Systems of Linear Algebraic Equations

Chapter #3

Errors

1. Residual vector

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^*$$

2. Relative Residual

$$\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} = \frac{\|\mathbf{b} - A\mathbf{x}^*\|}{\|\mathbf{b}\|}$$

3. Condition number

$$K(A) = \|A\|\|A^{-1}\|$$

4. Error

$$\|\mathbf{x} - \mathbf{x}^*\| \leq \|\mathbf{r}\|\|A^{-1}\|$$

5. Relative Error

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq K(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|},$$

Direct method

LU - Decomposition

Crout's method

Write:
A=IA

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Before changing any value of **A** move it to **I**

After m21 and m31

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

After m32

$$\underset{\mathbf{L}}{\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}} \underset{\mathbf{U}}{\begin{bmatrix} a_{11\,\text{new}} & a_{12\,\text{new}} & a_{13\,\text{new}} \\ 0 & a_{22\,\text{new}} & a_{23\,\text{new}} \\ 0 & 0 & a_{33\,\text{new}} \end{bmatrix}}$$

Make sure that diagonal of U is ones

Solve
Ly=b

$$\underset{L}{\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}} \underset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Solve
Ux=y

$$\underset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

Doolittles's method

Write:

$$A = LU$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{\mathbf{A}}$$

$$\underset{\mathbf{L}}{\begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}} \underset{\mathbf{U}}{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}$$

Solve
Ly=b

$$\underset{L}{\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}} \underset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Solve
Ux=y

$$\underset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

Gaussian by partial pivoting

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

let assume that

$$R_2 > R_1 > R_3$$

At partial pivoting arrangement descending

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}}, \quad R_{2\,\text{new}} = R_2 - m_{21} \cdot R_1$$

$$m_{31} = \frac{a_{31}}{a_{11}}, \quad R_{3\,\text{new}} = R_3 - m_{31} \cdot R_1$$

$$\begin{bmatrix} a_{11\,\text{new}} & a_{12\,\text{new}} & a_{13\,\text{new}} \\ 0 & a_{22\,\text{new}} & a_{23\,\text{new}} \\ 0 & a_{32\,\text{new}} & a_{33\,nw} \end{bmatrix}$$

$$m_{32} = \frac{a_{32\,new}}{a_{22\,new}}, \quad R_3\,\text{new} = R_3 - m_{32} \cdot R_2$$

$$\begin{bmatrix} a_{11\,\text{new2}} & a_{12\,\text{new2}} & a_{13\,\text{new2}} \\ 0 & a_{22\,\text{new2}} & a_{23\,\text{new2}} \\ 0 & 0 & a_{33\,\text{new2}} \end{bmatrix}$$

Simple Gaussian method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}}, \quad R_{2\,\text{new}} = R_2 - m_{21} \cdot R_1$$

$$m_{31} = \frac{a_{31}}{a_{11}}, \quad R_{3\,\text{new}} = R_3 - m_{31} \cdot R_1$$

$$\begin{bmatrix} a_{11\,\text{new}} & a_{12\,\text{new}} & a_{13\,\text{new}} \\ 0 & a_{22\,\text{new}} & a_{23\,\text{new}} \\ 0 & a_{32\,\text{new}} & a_{33\,nw} \end{bmatrix}$$

$$m_{32} = \frac{a_{32\,new}}{a_{22\,new}}, \quad R_3\,\text{new} = R_3 - m_{32} \cdot R_2$$

$$\begin{bmatrix} a_{11\,\text{new2}} & a_{12\,\text{new2}} & a_{13\,\text{new2}} \\ 0 & a_{22\,\text{new2}} & a_{23\,\text{new2}} \\ 0 & 0 & a_{33\,\text{new2}} \end{bmatrix}$$

Indirect method

Gauss Seidel method

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$$

That's how norm working

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 &+ a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \end{aligned}$$

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right] \end{aligned}$$

Matrix form of Gauss Seidel

$$A = L + D + U$$

$$\underset{L}{\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}} \underset{U}{\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}} \underset{D}{\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}}$$

$$T_G = -(L+D)^{-1}U \quad \text{and} \quad \mathbf{c}_G = (L+D)^{-1}\mathbf{b}$$

$$\mathbf{x}^{(k+1)} = T_G\mathbf{x}^{(k)} + \mathbf{c}_G$$

$$\|T\|_\infty < 1$$

Smaller $\|\mathbf{T}\|$ is faster to converge

Error bound

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$$

Jacobi method

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$$

That's how norm working

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 &+ a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \end{aligned}$$

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right] \end{aligned}$$

Matrix form of Jacobi

$$A = L + D + U$$

$$\underset{L}{\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}} \underset{U}{\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}} \underset{D}{\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}}$$

$$T_J = -D^{-1}(L+U) \quad \mathbf{c}_J = D^{-1}\mathbf{b}$$

$$\mathbf{x}^{(k+1)} = T_J\mathbf{x}^{(k)} + \mathbf{c}_J$$

$$\|T\|_\infty < 1$$

Smaller $\|\mathbf{T}\|$ is faster to converge

Error bound

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$$