Financial Mathematics

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Chapter 5: Variable interest rates: Term structure of interest rates

Main Content

1. Term structure of interest rates
2. Yield curve
3. Spot rates

Section 5.1: Term structure of interest rates

Term structure of interest rates. The relationship between yield and maturity. Note: the longer the term, the higher the rate.

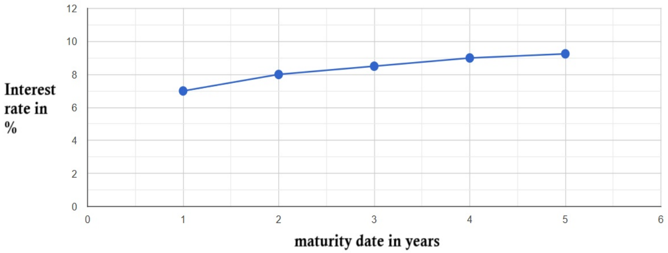
A yield curve is a graph that shows interest rates (vertical axis) versus (maturity date) duration of a investment/loan (horizontal axis).

Example. A bank offers certificates of deposit with the following interest rates:

length of the investment (in years)	1 year	2 years	3 years	4 years	5 years
Interest rate	7%	8%	8.5%	9%	9.25%

Graph the yield curve for these interest rates.

Solution:



Notes.

■ The interest rates appearing in the yield curve are called **the spot rates**. Thus, for Example 1, the spot rates are 7%, 8%, 8.5%, 9% and 9.25%.

■ **The j year spot rate (s_j)** is the rate of interest charged in a loan paid with a unique payment at the end of j years.

■ The j year spot rate s_j is as an effective annual rate of interest, the current j year interest factor (the j–year accumulation factor) is $(1 + s_j)^j$. Money invested now multiply by (1 + sj)j in j years.

■ For a zero coupon with face value F, the price is $P = F(1 + s_j)^{-j}$ and this implies $F = P(1 + s_j)^j$.

■ If the current interest rates follow the accumulation function is $a(t)$, $t > 0$, then $a(j) = (1 + s_j)^j$ and this implies $s_j = (a(j))^{1/j} - 1$.

Example. The following table lists prices of zero-coupon $ 1000 bonds with their respective maturities:

Number of years to maturity	1 year	2 years	5 years	10 years
Price	980.39	957.41	888.18	781.20

Calculate the 1year, 2year, 5year, and 10year spot rates of interest.

Solution: Remember: $P = F(1 + s_j)^{-j}$, so

$$(1000)(1 + s_1)^{-1} = 980.39 \Rightarrow (1 + s_1)^{-1} = \frac{980.39}{1000} \Rightarrow (1 + s_1)^1 = \frac{1000}{980.39} \Rightarrow s_1 = \frac{1000}{980.39} - 1 = 2\%$$

$$(1000)(1 + s_2)^{-2} = 957.41 \Rightarrow (1 + s_2)^{-2} = \frac{957.41}{1000} \Rightarrow (1 + s_2)^2 = \frac{1000}{957.41} \Rightarrow s_2 = (\frac{1000}{957.41})^{\frac{1}{2}} - 1 = 2.2\%$$

$$(1000)(1 + s_5)^{-5} = 888.18 \Rightarrow (1 + s_5)^{-5} = \frac{888.18}{1000} \Rightarrow (1 + s_5)^5 = \frac{1000}{888.18} \Rightarrow s_5 = (\frac{1000}{888.18})^{\frac{1}{5}} - 1 = 2.40\%$$

$$(1000)(1 + s_{10})^{-10} = 781.20 \Rightarrow (1 + s_{10})^{-10} = \frac{781.20}{1000} \Rightarrow (1 + s_{10})^{10} = \frac{1000}{781.20} \Rightarrow s_{10} = (\frac{1000}{781.20})^{\frac{1}{10}} - 1 = 2.50\%$$

- The present value at time zero of a cashflow

Time	0	1	2	...	n
Contributions	0	C_1	C_2	...	C_n

following the spot rates

Maturity Time	1	2	...	n
Spot rate	s_1	s_2	...	s_n

is given by the formula $PV = \sum_{j=1}^{n} C_j(1+s_j)^{-j}$

Example. A 2year 1000 par value 6% bond with semi-annual coupons using the spot rates:

Maturity time (in half years)	1	2	3	4
Nominal annual interest rate convertible semiannually	4%	5%	6%	7%

(1) Find the price of the bond.
(2) Find the annual effective yield rate of the previous bond, if bought at the price in (1).

Solution: (1) Remember: coupon payment $Fr = 1000 \times 3\% = 30$.

Time (in half years)	1	2	3	4
Payments	30	30	30	30+1000

The price of the bond is the present value of the payments:
$PV = (30)(1+\frac{0.04}{2})^{-1} + (30)(1+\frac{0.05}{2})^{-2} + (30)(1+\frac{0.06}{2})^{-3} + (30)(1+\frac{0.07}{2})^{-4} + (1000)(1+\frac{0.07}{2})^{-4} = 983.0059$

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(2) To find the yield rate, we solve for $i^{(2)}$ in the price equation: $P = Fr\, a_{\overline{n}|i} + C(1+i)^{-n}$.

$$983.0059 = (30)a_{\overline{4}|\frac{i^{(2)}}{2}} + 1000(1+\frac{i^{(2)}}{2})^{-4} \Rightarrow i^{(2)} = 6.92450\% \Rightarrow i = 7.0443\%$$

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