Financial Mathematics

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Main Content

1. Macaulay duration
2. Volatility or Modified duration.
3. Convexity

The duration (or Macaulays duration) of a cashflow is an average of the times when the payments of the cashflow are made:

Time in years	1	2	3	...	5
Contributions	C_1	C_2	C_3	...	C_n

with $C_j > 0$ for each $1 \leq j \leq m$, is defined as

$$\bar{d} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \sum_{j=1}^{n} j \frac{C_j\nu^j}{\sum_{k=1}^{n} C_k\nu^k} = \sum_{j=1}^{n} j\ w_j$$

where $w_j = \frac{C_j\nu^j}{\sum_{k=1}^{n} C_k\nu^k}$ satisfy $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$

Notes.

- The units of the duration are years.
- The Macaulay duration is a measure of the price sensitivity of a cashflow to interest rate changes.
- w_j is the fraction of the present value of contribution at time t over the present value of the whole cashflow.

Example. An investment pays 1000 at the end of year two and 1000 at the end of year 12. The annual effective rate of interest is 8%. Calculate the Macaulay duration for this investment.

Solution:

$$\bar{d} = \frac{\sum_{j=1}^{n} jC_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j} = \frac{(2)(1000)(1.08)^{-2} + (12)(1000)(1.08)^{-12}}{(1000)(1.08)^{-2} + (1000)(1.08)^{-12}} = 5.165633881 years$$

Theorem. Let $r > 0$. If the Macaulay duration of the cashflow

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

is $\bar{d}$, then the Macaulay duration of the cashflow

Time in years	1	2	3	...	n
Contributions	rC_1	rC_2	rC_3	...	rC_n

is $\bar{d}$.

Proof. The duration of the modified cashflow is

$$\frac{\sum_{j=1}^{n} jrC_j \nu^j}{\sum_{j=1}^{n} rC_j \nu^j} = \frac{\sum_{j=1}^{n} jC_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j} = \bar{d}$$

Example. The Macaulay duration of a 10year annuityimmediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10year annuityimmediate with annual payments of \$50000.

Solution: Note that $50000 = 1000 \times 50$, so from the above theory where $r = 50$, we have that the duration of both cash flows is 5.6 years.

Theorem. If the Macaulay duration of the cashflow

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

is $\bar{d}$, then the Macaulay duration of the cashflow

Time in years	$t+1$	$t+2$	$t+3$	...	$t+n$
Contributions	C_1	C_2	C_3	...	C_n

is $t + \bar{d}$.

Proof. The duration of the modified cashflow is

$$\frac{\sum_{j=1}^{n}(t+j)C_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{\sum_{j=1}^{n} tC_j\nu^j + \sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = t + \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = t + \bar{d}$$

Example. The Macaulay duration of a 10-year annuityimmediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10-year annuity-due with annual payments of \$5000.

Solution: Since the cashflow of an annuity-due is obtained from the cashflow of an annuity-immediate by translating payments 1 year, the answer is $5.6 - 1 = 4.6$ years.

Theorem. Suppose that two cashflows have durations $\bar{d}_1$ and $\bar{d}_2$, respectively, present values P_1 and P_2, respectively. Then, the duration of the combined cashflow is

$$\bar{d} = \frac{P_1 \ \bar{d}_1 + P_2 \ \bar{d}_2}{P_1 + P_2}$$

Proof. Suppose that the considered cashflows are

Time in years	1	2	3	...	n
Contributions	C_1	C_2	C_3	...	C_n

Time in years	1	2	3	...	n
Contributions	D_1	D_2	D_3	...	D_n

Then, the combined cashflow is

Time in years	1	2	3	...	n
Contributions	$C_1 + D_1$	$C_2 + D_2$	$C_3 + D_3$	...	$C_n + D_n$

We have that $P_1 = \sum_{j=1}^{n} C_j \nu^j$ and $P_2 = \sum_{j=1}^{n} D_j \nu^j$. By definition of duration,

$$\bar{d}_1 = \frac{\sum_{j=1}^{n} jC_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j} = \frac{\sum_{j=1}^{n} jC_j \nu^j}{P_1} \text{ and } \bar{d}_2 = \frac{\sum_{j=1}^{n} jD_j \nu^j}{\sum_{j=1}^{n} D_j \nu^j} = \frac{\sum_{j=1}^{n} jD_j \nu^j}{P_2}$$

Hence,

$$\bar{d} = \frac{\sum_{j=1}^{n} j(C_j + D_j)\nu^j}{\sum_{j=1}^{n}(C_j + D_j)\nu^j} = \frac{\sum_{j=1}^{n} jC_j \nu^j + \sum_{j=1}^{n} jD_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j + \sum_{j=1}^{n} D_j \nu^j} = \frac{\bar{d}_1 P_1 + \bar{d}_2 P_2}{P_1 + P_2}$$

Note that $\bar{d}_1 P_1 = \frac{\sum_{j=1}^{n} j \ C_j \nu^j}{P_1} P_1 = \sum_{j=1}^{n} jC_j \nu^j$ and $\bar{d}_2 P_2 = \frac{\sum_{j=1}^{n} j \ C_j \nu^j}{P_2} P_2 = \sum_{j=1}^{n} jC_j \nu^j$

Section 6.1: Macaulay duration

Note. By induction the previous formula holds for a combination of finitely many cashflows. Suppose that we have n cashflows. The j-the cashflow has present value P_j and duration . Then, the duration of the combined cashflow is

$$\frac{\sum_{j=1}^{n} P_j \bar{d}_j}{\sum_{j=1}^{n} P_j}$$

Example. An insurance has the following portfolio of investments:
(i) Bonds with a value of $1,520,000 and duration 4.5 years.
(ii) Stock dividends payments with a value of $1,600,000 and duration 14.5 years.
(iii) Certificate of deposits payments with a value of $2,350,000 and duration 2 years.
Calculate the duration of the portfolio of investments.

Solution: The duration of the portfolio is

$$\frac{\sum_{j=1}^{n} P_j \bar{d}_j}{\sum_{j=1}^{n} P_j} = \frac{(4.5)(1,520,000) + (14.5)(1,600,000) + (2)(2,350,000)}{1,520,000 + 1,600,000 + 2,350,000} = 6.351005484 years$$

Theorem. The Macaulay duration of a level payments annuityimmediate is $\bar{d} = \frac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}}$

Proof. We have that $\bar{d} = \frac{\sum_{j=1}^{n} jP\nu^j}{\sum_{j=1}^{n} P\nu^j} = \frac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}}$.

Example. Calculate Macaulay the duration of a 15-year annuity immediate with level payments if the current effective interest rate per annum is 5%.

Solution: The Macaulay the duration is

$$\bar{d} = \frac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}} = \frac{(Ia)_{\overline{15|}5\%}}{a_{\overline{15|}5\%}} = \frac{73.66768937}{10.37965804} = 7.097313716$$

Remember. $(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n\nu^n}{i}$

Theorem. The duration of a level payments perpetuity-immediate is

$$\bar{d} = \frac{1+i}{i}$$

Proof. We have that $\bar{d} = \frac{\sum_{j=1}^{\inf} jP\nu^j}{\sum_{j=1}^{\inf} P\nu^j} = \frac{(Ia)_{\overline{\inf|}i}}{a_{\overline{\inf|}i}} = \frac{\frac{1+i}{i^2}}{\frac{1}{i}} = \frac{1+i}{i}$

Example. Suppose that the Macaulay duration of a perpetuity immediate with level payments of 1000 at the end of each year is 21. Find the current effective rate of interest.

Solution: We have that $\bar{d} = \frac{1+i}{i} = 21 \Rightarrow 1+i = 21i \Rightarrow 20i = 1 \Rightarrow i = \frac{1}{20} = 5\%$

Theorem. The duration of n year bond with $r\%$ annual coupons, face value F and redemption value C is

$$\bar{d} = \frac{Fr(Ia)_{\overline{n|}i} + Cn\nu^n}{Fra_{\overline{n|}i} + C\nu^n}$$

Proof. We have the cashflow

Time in years	1	2	...	$n-1$	n
Contributions	Fr	Fr	...	Fr	$Fr + C$

the duration is

$$\bar{d} = \frac{Fr\sum_{j=1}^{n} j\nu^j + Cn\nu^j}{Fr\sum_{j=1}^{n} \nu^j + C\nu^j} = \frac{Fr(Ia)_{\overline{n|}i} + Cn\nu^n}{Fra_{\overline{n|}i} + C\nu^n}$$

Example. Megan buys a 10year 1000facevalue bond with a redemption value of 1200 which pay annual coupons at rate 7.5%. Calculate the Macaulay duration if the effective rate of interest per annum is 8%.

Solution: We have that $\bar{d} = \frac{Fr(Ia)_{\overline{n|}i} + Cn\nu^n}{Fra_{\overline{n|}i} + C\nu^n}$, so

$$\bar{d} = \frac{(1000)(0.075)(Ia)_{\overline{10|}8\%} + (1200)(10)(1.08)^{-10}}{(1000)(0.075)a_{\overline{10|}8\%} + (1200)(1.08)^{-10}} = \frac{(75)(32.68691288) + 5558.321857}{(75)(6.710081399) + 555.8321857} = 7.562958059$$