# Integral Calculus

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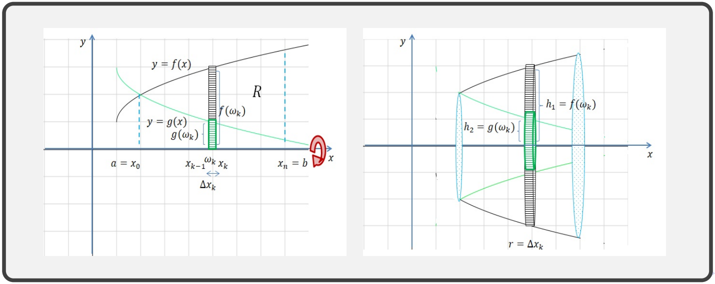
**Main Contents.**

# Volumes of Revolution Solids (Washer Method)

- Let $R$ be a region bounded by the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.
- Let $S$ be a solid generated by revolving the region $R$ about $x$-axis.
- The volume of the solid $S$ is equal to the difference between the volumes of the two solids generated by revolving the regions under the functions $f(x)$ and $g(x)$ about the $x$-axis as follows:

- The outer radius: $y_1 = f(x)$
- The inner radius: $y_2 = g(x)$
- The thickness: $dx$
- The volume of a washer is $dV = \pi\left[(\text{the outer radius})^2 - (\text{the inner radius})^2\right].$ thickness.
- This implies $dV = \pi\left[(f(x))^2 - (g(x))^2\right] dx$.
- Hence, the volume of the solid over the interval $[a, b]$ is

$$V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right] dx.$$

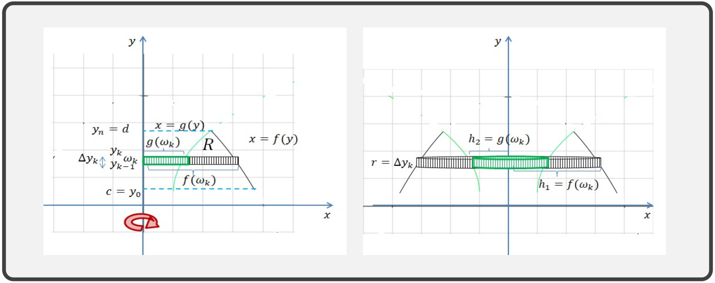


# Volumes of Revolution Solids (Washer Method)

- Similarly, let $R$ be a region bounded by the graphs of $f(y)$ and $g(y)$ such that $f(y) \geq g(y)$ for all $y \in [c, d]$ as shown in the figure.
- Let $S$ be a solid generated by revolving the region $R$ about $y$-axis.

- The outer radius: $x_1 = f(y)$
- The inner radius: $x_2 = g(y)$
- The thickness: $dy$
- The volume of a washer is $dV = \pi\left[(\text{the outer radius})^2 - (\text{the inner radius})^2\right].$ thickness.
- This implies $dV = \pi\left[(f(y))^2 - (g(y))^2\right] dy$.
- Hence, the volume of the solid over the interval $[c, d]$ is

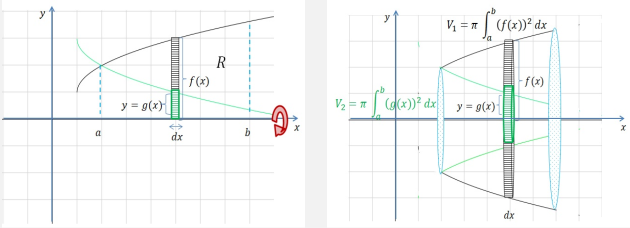
$$V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2\right] dy.$$



# Volumes of Revolution Solids (Washer Method)

**Theorem:**

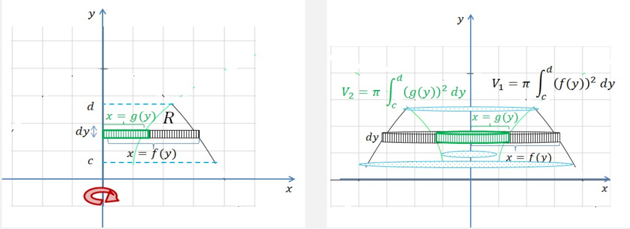
**(1)** If $R$ is a region bounded by the graphs of $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ such that $f \geq g$, the volume of the revolution solid generated by revolving $R$ about $x$-axis is




$$V = \pi \int_a^b \left( [f(x)]^2 - [g(x)]^2 \right) dx.$$

1. The two points (area boundaries) on the $x-$axis.
2. Rotation about the $x-$axis.
3. The rectangles are perpendicular to the axis of rotation ($x-$axis).

**(2)** If $R$ is a region bounded by the graphs of $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ such that $f \geq g$, the volume of the revolution solid generated by revolving $R$ about $y$-axis is

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# Volumes of Revolution Solids (Washer Method)

## Example

*Let R be a region bounded by the graphs of the functions* $y = x^2$ *and* $y = 2x$. *Evaluate the volume of the solid generated by revolving R about* $x$*-axis.*

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*Let $R$ be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving $R$ about $x$-axis.*

Solution: First, we check whether the graphs of the two functions are intersecting or not.

$$\begin{aligned} f(x) = g(x) \Rightarrow x^2 = 2x &\Rightarrow x^2 - 2x = 0 \\ &\Rightarrow x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

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## Example

*Let $R$ be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving $R$ about $x$-axis.*

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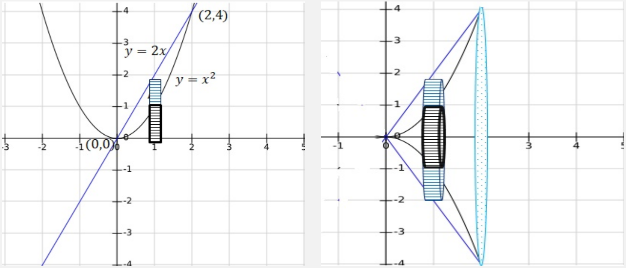
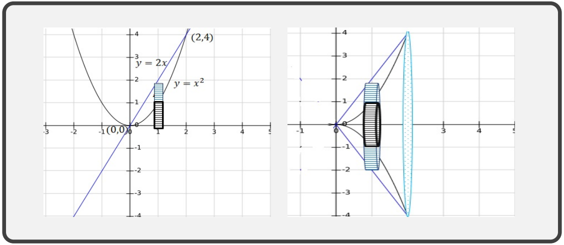
By substitution, we have that the two curves intersect in two points $(0, 0)$ and $(2, 4)$.



# Volumes of Revolution Solids (Washer Method)




The figure shows the region $R$ and the solid generated by revolving the region about the $x$-axis. The vertical rectangles generate a washer where

- the outer radius: $y_1 = 2x$,
- the inner radius: $y_2 = x^2$ and
- the thickness: $dx$.

The volume of the washer is $dV = \pi\left[(2x)^2 - (x^2)^2\right]\,dx$.

Hence, the volume of the solid over the interval $[0, 2]$ is

$$V = \pi\int_0^2 \left((2x)^2 - (x^2)^2\right)\,dx = \pi\int_0^2 (4x^2 - x^4)\,dx$$

$$= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{64}{15}\pi.$$

## Example

*Consider the same region as in the previous example enclosed by the graphs of* $y = x^2$ *and* $y = 2x$. *Revolve the region about* $y$*-axis instead and find the volume of the generated solid.*

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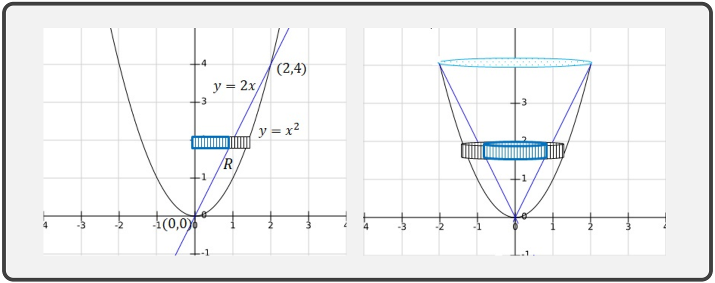
Solution: The figure shows the region $R$ and the solid generated by revolving the region about the $y$-axis.

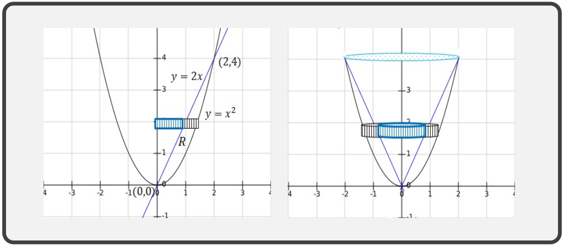



Since the revolution is about the $y$-axis, we need to rewrite the equations in term of $y$ i.e., $x_1 = f(y)$ and $x_2 = g(y)$.

$$y = x^2 \Rightarrow x = \sqrt{y} \Rightarrow f(y) = \sqrt{y}$$

$$y = 2x \Rightarrow x = \frac{y}{2} \Rightarrow g(y) = \frac{y}{2} .$$

# Volumes of Revolution Solids (Washer Method)




The two horizontal rectangles generate a washer where

- the outer radius: $x_1 = \sqrt{y}$,
- the inner radius: $x_2 = \frac{y}{2}$ and
- the thickness: $dy$.

The volume of the washer is $dV = \pi\left[(\sqrt{y})^2 - (\frac{y}{2})^2\right]\, dy$.

Hence, the volume of the solid over the interval $[0, 4]$ is

$$V = \pi \int_0^4 \left((\sqrt{y})^2 - (\frac{y}{2})^2\right)\, dy = \pi \int_0^4 \left(y - \frac{y^2}{4}\right)\, dy$$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$$

## Example

*Consider a region R bounded by the graphs of the functions* $y = \sqrt{x}$, $y = 6 - x$ *and x-axis. Revolve the region about y-axis and find the volume of the generated solid.*

# Volumes of Revolution Solids (Washer Method)

## Example

*Consider a region $R$ bounded by the graphs of the functions $y = \sqrt{x}$, $y = 6 - x$ and $x$-axis. Revolve the region about $y$-axis and find the volume of the generated solid.*

Solution: Since the revolution is about $y$-axis, we need to rewrite the functions in terms of $y$ i.e., $x = f(y)$ and $x = g(y)$.

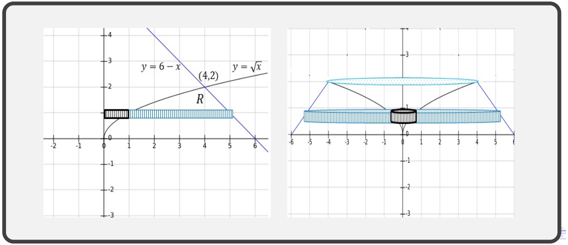
$$y = \sqrt{x} \Rightarrow x = y^2 = f(y) \text{ and } y = 6 - x \Rightarrow x = 6 - y = g(y).$$

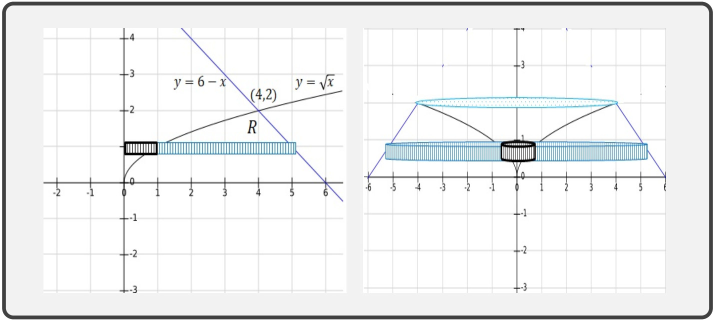
Now we see if the graphs of the two functions intersect:

$$f(y) = g(y) \Rightarrow y^2 = 6 - y \Rightarrow y^2 + y - 6 = 0 \Rightarrow (y+3)(y-2) = 0 \Rightarrow y = -3 \text{ or } y = 2$$

**Note.** Since $y = \sqrt{x}$, we ignore the value $y = -3$.

By substituting $y = 2$ into the two functions, we have $x = 4$. Thus, the two curves intersect in one point $(4, 2)$. The solid $S$ generated by revolving the region $R$ about $y$-axis is shown in the figure.






Also, the revolution is about the $y$-axis, so we have a horizontal rectangle that generates a washer where

- the outer radius: $x_1 = 6 - y$,
- the inner radius: $x_2 = y^2$ and
- the thickness: $dy$.

The volume of the washer is $dV = \pi\left[(6-y)^2 - (y^2)^2\right]\, dy$.

The volume of the solid over the interval $[0, 2]$ is

$$V = \pi \int_0^2 \left[(6-y)^2 - (y^2)^2\right]\, dy = \pi\left[-\frac{(6-y)^3}{3} - \frac{y^5}{5}\right]_0^2 = \pi\left[\left(-\frac{64}{3} - \frac{32}{5}\right) - \left(-\frac{216}{3} - 0\right)\right] = \frac{664}{15}\pi.$$

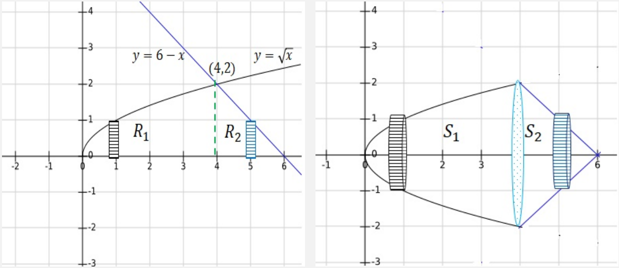
## Example

*Consider the same region as in the previous example enclosed by the graphs of* $y = \sqrt{x}$, $y = 6 - x$ *and* $x$*-axis. Revolve the region about* $x$*-axis instead and find the volume of the generated solid.*

## Example

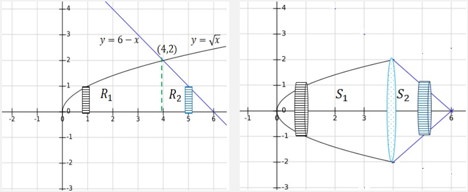
*Consider the same region as in the previous example enclosed by the graphs of $y = \sqrt{x}$, $y = 6 - x$ and $x$-axis. Revolve the region about $x$-axis instead and find the volume of the generated solid.*

Solution:




**Note.** The solid is made up of **two separate regions**: $R_1$ and $R_2$, and each requires its own integral. We use the disk method to evaluate the volume of the solid generated by revolving each region.

# Volumes of Revolution Solids (Washer Method)




**(1) Region $R_1$:** Revolution of $R_1$ about the $x$-axis generates a solid $S_1$ with a vertical disk of radius $y = \sqrt{x}$ and thickness $dx$.

$$V_1 = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \frac{\pi}{2} \left[ \, x^2 \, \right]_0^4 = 8\pi.$$

**(2) Region $R_2$:** Revolution of $R_2$ about the $x$-axis generates a solid $S_2$ with a vertical disk of radius $y = 6 - x$ and thickness $dx$.

$$V_2 = \pi \int_4^6 (6-x)^2 \, dx = \pi \int_4^6 (6-x)^2 \, dx = -\frac{\pi}{3} \left[ (6-x)^3 \right]_4^6 = \frac{8}{3}\pi.$$

The volume of the total solid:

$$\begin{aligned} V &= V_1 + V_2 \\ &= 8\pi + \frac{8}{3}\pi = \frac{32}{3}\pi. \end{aligned}$$