Financial Mathematics

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Main Content

1. Securities
2. Price of a bond
3. Book value and amortization schedules
4. Callable bonds

Section 4.1: Securities

Corporations need to obtain money in order to start projects.

■ One way to secure funding is to sell stock in the company, but that expands the ownership of the company, since every stockholder is an owner of a proportional share of the company.

■ Another way to get money is to borrow it. Corporations can borrow money from banks and other lenders, but a more widely used method for large-scale borrowing is to issue **bonds**.

■ **Financial assets or securities:** contracts arranged by corporation or a public institution with investors to raise money.

■ From the investors view point, securities are investment instruments, endorsed by a corporation, government, or other organization.

■ There are three categories of securities: (1) Bonds (2) Preferred stock (3) Common stock

■ A **bond** is an interest-bearing security which promises to pay a stated amount (or amounts) of money at some future date (or dates).

■ It is a formal certificate of indebtedness issued by a borrower. Bonds are commonly issued by corporations and government units as a means of raising money.

■ Bonds are redeemed at the end of a fixed period of time called the term of the bond. The end of the term of a bond is called the maturity date. Any date prior to or including the maturity date on which a bond may be redeemed is termed a redemption date.

■ Bonds with an infinite term are issued such bonds called perpetuals. Also, bonds may be issued with a term and such a bond is term a callable bond.

■ Think of the following three questions:
1) Given a desired yield rate of investor, what price should be paid for given bond?
2) Given the purchase price of a bond, what is the resulting yield rate to an investor?
3) What is the value of a bond on a given date after it has been purchased?

Section 4.2: Price of a bond

There are two kinds of bonds: **(A) accumulation bonds** and **(B) bonds with coupons:**

(A) **Accumulation bonds or zero coupon bond**: An accumulation bond is one in which the redemption price includes the original loan plus all accumulated interests. The borrower agrees to pay the loan plus interest at a unique date, called the redemption time.

(B) **Bonds with coupons:** The coupons are periodic payments made by the issuer of the bond prior to its redemption. For bonds with coupons, the borrower agrees to make period payments (coupons) plus a balloon payment (the redemption value) C at the maturity date.

For example: Company A needs \$100,000,000 to build new manufacturing facilities. The company creates a bond issue consisting of 100,000 bonds in denominations of \$1,000. This amount is called the face value or par value of the bond.

If the coupon rate of the bond r (for example a nominal interest rate of 10% convertible semiannually for 10 years), the bonds will pay the face value of \$1,000 along with the final interest payment and the redemption value.

The time at which the loan is repaid is called the maturity date (or redemption date).

Every bond has a face value (or par value) F. The coupon payment is $F \times r$. Here, r is the coupon rate per interest period.

Note. Often, the payments are semiannually and $2r$ is the annual nominal rate of interest convertibly semiannually.

A bond is called redeemable at par if $C = F$. Unless said otherwise we assume that a bond is redeemable at par.

Let n be the number of interest periods until the redemption date. Let i be the yield rate per interest period and P be the price of a bond (the amount that the lender pays loans). The cashflow for the borrower is

Contributions	P	$-Fr$	$-Fr$	$-Fr$	...	$-Fr-C$
Time	0	1	2	3	...	n

Section 4.2: Price of a bond

- **Variables for a bond:**
  - P : the price of a bond.
  - F : the par or face value of a bond.
  - C : the redemption value of a bond (money paid at redemption date to the holder of the bond) and in most cases $C = F$.
  - r : the coupon rate of a bond per coupon period (e.g., per semi-annual period).
  - Fr : the amount of a coupon.
  - i : the yield rate of the bond per coupon period (the investor earns).
  - $\nu = \frac{1}{1+i}$: the discount factor per coupon period.
  - n : the number of coupon payment periods (number of coupon periods to redemption).
  - $g = \frac{Fr}{C}$: the modified coupon rate of a bond (The coupon rate per unit of redemption value rather than per unit of par value).
  - $G = \frac{Fr}{i}$: the base amount of a bond.
  - $K = C\nu^n$: the present value, compounded at the yield rate, of the redemption value of a bond
  - $P - C$: the premium (if $P > C$).
  - $C - P$: the discount (if $C > P$).
  - $k = \frac{P-C}{C}$: premium as a fraction of redemption value.

- The basic formula for the price of a bond is

$$P = Fr\, a_{\overline{n}|i} + C(1+i)^{-n} = Fr\, a_{\overline{n}|i} + C\nu^n = Fra_{\overline{n}|i} + K$$

- The premium/discount formula for the price of a bond is

$$P = Fr\, a_{\overline{n}|i} + C\nu^n = Fra_{\overline{n}|i} + C(1 - ia_{\overline{n}|i}) = C + (Fr - Ci)a_{\overline{n}|i} = C + C(g-i)a_{\overline{n}|i}$$

- The base amount formula for the price of a bond is

$$P = Fr\, a_{\overline{n}|i} + C\nu^n = Gia_{\overline{n}|i} + C\nu^n = G(1-\nu^n) + C\nu^n = G + (C-G)\nu^n$$

- The Makeham formula for the price of a bond is

$$P = Fr\, a_{\overline{n}|i} + C\nu^n = Cg\frac{1-\nu^n}{i} + C\nu^n = \frac{g}{i}(C - C\nu^n) + C\nu^n = \frac{g}{i}(C-K) + K$$

Example. Find the price of a 10-year bond, redeemable at par, with face value of \$10,000 and coupon rate of 10%, convertible quarterly, that will yield 8%, convertible quarterly.

Solution: Using $P = Fr\, a_{\overline{n}|i} + C(1+i)^{-n}$
We know that

$$F = C = 10000, \quad n = (10)(4) = 40, \quad r = \frac{0.10}{4} = 0.025 \quad \text{and} \quad i = \frac{0.08}{4} = 0.02$$

The amount of a coupon $Fr = (10000)(0.025) = 250$
So, the price of the bond is

$$P = Fr\, a_{\overline{n}|i} + C(1+i)^{-n} = 250a_{\overline{40}|0.02} + 10000(1+0.02)^{-40} = 11367.77396$$

Notes.

- When the interest rate rises, the price of the bond falls. In the above example, if $i = \frac{0.11}{4} = 0.0275$, the price of the bond will decrease to 9398.05 and in this case the bond is sold at a discount of 601.95 (10000- 9398.05)

- When the interest rate decrease, the price of the bond increases. In the above example, if $i = \frac{0.07}{4} = 0.0175$, the price of the bond will increase to 12144.57 and in this case the bond is sold at a premium of 2144.57.

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Example. A 30 year bond matures at its face value of 10,000. It pays semiannual coupons of 600. Calculate the price of the bond if the annual nominal interest rate convertible semiannually is 7.5%.

Solution: We know that $F = C = 10000$, $n = (30)(2) = 60$

Remember. Fr : the amount of a coupon: $Fr = 600$. Also, we have $i = \frac{7.5\%}{2} = 3.75\%$.

The price of the bond is

$$P = Fr\, a_{\overline{n}|i} + C(1+i)^{-n} = 600a_{\overline{60}|0.0375} + 10000(1+0.0375)^{-60} = 15341.03109$$

Example. What is the price of a 5−year 100 par-value bond having quarterly coupons at a quarter rate of 1.5% that is bought to yield a nominal annual rate of 12% convertible monthly?

Solution: We know that

$$F = C = 100, \quad n = (5)(4) = 20, \quad r = 0.015 \quad \text{and} \quad Fr = (100)(0.015) = 1.5$$

Now, we have $i^{(12)} = 12\%$ but we need $\frac{i^{(4)}}{4}$.

Remember. $1 + i = (1 + \frac{i^{(m)}}{m})^m \Rightarrow i = (1 + \frac{12\%}{12})^{12} - 1 = 12.68250301\%$
Thus, $1 + i = (1 + \frac{i^{(4)}}{4})^4 \Rightarrow 1 + 12.68250301\% = (1 + \frac{i^{(4)}}{4})^4 \Rightarrow \frac{i^{(4)}}{4} = 3.0301\%$

The price of the bond is

$$P = Fr\, a_{\overline{n|}\,i} + C(1+i)^{-n} = 1.5a_{\overline{20|}\,3.0301\%} + 100(1.030301)^{-20} = 77.29919664$$

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Example. The price of a zero coupon 1000 face value bond is 599.4584. The yield rate convertible semiannually is 6.5%. Calculate the maturity date.

Solution: Let n be the maturity date in years. We have that

$$P = F\nu^n \Rightarrow 599.4584 = (1000)(1 + \frac{0.065}{2})^{-2n} \Rightarrow (1 + \frac{0.065}{2})^{-2n} = \frac{599.4584}{1000} \Rightarrow -2n\ln(1 + \frac{0.065}{2}) = \ln\left(\frac{599.4584}{1000}\right)$$

$$\Rightarrow n = \frac{\ln\left(\frac{599.4584}{1000}\right)}{-2\ln(1 + \frac{0.065}{2})} = 8$$

The maturity date in years is 8 years.

Example. What is the yield as an annual effective rate of interest on a 100 par-value 10-year bond with coupon rate 6%, convertible monthly, that is selling for 90?

Solution: We know that

$$F = C = 100, \quad n = (10)(12) = 120, \quad r = \frac{0.06}{12} = 0.005 \quad P = 90 \quad \text{and} \quad Fr = (100)(0.005) = 0.5$$

Using

$$P = Fr\, a_{\overline{n}|\,i} + C(1+i)^{-n}$$

we have $90 = (0.05)a_{\overline{120}|\,\frac{i^{(12)}}{12}} + 100(1 + \frac{i^{(12)}}{12})^{-12\times 10} \Rightarrow \frac{i^{(12)}}{12} = 0.618181404\%$

This implies $i^{(12)} = (12)(0.618181404\%) = 7.419376846\%$ and $i = 7.676949087\%$.

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Example. A 1000 par value 10-year bond with semiannual coupons and redeemable at 1200 is purchased to yield 8% convertible semiannually. The first coupon is 50. Each subsequent coupon is 3% greater than the preceding coupon. Find the price of the bond.

Solution: We know that $F = 1000, \quad C = 1200, \quad n = (10)(2) = 20, \quad i = \frac{8\%}{2}$ and $g = 3\%$

The cashflow of coupons is

Time (in half-years)	1	2	3	4	...	20
Coupons	50	50(1.03)	$50(1.03)^2$	$50(1.03)^3$	...	$50(1.03)^{19}$

Using $a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}$, the present value of the payments is

$$P = 50a^{g}_{\overline{20}|\frac{0.08}{2}} + (1200)(1+0.04)^{-20} = 878.5721 + 547.6643 = 1426.2364$$

Example. Company A puts its bonds up for sale on a day when investors in the marketplace are demanding a yield rate of 10.2% convertible semiannually. They want to buy each bond at a price that gives that yield. Over 10 years each bond will make 20 semi-annual payments of 50 and a final payment of 1,000 at the same time as the last payment of 50.

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Solution: We know that

$$i = \frac{10.2\%}{2} = 5.1\%, \quad n = (10)(2) = 20, \quad Fr = 50, \quad C = 1000$$

Using

$$P = Fr\ a_{\overline{n}|i} + C(1+i)^{-n}$$

we have

$$P = 50\ a_{\overline{20}|5.1\%} + (1000)(1+5.1\%)^{-20} = 987.64$$

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Exercise. Suppose Company A's bonds were sold on a day when the market interest rate was 9.8% convertible semi-annually. What is the price of an individual bond with a 1,000 face amount? Answer: 1,012.57

Section 4.3: Book value and amortization schedules

The **book value** of a bond at time k (B_k) is the present value of the payments to be made, i.e. the present value of the remaining $n-k$ coupons and the redemption value C. This is also the outstanding balance of the loan at that time.

Using **the prospective method**, the book value of a bond is B_k:

$$B_k = Fr\, a_{\overline{n-k}|i} + C\nu^{n-k} = Fr\, a_{\overline{n-k}|i} + C(1 - i\, a_{\overline{n-k}|i}) = C + (Fr - Ci)a_{\overline{n-k}|i} = C + C(g-i)a_{\overline{n-k}|i} \quad \text{Remember. } Fr = Cg$$

Note that, $B_0 = P$ and $B_n = C$

Using **the retrospective method**, the book value of a bond is

$$B_k = P(1+i)^k - Fr\, s_{\overline{k}|i}$$

The previous formula is equivalent to

$$P = B_k(1+i)^{-k} + Fr\, a_{\overline{k}|i}$$

B_k is the balance in the loan after the first k payments are made. the previous formula is interpreted as follows: the present value of B_k and the payments made until that moment equals the initial balance (the price of the bond).

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Example. Jack buys a 20 year bond with a par value of 4000 and coupon rate of 10%, convertible semiannually. He attains an annual yield of 5% convertible semiannually. The redemption value of the bond is 1200. Find the book value of the bond at the end of the 12th year.

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Solution: We know that

$$F = 4000, \quad r = \frac{10\%}{2} = 5\%, \quad i = \frac{5\%}{2} = 2.5\%, \quad Fr = (4000)(0.05) = 200, \quad n = (2)(20) = 40, \quad k = 24 \quad \text{and} \quad C = 1200$$

Using $B_k = Fr\, a_{\overline{n-k}|i} + C\nu^{n-k}$ where $k = (2)(12) = 24$, we have

$$B_{24} = (200)\, a_{\overline{16}|\frac{5\%}{2}} + (1200)(1 + \frac{5\%}{2})^{-16} = 3419.350452$$

Example. An n-year 5000 par value bond pays 6% annual coupons. At annual yield of 3%, the book value of the bond at the end of year 7 is 5520. Calculate the price of the bond.

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Solution: We know that

$$F = C = 5000, \quad r = 6\%, \quad i = 3\% \quad Fr = (5000)(0.06) = 300, \quad B_7 = 5520$$

Hence,

$$P = B_k(1+i)^{-k} + Fr\, a_{\overline{k}|i} = 5520(1.03)^{-7} + (300)a_{\overline{7}|3\%} = 6357.35$$

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Theorem. Inductive relation for book value

$$B_{k+1} = B_k(1+i) - Fr$$

Proof.

$$\begin{aligned} B_k(1+i) - Fr &= (Fr\ a_{\overline{n-k}|i} + C\nu^{n-k})(1+i) - Fr \\ &= Fr((1+i)a_{\overline{n-k}|i} - 1) + C\nu^{n-k-1} \\ &= Fr\ a_{\overline{n-k-1}|i} + C\nu^{n-k-1} \qquad : (1+i)\frac{1-\nu^{n-k}}{i} - 1 = \frac{(1+i) - \nu^{n-k-1} - i}{i} \\ &= Fr\ a_{\overline{n-(k+1)}|i} + C\nu^{n-(k+1)} = B_{k+1} \end{aligned}$$

B_k is the outstanding balance at time k. One period later, the principal has increased to $B_k(1+i)$, i.e. $B_k(1+i)$ is the outstanding balance immediately before the $(k+1)$-th payment is made. Immediately after the $(k+1)$-th payment to principal of Fr is made, the outstanding balance is $B_{k+1} = B_k(1+i) - Fr$.

Example. Consider a 30-year \$50,000 par-value bond with semiannual coupons, with $r = 0.03$, and yield rate 10%, convertible semiannually.
(i) Find the book value of the bond immediately after the 25-th coupon payment.
(ii) Find the book value of the bond immediately after the 26-th coupon payment.

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(i) Find the book value of the bond immediately after the 25-th coupon payment.
(ii) Find the book value of the bond immediately after the 26-th coupon payment.

Solution: We know that

$$F = C = 50000, \quad i = 0.05, \quad Fr = 50000(0.03) = 1500 \quad \text{and} \quad n = (2)(30) = 60$$

(i) Using

$$B_k = Fr\, a_{\overline{n-k|}i} + C\nu^{n-k}$$

we have

$$B_{25} = (1500)a_{\overline{35|}0.05} + 50000(1.05)^{-35} = 33625.80571$$

(ii) Using

$$B_{k+1} = B_k(1+i) - Fr$$

the book value of the bond immediately after the 26-th coupon payment is

$$B_{26} = B_{25}(1+0.05) - 1500 = 35307.096 - 1500 = 33807.096$$

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Example. An n–year \$50,000 par value with semiannual coupons, with r = 0.03, and yield rate 10%, convertible semiannually. If the book value of the bond immediately after the 25-th coupon payment is 33625.81, then B_{26} is equal to ...

(A) 81803.021 (B) 23501.021 (C) 43501.193 (D) 33807.096

General inductive relation for book value

By induction from the previous formula, we get that

$$B_{k+m} = B_k(1+i)^m - Fr\, s_{\overline{m}|i}$$

Notes.

- B_{k+m} is the outstanding balance immediately after the $k+m$ payment.
- $B_k(1+i)^m$ is the accrued balance at time $k+m$ of the outstanding balance immediately after the k payment.
- $Fr\, s_{\overline{m}|i}$ is the future value at time $k+m$ of the coupon payments $k+1, k+2, \ldots, k+m$, i.e. the coupons payments from when the outstanding was B_k until when the outstanding is B_{k+m}.

Example. An n-year 4000 par value bond with coupon rate 9%, convertible semiannually has an annual nominal yield of i, $i > 0$, convertible semiannually. The book value of the bond at the end of year 4 is 3812.13 and the book value at the end of year 7 is 3884.27. Calculate i.

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Solution: We have that $F = C = 4000$, $r = 4.5\%$ and $Fr = 4000(0.045) = 180$.

The end of year 4 is the end of the 8th period: the book value $B_8 = 3812.13$

The end of year 7 is the end of the 14th period: the book value $B_{14} = 3884.27$

Using

$$B_{k+m} = B_k(1+i)^m - Fr\, s_{\overline{m}|i}$$

we have

$$B_{8+6} = B_8(1+i)^6 - Fr\, s_{\overline{6}|i} \Rightarrow 3884.27 = (3812.13)(1+i)^6 - (180)\, s_{\overline{6}|i} \Rightarrow i = 10\%$$

Section 4.3: Book value and amortization schedules

- The amount of interest contained in the kth coupon:

$$I_k = iB_{k-1} = Fr - (Fr - Ci)\nu^{n-k+1} = Cg - C(g-i)\nu^{n-k+1}$$

Proof.

$$\begin{aligned} I_k = iB_{k-1} = iFr\, a_{\overline{n-k+1}|i} + iC\nu^{n-k+1} &= Fr(1-\nu^{n-k+1}) + iC\nu^{n-k+1} \\ &= Fr - (Fr - Ci)\nu^{n-k+1} \\ &= Cg - (Cg - Ci)\nu^{n-k+1} \\ &= Cg - C(g-i)\nu^{n-k+1} \end{aligned}$$

- The principal portion in the k–th coupon is $P_k = B_{k-1} - B_k = Fr - I_k$

Note that $P_k = Fr - I_k = Fr - \left(Fr - (Fr - Ci)\nu^{n-k+1}\right) = (Fr - Ci)\nu^{n-k+1} = C(g-i)\nu^{n-k+1}$

Proof.

$$P_k = B_{k-1} - B_k = B_{k-1} - (B_{k-1}(1+i) - Fr) \quad \text{Remember. } B_{k+1} = B_k(1+i) - Fr$$

$$P_k = B_{k-1} - B_{k-1} - iB_{k-1} + Fr$$

$$P_k = Fr - iB_{k-1}$$

$$P_k = Fr - I_k \quad \text{Remember. } I_k = iB_{k-1}$$

Notes.

- P_k is the change in the book value of the bond (principal adjustment) between times $k-1$ and k.
- P_k could be either negative, or zero or positive.
- P_k is the amortization in the kth payment.

Example. Kendal buys a 5000 par-value 10 year bond with coupon rate 8%, convertible semiannually to yield 4% converted semiannually. Find the amount of interest and principal in the 5-th coupon.

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Solution: We have that $F = C = 5000, \quad r = 0.04, \quad Fr = 200, \quad n = 20$ and $i = 2\%$.

Using $I_k = iB_{k-1} = Fr - (Fr - Ci)\nu^{n-k+1}$, we have

$$I_5 = 200 - \big(200 - (5000)(0.02)\big)(1.02)^{-(20+1-5)} = 127.1554$$

OR Find B_4, then $I_5 = (2\%)(6357.7709) = 127.1554$

Using $P_k = (Fr - Ci)\nu^{n+1-k}$, we have

$$P_5 = \big(200 - (5000)(0.02)\big)(1.02)^{-(20+1-5)} = 72.8446$$

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Example. For a bond pays annual coupons. The principal amortized in the first coupon is half that amortized in the 10-th coupon. Calculate the annual yield rate.

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Example. For a bond pays annual coupons. The principal amortized in the first coupon is half that amortized in the 10-th coupon. Calculate the annual yield rate.

Solution: We know that $P_1 = \frac{1}{2}P_{10}$ and using $P_k = (Fr - Ci)\nu^{n-k+1}$, we have

$$P_1 = \frac{1}{2}P_{10} \Rightarrow (Fr - Ci)\nu^n = \frac{1}{2}(Fr - Ci)\nu^{n-9} \Rightarrow \nu^n = \frac{1}{2}\nu^{n-9} \Rightarrow \nu^{-9} = 2 \Rightarrow (1+i)^9 = 2$$

$$\Rightarrow (1+i) = \sqrt[9]{2} \Rightarrow i = \sqrt[9]{2} - 1 = 8.005973889\%$$

Notes.

- The market value of a bond is the price at which a bond is bought/sold. When rates of interest change, the market value of a bond changes.

- The price of the bond moves in the opposite direction from interest rates as follows:

- As interest rates rise, the price of a bond falls. The market value is less than the book value, so we have an **unrealized capital loss**.

- As the interest rates decline, the price of a bond rises. The market value is bigger than the book value, so we have an **unrealized capital gain**.

Example. Oliver buys a ten-year 5000 face value bond with semiannual coupons at annual rate of 6%. He buys his bond to yield 8% compounded semiannually and immediately sell them to an investor to yield 4% compounded semiannually. What is Oliver's profit in this investment?

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Solution: We have that $F = C = 5000$, $r = 0.03$, $Fr = 150$ and $n = (2)(10) = 20$. Using $P = Fr\ a_{\overline{n}|i} + C\nu^n$

Oliver buys his bond for $P = (150)a_{\overline{20}|0.04} + 5000(1.04)^{-20} = 4320.484$

Oliver sells his bond for $P = (150)a_{\overline{20}|0.02} + 5000(1.02)^{-20} = 5817.572$

Oliver's profit is $5817.572 - 4320.484 = 1497.088$.

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Example. On January 1, 2000, Maxwell bought a 10-year \$5000 noncallable bond with coupons at 7% convertible semiannually. Maxwell bought the bond to yield 7%, compounded semiannually. On July 1, 2005, the market value of bonds is based on a 5% interest rate, compounded semiannually. Calculate the unrealized capital gain on July 1, 2005.

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Solution: We have that $F = 5000, \quad r = 0.035, \quad Fr = 175, \quad n = 20.$

On July 1, 2005, we have $k = 11$ i.e, there are 9 remaining coupons $(20 - 11 = 9)$. Thus, the book value of the bond is $B_{11} = 175\ a_{\overline{9}|3.5\%} + 5000(1.035)^{-9} = 5000$.

On July 1, 2005, the market value of the bond is $B_{11} = 175\ a_{\overline{9}|2.5\%} + 5000(1.025)^{-9} = 5398.543276$.

The unrealized capital gain is $5398.543276 - 5000 = 398.543276$.

Premium.

If the investor is paying more than the redemption value, i.e. if $P > C$, we say that the bond has being bought at premium. The premium is $P - C$. We have that

$$P = Fr\, a_{\overline{n}|i} + C\nu^n = Fr\, a_{\overline{n}|i} + C(1 - i a_{\overline{n}|i}) \Rightarrow P - C = (Fr - Ci)\, a_{\overline{n}|i} = C(g - i)\, a_{\overline{n}|i}$$

Remember. $g = \dfrac{Fr}{C} \Rightarrow Cg = Fr$

Notes.

- A bond has been bought at premium if and only if $g > i$. Also, for a bond bought at premium $Fr > Ci$ and

$$P = B_0 > B_1 > B_2 > \cdots > B_n = C$$

- $P_k = B_{k-1} - B_k = (Fr - Ci)\nu^{n+1-k}$ is the write-up in premium in the k-th coupon. The premium is $P - C = \sum_{j=1}^{n}(B_{j-1} - B_j)$.

Discount.

If the investor is paying less than the redemption value, i.e. if $P < C$, we say that the bond has being bought at discount. The discount is $C - P$. We have that

$$C - P = (Ci - Fr)\, a_{\overline{n}|i} = C(i - g)\, a_{\overline{n}|i}$$

Notes.

- A bond has been bought at discount if and only if $g < i$. Also, for a bond bought at discount $Fr < Ci$ and

$$P = B_0 < B_1 < B_2 < \cdots < B_n = C$$

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Example. A 10 year 50000 face value bond pays semiannual coupons of 3000. The bond is bought to yield a nominal annual interest rate of 6% convertible semiannually. Calculate the premium paid for the bond.

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Solution: We know that $F = C = 50000, \quad Fr = 3000, n = 20$, and $i = 3\%$. Note that $Fr > Ci$, so the bond is bought at premium.

The price of the bond is $P = Fr\, a_{\overline{n}|i} + C\nu^n = (3000)a_{\overline{20}|3\%} + (50000)(1.03)^{-20} = 72316.21229$.

The premium of the bond is $P - C = 72316.21229 - 50000 = 22316.21229$.

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Example. Oprah buys a 10000 par-value 15 year bond with 9% semiannual coupons to yield 5% converted semiannually.

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(i) Find the premium in the bond.

(ii) Find the write up in premium in the 8-th coupon.

Solution: (i) We have that $F = C = 10000$, $r = 0.045$, $Fr = 450$ $n = 30$ and $i = 2.5\%$. Note that $Fr > Ci$, so the bond is bought at premium.

Oprah buys her bond for $P = (450)\ a_{\overline{30}|2.5\%} + 10000(1.025)^{-30} = 14186.06$

The premium of the bond is $P - C = 14186.06 - 10000 = 4186.06$.

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Oprah buys her bond for $P = (450)\ a_{\overline{30}|2.5\%} + 10000(1.025)^{-30} = 14186.06$

The premium of the bond is $P - C = 14186.06 - 10000 = 4186.06$.

(ii) The write-up in premium in the 8-th coupon is $P_k = (Fr - Ci)\nu^{n+1-k} = (10000)(0.045 - 0.025)(1.025)^{-23} = 113.34$.

OR

$B_7 = (450)\ a_{\overline{23}|2.5\%} + 10000(1.025)^{-23} = 13466.42$

$B_8 = (450)\ a_{\overline{22}|2.5\%} + 10000(1.025)^{-22} = 13353.08$

and $B_7 - B_8 = 13466.42 - 13353.08 = 113.34$

- *Recall.* The write-up in premium in the k-th coupon is $P_k = B_{k-1} - B_k = (Fr - Ci)\nu^{n+1-k}$.

For a bond that is redeemable at face value ($F = C$),

- The amortization of premium in period k: $P_k = F(r-i)\nu^{n-k+1}$
- The amortization of premium in period $k+m$: $P_{k+m} = F(r-i)\nu^{n-(k+m)+1} = (1+i)^m F(r-i)\nu^{n-k+1} = (1+i)^m P_k$

- *Similarly.* The write-up in discount in the k-th coupon is $P_k = B_k - B_{k-1} = (Ci - Fr)\nu^{n+1-k}$.

For a bond that is redeemable at face value ($F = C$),

- The amortization of discount in period k: $P_k = F(i-r)\nu^{n-k+1}$
- The amortization of discount in period $k+m$: $P_{k+m} = F(i-r)\nu^{n-(k+m)+1} = (1+i)^m F(i-r)\nu^{n-k+1} = (1+i)^m P_k$

Example. A premium bond is purchased to yield 4% convertible semi–annually. The amount of premium amortized in the 2nd payment is 8.37. Find the amount of premium amortized in the 7th payment.

Section 4.3: Book value and amortization schedules

■ *Recall.* The write-up in premium in the k-th coupon is $P_k = B_{k-1} - B_k = (Fr - Ci)\nu^{n+1-k}$.

For a bond that is redeemable at face value ($F = C$),

■ The amortization of premium in period k: $P_k = F(r - i)\nu^{n-k+1}$

■ The amortization of premium in period $k + m$: $P_{k+m} = F(r - i)\nu^{n-(k+m)+1} = (1 + i)^m F(r - i)\nu^{n-k+1} = (1 + i)^m P_k$

■ *Similarly.* The write-up in discount in the k-th coupon is $P_k = B_k - B_{k-1} = (Ci - Fr)\nu^{n+1-k}$.

For a bond that is redeemable at face value ($F = C$),

■ The amortization of discount in period k: $P_k = F(i - r)\nu^{n-k+1}$

■ The amortization of discount in period $k + m$: $P_{k+m} = F(i - r)\nu^{n-(k+m)+1} = (1 + i)^m F(i - r)\nu^{n-k+1} = (1 + i)^m P_k$

Example. A premium bond is purchased to yield 4% convertible semi–annually. The amount of premium amortized in the 2nd payment is 8.37. Find the amount of premium amortized in the 7th payment.

Solution: We have $P_2 = 8.37$

The amortization of premium in period $k + m$: $P_{k+m} = (1 + i)^m P_k$

The amortization of premium in 7th period: $P_7 = P_{2+5} = (1 + i)^5 P_2 = (1 + \frac{4\%}{2})^5(8.37) = 9.24$

Remember.

- The price of the bond is $P = Fr\, a_{\overline{n}|i} + C\nu^n$ and the premium is $P - C$

- The book value of a bond is B_k:

$$B_k = Fr\, a_{\overline{n-k}|i} + C\nu^{n-k} = C + (Fr - Ci)a_{\overline{n-k}|i} = C + C(g-i)a_{\overline{n-k}|i} \qquad : \; Fr = Cg$$

Note that, $B_0 = P$ and $B_n = C$

- The amount of interest contained in the kth coupon: $I_k = iB_{k-1} = Fr - (Fr - Ci)\nu^{n-k+1} = Cg - C(g-i)\nu^{n-k+1}$
- The principal portion in the k–th coupon: $P_k = B_{k-1} - B_k = Fr - I_k = (Fr - Ci)\nu^{n-k+1} = C(g-i)\nu^{n-k+1}$

- The amortization schedule of a bond is

Time	Payment	Interest paid (iB_{k-1})	Principal repaid	Book value (B_k)
0	-	-	-	$P = C + (Fr - Ci)a_{\overline{n}|i}$
1	Fr	$Fr - (Fr - Ci)\nu^n$	$(Fr - Ci)\nu^n$	$C + (Fr - Ci)a_{\overline{n-1}|i}$
2	Fr	$Fr - (Fr - Ci)\nu^{n-1}$	$(Fr - Ci)\nu^{n-1}$	$C + (Fr - Ci)a_{\overline{n-2}|i}$
3	Fr	$Fr - (Fr - Ci)\nu^{n-2}$	$(Fr - Ci)\nu^{n-2}$	$C + (Fr - Ci)a_{\overline{n-3}|i}$
...	...	...	...	...
k	Fr	$Fr - (Fr - Ci)\nu^{n-k+1}$	$(Fr - Ci)\nu^{n-k+1}$	$C + (Fr - Ci)a_{\overline{n-k}|i}$
...	...	...	...	...
$n-1$	Fr	$Fr - (Fr - Ci)\nu^2$	$(Fr - Ci)\nu^2$	$C + (Fr - Ci)a_{\overline{1}|i}$
n	$Fr + C$	$Fr - (Fr - Ci)\nu$	$(Fr - Ci)\nu + C$	C

Example. A 3–year 1,000 par value bond pays semi–annual coupons at a 6% annual rate. It is sold at a yield of 5% convertible semi–annually. Construct a bond amortization schedule.

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Solution: We have that $F = C = 1000$, $r = \frac{6\%}{2} = 0.03$, $Fr = (1000)(0.03) = 30$, $i = \frac{5\%}{2} = 2.5\% = 0.025$ and $n = (2)(3) = 6$.

- Since $r > i$ (i.e., $Fr > Ci$), the bond has being bought at premium.
- The price of the bond: $P = Fra_{\overline{n}|i} + C\nu^n = 450\ a_{\overline{6}|2.5\%} + (1000)(1.025)^6 = 1,027.54$ and the premium is $P - C = 27.54$
- Thus, the buyer of the bond has an investment of 1,027.54 that pays interest at the yield rate of 2.5% (per half-year).

Time	Payment (Coupon) (Fr)	Redemption Value (C)	Interest paid $(I_k = iB_{k-1})$	Principal repaid $(P_k = Fr - I_k)$	Book value $(B_k = B_{k-1} - P_k)$	Remaining Premium $(P - C)$
0	-	-	-	-	$P = 1,027.54$	$\underbrace{27.54}_{(1,027.54-1,000)}$
1	30		$\underbrace{25.69}_{(iB_{k-1}=0.025\times 1,027.54)}$	$\underbrace{4.31}_{(30-25.69)}$	$\underbrace{1,023.23}_{(1,027.54-4.31)}$	$\underbrace{23.23}_{(1,023.23-1,000)}$
2	30	-	25.58	4.42	1,018.81	18.81
3	30	-	25.47	4.53	1,014.28	14.28
4	30	-	25.36	4.64	1,009.64	9.64
5	30	-	25.24	4.76	1,004.88	4.88
6	30	1,000	25.12	4.88	1,000.00	0

Section 4.3: Book value and amortization schedules

Example. A 3–year 1,000 par value bond pays semi-annual coupons at a 6% annual coupon rate. It is sold at a yield of 7% convertible semi–annually. Construct a bond amortization schedule.

Section 4.3: Book value and amortization schedules

Example. A 3–year 1,000 par value bond pays semi-annual coupons at a 6% annual coupon rate. It is sold at a yield of 7% convertible semi–annually. Construct a bond amortization schedule.

Solution: We have that $F = C = 1000$, $r = \frac{6\%}{2} = 0.03$, $Fr = (1000)(0.03) = 30$, $i = \frac{7\%}{2} = 3.5\% = 0.035$ and $n = (2)(3) = 6$.

- Since $r < i$ (i.e., $Fr < Ci$), the bond has being bought at discount.
- The price of the bond: $P = Fra_{\overline{n}|i} + C\nu^n = 450\ a_{\overline{6}|3.5\%} + (1000)(1.035)^6 = 973.36$ and the discount is $C - P = 26.64$
- Thus, the buyer of the bond has an investment of 973.36 that pays interest at the yield rate of 3.5% (per half-year).

Time	Payment (Coupon) (Fr)	Redemption Value (C)	Interest paid ($I_k = iB_{k-1}$)	Principal repaid ($P_k = I_k - Fr$)	Book value ($B_k = B_{k-1} + P_k$)	Remaining Discount ($C - P$)
0	-	-	-	-	$P = 973.36$	$\underbrace{26.64}_{(1{,}000-973.36)}$
1	30		$\underbrace{34.07}_{(iB_{k-1}=0.035\times 973.36)}$	$\underbrace{4.07}_{(34.07-30)}$	$\underbrace{977.43}_{(973.36+4.07)}$	$\underbrace{22.57}_{(1{,}000-977.43)}$
2	30	-	34.21	4.21	981.63	18.37
3	30	-	34.36	4.36	985.99	14.01
4	30	-	34.51	4.51	990.50	9.50
5	30	-	34.67	4.67	995.17	4.83
6	30	1,000	34.83	4.83	1,000.00	0

Example. A 1000 par-value 3-year bond pays 6%, convertible semiannually, and has a yield rate of 8%, convertible semiannually.
(i) What is the interest paid in the 3rd coupon?
(ii) What is the change in book value contained in the 3rd coupon?
(iii) Construct a bond amortization schedule.

Solution: We know that $F = C = 1000, \quad r = 0.03, \quad Fr = 30, \quad n = 6, i = 0.04$ and $Ci = 40$.
(i) The interest paid in the 3rd coupon is

$$I_3 = Fr - (Fr - Ci)\nu^{n-k+1} = 30 - (30 - 40)(1.04)^{-4} = 38.54804191$$

OR $I_k = iB_{k-1}$ i.e., $I_3 = iB_2$ where $B_k = Fra_{\overline{n-k}|i} + C\nu^{n-k}$

$$B_2 = (30)a_{\overline{6-2}|0.04} + 1000(1.04)^{-4} = 963.7010478$$

So, $I_3 = (0.04)(963.7010478) = 38.54804191$

(ii) Since $I_3 > Fr$, the book value increases in the 3rd coupon. Remember. If $Fr < Ci$, we have a discount bond and $P = B_0 < B_1 < B_2 < \cdots < B_n = C$.
The increase in the book value in the 3rd coupon is $I_3 - Fr = 38.54804191 - 30 = 8.54804191$.

OR The change in book value contained in the 3rd coupon is $B_3 - B_2$

$$B_3 = 30a_{\overline{3}|0.04} + 1000(1.04)^{-3} = 972.2490897$$

So,

$$B_3 - B_2 = 972.2490897 - 963.7010478 = 8.5480419$$

(iii) Homework

Section 4.4: Callable bonds

A **callable bond** is a bond which gives the issuer (not the investor) the right to redeem prior to its maturity date, under certain conditions. When issued, the **call provisions** explain when the bond can be redeemed and what the price will be. The earliest time to call the bond is named the **call date** . The **call price** is the amount of money the insurer must pay to buy the bond back. In most cases, there is some period of time during which the bond cannot be called. This period of time is named the **call protection period** .

If the bond is called immediately after the payment of k-th coupon, the present value of the obtained payments is

$$P_k = Fr\ a_{\overline{k|}i} + C\nu^k = C + (Fr - Ci)\ a_{\overline{k|}i} = C + C(g - i)\ a_{\overline{k|}i}$$

P_k is the price which the investor would pay for the bond assuming that the bond is called immediately after the k coupon. As smaller as P_k is, as worst for the lender is.

Assuming that the redemption value is a constant and that a bond can be called after any coupon payment:

(i) if $Fr > Ci$ (bond sells at a premium), P_k increases with k, and we assume the redemption date is the earliest possible.

(ii) if $Fr < Ci$ (bond sells at a discount), P_k decreases with k, and we assume that the redemption date is the latest possible.

Example. A 10–year 1,000–face bond has a 10% coupon rate payable semi–annually. It is called in 6 years. At what price should the investor buy the bond to assure a yield 8% convertible semi–annually?

Solution: We have that $F = C = 1000$, $r = 5\%$, $Fr = 50$, $n = 20$ and $i = 4\%$. Since $r > i$ (i.e., $Fr > Ci$), the bond has being bought at premium. This means the investor would price the bond using $n = 12$:
$P_{12} = (50)\ a_{\overline{12|}4\%} + (1000)\nu^{12} = (50)(9.39) + (1000)(0.9615)^{12} = 1,093.85$

Example. A 10–year 1,000–face bond has a 10% coupon rate payable semi–annually. It is called in 6 years. At what price should the investor buy the bond to assure a yield 12% convertible semi–annually?

Solution: We have that $F = C = 1000$, $r = 5\%$, $Fr = 50$, $n = 20$ and $i = 6\%$. Since $r < i$ (i.e., $Fr < Ci$), the bond has being bought at discount. This means the investor would price the bond using $n = 20$:
$P_{20} = (50)\ a_{\overline{20|}6\%} + (1000)\nu^{20} = (50)(11.47) + (1000)(0.9433)^{20} = 885.30$

Example. Consider a 100 par-value 8% bond with semiannual coupons callable at 120 on any coupon date starting 5 years after issue for the next 5 years, at 110 starting 10 years after issue for the next 5 years and maturing at 105 at the end of 15 years. What is the highest price which an investor can pay and still be certain of a yield of 9% converted semiannually.

Solution: We have that $F = 100, \quad r = 4\%, \quad Fr = 4, \quad n = 30$ and $i = 4.5\%$.

Let k be the number of the half year when the bond is called.

If $10 \leq k \leq 19$, then $C = 120$ and $Ci = (120)(0.045) = 5.4$. So, the bond sells at a discount. We have that lowest price which the investor can get is $P_{19} = 4\, a_{\overline{19}|4.5\%} + 120(1.045)^{-19} = 102.37$.

If $20 \leq k \leq 29$, then $C = 110$ and $Ci = (110)(0.045) = 4.95$. So, the bond sells at a discount. We have that lowest price which the investor can get is $P_{29} = 4\, a_{\overline{29}|4.5\%} + (110)(1.045)^{-29} = 94.78$.

If $k = 30$, then the price is $P_{30} = 4\, a_{\overline{30}|4.5\%} + 105(1.045)^{-30} = 93.19$.

We conclude that the highest price which an investor can pay and still be certain of a yield of 9% converted semiannually is 93.19.

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Example. Joshua paid 800 for a 15-year 1000 par value bond with semiannual coupons at a nominal annual rate of 4% convertible semiannually. The bond can be called at 1300 on any coupon date starting at the end of year 7. What is the minimum annual nominal rate convertible semiannually yield that Joshua could receive?

Solution: We have that $F = 1000, \quad P = 800, \quad C = 1300$, $r = 0.02$ and $n = 30$.

Since $P < C$, the bond was bought at discount. We assume that the redemption value is as late as possible.

From the equation $800 = 20\ a_{\overline{30}|i} + 1300(1+i)^{-30}$, we get that $i = 3.6760\%$ and $i^{(2)} = 7.3521\%$.