lines

The general linear equation in two variables x and y can be written in the form:

$$ax + by + c = 0 \quad \text{OR} \quad y = mx + b$$

where a, b and c are constants with a and b not both 0.

Example: $2x + y = 4$

$$a = 2, \;\; b = -1, \;\; c = -4$$

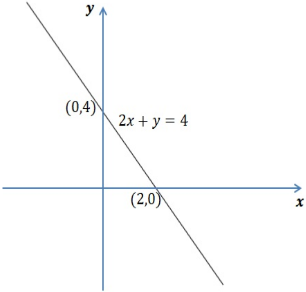
To plot the line, we rewrite the equation to become

$$y = -2x + 4$$

Then, we use the following table to make points on the plane:

x	0	2
y	4	0

The line $2x+y = 4$ passes through the points $(0, 4)$ and $(2, 0)$.



Review

- **Special cases of Lines**

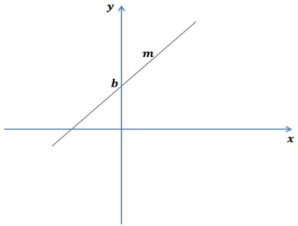
$y = mx + b$

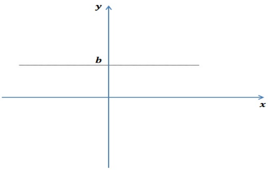
$y = b$

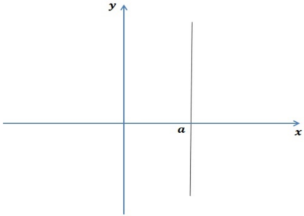
If $m = 0$, the line is horizontal.

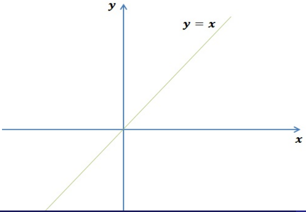
$x = a$

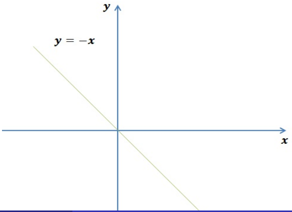
If m is undefined, the line is vertical.











Review

(2) Quadrature Functions $y = ax^2 + bx + c$

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Example: $y = 1 - x^2$

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(1) Intersection with x-axis: $y = 0$

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(2) Intersection with y-axis: $x = 0$

$$y = 1 - (0)^2 \Rightarrow y = 1$$

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The curve pass through the following points

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(3) First derivative test:

$$y' = -2x = 0$$

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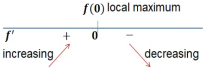

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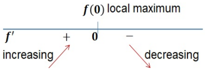

(4) Second derivative test:

$$y'' = -2$$

(2) Quadrature Functions $y = ax^2 + bx + c$

Example: $y = 1 - x^2$

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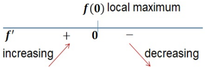
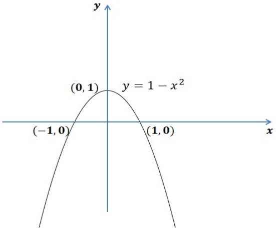
(3) First derivative test:

$$y' = -2x = 0 \Rightarrow x = 0$$



(4) Second derivative test:

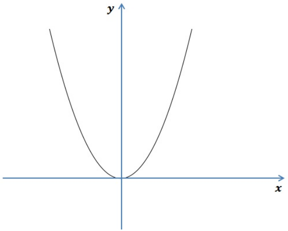
$$y'' = -2 \Rightarrow \quad \text{the curve concave downward}$$



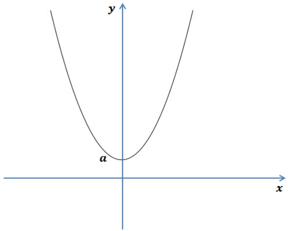
Review

- **Special cases of Quadrature Functions**

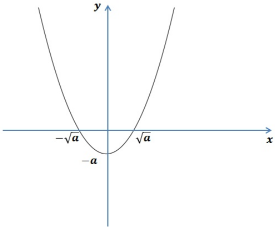
$y = x^2$



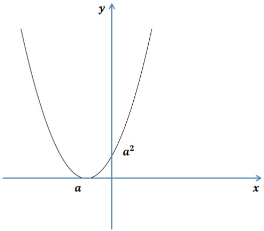
$y = x^2 + a$



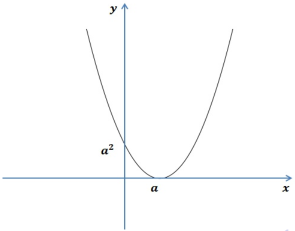
$y = x^2 - a$



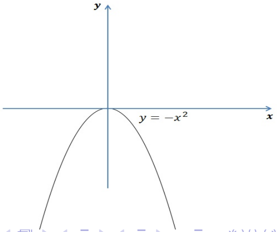
$y = (x + a)^2$



$y = (x - a)^2$

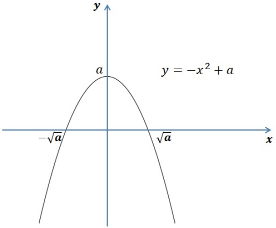


$y = -x^2$

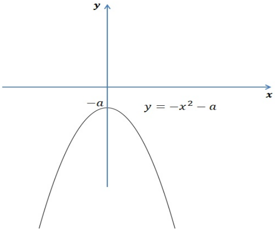


Review

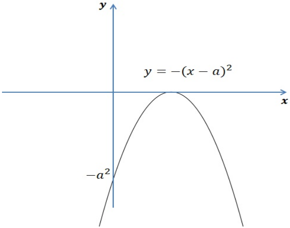
$y = -x^2 + a$



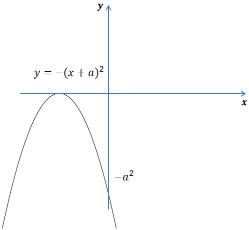
$y = -x^2 - a$



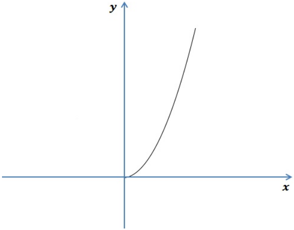
$y = -(x - a)^2$

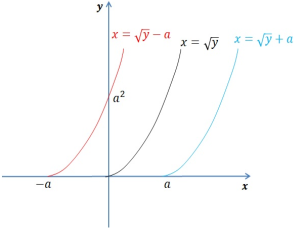


$y = -(x + a)^2$



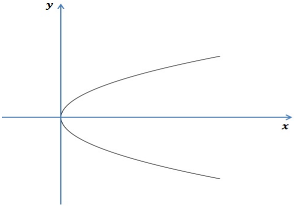
$x = \sqrt{y}$



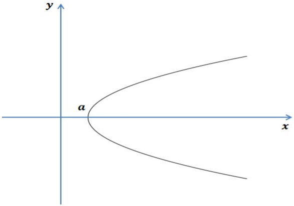


Review

$x = y^2$



$x = y^2 + a$



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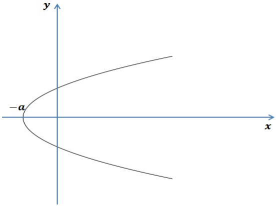
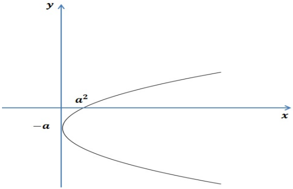
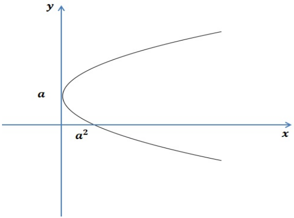


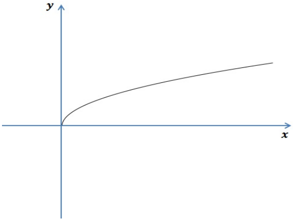
$x = (y + a)^2$



$x = (y - a)^2$

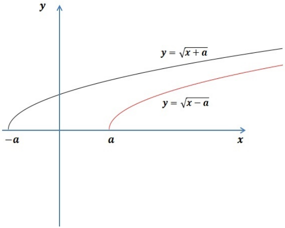


$y = \sqrt{x}$

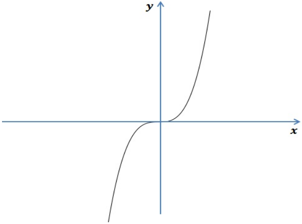


Review

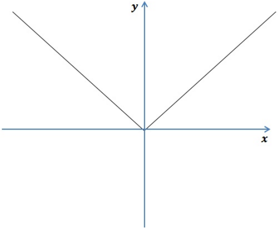
$y = \sqrt{x \pm a}$



$y = x^3$

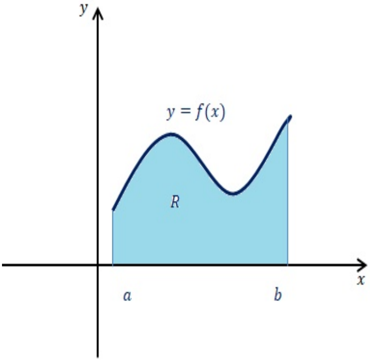


$y = |x|$



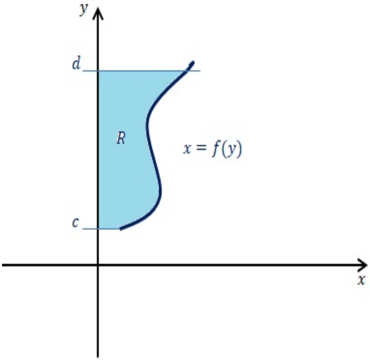
Areas

If $y = f(x)$ is a continuous function on $[a, b]$ and $f(x) \geq 0$ for every $x \in [a, b]$, then the area of the region bounded by the graph of f and x-axis from $x = a$ to $x = b$ is given by the integral:



$$A = \int_a^b f(x)\ dx$$

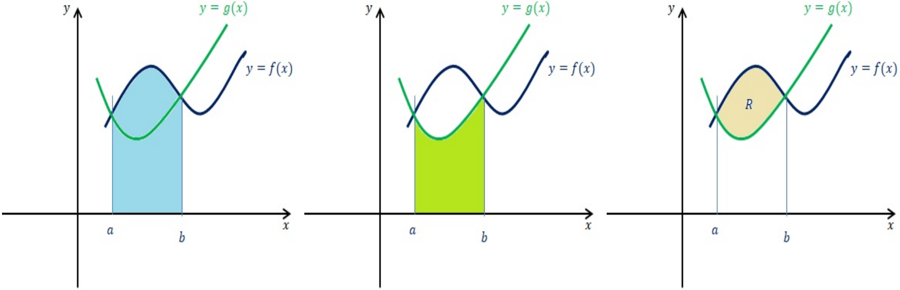
If $x = f(y)$ is a continuous function on $[c, d]$ and $f(y) \geq 0$ $\forall y \in [c, d]$, then the area of the region bounded by the graph of f and y-axis from $y = c$ to $y = d$ is given by the integral:



$$A = \int_c^d f(y)\ dy$$

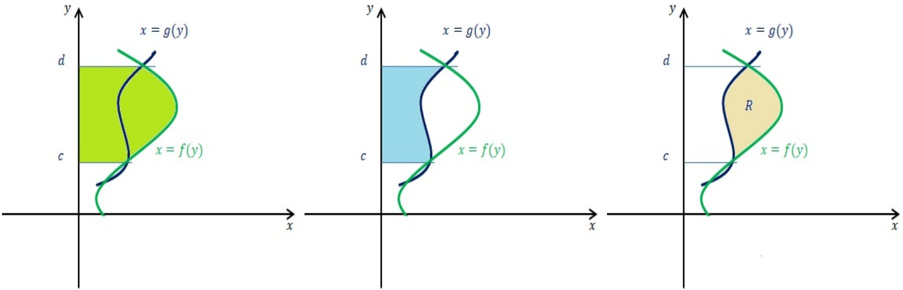
Areas

■ If the functions f and g are continuous and $f(x) \geq g(x)\ \forall x \in [a, b]$, then the area A of the region bounded by the graphs of f (the upper boundary of R) and g (the lower boundary of R) from $x = a$ to $x = b$ is subtracting the area of the region under g from the area of the region under f. This can be stated as follows:



$$A = \int_a^b \Big(f(x) - g(x) \Big)\ dx$$

If the functions f and g are continuous and $f(y) \geq g(y)\ \forall y \in [c, d]$, then the area A of the region bounded by the graphs of f (the right boundary of R) and g (the left boundary of R) from $y = c$ to $y = d$ is subtracting the area of the region bounded by $g(y)$ from the area of the region bounded by $f(y)$. This can be stated as follows:



$$A = \int_c^d \left(f(y) - g(y)\right)\, dy$$

Example

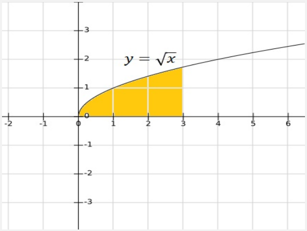
Sketch the region bounded by the graph of $y = \sqrt{x}$ *and* $x-$*axis from* $x = 0$ *to* $x = 3$*, then find its area.*

Example

Sketch the region bounded by the graph of $y = \sqrt{x}$ and $x-$axis from $x = 0$ to $x = 3$, then find its area.

Solution:
The area of the region is

$$A = \int_0^3 \sqrt{x}\, dx = \left[\frac{x^{3/2}}{\frac{3}{2}} \right]_0^3$$
$$= \frac{2}{3} \left[x^{3/2} \right]_0^3$$
$$= 2\sqrt{3}.$$



Example

Sketch the region bounded by the graph of $x = y + 1$ *and* $x-$*axis from* $y = -1$ *to* $y = 0$*, then find its area.*

Example

Sketch the region bounded by the graph of $x = y + 1$ and $x-$axis from $y = -1$ to $y = 0$, then find its area.

Solution:

x	0	1
y	-1	0

The line $x = y+1$ passes through the points $(0, -1)$ and $(1, 0)$.

Areas

Example

Sketch the region bounded by the graph of $x = y + 1$ and $x-$axis from $y = -1$ to $y = 0$, then find its area.

Solution:

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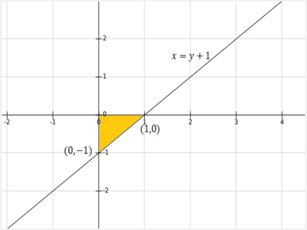


The line $x = y+1$ passes through the points $(0, -1)$ and $(1, 0)$.

The area of the region is

$$A = \int_{-1}^{0} (y + 1)\, dy$$

$$= \left[\frac{y^2}{2} + y \right]_{-1}^{0}$$

$$= \left[0 - \left(\frac{(-1)^2}{2} - 1\right) \right]$$

$$= \frac{1}{2}\ .$$

Example

Sketch the region bounded by the graph of $x = y + 1$ *and* $y-$*axis over the interval* $[-1, 1]$*, then find its area.*

Example

Sketch the region bounded by the graph of $x = y + 1$ and $y-$axis over the interval $[-1, 1]$, then find its area.

Solution:

x	0	1
y	-1	0

The line $x = y+1$ passes through the points $(0, -1)$ and $(1, 0)$.

Example

Sketch the region bounded by the graph of $x = y + 1$ and $y-$axis over the interval $[-1, 1]$, then find its area.

Solution:

x	0	1
y	-1	0

The line $x = y+1$ passes through the points $(0, -1)$ and $(1, 0)$.

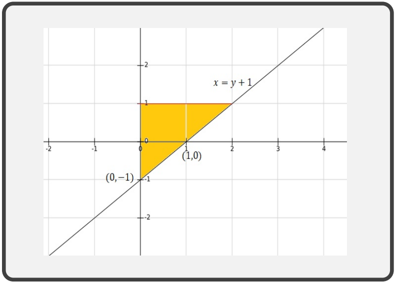
The area of the region is

$$A = \int_{-1}^{1} (y + 1)\ dy$$

$$= \left[\ \frac{y^2}{2} + y\ \right]_{-1}^{1}$$

$$= \left(\frac{(1)^2}{2} + 1\right) - \left(\frac{(-1)^2}{2} + (-1)\right)$$

$$= 2\ .$$



Example

Sketch the region bounded by the graph of $y = 2 - x^2$ *and* $x-$*axis, then find its area.*

Solution:

Areas

Example

Sketch the region bounded by the graph of $y = 2 - x^2$ and $x-$axis, then find its area.

Solution:

The area of the region is

$$
\begin{aligned}
A &= \int_{-\sqrt{2}}^{\sqrt{2}} (1 - x^2)\ dx \\
&= \Big[\ x - \frac{x^3}{3}\ \Big]_{-\sqrt{2}}^{\sqrt{2}} \\
&= (\sqrt{2} - \frac{(\sqrt{2})^3}{3}) - (\ - \sqrt{2} - \frac{(-\sqrt{2})^3}{3}) \\
&= \sqrt{2} + \sqrt{2} - \frac{(\sqrt{2})^3}{3} - \frac{(\sqrt{2})^3}{3} \\
&= 2\sqrt{2} - \frac{2(\sqrt{2})^3}{3}\ .
\end{aligned}
$$