

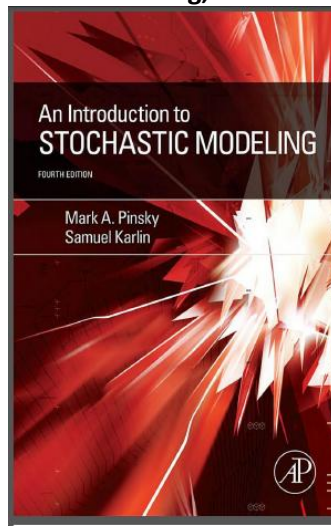


King Saud University
College of sciences
Department of Mathematics

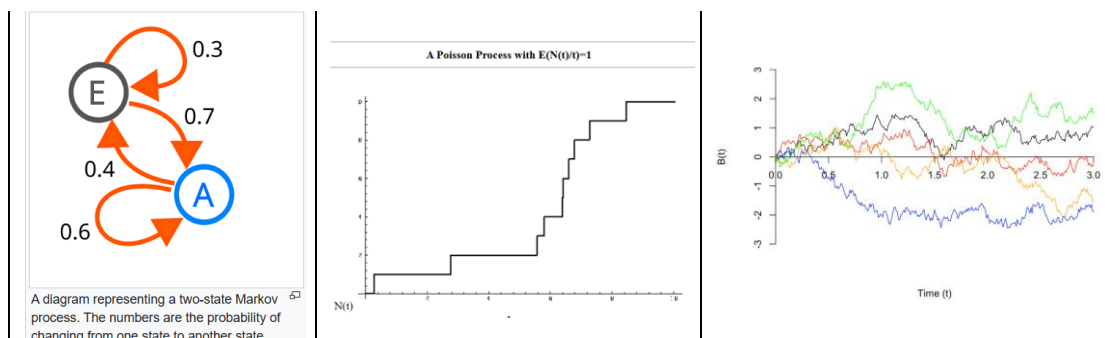
Math-380

Exercises (solved)

The exercises from :
An introduction to stochastic modelling, Mark A. Pinsky and Samuel Karlin



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1 Introduction

1.2 Probability Review*

1.2.1 Let A and B be arbitrary, not necessarily disjoint, events. Use the law of total probability to verify the formula

$$\Pr\{A\} = \Pr\{AB\} + \Pr\{AB^c\},$$

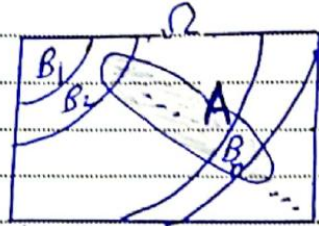
where B^c is the complementary event to B (i.e., B^c occurs if and only if B does not occur).

leads directly to the *law of total probability*: Let $A_1, A_2, \dots$ be disjoint events for which $\Omega = A_1 \cup A_2 \cup \dots$. Equivalently, exactly one of the events $A_1, A_2, \dots$ will occur. The law of total probability asserts that $\Pr\{B\} = \sum_{i=1}^{\infty} \Pr\{B \cap A_i\}$ for any event B . The law

pb 1.2.1 p.16 Textbook

prove that
 $\Pr(A) = \Pr(AB) + \Pr(AB^c)$

proof
 $\therefore \Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap B_i)$



where $\Omega = B_1 \cup B_2 \cup \dots$ and $B_i \cap B_j = \emptyset$ if $i \neq j$

For any two subsets A and B , we can write
 $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c)$

$\therefore \Pr(A) = \Pr(AB) + \Pr(AB^c)$

where $\Omega = B \cup B^c$ and $B \cap B^c = \emptyset$



1.2.2 Let A and B be arbitrary, not necessarily disjoint, events. Establish the general addition law

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{AB\}.$$

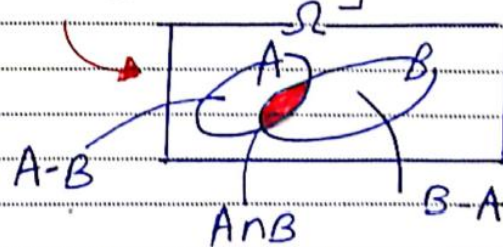
Hint: Apply the result of Exercise 1.2.1 to evaluate $\Pr\{AB^c\} = \Pr\{A\} - \Pr\{AB\}$. Then, apply the addition law to the disjoint events AB and AB^c , noting that $A = (AB) \cup (AB^c)$.

proof $\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$ for disjoint sets

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$\therefore \Pr(A \cup B) = \Pr[(A-B) \cup (A \cap B) \cup (B-A)]$

where $A-B$, $A \cap B$ and $B-A$ are disjoint sets



$\therefore \Pr(A \cup B) = \Pr(A-B) + \Pr(A \cap B) + \Pr(B-A)$

$= \Pr(A) - \Pr(A \cap B) + \Pr(A \cap B) + \Pr(B) - \Pr(A \cap B)$

$\therefore \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ ويعطى النتيجة
 $= \Pr(A) + \Pr(B) - \Pr(AB) \quad \underline{\underline{\#}}$

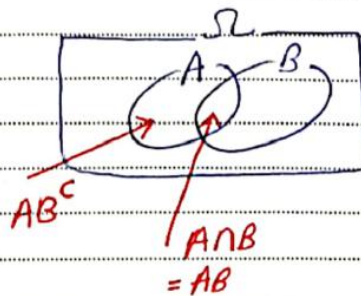
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 2/ Another solution

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prove that

$$pr(A \cup B) = pr(A) + pr(B) - pr(AB)$$



proof:

$$A \cup B = (A \cup B) \cap (B \cup B^c)$$

$$\therefore A \cup B = (A \cap B^c) \cup B \quad \text{distribution prop.}$$

$$\therefore A \cup B = (AB^c) \cup B \quad (\text{or you can deduce it directly from the opp. Venn diagram})$$

$$\therefore pr(A \cup B) = pr(AB^c) + pr(B) \quad \textcircled{1} \text{ where } AB^c \text{ and } B \text{ are disjoint events (mutually exclusive events)}$$

$$\therefore A = (AB) \cup (AB^c)$$

where AB and AB^c are disjoint events

$$\therefore pr(A) = pr(AB) + pr(AB^c)$$

$$\therefore pr(AB^c) = pr(A) - pr(AB) \quad \textcircled{2}$$

Substitute $\textcircled{2}$ in $\textcircled{1}$ we get

$$pr(A \cup B) = pr(A) - pr(AB) + pr(B)$$

$$\therefore pr(A \cup B) = pr(A) + pr(B) - pr(AB) \quad \#$$

1.2.3 (a) Plot the distribution function

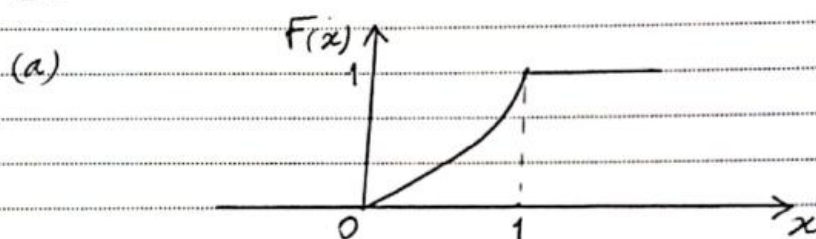
$$F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^3 & \text{for } 0 < x < 1, \\ 1 & \text{for } x \geq 1. \end{cases}$$

(b) Determine the corresponding density function $f(x)$ in the three regions (1) $x \leq 0$, (2) $0 < x < 1$, and (3) $1 \leq x$.

(c) What is the mean of the distribution?

(d) If X is a random variable following the distribution specified in (a), evaluate $\Pr\left\{\frac{1}{4} \leq X \leq \frac{3}{4}\right\}$.

Ans:



(b) $\therefore f(x) = \frac{dF}{dx}$

$$\Rightarrow f(x) = \begin{cases} 0 & , x \leq 0 \\ 3x^2 & , 0 < x < 1 \\ 0 & , x \geq 1 \end{cases} \quad \text{is the p.d.f}$$

(c) The mean is

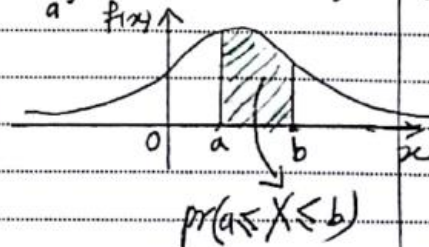
$$\begin{aligned} \mu = E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x (3x^2) dx = 3 \int_0^1 x^3 dx \\ &= 3 \left[\frac{x^4}{4} \right]_0^1 = \frac{3}{4} \end{aligned}$$

(d) $pr\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right)$

$$= F\left(\frac{3}{4}\right) - F\left(\frac{1}{4}\right)$$

$$= \left(\frac{3}{4}\right)^3 - \left(\frac{1}{4}\right)^3 = \frac{13}{32} \quad \#$$

Remember $pr(a \leq X \leq b)$
 $= \int_a^b f(x) dx = F(b) - F(a)$



1.2.7 Suppose X is a random variable having the probability density function

$$f(x) = \begin{cases} Rx^{R-1} & \text{for } 0 \leq x \leq 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where $R > 0$ is a fixed parameter.

(a) Determine the distribution function $F_X(x)$.

(b) Determine the mean $E[X]$.

(c) Determine the variance $\text{Var}[X]$.

then $f(x)$ is called the *probability density function* for the random variable X . If X has a probability density function $f(x)$, then X is continuous and

$$F(x) = \int_{-\infty}^x f(\xi) d\xi, \quad -\infty < x < \infty.$$

Ans: (a) The cumulative distn f_n cdf is

$$F_X(x) = \int_{-\infty}^x f(t) dt = R \int_0^x t^{R-1} dt$$

$$= R \left[\frac{t^R}{R} \right]_0^x = x^R, \quad 0 \leq x \leq 1$$

$$\Rightarrow F_X(x) = \begin{cases} 0, & x < 0 \\ x^R, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \quad \text{is the cdf}$$

clearly, $F(0) = 0$, $F(1) = 1$ and $0 \leq F(x) \leq 1$

(b) the mean, $E(X)$ is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x R x^{R-1} dx$$

$$= R \int_0^1 x^R dx = R \left[\frac{x^{R+1}}{R+1} \right]_0^1 = \frac{R}{R+1} \quad (1)$$

(c) The Variance, $V(X)$ is

$$V(X) = E(X^2) - \mu^2 \quad (2)$$

$$\therefore E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^1 x^2 R x^{R-1} dx$$

$$= R \int_0^1 x^{R+1} dx$$

$$\therefore E(X^2) = R \left[\frac{x^{R+2}}{R+2} \right]_0^1 = \frac{R}{R+2} \quad (3)$$

$\therefore$ By substituting, (1) and (3) in (2), we get

$$Var(X) = \frac{R}{R+2} - \frac{R^2}{(R+1)^2}$$

$$= R \left[\frac{1}{R+2} - \frac{R}{(R+1)^2} \right]$$

$$= R \left[\frac{(R+1)^2 - R(R+2)}{(R+2)(R+1)^2} \right]$$

$$= R \left[\frac{R^2 + 2R + 1 - R^2 - 2R}{(R+2)(R+1)^2} \right]$$

$$\therefore Var(X) = R / [(R+2)(R+1)^2] \quad \#$$

1.2.9 Determine the distribution function, mean, and variance corresponding to the triangular density.

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ 2-x & \text{for } 1 \leq x \leq 2, \\ 0 & \text{elsewhere.} \end{cases}$$

Ans: $F_X(x) = \int_{-\infty}^x f(t) dt$

for $0 \leq x \leq 1$

$$F_X(x) = \int_0^x t dt$$

$$= \left[\frac{t^2}{2} \right]_0^x = \boxed{\frac{x^2}{2}}$$

Now, $F(1) = \frac{1}{2}$

* for $1 \leq x \leq 2$

$$F_X(x) = \frac{1}{2} + \int_1^x (2-t) dt$$

$$= \frac{1}{2} + \left[2t - \frac{t^2}{2} \right]_1^x$$

$$= \frac{1}{2} + \left[(2x - \frac{x^2}{2}) - (2 - \frac{1}{2}) \right]$$

$$F_X(x) = \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$F_X(x) = \boxed{2x - \frac{x^2}{2} - 1}$$

Now, $F(2) = 2(2) - \frac{4}{2} - 1 = \boxed{1}$

as we expect

$$\therefore F_X(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{2} & , 0 \leq x \leq 1 \\ 2x - \frac{x^2}{2} - 1 & , 1 \leq x \leq 2 \\ 1 & , x > 2 \end{cases}$$

Problems

1.2.1 Thirteen cards numbered $1, \dots, 13$ are shuffled and dealt one at a time. Say a *match* occurs on deal k if the k th card revealed is card number k . Let N be the total number of matches that occur in the thirteen cards. Determine $E[N]$.

Hint: Write $N = \mathbf{1}\{A_1\} + \dots + \mathbf{1}\{A_{13}\}$ where A_k is the event that a match occurs on deal k .

2.1 $E[\mathbf{1}\{A_1\}] = \Pr\{A_1\} = \frac{1}{13}$. Similarly, $E[\mathbf{1}\{A_k\}] = \Pr\{A_k\} = \frac{1}{13}$ for $k = 1, \dots, 13$. Then, because the expected value of a sum is always the sum of the expected values, $E[N] = E[\mathbf{1}\{A_1\}] + \dots + E[\mathbf{1}\{A_{13}\}] = \frac{1}{13} + \dots + \frac{1}{13} = 1$.

1.2.2 Let N cards carry the distinct numbers $x_1, \dots, x_n$. If two cards are drawn at random without replacement, show that the correlation coefficient ρ between the numbers appearing on the two cards is $-1/(N-1)$.

2.2 Let X be the first number observed and let Y be the second. We use the identity $(\sum x_i)^2 = \sum x_i^2 + \sum_{i \neq j} x_i x_j$ several times.

$$E[X] = E[Y] = \frac{1}{N} \sum x_i;$$

$$\text{Var}[X] = \text{Var}[Y] = \frac{1}{N} \sum x_i^2 - \left(\frac{1}{N} \sum x_i \right)^2 = \frac{(N-1) \sum x_i^2 - \sum_{i \neq j} x_i x_j}{N^2};$$

$$E[XY] = \frac{\sum_{i \neq j} x_i x_j}{N(N-1)};$$

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = \frac{\sum_{i \neq j} x_i x_j - (N-1) \sum x_i^2}{N^2(N-1)}$$

$$\rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y} = -\frac{1}{N-1}.$$

1.2.3 A population having N distinct elements is sampled with replacement. Because of repetitions, a random sample of size r may contain fewer than r distinct elements. Let S_r be the sample size necessary to get r distinct elements. Show that

$$E[S_r] = N \left(\frac{1}{N} + \frac{1}{N-1} + \dots + \frac{1}{N-r+1} \right).$$

2.3 Write $S_r = \xi_1 + \dots + \xi_r$ where ξ_k is the number of additional samples needed to observe k distinct elements, assuming that $k-1$ distinct elements have already been observed. Then, defining $p_k = \Pr\{\xi_k = 1\} = 1 - \frac{k-1}{N}$ we have $\Pr\{\xi_k = n\} = p_k(1-p_k)^{n-1}$ for $n = 1, 2, \dots$ and $E[\xi_k] = \frac{1}{p_k}$. Finally, $E[S_r] = E[\xi_1] + \dots + E[\xi_r] = \frac{1}{p_1} + \dots + \frac{1}{p_r}$ will verify the given formula.

$n-1$

1.2.4 A fair coin is tossed until the first time that the same side appears twice in succession. Let N be the number of tosses required.

(a) Determine the probability mass function for N .

(b) Let A be the event that N is even and B be the event that $N \leq 6$. Evaluate $\Pr\{A\}$, $\Pr\{B\}$, and $\Pr\{AB\}$.

P_1

P_r

2.4 Using an obvious notation, the event $\{N = n\}$ is equivalent to either $\overbrace{HTH \dots HTH}^{n-1}$ or $\overbrace{THT \dots THH}^{n-1}$ so $P_r\{N = n\} = 2 \times \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}$ for $n = 2, 3, \dots$; $Pr\{N \text{ is even}\} = \sum_{n=2,4,\dots} \left(\frac{1}{2}\right)^{n-1} = \frac{2}{3}$ and $P_r\{N \leq 6\} = \sum_{n=2}^6 \left(\frac{1}{2}\right)^{n-1} = \frac{31}{32}$. $Pr\{N \text{ is even and } N \leq 6\} = \sum_{n=2,4,6} \left(\frac{1}{2}\right)^{n-1} = \frac{21}{32}$.

1.2.5 Two players, A and B, take turns on a gambling machine until one of them scores a success, the first to do so being the winner. Their probabilities for success on a single play are p for A and q for B, and successive plays are independent.

2.5 Using an obvious notation, the probability that A wins on the $2n + 1$ trial is $Pr\left\{\overbrace{A^c B^c \dots A^c B^c}^{n \text{ losses}} A\right\}$

$$= [(1-p)(1-q)]^n p, \quad n = 0, 1, \dots \quad Pr\{A \text{ wins}\} = \sum_{n=0}^{\infty} [(1-p)(1-q)]^n p = \frac{p}{1 - (1-p)(1-q)}.$$

$$Pr\{A \text{ wins on } 2n + 1 \text{ play} | A \text{ wins}\} = (1 - \pi)\pi^n \text{ where } \pi = (1-p)(1-q). \quad E[\# \text{ trials} | A \text{ wins}] = \sum_{n=0}^{\infty} (2n+1)(1-\pi)\pi^n = 1 + \frac{2\pi}{1-\pi} = \frac{1 + (1-p)(1-q)}{1 - (1-p)(1-q)} = \frac{2}{1 - (1-p)(1-q)} - 1.$$

1.2.6 A pair of dice is tossed. If the two outcomes are equal, the dice are tossed again, and the process repeated. If the dice are unequal, their sum is recorded. Determine the probability mass function for the sum.

2.6 Let N be the number of losses and let S be the sum. Then $Pr\{N = n, S = k\} = \left(\frac{1}{6}\right)^{n-1} \left(\frac{5}{6} p_k\right)$ where $p_3 = p_{11} = p_4 = p_{10} = \frac{1}{15}$; $p_5 = p_9 = p_6 = p_8 = \frac{2}{15}$ and $p_7 = \frac{3}{15}$. Finally $Pr\{S = k\} = \sum_{n=1}^{\infty} Pr\{N = n, S = k\} = p_k$. (It is *not* a correct argument to simply say $Pr\{S = k\} = Pr\{\text{Sum of 2 dice} = k | \text{Dice differ}\}$. Compare with Exercise II, 2.1.)

1.2.7 Let U and W be jointly distributed random variables. Show that U and W are independent if

$$Pr\{U > u \text{ and } W > w\} = Pr\{U > u\} Pr\{W > w\} \quad \text{for all } u, w.$$

Given a pair (X, Y) of random variables, their *joint distribution function* is the function F_{XY} of two real variables given by

$$F_{XY}(x, y) = F(x, y) = Pr\{X \leq x \text{ and } Y \leq y\}.$$

If it happens that $F(x, y) = F_X(x) \times F_Y(y)$ for every choice of x, y , then the random variables X and Y are said to be *independent*. If X and Y are independent and possess

variable X . The probability that the event occurs is written $\Pr\{X \leq x\}$. Allowing x to vary, this probability defines a function

$$F(x) = \Pr\{X \leq x\}, \quad -\infty < x < +\infty,$$

called the *distribution function* of the random variable X . Where several random vari-

Ans:

$\therefore U$ and W are jointly distributed r.v.s such that

$$\Pr\{U > u, W > w\}$$

$$= [1 - F_U(u)] [1 - F_W(w)] \quad \forall u, w \quad (1)$$

$$\Rightarrow \Pr\{U \leq u, W \leq w\}$$

$$= 1 - \Pr\{U > u \text{ or } W > w\}$$

$$= 1 - [\Pr\{U > u\} + \Pr\{W > w\} - \Pr\{U > u, W > w\}]$$

$$= 1 - [(1 - F_U(u)) + (1 - F_W(w)) - (1 - F_U(u))(1 - F_W(w))]$$

$$= 1 - [1 - F_U(u) + 1 - F_W(w) - 1 + F_U(u) + F_W(w) - F_U(u)F_W(w)]$$

$$= F_U(u) F_W(w) \quad (2)$$

$\therefore U$ and W are independent r.v.s

##

Note that

A and B are independent events iff $\Pr(A \cap B) = \Pr(A)\Pr(B)$

1.2.7 Let U and W be jointly distributed random variables. Show that U and W are independent if

$$\Pr\{U > u \text{ and } W > w\} = \Pr\{U > u\} \Pr\{W > w\} \quad \text{for all } u, w.$$

Since U and W are JD

$$\begin{aligned} \Pr\{U \leq u \text{ and } W \leq w\} &= 1 - \Pr\{U > u \text{ or } W > w\} \\ &= 1 - (\Pr\{U > u\} + \Pr\{W > w\} - \Pr\{U > u \text{ and } W > w\}) \\ &= 1 - (1 - F_U(u) + 1 - F_W(w) - [1 - F_U(u) - F_W(w) + F_U(u)F_W(w)]) \\ &= 1 - 1 + F_U(u) - 1 + F_W(w) + 1 - F_U(u) - F_W(w) + F_U(u)F_W(w) \\ &= F_U(u)F_W(w) \end{aligned}$$

2.7 We are given that (*) $\Pr\{U > u, W > w\} = [1 - F_U(u)][1 - F_W(w)]$ for all u, w . According to the definition for independence we wish to show that $\Pr\{U \leq u, W \leq w\} = F_U(u)F_W(w)$ for all u, w . Taking complements and using the addition law

$$\begin{aligned} \Pr\{U \leq u, W \leq w\} &= 1 - \Pr\{U > u \text{ or } W > w\} \\ &= 1 - [\Pr\{U > u\} + \Pr\{W > w\} - \Pr\{U > u, W > w\}] \\ &= 1 - [(1 - F_U(u)) + (1 - F_W(w)) - (1 - F_U(u))(1 - F_W(w))] \\ &= F_U(u)F_W(w) \quad \text{after simplification.} \end{aligned}$$

2.8. Suppose X is a random variable with finite mean μ and variance σ^2 , and $Y = a + bX$ for certain constants $a, b \neq 0$. Determine the mean and variance for Y .

2.8 (a) $E[Y] = E[a + bX] = \int (a + bx)dF_X(x) = a \int dF_X(x) + b \int x dF_X(x) = a + bE[X] = a + b\mu$. In words, (a) implies that the expected value of a constant times a random variable is the constant times the expected value of the random variable. So $E[b^2(X - \mu)^2] = b^2E[(X - \mu)^2]$.

(b) $\text{Var}[Y] = E[(Y - E[Y])^2] = E[(a + bX - a - b\mu)^2] = E[b^2(X - \mu)^2] = b^2E[(X - \mu)^2] = b^2\sigma^2$

1.2.9 Determine the mean and variance for the probability mass function

$$p(k) = \frac{2(n-k)}{n(n-1)} \quad \text{for } k = 1, 2, \dots, n.$$

2.9 Use the usual sums of numbers formula (See I, 6 if necessary) to establish

$$\sum_{k=1}^n k(n-k) = \frac{1}{6}n(n+1)(n-1); \text{ and}$$

$$\sum_{k=1}^n k^2(n-k) = n \sum k^2 - \sum k^3 = \frac{1}{12}n^2(n+1)(n-1), \text{ so}$$

$$E[X] = \frac{2}{n(n-1)} \sum k(n-k) = \frac{1}{3}(n+1)$$

$$E[X^2] = \frac{3}{n(n-1)} \sum k^2(n-k) = \frac{1}{6}n(n+1), \text{ and}$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{1}{18}(n+1)(n-2).$$

2.10. Random variables X and Y are independent and have the probability mass functions

$$p_X(0) = \frac{1}{3}, \quad p_Y(1) = \frac{1}{6},$$

$$p_X(3) = \frac{1}{3}, \quad p_Y(2) = \frac{1}{3},$$

$$p_Y(3) = \frac{1}{2}.$$

Determine the probability mass function of the sum $Z = X + Y$.

2.10 Observe, for example, $\Pr\{Z = 4\} = \Pr\{X = 3, Y = 1\} = (\frac{1}{3})(\frac{1}{6})$, using independence. Continuing in this manner,

z	$\Pr\{Z = z\}$	$\frac{1}{12}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{12}$	$\frac{5}{6}$	$\frac{6}{6}$
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1.2.11 Random variables U and V are independent and have the probability mass functions

$$p_U(0) = \frac{1}{3}, \quad p_V(1) = \frac{1}{2},$$

$$p_U(1) = \frac{1}{3}, \quad p_V(2) = \frac{1}{2}.$$

$$p_U(2) = \frac{1}{3},$$

Determine the probability mass function of the sum $W = U + V$.

A random variable X is called *discrete* if there is a finite or denumerable set of distinct values $x_1, x_2, \dots$ such that $a_i = \Pr\{X = x_i\} > 0$ for $i = 1, 2, \dots$ and $\sum_i a_i = 1$. The function

$$p(x_i) = p_X(x_i) = a_i \quad \text{for } i = 1, 2, \dots \quad (1.1)$$

is called the *probability mass function* for the random variable X and is related to the distribution function via

$$p(x_i) = F(x_i) - F(x_i -) \quad \text{and} \quad F(x) = \sum_{x_i \leq x} p(x_i).$$

Ans

To get the prob. mass fn of the sum $W = U + V$, we have to find the following probabilities

$$\Pr\{W=1\}, \Pr\{W=2\}, \Pr\{W=3\} \text{ and } \Pr\{W=4\}$$

$$\Pr\{W=1\} = \Pr\{U=0, V=1\} = \frac{1}{3} \left(\frac{1}{2}\right) = \frac{1}{6}$$

$$\begin{aligned} \Pr\{W=2\} &= \Pr\{U=0, V=2\} + \Pr\{U=1, V=1\} \\ &= \frac{1}{3} \left(\frac{1}{2}\right) + \frac{1}{3} \left(\frac{1}{2}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \Pr\{W=3\} &= \Pr\{U=1, V=2\} + \Pr\{U=2, V=1\} \\ &= \frac{1}{3} \left(\frac{1}{2}\right) + \frac{1}{3} \left(\frac{1}{2}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

$$\text{and } \Pr\{W=4\} = \Pr\{U=2, V=2\} = \frac{1}{3} \left(\frac{1}{2}\right) = \frac{1}{6}$$

$\therefore \Pr\{W=w\}$ is determined as

w	1	2	3	4	Σ
$\Pr\{W=w\}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	1

U	0	1	2
Pr	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

V	1	2
Pr	$\frac{1}{2}$	$\frac{1}{2}$

$$W = U + V$$

W	1	2	3	4
P	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

$$P(W=1) = \{U=0, V=1\} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$P(W=2) = \{U=0, V=2\} + \{U=1, V=1\} = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{6}$$

$$P(W=3) = \{U=1, V=2\} + \{U=2, V=1\} = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{6}$$

$$P(W=4) = \{U=2, V=2\} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

2.11 Observe, for example, $Pr\{W=z\} = Pr\{U=0, V=2\} + Pr\{U=1, V=1\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{3}$. Continuing in this manner, arrive at

W	1	2	3	4
$Pr\{W=w\}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

2.12. Let U , V , and W be independent random variables with equal variances σ^2 . Define $X = U + W$ and $Y = V - W$. Find the covariance between X and Y .

2.12 Changing any of the random variables by adding or subtracting a constant will not affect the covariance. Therefore, by replacing U with $U - E[U]$, if necessary, etc, we may assume, without

loss of generality that all of the means are zero. Because the means are zero, $Cov[X, Y] = E[XY] - E[X]E[Y] = E[XY] = E[UV - UW + VW - W^2] = -E[W^2] = -\sigma^2$. ($E[UV] = E[U]E[V] = 0$, etc.)

1.2.13 Let X and Y be independent random variables each with the uniform probability density function

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the joint probability density function of U and V , where $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$.

$$\begin{aligned}
2.13 \quad Pr\{v < V, U \leq u\} &= Pr\{v < X \leq u, v < Y \leq u\} \\
&= Pr\{v < X \leq u\} Pr\{v < Y \leq u\} \text{ (by independence)} \\
&= (u - v)^2 \\
&= \iint_{(u', v') : v < v' \leq u' \leq u} f_{u,v}(u', v') du' dv' \\
&= \int_v^u \left\{ \int_{v'}^u f_{u,v}(u', v') du' \right\} dv'.
\end{aligned}$$

The integrals are removed from the last expression by successive differentiation, first w.r.t. v (changing sign because v is a lower limit) than w.r.t. u . This tells us

$$f_{u,v}(u, v) = -\frac{d}{du} \frac{d}{dv} (u - v)^2 = 2 \quad \text{for } 0 < v \leq u \leq 1.$$

1.3 The Major Discrete Distributions

1.3.1 Consider tossing a fair coin five times and counting the total number of heads that appear. What is the probability that this total is three?

of elements of this event (total heads is three)

$$= C_3^5 = \binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \times 4 \times 3!}{3! \cdot 2 \times 1} = 10$$

OK for tossing a coin 5 times

of outcomes in the sample space is $2^5 = 32$

$$\Rightarrow \text{pr}(X=3) = \frac{10}{32} = \frac{5}{16}$$

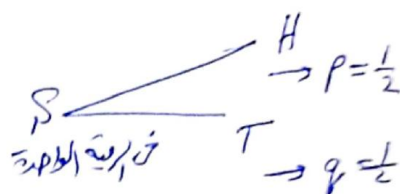
Not: $C_3^5 = 10$
كما في الفترة الأولى

pr (that the total of heads is three)

$$= \text{pr}(X=3)$$

$$= \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= 10 \left(\frac{1}{8}\right) \left(\frac{1}{4}\right) = \frac{10}{32} = \frac{5}{16}$$



1.3.2 A fraction $p = 0.05$ of the items coming off a production process are defective. If a random sample of 10 items is taken from the output of the process, what is the probability that the sample contains exactly one defective item? What is the probability that the sample contains one or fewer defective items?

$$pr(X=x) = \binom{n}{x} p^x q^{n-x}$$

$$X \sim \text{Bin}(10, 0.05)$$

$$a) \text{pr}(X=1) = \binom{10}{1} (0.05)^1 (0.95)^9$$

0.315/2

$$b) \text{pr}(X \leq 1) = \text{pr}(X=0) + \text{pr}(X=1)$$

$$= \binom{10}{0} (0.05)^0 (0.95)^{10} + 0.31512$$

$$= (0.95)^{10} + 0.31512$$

$$\therefore \text{pr}(X \leq 1) \approx 0.9139$$

1.3.3 A fraction $p = 0.05$ of the items coming off of a production process are defective. The output of the process is sampled, one by one, in a random manner. What is the probability that the first defective item found is the tenth item sampled?

X Counts # of failures prior to the first success

$$\Rightarrow X \sim \text{Geom}(p)$$

$$p = 0.05$$

$$p = 0.05$$
$$pr(X=10) = p(1-p)^{k-1}, k=1, 2, \dots$$

$$= 0.05 (1 - 0.05)^9$$

$$= 0.05(0.95)^9$$

$$= 0.0315$$

Variance

Variance & mean
الوسط والفرق

$E(X) = yp$, $Var(X) = \frac{1-p}{p^2}$

1.3.4 A Poisson distributed random variable X has a mean of $\lambda = 2$. What is the probability that X equals 2? What is the probability that X is less than or equal to 2?

$$pr(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

where, $X \sim \text{poisson}(2)$

a) $pr(X=2) = \frac{e^{-2} 2^2}{2!} = 2e^{-2} \approx 0.2707$

b) $pr(X \leq 2) = pr(X=0) + pr(X=1) + pr(X=2)$

$$= e^{-2} \left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} \right]$$

$$= e^{-2} (1 + 2 + 2)$$

$$= 5e^{-2}$$

$\therefore pr(X \leq 2) \approx 0.6767$

Problems

3.1. Suppose that X has a discrete uniform distribution on the integers $0, 1, \dots, 9$, and Y is independent and has the probability distribution $\Pr\{Y = k\} = a_k$ for $k = 0, 1, \dots$. What is the distribution of $Z = X + Y \pmod{10}$, their sum modulo 10?

3.1 Z has a discrete uniform distribution on $0, 1, \dots, 9$.

3.2. The *mode* of a probability mass function $p(k)$ is any value k^* for which $p(k^*) \geq p(k)$ for all k . Determine the mode(s) for

- (a) The Poisson distribution with parameter $\lambda > 0$.
- (b) The binomial distribution with parameters n and p .

3.2 In maximizing a continuous function, we often set the derivative equal to zero. In maximizing a function of a discrete variable, we equate the ratio of successive terms to one. More precisely, k^* is the smallest k for which $\frac{p(k+1)}{p(k)} < 1$, or, the smallest k for which $\frac{n-k}{k+1} \left(\frac{p}{1-p} \right) < 1$. Equivalently, (b)

$k^* = [(n+1)p]$ where $[x] = \text{greatest integer} \leq x$. For (a) let $n \rightarrow \infty, p \rightarrow 0, \lambda = np$. Then $k^* = [\lambda]$.

1.3.3 Let X be a Poisson random variable with parameter λ . Determine the probability that X is odd.

Ans:

$$\begin{aligned} \Pr\{X \text{ is odd}\} &= \sum_{k=1,3,5,\dots} \frac{\lambda^k e^{-\lambda}}{k!} \\ &= e^{-\lambda} \left[\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots \right] \\ &= e^{-\lambda} \sinh(\lambda) \\ &= e^{-\lambda} \cdot \frac{1}{2} (e^{\lambda} - e^{-\lambda}) \\ &= \frac{1}{2} (1 - e^{-2\lambda}) \end{aligned}$$

#

Remember
Maclaurin series of e^x , e^{-x} , $\sinh x$ and $\cosh x$

$$\begin{aligned} e^{\lambda} &= 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \\ e^{-\lambda} &= 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots \\ \sinh \lambda &= \frac{1}{2} (e^{\lambda} - e^{-\lambda}) \\ &= \lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots \\ \cosh \lambda &= \frac{1}{2} (e^{\lambda} + e^{-\lambda}) \\ &= 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \end{aligned}$$

3.3 Recall that $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots$ and $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots$ so that $\sinh \lambda = \frac{1}{2}(e^{\lambda} - e^{-\lambda}) = \lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots$. Then $\Pr\{X \text{ is odd}\} = \sum_{k=1,3,5,\dots} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sinh(\lambda) = \frac{1}{2}(1 - e^{-2\lambda})$.

3.4. Let U be a Poisson random variable with mean μ . Determine the expected value of the random variable $V = 1/(1 + U)$.

$$\begin{aligned} 3.4 \quad E[V] &= \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\lambda^k e^{-\lambda}}{k!} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} \\ &= \frac{1}{\lambda} e^{-\lambda} (e^{\lambda} - 1) = \frac{1}{\lambda} (1 - e^{-\lambda}). \end{aligned}$$

3.5. Let $Y = N - X$ where X has a binomial distribution with parameters N and p . Evaluate the product moment $E[XY]$ and the covariance $\text{Cov}[X, Y]$.

$$\begin{aligned} 3.5 \quad E[XY] &= E[X(N - X)] = NE[X] - E[X^2] \\ &= N^2 p - [Np(1 - p) + N^2 p^2] = N^2 p(1 - p) - Np(1 - p) \\ \text{Cov}[X, Y] &= E[XY] - E[X]E[Y] = -Np(1 - p). \end{aligned}$$

3.6. Suppose (X_1, X_2, X_3) has a multinomial distribution with parameters M and $\pi_i > 0$ for $i = 1, 2, 3$, with $\pi_1 + \pi_2 + \pi_3 = 1$.

- Determine the marginal distribution for X_1 .
- Find the distribution for $N = X_1 + X_2$.
- What is the conditional probability $\Pr\{X_1 = k | N = n\}$ for $0 \leq k \leq n$?

3.6 Your intuition should suggest the correct answers: (a) X_1 is binomially distributed with parameters M and π_1 ; (b) N is binomial with parameters M and $\pi_1 + \pi_2$; and (c) X_1 , given $N = n$, is conditionally binomial with parameters n and $p = \pi_1/(\pi_1 + \pi_2)$. To derive these correct answers formally, begin with

$$\Pr\{X_1 = i, X_2 = j, X_3 = k\} = \frac{M!}{i!j!k!} \pi_1^i \pi_2^j \pi_3^k; \quad i + j + k = M.$$

Since $k = M - (i + j)$

$$\Pr\{X_1 = i, X_2 = j\} = \frac{M!}{i!j!(M-i-j)!} \pi_1^i \pi_2^j \pi_3^{M-i-j}; \quad 0 \leq i + j \leq M.$$

$$\begin{aligned}
 (a) \Pr\{X_1 = i\} &= \sum_j \Pr\{X_1 = i, X_2 = j\} \\
 &= \frac{M!}{i!(M-i)!} \pi_1^i \sum_{j=0}^{M-i} \frac{(M-i)!}{j!(M-i-j)!} \pi_2^j \pi_3^{M-i-j} \\
 &= \binom{M}{i} \pi_1^i (\pi_2 + \pi_3)^{M-i}, \quad i = 0, 1, \dots, M.
 \end{aligned}$$

(b) Observe that $N = n$ if and only if $X_3 = M - n$. Apply the results of (a) to X_3 :

$$\Pr\{N = n\} = \Pr\{X_3 = M - n\} = \frac{M!}{n!(M-n)!} (\pi_1 + \pi_2)^n \pi_3^{M-n}$$

$$\begin{aligned}
 (c) \Pr\{X_1 = k | N = n\} &= \frac{\Pr\{X_1 = k, X_2 = n - k\}}{\Pr\{N = n\}} \\
 &= \frac{\frac{M!}{k!(M-n)!(n-k)!} \pi_1^k \pi_2^{n-k} \pi_3^{M-n}}{\frac{M!}{n!(M-n)!} (\pi_1 + \pi_2)^n \pi_3^{M-n}} \\
 &= \frac{n!}{k!(n-k)!} \left(\frac{\pi_1}{\pi_1 + \pi_2} \right)^k \left(\frac{\pi_2}{\pi_1 + \pi_2} \right)^{n-k}, \quad k = 0, 1, \dots, n.
 \end{aligned}$$

3.7. Let X and Y be independent Poisson distributed random variables having means μ and ν , respectively. Evaluate the convolution of their mass functions to determine the probability distribution of their sum $Z = X + Y$.

Ans: Let $Z = X + Y = n$ where X and Y are two independent Poisson r.v.s.
 The prob. mass fn of Z is

$$\begin{aligned}
 \Pr(Z = n) &= \sum_{k=0}^n \Pr\{X = k\} \Pr\{Y = n - k\} \\
 \Pr(Z = n) &= \sum_{k=0}^n \frac{\mu^k e^{-\mu}}{k!} \cdot \frac{\nu^{n-k} e^{-\nu}}{(n-k)!} = \frac{e^{-(\mu+\nu)}}{n!} \sum_{k=0}^n \frac{n! \mu^k \nu^{n-k}}{k! (n-k)!} \\
 \therefore \Pr(Z = n) &= \frac{e^{-(\mu+\nu)}}{n!} (\mu + \nu)^n \quad \text{By using Binomial Formula.} \\
 \therefore Z &\sim \text{Poisson}(\mu + \nu). \quad \#
 \end{aligned}$$

$$\begin{aligned}
3.7 \quad \Pr\{Z = n\} &= \sum_{k=0}^n \Pr\{X = k\} \Pr\{Y = n - k\} \\
&= \sum_{k=0}^n \frac{\mu^k e^{-\mu}}{k!} \frac{v^{(n-k)} e^{-v}}{(n-k)!} = e^{-(\mu+v)} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \mu^k v^{n-k} \\
&= \frac{e^{-(\mu+v)} (\mu + v)^n}{n!} \quad (\text{Using binomial formula.})
\end{aligned}$$

Z is Poisson distributed, parameter $\mu + v$.

3.8. Let X and Y be independent binomial random variables having parameters (N, p) and (M, p) , respectively. Let $Z = X + Y$.

- (a) Argue that Z has a binomial distribution with parameters $(N + M, p)$ by writing X and Y as appropriate sums of Bernoulli random variables.
- (b) Validate the result in (a) by evaluating the necessary convolution.

3.8 (a) X is the sum of N independent Bernoulli random variables, each with parameter p , and Y is the sum of M independent Bernoulli random variables each with the same parameter p . Z is the sum of $M + N$ independent Bernoulli random variables, each with parameter p .

(b) By considering the ways in which a committee of n people may be formed from a group comprised of M men and N women, establish the identity
$$\binom{M+N}{n} = \sum_{k=0}^n \binom{N}{k} \binom{M}{n-k}.$$

Then

$$\begin{aligned}
\Pr\{Z = n\} &= \sum_{k=0}^n \Pr\{X = k\} \Pr\{Y = n - k\} \\
&= \sum_{k=0}^n \binom{N}{k} p^k (1-p)^{N-k} \binom{M}{n-k} p^{n-k} (1-p)^{M-n+k} \\
&= \binom{M+N}{n} p^n (1-p)^{M+N-n} \quad \text{for } n = 0, 1, \dots, M+N.
\end{aligned}$$

Note:

$$\binom{N}{k} = 0 \quad \text{for } k > N.$$

3.9. Suppose that X and Y are independent random variables with the geometric distribution

$$p(k) = (1 - \pi)\pi^k \quad \text{for } k = 0, 1, \dots$$

Perform the appropriate convolution to identify the distribution of $Z = X + Y$ as a negative binomial.

$$\begin{aligned}
 3.9 \quad \Pr\{X + Y = n\} &= \sum_{k=0}^n \Pr\{X = k, Y = n - k\} = \sum_{k=0}^n (1 - \pi)\pi^k (1 - \pi)\pi^{n-k} \\
 &= (1 - \pi)^2 \pi^n \sum_{k=0}^n 1 = (n + 1)(1 - \pi)^2 \pi^n \text{ for } n \geq 0.
 \end{aligned}$$

3.10. Determine numerical values to three decimal places for $\Pr\{X = k\}$, $k = 0, 1, 2$, when

- (a) X has a binomial distribution with parameters $n = 10$ and $p = 0.1$.
- (b) X has a binomial distribution with parameters $n = 100$ and $p = 0.01$.
- (c) X has a Poisson distribution with parameter $\lambda = 1$.

3.10	k	Binomial $n = 10, p = .1$	Binomial $n = 100, p = .01$	Poisson $\lambda = 1$
	0	.349	.366	.368
	1	.387	.370	.368
	2	.194	.185	.184

3.11. Let X and Y be independent random variables sharing the geometric distribution whose mass function is

$$p(k) = (1 - \pi)\pi^k \quad \text{for } k = 0, 1, \dots,$$

where $0 < \pi < 1$. Let $U = \min\{X, Y\}$, $V = \max\{X, Y\}$, and $W = V - U$. Determine the joint probability mass function for U and W and show that U and W are independent.

$$3.11 \quad \Pr\{U = u, W = 0\} = \Pr\{X = u, Y = u\} = (1 - \pi)^2 \pi^{2u}, \quad u \geq 0.$$

$$\Pr\{U = u, W = w > 0\} = \Pr\{X = u, Y = u + w\} + \Pr\{Y = u, X = u + w\} = 2(1 - \pi)^2 \pi^{2u+w}$$

$$\Pr\{U = u\} = \sum_{w=0}^{\infty} \Pr\{U = u, W = w\} = \pi^{2u}(1 - \pi^2).$$

$$\Pr\{W = 0\} = \sum_{u=0}^{\infty} \Pr\{U = u, W = 0\} = (1 - \pi)^2 / (1 - \pi^2).$$

$$\Pr\{W = w > 0\} = 2[(1 - \pi)^2 / (1 - \pi^2)] \pi^w, \text{ and}$$

$$\Pr\{U = u, W = w\} = \Pr\{U = u\} \Pr\{W = w\} \text{ for all } u, w.$$

3.12. Suppose that the telephone calls coming into a certain switchboard during a one-minute time interval follow a Poisson distribution with mean $\lambda = 4$. If the switchboard can handle at most 6 calls per minute, what is the probability that the switchboard will receive more calls than it can handle during a specified one-minute interval?

3.12 Let X = number of calls to switch board in a minute. $Pr\{X \geq 7\} = 1 - \sum_{k=0}^6 \frac{4^k e^{-4}}{k!} = .111.$

3.13. Suppose that a sample of 10 is taken from a day's output of a machine that produces parts of which 5 percent are normally defective. If 100 percent of a day's production is inspected whenever the sample of 10 gives 2 or more defective parts, then what is the probability that 100 percent of a day's production will be inspected? What assumptions did you make?

Ans:
Assume that the inspected items are independently defective or good.
Let X be # of defects in a sample

$$Pr\{X=0\} = (0.95)^{10} = 0.5987$$

$$Pr\{X=1\} = \binom{10}{1} (0.95)^9 (0.05)^1 \quad \text{Binomial dist'n}$$

$$= 10 (0.95)^9 (0.05)$$

$$= 0.3151$$

$$Pr\{X \geq 2\} = 1 - (0.5987 + 0.3151)$$

$$= 0.0862 = 8.62\% \text{ which is the probability that 100\% of a day's production will be inspected.}$$

3.13 Assume that inspected items are independently defective or good. Let X = # of defects in sample.

$$Pr\{X = 0\} = (.95)^{10} = .599$$

$$Pr\{X = 1\} = 10(.95)^9(.05) = .315$$

$$Pr\{X \geq 2\} = 1 - (.599 + .315) = .086.$$

3.14. Suppose that a random variable Z has the geometric distribution

$$p_Z(k) = p(1 - p)^k \quad \text{for } k = 0, 1, \dots,$$

where $p = 0.10$.

- (a) Evaluate the mean and variance of Z .
- (b) What is the probability that Z strictly exceeds 10?

3.14 (a) $E[Z] = \frac{1-p}{p} = 9, \text{ } Var[Z] = \frac{1-p}{p^2} = 90$

(b) $Pr\{Z > 10\} = (.9)^{10} = .349.$

3.15. Suppose that X is a Poisson distributed random variable with mean $\lambda = 2$. Determine $Pr\{X \leq \lambda\}$.

Ans:

$$Pr\{X \leq 2\} = \sum_{x=0}^2 \frac{e^{-2} 2^x}{x!}$$

$$= e^{-2} \left[1 + \frac{2}{1!} + \frac{2^2}{2!} \right]$$

$\therefore Pr\{X \leq 2\} = 5e^{-2} = 0.6767$

$X \sim \text{Poisson}(\lambda)$
 $\Rightarrow P(x) = \frac{e^{-\lambda} \lambda^x}{x!},$
 $x = 0, 1, 2, \dots$

3.15 $Pr\{X \leq 2\} = \left(1 + 2 + \frac{2^2}{2} \right) e^{-2} = 5e^{-2} = .677.$

3.16. Consider the generalized geometric distribution defined by

$$p_k = b(1 - p)^k \quad \text{for } k = 1, 2, \dots,$$

and

$$p_0 = 1 - \sum_{k=1}^{\infty} p_k,$$

where $0 < p < 1$ and $p \leq b \leq p/(1 - p)$.

- (a) Evaluate p_0 in terms of b and p .
- (b) What does the generalized geometric distribution reduce to when $b = p$? When $b = p/(1 - p)$?
- (c) Show that $N = X + Z$ has the generalized geometric distribution when X is a Bernoulli random variable for which $\Pr\{X = 1\} = \alpha$, $0 < \alpha < 1$, and Z independently has the usual geometric distribution given in (3.4).

3.16 (a) $p_0 = 1 - b \sum_{k=1}^{\infty} (1 - p)^k = 1 - b \left(\frac{1 - p}{p} \right).$

(b) When $b = p$, then p_k is given by (3.4).

When $b = \frac{p}{1 - p}$, then p_k is given by (3.5).

(c)
$$\begin{aligned} \Pr\{N = n > 0\} &= \Pr\{X = 0, Z = n\} + \Pr\{X = 1, Z = n - 1\} \\ &= (1 - \alpha)p(1 - p)^n + \alpha p(1 - p)^{n-1} \\ &= [(1 - \alpha)p + \alpha p/(1 - p)](1 - p)^n \end{aligned}$$

So $b = (1 - \alpha)p + \alpha p/(1 - p)$.

1.4 Important Continuous Distributions

1.4.1 The lifetime, in years, of a certain class of light bulbs has an exponential distribution with parameter $\lambda = 2$. What is the probability that a bulb selected at random from this class will last more than 1.5 years? What is the probability that a bulb selected at random will last exactly 1.5 years?

A nonnegative random variable T is said to have an exponential distribution with parameter $\lambda > 0$ if the probability density function is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \quad (1.38)$$

The corresponding distribution function is

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad (1.39)$$

$T \sim \exp(2), \lambda = 2$

$\Rightarrow$ The pdf is $f_T(t) = \lambda e^{-\lambda t}$
and cdf is $F_T(t) = 1 - e^{-\lambda t}$

$\Rightarrow F_T(t) = 1 - e^{-2t}$
 $= \text{pr}(T \leq t)$

a) $\text{pr}(T > 1.5) = 1 - [1 - e^{-2(1.5)}]$
 $= e^{-3}$
 ≈ 0.0498

Another solution

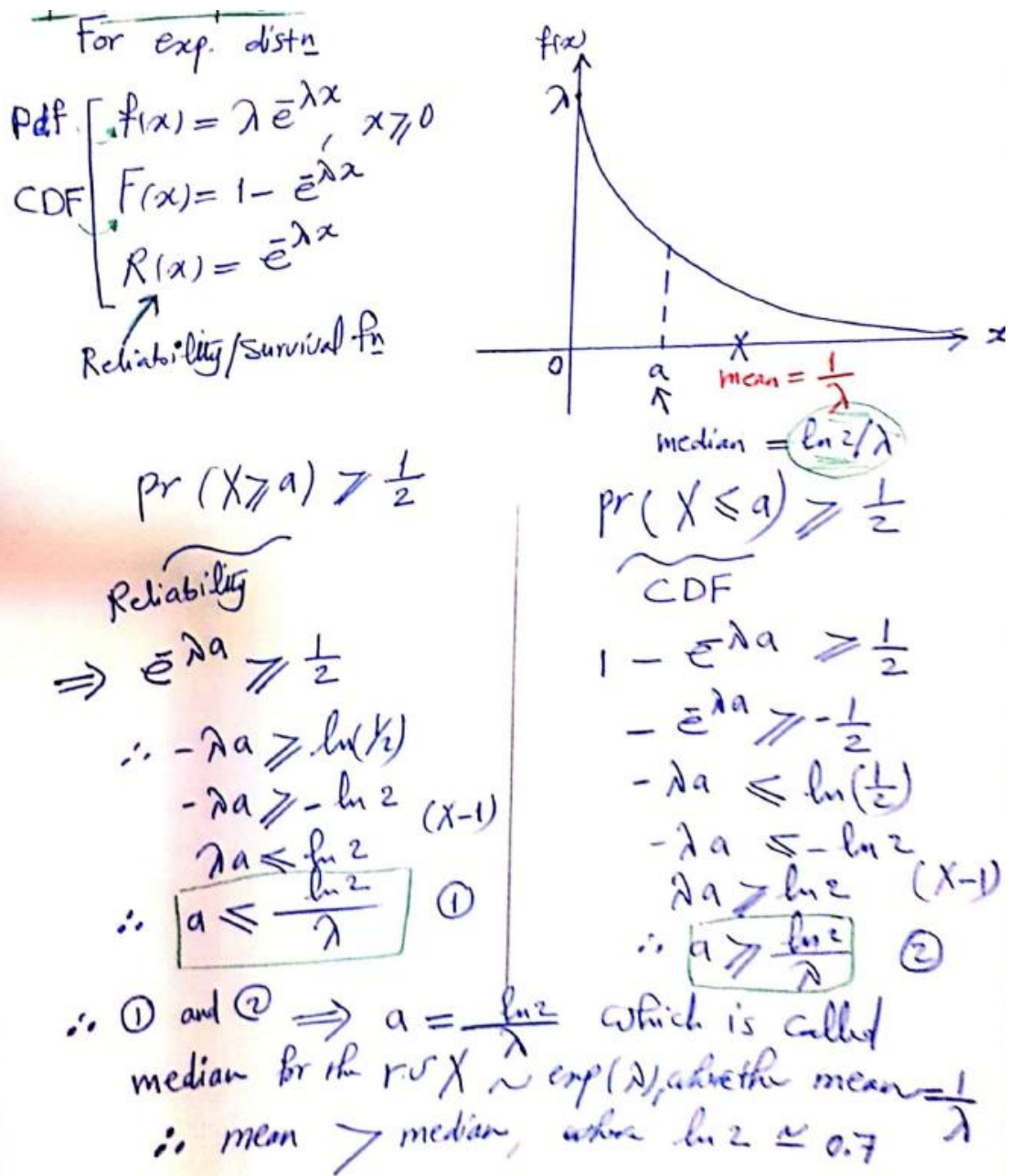
Reliability fn (Survival fn) is given by

$$R(t) = \text{pr}(T > t)$$
$$R(t) = e^{-\lambda t} = e^{-2t}, \lambda = 2$$
$$R(1.5) = e^{-2(1.5)} = e^{-3} \approx 0.0498$$

b) $\text{pr}(T = 1.5) = 0$

Note that:
For continuous r.v. T , $\text{pr}(T = t) = 0 \quad \forall t \in \mathbb{R}$

1.4.2 The median of a random variable X is any value a for which $\Pr\{X \leq a\} \geq \frac{1}{2}$ and $\Pr\{X \geq a\} \geq \frac{1}{2}$. Determine the median of an exponentially distributed random variable with parameter λ . Compare the median to the mean.



1.4.3 The lengths, in inches, of cotton fibers used in a certain mill are exponentially distributed random variables with parameter λ . It is decided to convert all measurements in this mill to the metric system. Describe the probability distribution of the length, in centimeters, of cotton fibers in this mill.

For exponential dist'n

$$E(X) = \frac{1}{\lambda} \text{ inch}$$

$$= \frac{2.54}{\lambda} \text{ cm}$$

Mean of lengths of Cotton fibers

where 1 inch = 2.54 cm

$$\Rightarrow X \sim \exp\left(\frac{\lambda}{1}\right) = \exp(\lambda) \text{ in mill system}$$

but,

$$X \sim \exp\left(\frac{\lambda}{2.54}\right) \text{ in metric system}$$

1.4.4 Twelve independent random variables, each uniformly distributed over the interval $(0, 1]$, are added, and 6 is subtracted from the total. Determine the mean and variance of the resulting random variable.

We have $X_k \sim \text{uniform}(0, 1)$, $k=1, 2, \dots, 12$

$$E(X_k) = \frac{1}{2}(a+b) = \frac{1}{2}(0+1) = \frac{1}{2}$$

$$\text{Var}(X_k) = \frac{1}{12}(b-a)^2 = \frac{1}{12}(1-0)^2 = \frac{1}{12}$$

For $Z = X_1 + X_2 + \dots + X_{12} - 6$,

Mean $E(Z) = 12\left(\frac{1}{2}\right) - 6 = 0$

where $E(X_1) = E(X_2) = \dots = E(X_{12}) = \frac{1}{2}$, $E(6) = 6$

and $\text{Var}(Z) = 12\left(\frac{1}{12}\right) - 0 = 1$

where $\text{Var}(X_1) = \text{Var}(X_2) = \dots = \text{Var}(X_{12}) = \frac{1}{12}$

$\text{Var}(6) = 0$

1.4.5 Let X and Y have the joint normal distribution described in equation (1.47). What value of α minimizes the variance of $Z = \alpha X + (1 - \alpha)Y$? Simplify your result when X and Y are independent.

Ans: $Z = \alpha X + (1-\alpha)Y$

$$\text{Var}(Z) = \alpha^2 \sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$$

$$\text{Var}(Z) = \alpha^2 \sigma_X^2 + (2\alpha - 2\alpha^2)\rho\sigma_X\sigma_Y + (1 - 2\alpha + \alpha^2)\sigma_Y^2$$

To minimize $V = \text{Var}(Z)$ by using α

Diff. V w.r.t α

and then equating by zero, i.e. let $\frac{\partial V}{\partial \alpha} = 0$ as follows.

$$2\alpha\sigma_X^2 + (2 - 4\alpha)\rho\sigma_X\sigma_Y + (-2 + 2\alpha)\sigma_Y^2 = 0$$

$$\therefore 2\alpha\sigma_X^2 - 4\alpha\rho\sigma_X\sigma_Y + 2\alpha\sigma_Y^2 = 2\sigma_Y^2 - 2\rho\sigma_X\sigma_Y$$

$$\therefore 2\alpha(\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2) = 2(\sigma_Y^2 - \rho\sigma_X\sigma_Y)$$

$$\therefore \alpha = \alpha^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}, \quad -1 \leq \rho < 1$$

• For independent r.v.s X and Y , $\rho = 0$, in this case

$$\alpha^* = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} \quad \#$$

1.4.7 Given independent exponentially distributed random variables S and T with common parameter λ , determine the probability density function of the sum $R = S + T$ and identify its type by name.

Given independent exponentially distributed random variables S and T with common parameter λ , determine the probability density function of the sum $R = S + T$ and identify its type by name.

Ans: $\because S, T \sim \exp(\lambda)$

$$\therefore R = S + T$$

$$\therefore R \sim \text{Gamma}(2, \lambda)$$

$$\therefore f_R(r) = \frac{\lambda^2}{\Gamma(2)} r^{2-1} e^{-\lambda r}, \quad r \geq 0$$

$$\therefore f_R(r) = \frac{\lambda^2}{1!} r e^{-\lambda r}$$

$$\therefore f_R(r) = \lambda^2 r e^{-\lambda r}, \quad r \geq 0$$

which is the Gamma prob. density f_n .

Remember

$$f_X(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}, \quad x \geq 0$$

$$\begin{matrix} X \rightarrow R \\ x \rightarrow r \end{matrix}$$

Problems

4.1. Evaluate the moment $E[e^{\lambda Z}]$, where λ is an arbitrary real number and Z is a random variable following a standard normal distribution, by integrating

$$E[e^{\lambda Z}] = \int_{-\infty}^{+\infty} e^{\lambda z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

Hint: Complete the square $-\frac{1}{2}z^2 + \lambda z = -\frac{1}{2}[(z - \lambda)^2 - \lambda^2]$ and use the fact that

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(z - \lambda)^2/2} dz = 1.$$

$$4.1 \quad E[e^{\lambda z}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2 + \lambda z} dz = e^{\frac{1}{2}\lambda^2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z-\lambda)^2} dz \right\} = e^{\frac{1}{2}\lambda^2}.$$

4.2. Let W be an exponentially distributed random variable with parameter θ and mean $\mu = 1/\theta$.

- (a) Determine $\Pr\{W > \mu\}$.
- (b) What is the mode of the distribution?

$$4.2 \quad (a) \quad \Pr\left\{W > \frac{1}{\theta}\right\} = e^{-\theta/\theta} = e^{-1} = .368 \dots$$

$$(b) \quad \text{Mode} = 0$$

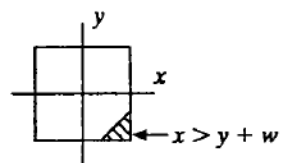
4.3. Let X and Y be independent random variables uniformly distributed over the interval $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ for some fixed θ . Show that $W = X - Y$ has a distribution that is independent of θ with density function

$$f_w(w) = \begin{cases} 1 + w & \text{for } -1 \leq w < 0, \\ 1 - w & \text{for } 0 \leq w \leq 1, \\ 0 & \text{for } |w| > 1. \end{cases}$$

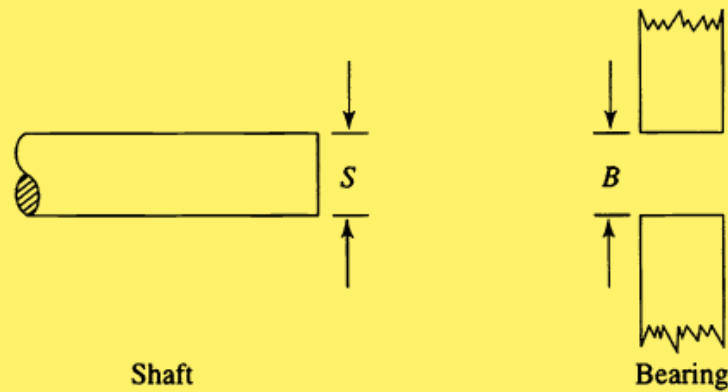
4.3 $X - \theta$ and $Y - \theta$ are both uniform over $[-\frac{1}{2}, \frac{1}{2}]$, independent of θ , and $W = X - Y = (X - \theta) - (Y - \theta)$. Therefore the distribution of W is independent of θ and we may determine it assuming $\theta = 0$. Also, the density of W is symmetric since that of both X and Y are.

$$\Pr\{W > w\} = \Pr\{X > Y + w\} = \frac{1}{2}(1 - w)^2, \quad w > 0$$

So $f_w(w) = 1 - w$ for $0 \leq w \leq 1$
and $f_w(w) = 1 - |w|$ for $-1 \leq w \leq +1$



4.4. Suppose that the diameters of bearings are independent normally distributed random variables with mean $\mu_B = 1.005$ inch and variance $\sigma_B^2 = (0.003)^2$ inch². The diameters of shafts are independent normally distributed random variables having mean $\mu_S = 0.995$ inch and variance $\sigma_S^2 = (0.004)^2$ inch².



Let S be the diameter of a shaft taken at random and let B be the diameter of a bearing.

- What is the probability $\Pr\{S > B\}$ of interference?
- What is the probability of one or fewer interferences in 20 random shaft-bearing pairs?

Hint: The clearance, defined by $C = B - S$, is normally distributed (why?), and interference occurs only if $C < 0$.

$$4.4 \quad \mu_c = -.010; \sigma_c^2 = (.005)^2, \Pr\{C < 0\} = \Pr\left\{\frac{C - .010}{.005} < \frac{-.010}{.005}\right\} = \Pr\{Z < -2\} = .0228.$$

4.5. If X follows an exponential distribution with parameter $\alpha = 2$, and independently, Y follows an exponential distribution with parameter $\beta = 3$, what is the probability that $X < Y$?

$$4.5 \quad \Pr\{X < Y\} = \int_0^\infty \left\{ \int_x^\infty 3e^{-3y} dy \right\} 2e^{-2x} dx = \frac{2}{5}.$$

1.5 Some Elementary Exercises

1.5.7 Let $X_1, X_2, \dots, X_n$ be independent random variables that are exponentially distributed with respective parameters $\lambda_1, \lambda_2, \dots, \lambda_n$. Identify the distribution of the minimum $V = \min\{X_1, X_2, \dots, X_n\}$.

Hint: For any real number v , the event $\{V > v\}$ is equivalent to $\{X_1 > v, X_2 > v, \dots, X_n > v\}$.

Given $X_1 \sim \exp(\lambda_1), X_2 \sim \exp(\lambda_2), \dots, X_n \sim \exp(\lambda_n)$ are independent r.v.s

To get distribution of V s.t $V = \min(X_1, X_2, X_3, \dots, X_n)$

Let $\min V = v, \quad v \in \mathbb{R}$

$$\begin{aligned} \therefore \Pr(V > v) &= \Pr(X_1 > v) \Pr(X_2 > v) \dots \Pr(X_n > v) \\ &= e^{-\lambda_1 v} e^{-\lambda_2 v} \dots e^{-\lambda_n v} \end{aligned}$$

$$\therefore \Pr(V > v) = e^{-\left(\sum_{i=1}^n \lambda_i\right)v}$$

$$\therefore V \sim \exp\left(\sum_i \lambda_i\right)$$

$\therefore V$ is exponentially distributed with parameter $\sum_{i=1}^n \lambda_i$

Problems

5.1. Let $X_1, X_2, \dots$ be independent and identically distributed random variables having the cumulative distribution function $F(x) = \Pr\{X \leq x\}$. For a fixed number ξ , let N be the first index k for which $X_k > \xi$. That is, $N = 1$ if $X_1 > \xi$; $N = 2$ if $X_1 \leq \xi$ and $X_2 > \xi$; etc. Determine the probability mass function for N .

$$\mathbf{5.1} \quad \Pr\{N > k\} = \Pr\{X_1 \leq \xi, \dots, X_k \leq \xi\} = [F(\xi)]^k, \quad k = 0, 1, \dots$$

$$\Pr\{N = k\} = \Pr\{N > k - 1\} - \Pr\{N > k\} = [1 - F(\xi)]F(\xi)^{k-1}, \quad k = 1, 2, \dots$$

5.2. Let $X_1, X_2, \dots, X_n$ be independent random variables, all exponentially distributed with the same parameter λ . Determine the distribution function for the minimum $Z = \min\{X_1, \dots, X_n\}$.

$$\begin{aligned} 5.2 \quad \Pr\{Z > z\} &= \Pr\{X_1 > z, \dots, X_n > z\} = \Pr\{X_1 > z\} \cdot \dots \cdot \Pr\{X_n > z\} \\ &= e^{-\lambda z} \cdot \dots \cdot e^{-\lambda z} = e^{-n\lambda z}, \quad z > 0. \end{aligned}$$

Z is exponentially distributed, parameter $n\lambda$.

5.3. Suppose that X is a discrete random variable having the geometric distribution whose probability mass function is

$$p(k) = p(1 - p)^k \quad \text{for } k = 0, 1, \dots$$

- (a) Determine the upper tail probabilities $\Pr\{X > k\}$ for $k = 0, 1, \dots$
- (b) Evaluate the mean via $E[X] = \sum_{k=0}^{\infty} \Pr\{X > k\}$.

$$5.3 \quad \Pr\{X > k\} = \sum_{l=k+1}^{\infty} p(1 - p)^l = (1 - p)^{k+1}, \quad k = 0, 1, \dots$$

$$E[X] = \sum_{k=0}^{\infty} \Pr\{X > k\} = \frac{1 - p}{p}.$$

5.4. Let V be a continuous random variable taking both positive and negative values and whose mean exists. Derive the formula

$$E[V] = \int_0^{\infty} [1 - F_V(v)] dv - \int_{-\infty}^0 F_V(v) dv.$$

5.4 Write $V = V^+ - V^-$ when $V^+ = \max\{V, 0\}$ and $V^- = \max\{-V, 0\}$. Then $\Pr\{V^+ > v\} = 1 - F_V(v)$ and $\Pr\{V^- > v\} = F_V(-v)$ for $v > 0$. Use (5.3) on V^+ and V^- together with $E[V] = E[V^+] - E[V^-]$. Mean does not exist if $E[V^+] = E[V^-] = \infty$.

5.5. Show that

$$E[W^2] = \int_0^{\infty} 2y[1 - F_W(y)] dy$$

for a nonnegative random variable W .

$$5.5 \quad E[W^2] = \int_0^{\infty} \Pr\{W^2 > t\} dt = \int_0^{\infty} [1 - F_W(\sqrt{t})] dt = \int_0^{\infty} 2y[1 - F_W(y)] dy \quad \text{by letting } y = \sqrt{t}.$$

5.6. Determine the upper tail probabilities $\Pr\{V > t\}$ and mean $E[V]$ for a random variable V having the exponential density

$$f_V(v) = \begin{cases} 0 & \text{for } v < 0, \\ \lambda e^{-\lambda v} & \text{for } v \geq 0, \end{cases}$$

where λ is a fixed positive parameter.

$$5.6 \quad \Pr\{V > t\} = \int_t^\infty \lambda e^{-\lambda v} dv = e^{-\lambda t}; \quad E[V] = \int_0^\infty \Pr\{V > t\} dt = \frac{1}{\lambda} \int_0^\infty \lambda e^{-\lambda t} dt = \frac{1}{\lambda}.$$

5.7. Let $X_1, X_2, \dots, X_n$ be independent random variables that are exponentially distributed with respective parameters $\lambda_1, \lambda_2, \dots, \lambda_n$. Identify the distribution of the minimum $V = \min\{X_1, X_2, \dots, X_n\}$.

Hint: For any real number v , the event $\{V > v\}$ is equivalent to $\{X_1 > v, X_2 > v, \dots, X_n > v\}$.

$$5.7 \quad \Pr\{V > v\} = \Pr\{X_1 > v, \dots, X_n > v\} = \Pr\{X_1 > v\} \cdots \Pr\{X_n > v\} \\ = e^{-\lambda_1 v} \cdots e^{-\lambda_n v} = e^{-(\lambda_1 + \dots + \lambda_n)v}, \quad v > 0.$$

V is exponentially distributed, parameter $\sum \lambda_i$.

5.8. Let $U_1, U_2, \dots, U_n$ be independent uniformly distributed random variables on the unit interval $[0, 1]$. Define the minimum $V_n = \min\{U_1, U_2, \dots, U_n\}$.

(a) Show that $\Pr\{V_n > v\} = (1 - v)^n$ for $0 \leq v \leq 1$.

(b) Let $W_n = nV_n$. Show that $\Pr\{W_n > w\} = [1 - (w/n)]^n$ for $0 \leq w \leq n$, and thus

$$\lim_{n \rightarrow \infty} \Pr\{W_n > w\} = e^{-w} \quad \text{for } w \geq 0.$$

5.8 Spares	3	2	1	0
A			*	*
B	*		*	
Mean	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$

Expected flash light operating duration = $\frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{2}{\lambda} = 2$ Expected battery operating durations!

5.9. A flashlight requires two good batteries in order to shine. Suppose, for the sake of this academic exercise, that the lifetimes of batteries in use are independent random variables that are exponentially distributed with parameter $\lambda = 1$. Reserve batteries do not deteriorate. You begin with five fresh batteries. On average, how long can you shine your light?

On average, you can shine your light for **2** units of time.

Step 1: Identify the Distribution of Battery Failures

The flashlight requires two batteries to operate. Let the lifetimes of the batteries be $X_1, X_2, \dots, X_5$, which are independent and identically distributed (i.i.d.) exponential random variables with parameter $\lambda = 1$. The time until the first battery failure occurs while the flashlight is on is the minimum of the lifetimes of the two batteries currently in use.

Step 2: Determine the Expected Time Between Failures

If two batteries are in use, each failing at rate $\lambda = 1$, the time T until the first failure follows an exponential distribution with a rate equal to the sum of the individual rates:

$$\lambda_{total} = \lambda + \lambda = 2(1) = 2$$

The expected time for one such interval is:

$$E[T_i] = \frac{1}{\lambda_{total}} = \frac{1}{2}$$

Due to the **memoryless property** of the exponential distribution, when one battery is replaced by a fresh one, the remaining battery in the flashlight acts as if it were new. Therefore, every interval until a failure occurs (as long as a spare is available) has an expected duration of $1/2$.

Step 3: Calculate the Total Number of Intervals

The process starts with 5 batteries. The flashlight continues to shine as long as there are at least 2 functional batteries. Each failure reduces the count of functional batteries by 1:

1. **First failure:** 5 batteries $\rightarrow$ 4 batteries remaining.
2. **Second failure:** 4 batteries $\rightarrow$ 3 batteries remaining.
3. **Third failure:** 3 batteries $\rightarrow$ 2 batteries remaining.
4. **Fourth failure:** 2 batteries $\rightarrow$ 1 battery remaining.
The flashlight stops after the fourth failure because only 1 battery is left, which is insufficient to power the device. There are 4 intervals of operation.

Step 4: Calculate the Total Expected Shine Time

The total time T_{total} is the sum of the four independent intervals T_1, T_2, T_3, T_4 :

$$E[T_{total}] = E[T_1] + E[T_2] + E[T_3] + E[T_4]$$

$$E[T_{total}] = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$$

Answer:

The average time the flashlight can shine is **2** units of time.

2 Conditional Probability and Conditional Expectation

2.1 The Discrete Case

Problems

1.1. Let M have a binomial distribution with parameters N and p . Conditioned on M , the random variable X has a binomial distribution with parameters M and π .

- (a) Determine the marginal distribution for X .
- (b) Determine the covariance between X and $Y = M - X$.

$$\begin{aligned} 1.1 \text{ (a)} \quad Pr\{X = k\} &= \sum_{m=k}^N \frac{N!}{m!(N-m)!} p^m (1-p)^{N-m} \frac{m!}{k!(m-k)!} \pi^k (1-\pi)^{m-k} \\ &= \binom{N}{k} (\pi p)^k (1-\pi p)^{N-k} \quad \text{for } k = 0, 1, \dots, N. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad E[XY] &= E[MX] - E[X]^2; \quad E[M] = Np; \\ E[M^2] &= N^2 p^2 + Np(1-p); \quad E[X|M] = M\pi \\ E[X] &= N\pi p; \quad E[X^2|M] = M^2 \pi^2 + M\pi(1-\pi); \\ E[X^2] &= (N\pi p)^2 + N\pi^2 p(1-p) + N\pi p(1-\pi), \\ E[MX] &= E\{ME[X|M]\} = \pi[N^2 p^2 + Np(1-p)] \\ Cov[X, Y] &= E[X(M-X)] - E[X]E[M-X] = -Np^2 \pi(1-\pi). \end{aligned}$$

1.2. A card is picked at random from N cards labeled $1, 2, \dots, N$, and the number that appears is X . A second card is picked at random from cards numbered $1, 2, \dots, X$ and its number is Y . Determine the conditional distribution of X given $Y = y$, for $y = 1, 2, \dots$.

$$1.2 \quad Pr\{X = x|Y = y\} = \frac{1/x}{[1/y + \dots + 1/N]} \quad \text{for } 1 \leq y \leq x \leq N.$$

1.3. Let X and Y denote the respective outcomes when two fair dice are thrown. Let $U = \min\{X, Y\}$, $V = \max\{X, Y\}$, and $S = U + V$, $T = V - U$.

- (a) Determine the conditional probability mass function for U given $V = v$.
 (b) Determine the joint mass function for S and T .

$$\begin{aligned} \mathbf{1.3 (a)} \quad \Pr\{U = u | V = v\} &= \frac{2}{2v-1} \quad \text{for } 1 \leq u < v \leq 6 \\ &= \frac{2}{2v-1} \quad \text{for } 1 \leq u = v \leq 6. \end{aligned}$$

1.3 (b)

	$\Pr\{S = s, T = t\}$										
t/s	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$	
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			
4					$\frac{2}{36}$		$\frac{2}{36}$				
5						$\frac{2}{36}$					

8

1.4. Suppose that X has a binomial distribution with parameters $p = \frac{1}{2}$ and N , where N is also random and follows a binomial distribution with parameters $q = \frac{1}{4}$ and $M = 20$. What is the mean of X ?

$$\mathbf{1.4} \quad E[X|N] = \frac{N}{2}; \quad E[X] = 20 \times \frac{1}{4} \times \frac{1}{2} = \frac{5}{2}.$$

1.5. A nickel is tossed 20 times in succession. Every time that the nickel comes up heads, a dime is tossed. Let X count the number of heads appearing on tosses of the dime. Determine $\Pr\{X = 0\}$.

$$\mathbf{1.5} \quad \Pr\{X = 0\} = \left(\frac{3}{4}\right)^{20} = .00317.$$

1.6. A dime is tossed repeatedly until a head appears. Let N be the trial number on which this first head occurs. Then a nickel is tossed N times. Let X count the number of times that the nickel comes up tails. Determine $\Pr\{X = 0\}$, $\Pr\{X = 1\}$, and $E[X]$.

$$\mathbf{1.6} \quad \Pr\{N = n\} = \left(\frac{1}{2}\right)^n \quad n = 1, 2, \dots$$

$$\Pr\{X = k|N = n\} = \binom{n}{k} \left(\frac{1}{2}\right)^n \quad 0 \leq k \leq n$$

$$\Pr\{X = k\} = \sum_{n=1}^{\infty} \binom{n}{k} \left(\frac{1}{2}\right)^n$$

$$\Pr\{X = 0\} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}$$

$$\Pr\{X = 1\} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{2}{2} = 1.$$

$$E[X|N] = \frac{N}{2}; \quad E[X] = \frac{1}{2} E[N] = 1.$$

1.7. The probability that an airplane accident that is due to structural failure is correctly diagnosed is 0.85, and the probability that an airplane accident that is not due to structural failure is incorrectly diagnosed as being due to structural failure is 0.35. If 30 percent of all airplane accidents are due to structural failure, then find the probability that an airplane accident is due to structural failure given that it has been diagnosed as due to structural failure.

$$\mathbf{1.7} \quad \Pr\{\text{True S.F.}|\text{Diag. S.F.}\} = \frac{.30(.85)}{.30(.85) + .70(.35)} = .51$$

1.8. Initially an urn contains one red and one green ball. A ball is drawn at random from the urn, observed, and then replaced. If this ball is red, then an additional red ball is placed in the urn. If the ball is green, then a green ball is added. A second ball is drawn. Find the conditional probability that the first ball was red given that the second ball drawn was red.

$$1.8 \quad \Pr\{\text{First Red}|\text{Second Red}\} = \frac{\Pr\{\text{Both Red}\}}{\Pr\{\text{Second Red}\}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)}{\frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)} = \frac{2}{3}.$$

1.9. Let N have a Poisson distribution with parameter $\lambda = 1$. Conditioned on $N = n$, let X have a uniform distribution over the integers $0, 1, \dots, n+1$. What is the marginal distribution for X ?

$$\begin{aligned} 1.9 \quad \Pr\{X = k\} &= \sum_{n=k-1}^{\infty} \Pr\{X = k|N = n\} \Pr\{N = n\} \\ &= \sum_{n=k-1}^{\infty} \frac{1}{n+2} \frac{e^{-1}}{n!} = \sum_{n=k-1}^{\infty} e^{-1} \left[\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right] = \frac{e^{-1}}{k!} \\ &\quad \text{for } k = 0, 1, \dots \quad (\text{Poisson, mean} = 1). \end{aligned}$$

1.10. *Do men have more sisters than women have?* In a certain society, all married couples use the following strategy to determine the number of children that they will have: If the first child is a girl, they have no more children. If the first child is a boy, they have a second child. If the second child is a girl, they have no more children. If the second child is a boy, they have exactly one additional child. (We ignore twins, assume sexes are equally likely, and the sex of distinct children are independent random variables, etc.) (a) What is the probability distribution for the number of children in a family? (b) What is the probability distribution for the number of girl children in a family? (c) A male child is chosen at random from all of the male children in the population. What is the probability distribution for the number of sisters of this child? What is the probability distribution for the number of his brothers?

1.10 A “typical” distribution of 8 families would break down as follows:

Family	#1	#2	#3	#4	#5	#6	#7	#8
Children	{G}	{G}	{G}	{G}	{B, G}	{B, G}	{B, B, G}	{B, B, B}

(a) Let N = the number of children in a family

$$Pr\{N = 1\} = \frac{1}{2}, \quad Pr\{N = 2\} = Pr\{N = 3\} = \frac{1}{4}.$$

(b) Let X = the number of girl children in a family.

$$Pr\{X = 0\} = \frac{1}{8}, \quad Pr\{X = 1\} = \frac{7}{8}$$

(c) Family #8 is three times as likely, and family #7 is twice as likely to be chosen as families #5 and #6.

$$Pr\{\text{No sisters}\} = Pr\{\#8\} = \frac{3}{7}.$$

$$Pr\{\text{One sister}\} = \frac{4}{7}.$$

$$Pr\{\text{Two brothers}\} = Pr\{\#8\} = \frac{3}{7}$$

$$Pr\{\text{One brother}\} = Pr\{\#7\} = \frac{2}{7}$$

$$Pr\{\text{No brothers}\} = \frac{2}{7}.$$

2.2 The Dice Game Craps

Problems

2.1. Let $X_1, X_2, \dots$ be independent identically distributed positive random variables whose common distribution function is F . We interpret $X_1, X_2, \dots$ as successive bids on an asset offered for sale. Suppose that the policy is followed of accepting the first bid that exceeds some prescribed number A . Formally, the accepted bid is X_N , where

$$N = \min\{k \geq 1; X_k > A\}.$$

Set $\alpha = \Pr\{X_1 > A\}$ and $M = E[X_N]$.

(a) Argue the equation

$$M = \int_A^\infty x dF(x) + (1 - \alpha)M$$

by considering the possibilities, either the first bid is accepted, or it is not.

(b) Solve for M , thereby obtaining

$$M = \alpha^{-1} \int_A^\infty x dF(x).$$

(c) When X_1 has an exponential distribution with parameter λ , use the memoryless property to deduce $M = A + \lambda^{-1}$.

(d) Verify this result by calculation in (b).

2.1 (a) The bid $X_1 = x$ is accepted if $x > A$; Otherwise, if $0 \leq x \leq A$, one is back where one started due to the independent and identically distributed nature of the bids.

$$E[X_N | X_1 = x] = \begin{cases} x & \text{if } x > A; \\ M & \text{if } 0 \leq x \leq A \end{cases}$$

(b) The given equation results from the law of total probability:

$$E[X_N] = \int_0^\infty E[X_N | X_1 = x] dF(x).$$

(c) When X is exponentially distributed the conditional distribution of X , given that $X > A$, is the same as the distribution of $A + X$.

(d) When $dF(x) = \lambda e^{-\lambda x} dx$, then

$$\alpha = e^{-\lambda A} \quad \text{and} \quad \int_A^\infty x \lambda e^{-\lambda x} dx = A\alpha + \alpha \int_0^\infty y \lambda e^{-\lambda y} dy = \alpha \left(A + \frac{1}{\lambda} \right) \quad \text{and} \quad M = A + \frac{1}{\lambda}.$$

2.2. Consider a pair of dice that are unbalanced by the addition of weights in the following manner: Die #1 has a small piece of lead placed near the four side, causing the appearance of the outcome 3 more often than usual, while die #2 is weighted near the three side, causing the outcome 4 to appear more often than usual. We assign the probabilities

Die #1

$$p(1) = p(2) = p(5) = p(6) = 0.166667,$$

$$p(3) = 0.186666,$$

$$p(4) = 0.146666;$$

Die #2

$$p(1) = p(2) = p(5) = p(6) = 0.166667,$$

$$p(3) = 0.146666,$$

$$p(4) = 0.186666.$$

Determine the win probability if the game of craps is played with these loaded dice.

2.2 The distribution for the sum is

$$\begin{aligned} p(2) &= p(12) = \frac{1}{36} & p(5) &= p(9) = \frac{4}{36} \\ p(3) &= p(11) = \frac{2}{36} & p(6) &= p(8) = \frac{5}{36} - (.02)^2 \\ p(4) &= p(10) = \frac{3}{36} & p(7) &= \frac{6}{36} + 2(.02)^2 \\ Pr\{\text{Win}\} &= .4924065. \end{aligned}$$

2.3 Random Sums

2.3.3 Suppose that upon striking a plate a single electron is transformed into a number N of electrons, where N is a random variable with mean μ and standard deviation σ . Suppose that each of these electrons strikes a second plate and releases further electrons, independently of each other and each with the same probability distribution as N . Let Z be the total number of electrons emitted from the second plate. Determine the mean and variance of Z .

$E(\xi_k) = \mu, \text{Var}(\xi_k) = \sigma^2$
 $E(N) = \mu, \text{Var}(N) = \sigma^2$
 $Z = \xi_1 + \xi_2 + \dots + \xi_k + \dots + \xi_N$
 is the random sum
 $E(Z) = \mu(\mu) = \mu^2$
 $\text{Var}(Z) = \mu\sigma^2 + \mu^2\sigma^2$
 $\therefore \boxed{\text{Var}(Z) = \mu\sigma^2(1 + \mu)}$

$E(X) = \mu v$
 $\text{Var}(X) = v\sigma^2 + \mu^2\tau^2$
 where $X = \xi_1 + \xi_2 + \dots + \xi_N$

2.3.5 The number of accidents occurring in a factory in a week is a Poisson random variable with mean 2. The number of individuals injured in different accidents is independently distributed, each with mean 3 and variance 4. Determine the mean and variance of the number of individuals injured in a week.

Let us assume that ξ_k and N have the finite moments

$$\begin{aligned} E[\xi_k] &= \mu, & \text{Var}[\xi_k] &= \sigma^2, \\ E[N] &= v, & \text{Var}[N] &= \tau^2, \end{aligned} \quad (2.29)$$

and determine the mean and variance for $X = \xi_1 + \dots + \xi_N$ as defined in (2.22). The derivation provides practice in manipulating conditional expectations, and the results,

$$E[X] = \mu v, \quad \text{Var}[X] = v\sigma^2 + \mu^2\tau^2, \quad (2.30)$$

Let N is # accidents in a week, and J_k is # individuals injured for k^{th} accident. $X = J_1 + J_2 + \dots + J_N, N > 0$

$N \sim \text{poisson}(2)$

$$E(N) = 2, \quad \text{Var}(N) = 2$$

$$J_k: E(J_k) = 3, \quad \text{Var}(J_k) = 4$$

$$\therefore E(X) = \mu_V = 3(2) = 6,$$

$$\text{Var}(X) = \sigma_V^2 + \mu^2 \tau^2 = 2(4) + 9(2) = 26 \quad \#$$

Problems

3.1. The following experiment is performed: An observation is made of a Poisson random variable N with parameter λ . Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials.

- Formulate Z as a random sum and thereby determine its mean and variance.
- What is the distribution of Z ?

$N \sim \text{poisson}(\lambda)$, $\mathcal{J}_k \sim \text{Bernoulli}(p)$
 $\Rightarrow E(N) = \mu = \lambda$, $\Rightarrow E(\mathcal{J}_k) = \mu = p$
 $\text{Var}(N) = \tau^2 = \lambda$, $\text{Var}(\mathcal{J}_k) = \sigma^2 = p(1-p)$

For $Z = \mathcal{J}_1 + \mathcal{J}_2 + \dots + \mathcal{J}_N$, $N \geq 0$

$E(Z) = \mu \sigma^2 + \mu^2 \tau^2$
 $= \lambda p(1-p) + p^2 \lambda$
 $\therefore \text{Var}(Z) = \lambda p - \cancel{\lambda p^2} + \cancel{\lambda p^2} + p^2 \lambda$

$\therefore E(Z) = \lambda p$, $\text{Var}(Z) = \lambda p$

$\therefore Z \sim \text{poisson}(\lambda p)$ #

3.1 (a) $\nu = \tau^2 = \lambda$, $\mu = p$, $\sigma^2 = p(1-p)$

$E[Z] = \lambda p$; $\text{Var}[Z] = \lambda p(1-p) + \lambda p^2 = \lambda p$.

(b) Z has the Poisson distribution.

3.2. For each given p , let Z have a binomial distribution with parameters p and N . Suppose that N is itself binomially distributed with parameters q and M . Formulate Z as a random sum and show that Z has a binomial distribution with parameters pq and M .

3.2 $\Pr\{Z = k\} = \sum_{n=k}^M \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{M!}{n!(M-n)!} q^n (1-q)^{M-n}$
 $= \binom{M}{k} (qp)^k (1-qp)^{M-k} \quad \text{for } k = 0, 1, \dots, M.$

2.3.3 Suppose that $\xi_1, \xi_2, \dots$ are independent and identically distributed with $\Pr\{\xi_k = \pm 1\} = \frac{1}{2}$. Let N be independent of $\xi_1, \xi_2, \dots$ and follow the geometric probability mass function

$$p_N(k) = \alpha(1 - \alpha)^k \quad \text{for } k = 0, 1, \dots,$$

where $0 < \alpha < 1$. Form the random sum $Z = \xi_1 + \dots + \xi_N$.

(a) Determine the mean and variance of Z .

(b) Evaluate the higher moments $m_3 = E[Z^3]$ and $m_4 = E[Z^4]$.

Hint: Express Z^4 in terms of the ξ_i 's where $\xi_i^2 = 1$ and $E[\xi_i \xi_j] = 0$.

Ans: we have

ξ_k	-1	1
$\Pr(\xi_k)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\therefore \mu = E(\xi_k) = \sum \xi_k \Pr(\xi_k)$$

$$\therefore \mu = -1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 0$$

$$E(\xi_k^2) = (-1)^2\left(\frac{1}{2}\right) + 1^2\left(\frac{1}{2}\right) = 1$$

$$\text{and } \sigma^2 = \text{Var}(\xi_k) = E(\xi_k^2) - [E(\xi_k)]^2 = 1 - 0 = 1$$

$$\therefore N \sim \text{geom}(\alpha), \quad p_N(k) = \alpha(1 - \alpha)^k, \quad k = 0, 1, \dots$$

$$\therefore \nu = E(N) = \frac{1 - \alpha}{\alpha}, \quad \tau^2 = \text{Var}(N) = \frac{1 - \alpha}{\alpha^2}$$

$$\therefore Z = \xi_1 + \xi_2 + \dots + \xi_N$$

$$\therefore E(Z) = \mu \nu = 0 \left(\frac{1 - \alpha}{\alpha} \right) = 0$$

$$\text{and } \text{Var}(Z) = \nu \sigma^2 + \mu^2 \tau^2 = \frac{1 - \alpha}{\alpha} (1) + 0 \left(\frac{1 - \alpha}{\alpha^2} \right)$$

$$\therefore \text{Var}(Z) = \frac{1 - \alpha}{\alpha}$$

#

$X \sim \text{geom}(p)$
 $\Rightarrow \Pr(X = k) = p(1 - p)^k, k = 0, 1, \dots$
 $\Rightarrow E(X) = \frac{1 - p}{p}$
 $\text{Var}(X) = \frac{1 - p}{p^2}$

3.3 (a) $\mu = 0, \sigma^2 = 1, \nu = \frac{1 - \alpha}{\alpha}, \tau^2 = \frac{1 - \alpha}{\alpha^2}; E[Z] = 0, \text{Var}[Z] = \frac{1 - \alpha}{\alpha}.$

(b) $E[Z^3] = 0, E[Z^4] = E[N^2 + 2N(N - 1)]$

$$= 3 \left(\frac{(1 - \alpha)^2}{\alpha^2} + \frac{1 - \alpha}{\alpha^2} \right) - 2 \left(\frac{1 - \alpha}{\alpha} \right) = \frac{(1 - \alpha)(6 - 5\alpha)}{\alpha^2}$$

3.4. Suppose $\xi_1, \xi_2, \dots$ are independent and identically distributed random variables having mean μ and variance σ^2 . Form the random sum $S_N = \xi_1 + \dots + \xi_N$.

- Derive the mean and variance of S_N when N has a Poisson distribution with parameter λ .
- Determine the mean and variance of S_N when N has a geometric distribution with mean $\lambda = (1 - p)/p$.
- Compare the behaviors in (a) and (b) as $\lambda \rightarrow \infty$.

3.4 (a) $E[N] = \text{Var}[N] = \lambda; \quad E[S_N] = \lambda\mu \quad \text{Var}[S_N] = \lambda(\mu^2 + \sigma^2)$

(b) $E[N] = \lambda, \quad \text{Var}[N] = \frac{\lambda}{p} = \lambda(1 + \lambda)$

$$E[S_N] = \lambda\mu \quad \text{Var}[S_N] = \lambda(\mu^2 + \sigma^2) + \lambda^2\mu^2$$

(c) For large λ , $\text{Var}[S_N] \propto \lambda$ in (a), while $\text{Var}[S_N] \propto \lambda^2$ in (b).

2.3.5 To form a slightly different random sum, let $\xi_0, \xi_1, \dots$ be independent identically distributed random variables and let N be a nonnegative integer-valued random variable, independent of $\xi_0, \xi_1, \dots$. The first two moments are

$$E[\xi_k] = \mu, \quad \text{Var}[\xi_k] = \sigma^2,$$

$$E[N] = v, \quad \text{Var}[N] = \tau^2.$$

Determine the mean and variance of the random sum $Z = \xi_0 + \dots + \xi_N$.

$$\begin{aligned} \therefore E(\xi_k) &= \mu, \quad \text{Var}(\xi_k) = \sigma^2 \\ \text{and } \therefore E(N+1) &= E(N) + 1 = v + 1, \\ \text{Var}(N+1) &= \text{Var}(N) + \text{Var}(1) = \tau^2 + 0 = \tau^2 \end{aligned}$$

$$\therefore \text{For the random sum } X = \underbrace{\xi_0 + \xi_1 + \dots + \xi_N}_{N+1 \text{ terms}}$$

$$\begin{aligned} E(X) &= \mu(v+1), \\ \text{Var}(X) &= (v+1)\sigma^2 + \mu^2\tau^2 \end{aligned}$$

$\neq$

3.5 $E[Z] = (1 + \frac{v}{\lambda})\mu; \quad \text{Var}[Z] = v\sigma^2 + \mu^2\tau^2.$

2.4 Conditioning on a Continuous Random Variable*

Problems

4.1. Suppose that the outcome X of a certain chance mechanism depends on a parameter p according to $\Pr\{X = 1\} = p$ and $\Pr\{X = 0\} = 1 - p$, where $0 \leq p \leq 1$. Suppose that p is chosen at random, uniformly distributed over the unit interval $[0, 1]$, and then that two independent outcomes X_1 and X_2 are observed. What is the unconditional correlation coefficient between X_1 and X_2 ?

Note: Conditionally independent random variables may become dependent if they share a common parameter.

$$\begin{aligned} 4.1 \quad E[X_1] &= E[X_2] = \int_0^1 p dp = \frac{1}{2} \\ E[X_1 X_2] &= \int_0^1 p^2 dp = \frac{1}{3} \\ \text{Cov}[X_1, X_2] &= E[X_1 X_2] - E[X_1]E[X_2] = \frac{1}{12}. \end{aligned}$$

4.2. Let N have a Poisson distribution with parameter $\lambda > 0$. Suppose that, conditioned on $N = n$, the random variable X is binomially distributed with parameters $N = n$ and p . Set $Y = N - X$. Show that X and Y have Poisson distributions with respective parameters λp and $\lambda(1 - p)$ and that X and Y are independent.

Note: Conditionally dependent random variables may become independent through randomization.

$$\begin{aligned}
4.2 \quad Pr\{X=i, Y=j\} &= Pr\{X=i, N=i+j\} = Pr\{X=i|N=i+j\}Pr\{N=i+j\} \\
&= \binom{i+j}{i} p^i (1-p)^j \lambda^{i+j} e^{-\lambda} / (i+j)! \\
&= \left\{ \frac{(\lambda p)^i e^{-\lambda p}}{i!} \right\} \left\{ \frac{[\lambda(1-p)^j e^{-\lambda(1-p)}]}{j!} \right\}, \quad i, j \geq 0. \\
Pr\{X=i\} &= \sum_{j=0}^{\infty} Pr\{X=i, Y=j\} = \frac{(\lambda p)^i e^{-\lambda p}}{i!}, \\
Pr\{Y=j\} &= \frac{[\lambda(1-p)]^j e^{-\lambda(1-p)}}{j!}
\end{aligned}$$

and $Pr\{X=i, Y=j\} = Pr\{X=i\}Pr\{Y=j\}$ for all i, j .

2.4.3 Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, following an exponential density with parameter θ .

(a) What is the marginal distribution of X ?

(b) Determine the conditional density for λ given $X = k$.

Ans:

(a)
$$\begin{aligned} \text{pr}(X=k) &= \int_0^{\infty} \text{pr}(k|\lambda) f(\lambda) d\lambda \\ &= \int_0^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \theta e^{-\theta \lambda} d\lambda \\ &= \frac{\theta}{k!} \int_0^{\infty} \lambda^k e^{-(1+\theta)\lambda} d\lambda \\ \text{pr}(X=k) &= \frac{\theta}{k!} \int_0^{\infty} \left(\frac{u}{1+\theta}\right)^k e^{-u} \frac{du}{1+\theta}, \quad u = (1+\theta)\lambda \\ &= \frac{\theta}{k!} \left(\frac{1}{1+\theta}\right)^k \frac{1}{1+\theta} \int_0^{\infty} u^k e^{-u} du \\ &= \frac{\theta}{k!} \left(\frac{1}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right) \Gamma(k+1) \\ &= \frac{\theta}{k!} \left(\frac{1}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right) k! \\ \text{pr}(X=k) &= \left(\frac{\theta}{1+\theta}\right) \left(\frac{1}{1+\theta}\right)^k, \quad k=0, 1, \dots \end{aligned}$$

which is the prob. mass fn for geometric distn with parameter $\frac{\theta}{1+\theta}$

i.e. $X \sim \text{geom}\left(\frac{\theta}{1+\theta}\right)$ $\neq$

(b)
$$\begin{aligned} \because P_{\lambda, X}(\lambda, k) &= f_{\lambda|X}(\lambda|k) P_X(k) \\ &= \frac{P(k|\lambda) f(\lambda)}{X! \lambda} \end{aligned}$$

$X \sim \text{geom}(p)$
 $\Rightarrow \text{pr}(X=k)$
 $= p(1-p)^k$
 $, k=0, 1, 2, \dots$

$\left\{ \begin{array}{l} p \rightarrow \frac{\theta}{1+\theta} \\ 1-p \rightarrow 1 - \frac{\theta}{1+\theta} \\ \quad = \frac{1+\theta-\theta}{1+\theta} \\ \quad = \frac{1}{1+\theta} \end{array} \right.$

ity density function $f_{X,Y}(x,y)$. We define the conditional probability density function $f_{X|Y}(x|y)$ for the random variable X given that $Y = y$ by the formula

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0, \quad (2.36)$$

$$\begin{aligned} \therefore f_{\lambda|X}(\lambda|k) &= \frac{P_{X|\lambda}(k|\lambda) P(\lambda)}{P(k)} \\ &= \frac{\frac{e^{-\lambda} \lambda^k}{k!} \cdot \theta e^{-\theta \lambda}}{\left(\frac{\theta}{1+\theta}\right) \left(\frac{1}{1+\theta}\right)^k} \end{aligned}$$

$$\therefore f_{\lambda|X}(\lambda|k) = \frac{(1+\theta)^{k+1} \lambda^k e^{-(1+\theta)\lambda}}{k!}$$

$$\therefore f_{\lambda|X}(\lambda|k) = \frac{(1+\theta)^{k+1} \lambda^k e^{-(1+\theta)\lambda}}{\Gamma(k+1)}$$

which is the gamma prob. density fn with parameters $(k+1, 1+\theta)$

Note that

$$\begin{aligned} x &\rightarrow \lambda \\ \lambda &\rightarrow 1+\theta \\ n &\rightarrow k+1 \end{aligned}$$

for gamma pdf

Remember

$X \sim \text{gamma}(n, \lambda)$

$$\Rightarrow f(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}, \quad x \geq 0$$

where $\Gamma(n) = (n-1)!$, $n = 1, 2, \dots$

$$\begin{aligned} 4.3 \text{ (a)} \quad \Pr\{X=i\} &= \int_0^\infty \frac{\lambda^i e^{-\lambda}}{i!} \theta e^{-\theta \lambda} d\lambda = \frac{\theta}{i!} \int_0^\infty \lambda^i e^{-(1+\theta)\lambda} d\lambda = \left(\frac{\theta}{1+\theta}\right) \left(\frac{1}{1+\theta}\right)^i, \\ i &= 0, 1, \dots \text{ (Geometric distribution).} \end{aligned}$$

$$\text{(b)} \quad f(\lambda|X=k) = \frac{(1+\theta)^{k+1} \lambda^k e^{-(1+\theta)\lambda}}{k!} \quad \text{(Gamma).}$$

4.4. Suppose X and Y are independent random variables having the same Poisson distribution with parameter λ , but where λ is also random, being exponentially distributed with parameter θ . What is the conditional distribution for X given that $X + Y = n$?

$$\begin{aligned}
 4.4 \quad \Pr\{X=i, Y=j\} &= \frac{\theta}{i!j!} \int_0^\infty \lambda^{i+j} e^{-(\lambda+\theta)\lambda} d\lambda \\
 &= \binom{i+j}{i} \left(\frac{\theta}{2+\theta} \right) \left(\frac{1}{2+\theta} \right)^{i+j}, \quad i, j \geq 0. \\
 \Pr\{X=i, N=n\} &= \binom{n}{i} \left(\frac{\theta}{2+\theta} \right) \left(\frac{1}{2+\theta} \right)^n \quad 0 \leq i \leq n. \\
 \Pr\{N=n\} &= \left(\frac{\theta}{2+\theta} \right) \left(\frac{1}{2+\theta} \right)^n \sum_{i=0}^n \binom{n}{i} = \left(\frac{\theta}{2+\theta} \right) \left(\frac{2}{2+\theta} \right)^n, \quad n \geq 0. \\
 \Pr\{X=i | N=n\} &= \binom{n}{i} \left(\frac{1}{2} \right)^n, \quad 0 \leq i \leq n.
 \end{aligned}$$

4.5. Let X and Y be jointly distributed random variables whose joint probability mass function is given in the following table:

		x		
		-1	0	1
y	-1	$\frac{1}{9}$	$\frac{2}{9}$	0
	0	0	$\frac{1}{9}$	$\frac{2}{9}$
	1	$\frac{2}{9}$	0	$\frac{1}{9}$

$$p(x, y) = \Pr\{X = x, Y = y\}$$

Show that the covariance between X and Y is zero even though X and Y are not independent.

4.5 $\Pr\{X = 0, Y = 1\} = 0 \neq \Pr\{X = 0\}\Pr\{Y = 1\}$ so X and Y are *NOT* independent.

$$E[X] = E[Y] = 0, \quad \text{Cov}[X, Y] = E[XY] = \frac{1}{9} + \frac{1}{9} - \frac{2}{9} = 0.$$

4.6. Let $X_0, X_1, X_2, \dots$ be independent identically distributed nonnegative random variables having a continuous distribution. Let N be the first index k for which $X_k > X_0$. That is, $N = 1$ if $X_1 > X_0$, $N = 2$ if $X_1 \leq X_0$ and $X_2 > X_0$, etc. Determine the probability mass function for N and the mean $E[N]$. (Interpretation: $X_0, X_1, \dots$ are successive offers or bids on a car that you are trying to sell. Then N is the index of the first bid that is better than the initial bid.)

4.6 $Pr\{N > n\} = Pr\{X_0 \text{ is the largest among } \{X_0, \dots, X_n\}\} = \frac{1}{n+1}$ since each random variable has the same chance to be the largest. Or, one can integrate $Pr\{N > n\} = \int_0^\infty [1 - F(x)]^n f(x) dx = - \int_0^\infty [1 - F(x)]^n d[1 - F(x)] = \int_0^1 y^n dy = \frac{1}{n+1}$

$$Pr\{N = n\} = Pr\{N > n-1\} - Pr\{N > n\} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

$$E[N] = \sum_{n=0}^{\infty} Pr\{N > n\} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty.$$

2.4.7 Suppose that X and Y are independent random variables, each having the same exponential distribution with parameter α . What is the conditional probability density function for X , given that $Z = X + Y = z$?

Ans: $f_{X|Z}(x|z) = \frac{f_{X,Y}(x,y)}{f_Z(z)} \quad (1)$

$\therefore X \sim \exp(\alpha), Y \sim \exp(\alpha)$
 and $Z = X + Y$
 $\therefore Z \sim \text{gamma}(2, \alpha)$

$\therefore f_Z(z) = \frac{\alpha^2}{\Gamma(2)} z e^{-\alpha z}$

$\therefore f_Z(z) = \alpha^2 z e^{-\alpha z} \quad (2)$

$\therefore X$ and Y are independent r.v.s

$\therefore f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$\therefore f_{X,Y}(x,y) = \alpha^2 e^{-\alpha(x+y)}$

$\therefore f_{X,Y}(x,y) = \alpha^2 e^{-\alpha z}$

which equals the density for $f_{X,Z}(x,z)$ for $0 \leq x \leq z$

$\therefore f_{X,Z}(x,z) = \alpha^2 e^{-\alpha z} \quad (3)$

Substitute (2) and (3) in (1), we get

$f_{X|Z}(x|z) = \frac{\alpha^2 e^{-\alpha z}}{\alpha^2 z e^{-\alpha z}} = \frac{1}{z}, 0 \leq x \leq z$

$\therefore X|Z$ is uniformly distributed over $[0, z]$. $\#$

4.7 $f_{X,Z}(x,z) = \alpha^2 e^{-\alpha z}$ for $0 \leq x \leq z$.

$$f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \frac{\alpha^2 e^{-\alpha z}}{\alpha^2 z e^{-\alpha z}} = \frac{1}{z}, \quad 0 < x < z.$$

X is, conditional on $Z = z$, uniformly distributed over the interval $[0, z]$.

4.8. Let X and Y have the joint normal density given in I, (4.16). Show that the conditional density function for X , given that $Y = y$, is normal with moments

$$\mu_{X|Y} = \mu_X + \frac{\rho\sigma_X}{\sigma_Y}(y - \mu_Y)$$

and

$$\sigma_{X|Y} = \sigma_X \sqrt{1 - \rho^2}.$$

4.8 The key algebraic step in simplifying the joint divided by the marginal is

$$\begin{aligned} Q(x, y) - \left(\frac{y - \mu_Y}{\sigma_Y} \right)^2 &= \frac{1}{1 - \rho^2} \left\{ \left(\frac{x - \mu_X}{\sigma_X} \right)^2 - 2\rho \left(\frac{x - \mu_X}{\sigma_X} \right) \left(\frac{y - \mu_Y}{\sigma_Y} \right) + \rho^2 \left(\frac{y - \mu_Y}{\sigma_Y} \right)^2 \right\} \\ &= \frac{1}{1 - \rho^2} \left\{ \left(\frac{x - \mu_X}{\sigma_X} \right) - \rho \left(\frac{y - \mu_Y}{\sigma_Y} \right) \right\}^2 \\ &= \frac{1}{(1 - \rho^2)\sigma_X^2} \left\{ x - \mu_X - \rho \left(\frac{\sigma_X}{\sigma_Y} \right) (y - \mu_Y) \right\}^2 \end{aligned}$$

2.5 Martingales*

2.5.2 Let $U_1, U_2, \dots$ be independent random variables each uniformly distributed over the interval $(0, 1]$. Show that $X_0 = 1$ and $X_n = 2^n U_1 \cdots U_n$ for $n = 1, 2, \dots$ defines a martingale.

Solution for 2.5.2

For $i \geq 1$, U_i independent identically distributed
with $U_i \sim \text{Uniform}(0, 1]$

$X_0 = 1$, $X_n = 2^n \cdot U_1 \cdots U_n$, $n = 1, 2, \dots$

To prove X_n is a martingale, we have to prove:

① $E[|X_n|] < \infty$

and

② $E[X_{n+1} | X_0, \dots, X_n] = X_n$

For ①, as X_n is non-negative random variable,

$$E[|X_n|] = E[X_n] = E[2^n \cdot U_1 \cdots U_n]$$

$$= 2^n \cdot E[U_1] \cdots E[U_n]$$

As U_i 's are independent, $E[U_1 \cdots U_n] = E[U_1] \cdots E[U_n]$

Remember if $U \sim \text{Uniform}(a, b)$ then $E(U) = \frac{1}{2}(a+b)$
So, $E[U_i] = \frac{1}{2}(0+1) = \frac{1}{2}$

$$= 2^n \cdot \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n\text{-times}}$$

$$= \frac{2^n}{2^n} = 1$$

$\therefore E[|X_n|] = 1 < \infty$ (finite).

② $E[X_{n+1} | X_0, \dots, X_n]$

$$= E[2^{n+1} U_1 \cdots U_n \cdot U_{n+1} | X_0, \dots, X_n]$$

$$= E[2^n \cdot U_1 \cdots U_n \cdot 2 \cdot U_{n+1} | X_0, \dots, X_n]$$

$$= E[2^n \cdot U_1 \dots U_n \cdot 2 \cdot U_{n+1} \mid X_0, \dots, X_n]$$

$$= \underbrace{2^n \cdot U_1 \dots U_n}_{\substack{\text{is out because they are determined by} \\ X_0, \dots, X_n}} \cdot E[2 \cdot U_{n+1} \mid X_0, \dots, X_n]$$

$$= 2^n \cdot U_1 \dots U_n \cdot 2 E[U_{n+1} \mid X_0, \dots, X_n]$$

$$= 2^n \cdot U_1 \dots U_n \cdot 2 \underbrace{E[U_{n+1}]}_{\substack{\text{because } U_{n+1} \text{ is independent} \\ \text{of } U_1, \dots, U_n \text{ and} \\ \text{so independent of } X_0, \dots, X_n. \\ \text{Remember } E[X \mid Y] = E[X] \text{ if } X \text{ is} \\ \text{indep. of } Y.}}$$

$$= 2^n \cdot U_1 \dots U_n \cdot 2 \cdot \underbrace{\frac{1}{2}}_{\text{as } E[U_i] = \frac{1}{2}}$$

$$= 2^n \cdot U_1 \dots U_n$$

$$= X_n$$

So, we proved the second condition,
and hence X_n is a martingale.

2.5.3 Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \varepsilon_1 + \dots + \varepsilon_n$ be the sum of n independent random variables, each exponentially distributed with mean $E[\varepsilon] = 1$. Show that

$$X_n = 2^n \exp(-S_n), \quad n \geq 0$$

defines a martingale.

$S_0 = 0$, $S_n = \xi_1 + \dots + \xi_n$, $n \geq 1$
 where ξ_i are independent identically distributed
 with $\xi_i \sim \exp(1)$ (since $E[\xi_i] = 1$)

* Remember when $X \sim \exp(\lambda)$, then $E[X] = 1/\lambda$.
 we should prove that $X_n = 2^n e^{-S_n}$, $n \geq 0$
 is a martingale.

□ X_n is a non-negative r.v., so

$$E[|X_n|] = E[X_n]$$

use if $X \sim \exp(\lambda)$
 $E[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx$

$$= E[2^n e^{-S_n}]$$

$$E[\bar{e}^X] = \int_0^\infty \bar{e}^x \cdot \lambda \bar{e}^{2x} dx$$
 because if $X \sim \text{Exp}(\lambda)$

$$\begin{aligned}
 * E[\bar{e}^{\xi_i}] &= \int_0^\infty \bar{e}^x \bar{e}^x dx \\
 &= \int_0^\infty \bar{e}^{2x} dx \\
 &= -\frac{1}{2} \bar{e}^{2x} \Big|_0^\infty \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 &= E[2^n \bar{e}^{S_n}] \\
 &= 2^n E[\bar{e}^{\xi_1} \cdot \bar{e}^{\xi_2} \cdots \bar{e}^{\xi_n}] \\
 &= 2^n \underbrace{E[\bar{e}^{\xi_1}] \cdots E[\bar{e}^{\xi_n}]}_{\text{Since } \xi_i\text{'s are independent}}
 \end{aligned}$$

So, $E[X_n] = 1 < \infty$.

[2] Prove that $E[X_{n+1} | X_0, \dots, X_n] = X_n$

$$\begin{aligned}
 &E[2^{n+1} \bar{e}^{S_{n+1}} | X_0, \dots, X_n] \\
 &= E[2^n \cdot \bar{e}^{S_n} \cdot 2 \cdot \bar{e}^{\xi_{n+1}} | X_0, \dots, X_n] \\
 &= 2^n \cdot \bar{e}^{S_n} \cdot E[2 \bar{e}^{\xi_{n+1}} | X_0, \dots, X_n] \\
 &\quad \uparrow \text{because } 2^n \cdot \bar{e}^{S_n} \text{ is determined by } X_0, \dots, X_n \\
 &= 2^n \cdot \bar{e}^{S_n} \cdot 2 \cdot \underbrace{E[\bar{e}^{\xi_{n+1}}]}_{\text{Since } \xi_{n+1} \text{ is independent of } X_0, \dots, X_n} \\
 &= 2^n \cdot \bar{e}^{S_n} \cdot 2 \cdot \frac{1}{2} \quad \leftarrow \text{as } E[\bar{e}^{\xi_i}] = \frac{1}{2} \\
 &= 2^n \cdot \bar{e}^{S_n} \\
 &= X_n
 \end{aligned}$$

So, X_n is a martingale.

2.5.4 Let $\xi_1, \xi_2, \dots$ be independent Bernoulli random variables with parameter $p, 0 < p < 1$. Show that $X_0 = 1$ and $X_n = p^{-n} \xi_1 \cdots \xi_n, n = 1, 2, \dots$, defines a nonnegative martingale. What is the limit of X_n as $n \rightarrow \infty$?

ξ_i are independent identically distributed
with $\xi_i \sim \text{Bernolli}(p)$ $0 < p < 1$

$$X_0 = 1, \quad X_n = p^{-n} \cdot \xi_1 \cdots \xi_n, \quad n=1,2,\dots$$

$$\textcircled{1} E[X_n] = E[p^{-n} \cdot \xi_1 \cdots \xi_n]$$

$$= p^{-n} E[\xi_1] \cdots E[\xi_n]$$

as ξ_i s are independent

As : If $X \sim \text{Bernolli}(p)$
then $E[X] = p$

$$= p^{-n} \cdot \underbrace{p \cdots p}_{n\text{-times}}$$

$$= p^{-n} \cdot p^n$$

$$= p^0 = 1 < \infty$$

$$\textcircled{2} E[X_{n+1} | X_0, \dots, X_n]$$

$$= E[p^{-(n+1)} \xi_1 \cdots \xi_{n+1} | X_0, \dots, X_n]$$

$$= E[p^{-n} \xi_1 \cdots \xi_n \cdot p^{-1} \cdot \xi_{n+1} | X_0, \dots, X_n]$$

$$= \underbrace{p^{-n} \xi_1 \cdots \xi_n}_{\text{is out because it is determined by } X_0, \dots, X_n} E[p^{-1} \cdot \xi_{n+1} | X_0, \dots, X_n]$$

is out because it is determined by $X_0, \dots, X_n$.

$$= p^{-n} \xi_1 \cdots \xi_n \cdot p^{-1} E[\xi_{n+1} | X_0, \dots, X_n]$$

$$= p^{-n} \xi_1 \cdots \xi_n \cdot p^{-1} \cdot \underbrace{E[\xi_{n+1}]}$$

as ξ_{n+1} is independent
of $X_0, \dots, X_n$

$$= p^{-n} \xi_1 \cdots \xi_n \cdot \cancel{p^{-1}} \cdot \cancel{p}$$

$$= p^{-n} \xi_1 \cdots \xi_n$$

$$= X_n$$

So, X_n is a martingale.

Problems

5.1. Use the law of total probability for conditional expectations $E[E\{X|Y, Z\}|Z] = E[X|Z]$ to show

$$E[X_{n+2}|X_0, \dots, X_n] = E[E\{X_{n+2}|X_0, \dots, X_{n+1}\}|X_0, \dots, X_n].$$

Conclude that when X_n is a martingale.

$$E[X_{n+2}|X_0, \dots, X_n] = X_n.$$

5.1 Following the suggestion

$$\begin{aligned} E[X_{n+2}|X_0, \dots, X_n] &= E[E\{X_{n+2}|X_0, \dots, X_{n+1}\}|X_0, \dots, X_n] \\ &= E[X_{n+1}|X_0, \dots, X_n] = X_n \end{aligned}$$

• pb 2.5.1 Use the law of total probability for conditional expectations p.77

$E[E\{X|Y, Z\}|Z] = E[X|Z]$ to show

$$E[X_{n+2}|X_0, \dots, X_n] = E[E\{X_{n+2}|X_0, \dots, X_{n+1}\}|X_0, \dots, X_n] \quad (1)$$

conclude that when X_n is a martingale,

$$E[X_{n+2}|X_0, \dots, X_n] = X_n \quad (2)$$

Ans: First, note that from the properties of conditional expectation

$$E[h(Y)|Y=y] = h(y)$$

↑ determined by Y

Similarly, if Z is determined by $X_0, \dots, X_n$, then

$$E[Z|X_0, \dots, X_n] = Z$$

* Let $Z = X_0, \dots, X_n$ and $Y = X_{n+1}$ in R.H.S of (2)

$$\text{then } E[E\{X_{n+2}|X_0, \dots, X_{n+1}\}|X_0, \dots, X_n]$$

$$= E[E\{X_{n+2}|Z, Y\}|Z] = E[X_{n+2}|Z] \quad \text{by using (1)}$$

$$\begin{aligned} &\therefore E[E\{X_{n+2} | X_0, \dots, X_{n+1}\} | X_0, \dots, X_n] \\ &= E[X_{n+2} | X_0, \dots, X_n] \end{aligned}$$

Thus, formula ② is already proved $\#$

Now, if X_n is a martingale, then for $n \geq 0$

$$E[X_{n+1} | X_0, \dots, X_n] = X_n \quad (4)$$

$$\text{So, } E[X_{n+2} | X_0, \dots, X_{n+1}] = X_{n+1} \quad (5)$$

Substitute ⑤ in ②, We get

$$\begin{aligned} E[X_{n+2} | X_0, \dots, X_n] &= E[X_{n+1} | X_0, \dots, X_n] \\ &= X_n \text{ by using ④} \end{aligned}$$

Thus, formula ③ is also proved $\#$.

Solution

① Remember

$$E[Z | Y] = Z$$

if Z is determined by Y

$$E[h(y) | Y=y] = h(y)$$

because $h(y)$ is determined by $Y=y$.

Similarly, if $\underline{Z}$ is determined by $X_0, \dots, X_n$ then

$$E[Z | X_0, \dots, X_n] = Z$$

Following law *, let $Z = X_0, \dots, X_n$ and $Y = X_{n+1}$

then

$$\text{R.H.S.} = E[E[X_{n+2} | X_0, \dots, X_n, X_{n+1}] | X_0, \dots, X_n]$$

$$= E[E[X_{n+2} | Z, Y] | Z]$$

$$\stackrel{\text{by } (*)}{=} E[X_{n+2} | Z]$$

$$= E[X_{n+2} | X_0, \dots, X_n]$$

$$= \text{L.H.S.}$$

(b) If X_n is a martingale,

$$E[X_{n+1} | X_0, \dots, X_n] = X_n \quad \dots (1)$$

Let $n \rightarrow n+1$

$$E[X_{n+2} | X_0, \dots, X_{n+1}] = X_{n+1} \quad \dots (2)$$

put (2) in (a)

$$\begin{aligned} \text{put (2) in (a)} \\ (a) \quad E[X_{n+2} | X_0, \dots, X_n] \\ = E[E[X_{n+2} | X_0, \dots, X_{n+1}] | X_0, \dots, X_n] \end{aligned}$$

$$E[X_{n+2} | X_0, \dots, X_n]$$

$$= E[X_{n+1} | X_0, \dots, X_n]$$

by defn of martingale

$$X_n$$

5.2. Let $U_1, U_2, \dots$ be independent random variables each uniformly distributed over the interval $(0, 1]$. Show that $X_0 = 1$ and $X_n = 2^n U_1 \cdots U_n$ for $n = 1, 2, \dots$ defines a martingale.

$$\begin{aligned}
 5.2 \quad E[X_{n+1} | X_0, \dots, X_n] &= E[2U_{n+1}X_n | X_n] \\
 &= X_n E[2U_{n+1}] = X_n.
 \end{aligned}$$

Ans: To prove that $\check{X}_n$ is a martingale, we have to prove that:

$$\boxed{1} \quad E[|X_n|] < \infty$$

$$\boxed{2} \quad E[X_{n+1} | X_0, \dots, X_n] = X_n$$

For 1, as X_n is a non-negative random variable.

$$\begin{aligned} E[|X_n|] &= E[X_n] \\ &= E[2^n U_1 U_2 \dots U_n] \quad \text{given in pb.} \\ &= 2^n E(U_1) E(U_2) \dots E(U_n) \\ &= 2^n \underbrace{\frac{1}{2} \cdot \frac{1}{2} \dots \frac{1}{2}}_{n \text{ times}} \end{aligned}$$

$$\therefore E[|X_n|] = \frac{2^n}{2^n} = 1 < \infty \quad \text{(finite)} \quad \textcircled{1}$$

Remember
If $U \sim \text{Uniform}(a, b)$
 $\Rightarrow E(U) = \frac{1}{2}(a+b)$
 $\therefore E(U_i) = \frac{1}{2}(0+1) = \frac{1}{2}$

For 2,

$$\begin{aligned} E[X_{n+1} | X_0, \dots, X_n] \\ &= E[2^{n+1} U_1 \dots U_n U_{n+1} | X_0, \dots, X_n] \\ &= E[2^n U_1 \dots U_n \cdot 2 U_{n+1} | X_0, \dots, X_n] \\ &= 2^n U_1 \dots U_n E[2 U_{n+1} | X_0, \dots, X_n] \end{aligned}$$

is out because they are determined
by $X_0, X_1, \dots, X_n$

$$\begin{aligned} &= 2^n U_1 \dots U_n \cdot 2 E[U_{n+1} | X_0, \dots, X_n] \\ &= 2^n U_1 \dots U_n \cdot 2 E[U_{n+1}] \quad \text{because } U_{n+1} \text{ is independent of } X_0, \dots, X_n \\ &= 2^n U_1 \dots U_n \cdot 2 \cdot \frac{1}{2} \quad \text{where } E[U_i] = \frac{1}{2} \\ &= X_n, \quad n = 1, 2, \dots \quad \textcircled{2} \end{aligned}$$

$\therefore$ From $\textcircled{1}$, $\textcircled{2}$, we proved that $X_n, n = 0, 1, 2, \dots$ is a martingale.

#

$$\textcircled{1} E[|X_n|] < \infty ?$$

X_n is a non-negative r.v.

$$E[|X_n|] = E[X_n] \\ = E[2^n \cdot U_1 \dots U_n]$$

$$= 2^n E[U_1 \dots U_n] \leftarrow$$

$$= 2^n E[U_1] \dots E[U_n]$$

$$= 2^n \cdot \underbrace{\frac{1}{2} \cdot \dots \cdot \frac{1}{2}}_{n \text{ times}}$$

$$= 2^n \cdot \frac{1}{2} \cdot \dots \cdot \frac{1}{2}$$

Since $U_i \sim \text{Uniform}(0,1)$
 $E(U_i) = \frac{0+1}{2} = \frac{1}{2}$
 $= 2^n \cdot \left(\frac{1}{2}\right)^n = 1 < \infty$

$$\textcircled{2} E[X_{n+1} | X_0, \dots, X_n] = X_n ?$$

$$E[2^{n+1} \cdot U_1 \dots U_n \cdot U_{n+1} | X_0, \dots, X_n]$$

* $E[Z(Y)] = Z$
 if Z is determined by Y

* $E[Z | Y] = E[Z]$
 if $Z \perp Y$ indep.

$$= E[\underbrace{2^n \cdot 2}_{Z} \cdot \underbrace{U_1 \dots U_n \cdot U_{n+1}}_{Z} | X_0, \dots, X_n]$$

$$= 2^n \cdot U_1 \dots U_n \cdot 2 E[U_{n+1} | X_0, \dots, X_n]$$

$2^n \cdot U_1 \dots U_n$
 U_i 's indep.
 $\Rightarrow U_{n+1}$ is indep of X_n

$$= 2^n \cdot U_1 \dots U_n \cdot 2 \cdot E[U_{n+1}]$$

$$= 2^n \cdot U_1 \cdot \dots \cdot U_n \cdot 2 \cdot \frac{1}{2}$$

$$= \underline{2^n \cdot U_1 \cdot \dots \cdot U_n} = \underline{X_n}$$

5.3 Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \varepsilon_1 + \dots + \varepsilon_n$ be the sum of n independent random variables, each exponentially distributed with mean $E[\varepsilon] = 1$. Show that

$$X_n = 2^n \exp(-S_n), n \geq 0$$

defines a martingale.

$$\begin{aligned} 5.3 \quad E[X_{n+1} | X_0, \dots, X_n] &= E[2e^{-\varepsilon_{n+1}} X_n | X_n] \\ &= X_n E[2e^{-\varepsilon_{n+1}}] = X_n. \end{aligned}$$

5.4. Let $\xi_1, \xi_2, \dots$ be independent Bernoulli random variables with parameter p , $0 < p < 1$. Show that $X_0 = 1$ and $X_n = p^{-n} \xi_1 \cdot \dots \cdot \xi_n$, $n = 1, 2, \dots$, defines a nonnegative martingale. What is the limit of X_n as $n \rightarrow \infty$?

$$\begin{aligned} 5.4 \quad E[X_{n+1} | X_0, \dots, X_n] &= E\left[X_n \frac{\xi_{n+1}}{p} \middle| X_n\right] \\ &= X_n E\left[\frac{\xi_{n+1}}{p}\right] = X_n \quad \lim_{n \rightarrow \infty} X_n = 0. \end{aligned}$$

5.5. Consider a stochastic process that evolves according to the following laws: If $X_n = 0$, then $X_{n+1} = 0$, whereas if $X_n > 0$, then

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{1}{2} \\ X_n - 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

- (a) Show that X_n is a nonnegative martingale.
 (b) Suppose that $X_0 = i > 0$. Use the maximal inequality to bound

$$\Pr\{X_n \geq N \text{ for some } n \geq 0 | X_0 = i\}.$$

Note: X_n represents the fortune of a player of a fair game who wagers \$1 at each bet and who is forced to quit if all money is lost ($X_n = 0$). This *gambler's ruin* problem is discussed fully in III, Section 5.3.

$$5.5 \text{ (b) } \Pr\{X_n \geq N \text{ for some } n \geq 0 | X_0 = i\} \leq \frac{E(X_0)}{N} = \frac{i}{N}. \text{ (In fact, equality holds. See III, 5.3)}$$

3 Markov Chains: Introduction

3.1 Definitions

3.1.1 A Markov chain $X_0, X_1, \dots$ on states 0, 1, 2 has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix} \end{matrix}$$

and initial distribution $p_0 = \Pr\{X_0 = 0\} = 0.3$, $p_1 = \Pr\{X_0 = 1\} = 0.4$, and $p_2 = \Pr\{X_0 = 2\} = 0.3$. Determine $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\}$.

pb 3.1.1 p. 81
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Given $P = \begin{matrix} & \begin{matrix} \text{states} \\ 0 & 1 & 2 \end{matrix} \\ \begin{matrix} \text{states} \\ 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix} \end{matrix} \Rightarrow \text{Transition prob. Mx}$

$p_0 = \Pr\{X_0 = 0\} = 0.3$, $p_1 = \Pr\{X_0 = 1\} = 0.4$ and $p_2 = \Pr\{X_0 = 2\} = 0.3$

$\Rightarrow$ initial distribution

Determine $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\}$?

Ans: $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\}$

$= p_0 P_{01} P_{12} = p_0 P_{01} P_{12}, p_0 = \Pr\{X_0 = 0\} = 0.3$

$= 0.3(0.2)(0) = 0$

state 0 $\leadsto$ can't go to 1

$\Pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$

$= p_1 P_{10} P_{02}$

$p_1 = \Pr\{X_0 = 1\} = 0.4$

$= 0.4(0.9)(0.7)$

$= 0.252$

state 1 $\leadsto$ can go to 0

3.1.2 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix} \end{matrix}.$$

Determine the conditional probabilities

$$\Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}.$$

Pb 3.1.2 p. 81
Textbook

states

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{bmatrix} \end{matrix}$$

① $\Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\}$

$= \Pr\{X_3 = 1 | X_2 = 1, X_1 = 0\} \cdot \Pr\{X_2 = 1 | X_1 = 0\}$
 $= \Pr\{X_3 = 1 | X_2 = 1\} \cdot \Pr\{X_2 = 1 | X_1 = 0\}$
 $= P_{11} P_{01} = 0.6(0.2) = 0.12$

prop. of cond. prob.
Markov prop.

ii) $\Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}$

$= \Pr\{X_2 = 1 | X_1 = 1, X_0 = 0\} \cdot \Pr\{X_1 = 1 | X_0 = 0\}$
 $= \Pr\{X_2 = 1 | X_1 = 1\} \cdot \Pr\{X_1 = 1 | X_0 = 0\}$
 $= P_{11} P_{01} = 0.6(0.2) = 0.12$

prop. of cond. prob.
Markov prop.

#

Solution. (1) By definition of the conditional probability, we have

$$\mathbb{P}\{X_2 = 1, X_3 = 1 | X_1 = 0\} = \frac{\mathbb{P}\{X_2 = 1, X_3 = 1, X_1 = 0\}}{\mathbb{P}\{X_1 = 0\}}. \quad (3.1.10)$$

For the nominator in the right-hand-side of the latter, again by the formula of the conditional probability, we have (after rearranging terms)

$$\mathbb{P}\{X_3 = 1, X_2 = 1, X_1 = 0\} = \mathbb{P}\{X_3 = 1 | X_1 = 0, X_2 = 1\} \mathbb{P}\{X_2 = 1, X_1 = 0\} \quad (3.1.11)$$

By the Markov property (3.1.1), the first term in the right-hand-side of (3.1.11) simplifies to

$$\mathbb{P}\{X_3 = 1 | X_1 = 0, X_2 = 1\} = \mathbb{P}\{X_3 = 1 | X_2 = 1\}, \quad (3.1.12)$$

while again by the conditional probability formula, the second term of the right-hand-side of (3.1.10) satisfies

$$\mathbb{P}\{X_2 = 1 | X_1 = 0\} = \frac{\mathbb{P}\{X_2 = 1, X_1 = 0\}}{\mathbb{P}\{X_1 = 0\}}. \quad (3.1.13)$$

Combining the latter formulas (3.1.10), (3.1.11), (3.1.12) and (3.1.13), we obtain

$$\mathbb{P}\{X_2 = 1, X_3 = 1 | X_1 = 0\} = \mathbb{P}\{X_3 = 1 | X_2 = 1\} \cdot \mathbb{P}\{X_2 = 1 | X_1 = 0\} = P_{11}P_{01}.$$

Looking at the given transition matrix:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0.7 & \mathbf{0.2} & 0.1 \\ 0 & \mathbf{0.6} & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix} \end{matrix}$$

we infer that $\mathbb{P}\{X_2 = 1, X_3 = 1 | X_1 = 0\} = P_{11}P_{01} = (0.6)(0.2) = 0.12$.

Another short solution using the formula (3.1.4):

By definition of the conditional probability, we have

$$\begin{aligned} \mathbb{P}\{X_2 = 1, X_3 = 1 | X_1 = 0\} &= \frac{\mathbb{P}\{X_1 = 0, X_2 = 1, X_3 = 1\}}{\mathbb{P}\{X_1 = 0\}}, \\ &= \frac{\mathbb{P}\{X_1 = 0\} \cdot P_{01}P_{11}}{\mathbb{P}\{X_1 = 0\}}, \\ &= P_{01}P_{11}, \\ &= (0.2)(0.6) = 0.12. \end{aligned}$$

(2) For the second required quantity, we have

$$\begin{aligned} \mathbb{P}\{X_1 = 1, X_2 = 1 | X_0 = 0\} &= \frac{\mathbb{P}\{X_0 = 0, X_1 = 1, X_2 = 1\}}{\mathbb{P}\{X_0 = 0\}}, \\ &= \frac{\mathbb{P}\{X_0 = 0\} \cdot P_{01}P_{11}}{\mathbb{P}\{X_0 = 0\}}, \\ &= P_{01}P_{11}, \\ &= (0.2)(0.6) = 0.12. \blacksquare \end{aligned}$$

3.1.3 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix} \end{matrix}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$.

Let $\Pr\{X_0 = i\} = p_i$.

$$\begin{aligned} \Pr\{X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} \\ = p_{i_0} P_{i_0, i_1} \cdots P_{i_{n-2}, i_{n-1}} P_{i_{n-1}, i_n}. \end{aligned} \quad (3.8)$$

$$\begin{aligned} \therefore \Pr\{X_0 = 1, X_1 = 0, X_2 = 2\} &= p_1 P_{10} P_{02}, \quad p_1 = \Pr\{X_0 = 1\} = 1 \\ &= 1(0.3)(0.1) \end{aligned}$$

$$\therefore \Pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = 0.03$$

Solution. Using Formula (3.1.4), we obtain

$$\mathbb{P}\{X_0 = 1, X_1 = 0, X_2 = 2\} = p_1 P_{10} P_{02}. \quad (3.1.14)$$

From the given data, since it is **known** that the process starts in the state $X_0 = 1$, this means that this event is **certain** and thus its probability is equal to 1, so for the initial state

we have $p_1 = \mathbb{P}\{X_0 = 1\} = 1$, while from the transition matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix} \end{matrix}$$

we see that $P_{02} = 0.1$ and $P_{10} = 0.3$. Substituting the obtained values in formula (3.1.14), we obtain $\mathbb{P}\{X_0 = 1, X_1 = 0, X_2 = 2\} = p_1 P_{10} P_{02} = (1).(0.3).(0.1) = 0.03$. ■

" " $P_{(X_0=1)}=1$
 If it is known that the process starts in state $X_0 = 1$, determine the probability
 $\Pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$.

$$P\{X_0=1, X_1=0, X_2=2\} = P_1 P_{10} P_{02} = 1(0.3)(0.1) = 0.03$$

3.1.4 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \left\| \begin{matrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{matrix} \right\| \end{matrix}.$$

Determine the conditional probabilities

$$\Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\}.$$

Markov property is that

$$\begin{aligned} \Pr\{X_{n+1} = j | X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = i\} \\ = \Pr\{X_{n+1} = j | X_n = i\} \end{aligned} \quad (3.1)$$

for all time points n and all states $i_0, \dots, i_{n-1}, i, j$.

$$P_{ij}^{n, n+1} = \Pr\{X_{n+1} = j | X_n = i\}.$$

limit our discussion to this case. Then, $P_{ij}^{n, n+1} = P_{ij}$ is independent of n , and P_{ij} is the conditional probability that the state value undergoes a transition from i to j in one trial. It is customary to arrange these numbers P_{ij} in a *matrix*, in the infinite square array

$$\begin{aligned} \Pr\{X_{n+1} = j_1, \dots, X_{n+m} = j_m | X_0 = i_0, \dots, X_n = i_n\} \\ = \Pr\{X_{n+1} = j_1, \dots, X_{n+m} = j_m | X_n = i_n\} \end{aligned} \quad (3.9)$$

for all time points n, m and all states $i_0, \dots, i_n, j_1, \dots, j_m$. In other words, once (3.9) is established for the value $m = 1$, it holds for all $m \geq 1$ as well.

First, to find $\Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}$

$$\begin{aligned} & \Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\} \\ &= \Pr\{X_2 = 1 | X_1 = 1, X_0 = 0\} \cdot \Pr\{X_1 = 1 | X_0 = 0\} \text{ Conditional Prob. Property} \\ &= \Pr\{X_2 = 1 | X_1 = 1\} \cdot \Pr\{X_1 = 1 | X_0 = 0\} = P_{11}P_{01} \text{ Markov definition} \\ &= 0.2(0.1) \\ &= 0.02 \end{aligned}$$

Second, to find $\Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\}$

$$\begin{aligned} & \Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\} \\ &= \Pr\{X_3 = 1 | X_2 = 1, X_1 = 0\} \cdot \Pr\{X_2 = 1 | X_1 = 0\} \text{ Conditional Prob. Property} \\ &= \Pr\{X_3 = 1 | X_2 = 1\} \cdot \Pr\{X_2 = 1 | X_1 = 0\} = P_{11}P_{01} \text{ Markov definition} \\ &= 0.2(0.1) \\ &= 0.02 \end{aligned}$$

3.1.4 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{matrix}$$

Handwritten notes: $P\{X_1=1, X_2=1 | X_0=0\}$, $P\{X_1=1 | X_0=0\}$, $P\{X_2=1 | X_1=1, X_0=0\}$, $P\{X_2=1 | X_1=0\}$, P_{11} , $P_{01} = (0.2)(0.1) = 0.02$

Determine the conditional probabilities

$$\Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\}.$$

$$P\{X_1 = 1, X_2 = 1 | X_0 = 0\}$$

$$= P\{X_2 = 1 | X_1 = 1, X_0 = 0\} * P\{X_1 = 1 | X_0 = 0\}$$

↳ by Markov

$$= P\{X_2 = 1 | X_1 = 1\} * P\{X_1 = 1 | X_0 = 0\}$$

$$= P_{11} * P_{01} = (0.2)(0.1) = 0.02$$

$$P\{X_2 = 1, X_3 = 1 | X_1 = 0\}$$

$$= P\{X_3 = 1 | X_2 = 1, X_1 = 0\} * P\{X_2 = 1 | X_1 = 0\}$$

↳ by Markov

$$= P\{X_3 = 1 | X_2 = 1\} * P\{X_2 = 1 | X_1 = 0\}$$

$$= P_{11} * P_{01} = (0.2)(0.1) = 0.02$$

$$P\{X_1 = 1, X_2 = 1 | X_0 = 0\} = P\{X_2 = 1 | X_1 = 1, X_0 = 0\} P\{X_1 = 1 | X_0 = 0\} = P\{X_2 = 1 | X_1 = 1\} P\{X_1 = 1 | X_0 = 0\} = P_{11} P_{01} = (0.2)(0.1) = 0.02$$

$$P\{X_2 = 1, X_3 = 1 | X_1 = 0\} = P\{X_3 = 1 | X_2 = 1, X_1 = 0\} P\{X_2 = 1 | X_1 = 0\} = P\{X_3 = 1 | X_2 = 1\} P\{X_2 = 1 | X_1 = 0\} = P_{11} P_{01} = (0.2)(0.1) = 0.02$$

3.1.5 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{vmatrix} \end{matrix}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities

$$\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}.$$

3.1.5 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{vmatrix} \end{matrix}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities

① $\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and ② $\Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}.$

① $P_{\sum X_i = 1, X_1 = 1, X_2 = 0}$

$$= P_{11} P_{10} = (0.5)(0.1)(0.5) = 0.025$$

② $P_{\sum X_i = 1, X_2 = 1, X_3 = 0}$

$$= P_{11} P_{10} = (0.15)(0.1)(0.5) = 0.0075$$

$$P_{\sum X_i = 13} = P_{\sum X_i = 1 \mid X_0 = 13} P_{\sum X_i = 13} + P_{\sum X_i = 1 \mid X_0 = 03} P_{\sum X_i = 03}$$

$$= P_{11} P_1 + P_{01} P_0$$

$$= (0.1)(0.5) + (0.2)(0.5) = 0.05 + 0.25 = 0.3$$

$$P_{\sum X_i = 1, X_1 = 1, X_2 = 0} = P_{11} P_{10} = (0.5)(0.1)(0.5) = 0.025 \quad P_1 = P_{\sum X_i = 13} = 0.5$$

$$P_{\sum X_i = 1, X_2 = 1, X_3 = 0} = P_{11} P_{10} = (0.15)(0.1)(0.5) = 0.0075 \quad P_1 = P_{\sum X_i = 13} = P_{\sum X_i = 1 \mid X_0 = 03} P_{\sum X_i = 03}$$

$$P_1 = P_{\sum X_i = 13} = P_{\sum X_i = 1 \mid X_0 = 03} P_{\sum X_i = 03} + P_{\sum X_i = 1 \mid X_0 = 13} P_{\sum X_i = 13} = P_{01}(0.5) + P_{11}(0.5) = (0.2)(0.5) + (0.1)(0.5) = 0.15$$

Problems

3.1.1 A simplified model for the spread of a disease goes this way: The total population size is $N = 5$, of which some are diseased and the remainder are healthy. During any single period of time, two people are selected at random from the population and assumed to interact. The selection is such that an encounter between any pair of individuals in the population is just as likely as between any other pair. If one of these persons is diseased and the other not, with probability $\alpha = 0.1$ the disease is transmitted to the healthy person. Otherwise, no disease transmission takes place. Let X_n denote the number of diseased persons in the population at the end of the n th period. Specify the transition probability matrix.

$$\mathbf{1.1} \quad P_{55} = 1, P_{k,k+1} = \alpha \frac{\binom{k}{1} \binom{5-k}{1}}{\binom{5}{2}}, \quad k = 1, 2, 3, 4$$

$P_{ij} = 0$, otherwise.

3.1.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α . Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the n th stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(a) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(b) Determine the probability that a correct signal is received at stage 2.

Hint: This is $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$.

The transition probability matrix can be written as

$$P = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{vmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{vmatrix} \end{matrix}$$

(i) The probability that no error occurs up to stage $n = 2$ is given as follows.

$$p_0 = \Pr\{X_0 = 0\} = 1$$

$$\begin{aligned} \Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} &= p_0 P_{00} P_{00} \\ &= 1 \times (1-\alpha) \times (1-\alpha) \\ &= (1-\alpha)^2 \end{aligned}$$

(ii) The probability that a correct signal is received at stage 2 is given as follows.

$$\begin{aligned} &\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\} \\ &= p_0 P_{00} P_{00} + p_0 P_{01} P_{10} \\ &= (1-\alpha)^2 + \alpha^2 \\ &= 1 - 2\alpha + 2\alpha^2 \end{aligned}$$

① $\Pr\{X_0=0, X_1=0, X_2=0\}$ $P = \begin{vmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{vmatrix}$

$$= p_0 P_{00} P_{00} = (1)(1-\alpha)(1-\alpha) = 1 - 2\alpha + \alpha^2$$

② $\Pr\{X_0=0, X_1=0, X_2=0\} + \Pr\{X_0=0, X_1=1, X_2=0\}$

$$= p_0 P_{00} P_{00} + p_0 P_{01} P_{10}$$

$$= 1 - 2\alpha + \alpha^2 + (1)\alpha\alpha$$

$$= 1 - 2\alpha + 2\alpha^2$$

a) $\Pr\{X_0=0, X_1=0, X_2=0\} = p_0 P_{00} P_{00} = (1)(1-\alpha)(1-\alpha) = (1-\alpha)^2 = 1 - 2\alpha + \alpha^2$

b) $\Pr\{X_2=0\} = \Pr\{X_2=0, X_1=0, X_0=0\} + \Pr\{X_2=0, X_1=1, X_0=0\} = p_0 P_{00} P_{00} + p_0 P_{01} P_{10} = (1)(1-\alpha)^2 + (1)\alpha^2 = 1 - 2\alpha + 2\alpha^2$

1.2 (a) $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00}^2 = 1(1 - \alpha)^2 = (1 - \alpha)^2.$

(b) $\Pr\{X_0 = 0, X_2 = 0\} = P_{00}P_{00} + P_{01}P_{10} = (1 - \alpha)^2 + \alpha^2.$

3.1.3 Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1 - \alpha$. Similarly, a defective item is followed by another defective item with probability β and is followed by a good item with probability $1 - \beta$. If the first item is good, what is the probability that the first defective item to appear is the fifth item?

Given: let $G \rightarrow$ for good product item
 $D \rightarrow$ " bad " "
 $P_{GG} = \alpha$, $P_{GD} = 1 - \alpha$
 $P_{DD} = \beta$, $P_{DG} = 1 - \beta$

Ques: IF the 1st item is good,
 What's the prob. that the first defective item to appear is the fifth item?

Ans: $\Pr\{X_2 = G, X_3 = G, X_4 = G, X_5 = D | X_1 = G\}$
 $= \Pr\{X_5 = D | X_4 = G, X_3 = G, X_2 = G, X_1 = G\}$
 $\cdot \Pr\{X_4 = G | X_3 = G, X_2 = G, X_1 = G\}$
 $\cdot \Pr\{X_3 = G | X_2 = G, X_1 = G\} \cdot \Pr\{X_2 = G | X_1 = G\}$
 $= P_{GD} P_{GG}^3 = (1 - \alpha) \alpha^3 = \alpha^3 (1 - \alpha)$
 By using Cond. prob. prop.
 By using Markov defn

Solution. Label the event "defective production" by the state 0, and the event "good production" by the state 1. So the entries of the transition matrix (conditional probabilities) are as follows:

$$P_{00} = \beta; P_{01} = 1 - \beta; P_{10} = 1 - \alpha; P_{11} = \alpha.$$

Now, by definition of the conditional probability and using the Formula (3.1.4), we obtain

$$\begin{aligned} \Pr\{X_2 = 1, X_3 = 1, X_4 = 1, X_5 = 0 | X_1 = 1\} &= \frac{\Pr\{X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1, X_5 = 0\}}{\Pr\{X_1 = 1\}}, \\ &= \frac{\Pr\{X_1 = 1\} P_{11} P_{11} P_{11} P_{10}}{\Pr\{X_1 = 1\}}, \\ &= P_{11}^3 P_{10}, \\ &= \alpha^3 (1 - \alpha). \end{aligned}$$

Remark: Observe that this Markov chain has the following transition probability matrix:

$$P = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} \beta & 1 - \beta \\ 1 - \alpha & \alpha \end{pmatrix} \end{matrix}$$

A reverse choice of the state labels 0, 1 is also possible. ■

1.3 $\Pr\{X_2 = G, X_3 = G, X_4 = G, X_5 = D | X_1 = G\} = P_{GG}^3 P_{GD} = \alpha^3 (1 - \alpha).$

3.1.4 The random variables $\xi_1, \xi_2, \dots$ are independent and with the common probability mass function

$k =$	0	1	2	3
$\Pr\{\xi = k\} =$	0.1	0.3	0.2	0.4

Set $X_0 = 0$, and let $X_n = \max\{\xi_1, \dots, \xi_n\}$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{03} & \cdots \\ P_{10} & P_{11} & P_{12} & P_{13} & \cdots \\ P_{20} & P_{21} & P_{22} & P_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2} & P_{i3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

For transition probability matrix of a Markov chain

The elements of first row are given by

$$P_{0,0} = \Pr\{X_1 = 0\} = p_0 = 0.1$$

$$P_{0,1} = \Pr\{X_1 = 1\} = p_1 = 0.3$$

$$P_{0,2} = \Pr\{X_1 = 2\} = p_2 = 0.2$$

$$P_{0,3} = \Pr\{X_1 = 3\} = p_3 = 0.4$$

The elements of second row are given by

$$P_{1,0} = 0 \text{ where } X_n \text{ cannot decrease}$$

$$P_{1,1} = \Pr\{X_n = 1 | X_{n-1} = 1\} = \Pr\{\xi \leq 1\} = 0.1 + 0.3 = 0.4$$

$$P_{1,2} = \Pr\{X_n = 2 | X_{n-1} = 1\} = \Pr\{\xi = 2\} = 0.2$$

$$P_{1,3} = \Pr\{X_n = 3 | X_{n-1} = 1\} = \Pr\{\xi = 3\} = 0.4$$

The elements of third row are given by

$$P_{2,0} = P_{2,1} = 0 \text{ where } X_n \text{ cannot decrease}$$

$$P_{2,2} = \Pr\{X_n = 2 | X_{n-1} = 2\} = \Pr\{\xi \leq 2\} = 0.1 + 0.3 + 0.2 = 0.6$$

$$P_{2,3} = \Pr\{X_n = 3 | X_{n-1} = 2\} = \Pr\{\xi = 3\} = 0.4$$

The elements of fourth row are given by

$$P_{3,0} = P_{3,1} = P_{3,2} = 0 \text{ where } X_n \text{ cannot decrease}$$

$$P_{3,3} = \Pr\{X_n = 3 | X_{n-1} = 3\} = \Pr\{\xi \leq 3\} = 0.1 + 0.3 + 0.2 + 0.4 = 1$$

The transition probability matrix will be of the form

$$\begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{matrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{matrix} \right\| \end{matrix}$$

$$p_{00} = P\{X_n = 0 \mid X_{n-1} = 0\} = P\{\max\{\sum_{i=1}^n \epsilon_i \leq 0\} = P\{\sum \epsilon_i = 0\} = 0.1$$

$$p_{01} = P\{X_n = 1 \mid X_{n-1} = 0\} = P\{\sum \epsilon_i = 1\} = 0.3$$

$$p_{02} = P\{X_n = 2 \mid X_{n-1} = 0\} = P\{\sum \epsilon_i = 2\} = 0.2$$

$$p_{03} = P\{X_n = 3 \mid X_{n-1} = 0\} = P\{\sum \epsilon_i = 3\} = 0.4$$

$$p_{10} = P\{X_n = 0 \mid X_{n-1} = 1\} = 0$$

$$p_{11} = P\{X_n = 1 \mid X_{n-1} = 1\} = P\{\sum \epsilon_i \leq 1\}$$

$$= 0.4$$

$$p_{12} = P\{X_n = 2 \mid X_{n-1} = 1\} = P\{\sum \epsilon_i = 2\} = 0.2$$

$$p_{13} = P\{X_n = 3 \mid X_{n-1} = 1\} = P\{\sum \epsilon_i = 3\} = 0.4$$

$$p_{20} = P\{X_n = 0 \mid X_{n-1} = 2\} = 0$$

$$p_{21} = P\{X_n = 1 \mid X_{n-1} = 2\} = 0$$

$$p_{22} = P\{X_n = 2 \mid X_{n-1} = 2\} = P\{\sum \epsilon_i \leq 2\} = 0.6$$

$$p_{23} = P\{X_n = 3 \mid X_{n-1} = 2\} = P\{\sum \epsilon_i = 3\} = 0.4$$

$$p_{30} = P\{X_n = 0 \mid X_{n-1} = 3\} = 0$$

$$p_{31} = P\{X_n = 1 \mid X_{n-1} = 3\} = 0$$

$$p_{32} = P\{X_n = 2 \mid X_{n-1} = 3\} = 0$$

$$p_{33} = P\{X_n = 3 \mid X_{n-1} = 3\} = P\{\sum \epsilon_i \leq 3\} = 1$$

	0	1	2	3
0	0.1	0.3	0.2	0.4
1	0	0.4	0.2	0.4
2	0	0	0.6	0.4
3	0	0	0	1

$$p_{00} = P\{X_1 = 0 | X_0 = 0\} = P\{\xi_1 = 0\} = 0.1$$

$$p_{01} = P\{X_1 = 1 | X_0 = 0\} = P\{\xi_1 = 1\} = 0.3$$

$$p_{02} = P\{X_1 = 2 | X_0 = 0\} = P\{\xi_1 = 2\} = 0.2$$

$$p_{03} = P\{X_1 = 3 | X_0 = 0\} = P\{\xi_1 = 3\} = 0.4$$

$$p_{10} = P\{X_n = 0 | X_{n-1} = 1\} = 0$$

$$p_{11} = P\{X_n = 1 | X_{n-1} = 1\} = P\{\xi_n \leq 1\} = 0.1 + 0.3 = 0.4$$

$$p_{12} = P\{X_n = 2 | X_{n-1} = 1\} = P\{\xi_n = 2\} = 0.2$$

$$p_{13} = P\{X_n = 3 | X_{n-1} = 1\} = P\{\xi_n = 3\} = 0.4$$

$$p_{20} = P\{X_n = 0 | X_{n-1} = 2\} = 0$$

$$p_{21} = P\{X_n = 1 | X_{n-1} = 2\} = 0$$

$$p_{22} = P\{X_n = 2 | X_{n-1} = 2\} = P\{\xi_n \leq 2\} = 0.6$$

$$p_{23} = P\{X_n = 3 | X_{n-1} = 2\} = P\{\xi_n = 3\} = 0.4$$

$$p_{30} = P\{X_n = 0 | X_{n-1} = 3\} = 0$$

$$p_{31} = P\{X_n = 1 | X_{n-1} = 3\} = 0$$

$$p_{32} = P\{X_n = 2 | X_{n-1} = 3\} = 0$$

$$p_{33} = P\{X_n = 3 | X_{n-1} = 3\} = P\{\xi_n \leq 3\} = 1$$

$$\Rightarrow P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$P = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix} \end{matrix}$$

$$P^2 = \begin{bmatrix} 0.25+0.5 & 0.25+0 \\ 0.5+0 & 0.5+0 \end{bmatrix} = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix} \end{matrix}$$

$$P\{X_5 = 2 | X_3 = 1\} = p_{12}^2 = 0.25$$

1.4

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} .1 & .3 & .2 & .4 \\ 0 & .4 & .2 & .4 \\ 0 & 0 & .6 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

3.2 Transition Probability Matrices of a Markov Chain

3.2.1 A Markov chain $\{X_n\}$ on the states 0, 1, 2 has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \end{matrix}.$$

- (a) Compute the two-step transition matrix P^2 .
 (b) What is $\Pr\{X_3 = 1 | X_1 = 0\}$?
 (c) What is $\Pr\{X_3 = 1 | X_0 = 0\}$?

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$$P^2 = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{bmatrix}$$

$$P^2 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.47 & 0.13 & 0.40 \\ 0.42 & 0.14 & 0.44 \\ 0.26 & 0.17 & 0.57 \end{bmatrix} \end{matrix}$$

b) what's the prob. $\{X_3 = 1 | X_1 = 0\}$
 $\Pr\{X_3 = 1 | X_1 = 0\} = P_{01}^2 = 0.13$

c) what's the prob. $\{X_3 = 1 | X_0 = 0\}$
 $\Pr\{X_3 = 1 | X_0 = 0\} = P_{01}^3$

$P^3 = P \cdot P^2$

$$P^3 = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 0.47 & 0.13 & 0.40 \\ 0.42 & 0.14 & 0.44 \\ 0.26 & 0.17 & 0.57 \end{bmatrix}$$

$$P^3 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.313 & 0.16 & 0.527 \\ 0.334 & 0.156 & 0.51 \\ 0.402 & 0.143 & 0.455 \end{bmatrix} \end{matrix}$$

$\therefore P_{01}^3 = 0.16$

OR $P_{01}^3 = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.13 & 0.14 & 0.17 \end{bmatrix} = 0.16$

Solution. (1) We calculate the square of the matrix $\mathbf{P}$:

$$\mathbf{P}^2 = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0.47 & 0.13 & 0.40 \\ 0.42 & 0.14 & 0.44 \\ 0.26 & 0.17 & 0.57 \end{pmatrix} \end{matrix}$$

(2) From (1), we have

$$\mathbf{P}^2 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0.47 & \mathbf{0.13} & 0.40 \\ 0.42 & 0.14 & 0.44 \\ 0.26 & 0.17 & 0.57 \end{pmatrix} \end{matrix}$$

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Thus, we see that $\mathbb{P}[X_3 = 1 | X_1 = 0] = P_{01}^{(2)} = 0.13$.

(3) We calculate the cubic power of the matrix $\mathbf{P}$:

$$\mathbf{P}^3 = \mathbf{P} \cdot \mathbf{P}^2 = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.47 & 0.13 & 0.40 \\ 0.42 & 0.14 & 0.44 \\ 0.26 & 0.17 & 0.57 \end{pmatrix} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0.313 & \mathbf{0.16} & 0.527 \\ 0.334 & 0.156 & 0.51 \\ 0.402 & 0.143 & 0.455 \end{pmatrix} \end{matrix}$$

It follows that $\mathbb{P}[X_3 = 1 | X_0 = 0] = P_{01}^{(3)} = 0.16$. ■

3.2.2 A particle moves among the states 0, 1, 2 according to a Markov process whose transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix} \end{matrix}.$$

Let X_n denote the position of the particle at the n th move. Calculate $\Pr\{X_n = 0 | X_0 = 0\}$ for $n = 0, 1, 2, 3, 4$.

$$P_{ij}^{(n)} = \Pr\{X_{m+n} = j | X_m = i\}.$$

Ans:

At $n = 0$

$$\Pr\{X_0 = 0 | X_0 = 0\} = P_{00}^0 = 1$$

At $n = 1$

$$\Pr\{X_1 = 0 | X_0 = 0\} = P_{00}^1 = 0$$

At $n = 2$

$$\Pr\{X_2 = 0 | X_0 = 0\} = P_{00}^2$$

$$P^2 = P.P$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$\therefore pr\{X_2 = 0 | X_0 = 0\} = P_{00}^2 = 0.5$$

At $n = 3$

$$pr\{X_3 = 0 | X_0 = 0\} = P_{00}^3$$

$$P^3 = P.P^2$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0.375 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.375 & 0.25 \end{bmatrix}$$

$$\therefore pr\{X_3 = 0 | X_0 = 0\} = P_{00}^3 = 0.25$$

Finally, at $n = 4$

$$pr\{X_4 = 0 | X_0 = 0\} = P_{00}^4$$

$$P^4 = P^2.P^2$$

$$= \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix} \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3750 & 0.3125 & 0.3125 \\ 0.3125 & 0.3750 & 0.3125 \\ 0.3125 & 0.3125 & 0.3750 \end{bmatrix}$$

$$\therefore pr\{X_4 = 0 | X_0 = 0\} = P_{00}^4 = 0.375$$

$$P = \begin{bmatrix} 0 & 1 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad P^3 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Calculate $P\{X_n=0 \mid X_0=0\}$, $n=0,1,2,3,4$

$$P\{X_0=0 \mid X_0=0\} = 1$$

$$P\{X_1=0 \mid X_0=0\} = P_{00} = 0$$

$$P\{X_2=0 \mid X_0=0\} = P_{00}^{(2)} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P\{X_3=0 \mid X_0=0\} = P_{00}^{(3)} = P_{00} P_{00}^{(2)} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P\{X_4=0 \mid X_0=0\} = P_{00}^{(4)} = P_{00} P_{00}^{(3)} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{8} \end{bmatrix} = \frac{3}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P^2 = PP = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad P^3 = P P^2 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad P^4 = P P^3 = \begin{bmatrix} 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{5}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$n=0: P\{X_0=0 \mid X_0=0\} = P_{00} = 1$$

$$n=1: P\{X_1=0 \mid X_0=0\} = P_{00}^{(1)} = 0$$

$$n=2: P\{X_2=0 \mid X_0=0\} = P_{00}^{(2)} = P_{00} P_{00}^{(1)} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = 0 + 0.25 + 0.25 = 0.5 = \frac{1}{2}$$

$$n=3: P\{X_3=0 \mid X_0=0\} = P_{00}^{(3)} = P_{00} P_{00}^{(2)} = 0.25 = \frac{1}{4}$$

$$n=4: P\{X_4=0 \mid X_0=0\} = P_{00}^{(4)} = 0.375 = \frac{3}{8}$$

3.2.4 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \end{matrix}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_2 = 2\}$.

Solution. First, observe that, from the given data, since it is **known** that the process starts in the state $X_0 = 1$, this means that this event is **certain** and thus its probability is equal to 1, so for the initial state we have $p_1 = \mathbb{P}\{X_0 = 1\} = 1$. It follows that $p_0 = \mathbb{P}\{X_0 = 0\} = 0$ and $p_2 = \mathbb{P}\{X_0 = 2\} = 0$, because $p_0 + p_1 + p_2 = 0 + 1 + 0 = 1$. Consequently, using Formula (3.1.17), we obtain

$$\mathbb{P}\{X_2 = 2\} = \sum_{k=0}^2 p_k P_{k2}^{(2)} = 0 \times P_{02}^{(2)} + 1 \times P_{12}^{(2)} + 0 \times P_{22}^{(2)} = P_{12}^{(2)}. \quad (3.1.18)$$

So, we need only to calculate the entry $P_{12}^{(2)}$ of the square $\mathbf{P}^2$ of the matrix $\mathbf{P}$:

$$\mathbf{P}^2 = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix} \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} . & . & . \\ . & . & 0.35 \\ . & . & . \end{bmatrix} \end{matrix}$$

Substituting the obtained entry $P_{12}^{(2)}$ in (3.1.18), we obtain

$$\mathbb{P}\{X_2 = 2\} = P_{12}^{(2)} = 0.35. \blacksquare$$

Ans: $\Pr\{X_2 = 2\} = \Pr\{X_2 = 2 | X_0 = 1\} \cdot \Pr\{X_0 = 1\}$
 $\Pr\{X_2 = 2\} = P_{12}^2 \cdot P_1 = P_{12}^2 \cdot 1$
 where $P_1 = \Pr\{X_0 = 1\} = 1$

$$P^2 = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.49 & 0.28 & 0.23 \\ 0.43 & 0.22 & 0.35 \\ 0.47 & 0.2 & 0.33 \end{bmatrix}$$

 $\Rightarrow \Pr\{X_2 = 2\} = P_{12}^2 = 0.35$

3.2.5 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{matrix}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1 | X_1 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1 | X_0 = 0\}.$$

Solution:

$$P_r(X_3 = 1 | X_1 = 0) = P_{01}^2 = 0.27$$

$$P^2 = \begin{bmatrix} 0.27 & 0.27 & 0.46 \\ 0.24 & 0.24 & 0.52 \\ 0.21 & 0.21 & 0.58 \end{bmatrix}$$

$$P_r(X_2 = 1 | X_0 = 0) = P_{01}^2 = 0.27$$

Ans:

$$P^2 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{matrix} \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{matrix}$$

$$\begin{aligned} \text{a) } \Pr\{X_3 = 1 | X_1 = 0\} &= P_{01}^2 \\ &= [0.1 \quad 0.1 \quad 0.8] \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \\ &= 0.27 \end{aligned}$$

(without obtaining all entries of the $M_{11} P^2$)

$$\begin{array}{r} 0.01 \\ + 0.02 \\ + 0.24 \\ \hline 0.27 \end{array}$$

$$\begin{aligned} \text{b) } \Pr\{X_2 = 1 | X_0 = 0\} &= P_{01}^2 = 0.27 \end{aligned}$$

==

$$\begin{aligned}
 P\{X_3=1 \mid X_1=0\} &= P_{01}^{(2)} = P_{0row} P_{1col} = [0.1 \quad 0.1 \quad 0.8] \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \\
 &= 0.01 + 0.02 + 0.24 \\
 &= 0.27
 \end{aligned}$$

$$P\{X_2=1 \mid X_0=0\} = P_{01}^{(2)} = P_{0row} P_{1col} = 0.27$$

$$P\{X_3=1 \mid X_1=0\} = P_{01}^{(2)} = P_{0row} P_{1col} = [0.1 \quad 0.1 \quad 0.8] \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} = 0.01 + 0.02 + 0.24 = 0.27$$

$$P\{X_2=1 \mid X_0=0\} = P_{01}^{(2)} = P_{0row} P_{1col} = [0.1 \quad 0.1 \quad 0.8] \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} = 0.01 + 0.02 + 0.24 = 0.27$$

3.2.6 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \left\| \begin{matrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{matrix} \right\| \end{matrix}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

$$\begin{aligned}
 \text{i) } \because \Pr\{X_2 = 0\} &= \Pr\{X_2 = 0 \mid X_0 = 0\} \Pr\{X_0 = 0\} \\
 &\quad + \Pr\{X_2 = 0 \mid X_0 = 1\} \Pr\{X_0 = 1\} \\
 \therefore \Pr\{X_2 = 0\} &= P_{00}^2 P_0 + P_{10}^2 P_1, \quad P_0 = 0.5, \quad P_1 = 0.5
 \end{aligned}$$

$$\begin{aligned}
P^2 &= \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \\
&= \begin{bmatrix} 0.44 & 0.18 & 0.38 \\ 0.40 & 0.19 & 0.41 \\ 0.40 & 0.18 & 0.42 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore pr\{X_2 = 0\} &= (0.44)(0.5) + (0.4)(0.5) \\
&= 0.42
\end{aligned}$$

$$\begin{aligned}
\text{ii) } \because pr\{X_3 = 0\} &= pr\{X_3 = 0 | X_0 = 0\} pr\{X_0 = 0\} \\
&\quad + pr\{X_3 = 0 | X_0 = 1\} pr\{X_0 = 1\} \\
&= P_{00}^3 P_0 + P_{10}^3 P_1, \quad P_0 = 0.5, \quad P_1 = 0.5
\end{aligned}$$

$$\begin{aligned}
P^3 &= \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 0.44 & 0.18 & 0.38 \\ 0.40 & 0.19 & 0.41 \\ 0.40 & 0.18 & 0.42 \end{bmatrix} \\
&= \begin{bmatrix} 0.4120 & 0.1820 & 0.4060 \\ 0.4200 & 0.1810 & 0.3990 \\ 0.4200 & 0.1820 & 0.3980 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore pr\{X_3 = 0\} &= (0.412)(0.5) + (0.42)(0.5) \\
&= 0.416
\end{aligned}$$

3.2.6 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \end{matrix}$$

$P_{\sum X_0=0}$ $P_{\sum X_0=1}$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

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$$P^2 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.44 & 0.18 & 0.38 \\ 0.4 & 0.19 & 0.41 \\ 0.4 & 0.18 & 0.42 \end{bmatrix} \end{matrix}$$

$$\begin{aligned} P_{\sum X_2=0} &= P_{\sum X_2=0 | X_0=0} * P_{\sum X_0=0} + P_{\sum X_2=0 | X_0=1} * P_{\sum X_0=1} \\ &= P_{00}^{(2)} P_0 + P_{10}^{(2)} P_1 \\ &= (0.44)(0.5) + (0.4)(0.5) \\ &= 0.22 + 0.2 = 0.42 \end{aligned}$$

$$P_{\sum X_3=0} = P_{\sum X_3=0 | X_0=0} * P_{\sum X_0=0} + P_{\sum X_3=0 | X_0=1} * P_{\sum X_0=1}$$

$$\begin{aligned} &= P_{00}^{(3)} P_0 + P_{10}^{(3)} P_1 \\ &= (0.412)(0.5) + (0.42)(0.5) \\ &= 0.416 \end{aligned}$$

$$P^3 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.412 & 0.182 & 0.406 \\ 0.42 & 0.181 & 0.399 \\ 0.42 & 0.182 & 0.398 \end{bmatrix} \end{matrix}$$

$$P_{\sum X_2=0} = P_{\sum X_2=0 | X_0=0} P(X_0=0) + P_{\sum X_2=0 | X_0=1} P(X_0=1) = P_{00}^{(2)}(0.5) + P_{10}^{(2)}(0.5) = (0.44)(0.5) + (0.4)(0.5) = 0.42$$

$$P_{\sum X_3=0} = P_{\sum X_3=0 | X_0=0} P(X_0=0) + P_{\sum X_3=0 | X_0=1} P(X_0=1) = P_{00}^{(3)}(0.5) + P_{10}^{(3)}(0.5) = (0.412)(0.5) + (0.42)(0.5) = 0.416$$

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \end{matrix}$$

$$P^2 = P \cdot P = \begin{bmatrix} 0.44 & 0.18 & 0.38 \\ 0.4 & 0.19 & 0.41 \\ 0.4 & 0.18 & 0.42 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0.412 & 0.182 & 0.406 \\ 0.42 & 0.181 & 0.399 \\ 0.42 & 0.182 & 0.398 \end{bmatrix}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

$$P_{\sum X_2=0} = P_{\sum X_2=0 | X_0=0} P(X_0=0) + P_{\sum X_2=0 | X_0=1} P(X_0=1) = P_{00}^{(2)}(0.5) + P_{10}^{(2)}(0.5) = (0.44)(0.5) + (0.4)(0.5) = 0.42$$

$$P_{\sum X_3=0} = P_{\sum X_3=0 | X_0=0} P(X_0=0) + P_{\sum X_3=0 | X_0=1} P(X_0=1) = P_{00}^{(3)}(0.5) + P_{10}^{(3)}(0.5) = (0.412)(0.5) + (0.42)(0.5) = 0.416$$

Problems

3.2.1 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{array} \right\| \end{matrix}.$$

Suppose that the initial distribution is $p_i = \frac{1}{4}$ for $i = 0, 1, 2, 3$. Show that $\Pr\{X_n = k\} = \frac{1}{4}, k = 0, 1, 2, 3$, for all n . Can you deduce a general result from this example?

2.1 Observe that the columns of $\mathbf{P}$ sum to one. Then $p_i^{(0)} = 1/4$ for $i = 0, 1, 2, 3$ and by induction

$$p_i^{(n+1)} = \sum_{k=0}^3 p_k^{(n)} P_{ki} = \frac{1}{4} \sum_{k=0}^3 P_{ki} = \frac{1}{4}.$$

3.2.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α . Let X_0 be the signal that is sent, and let X_n be the signal that is received at the n th stage. Suppose X_n is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, ($0 < \alpha < 1$). Determine $\Pr\{X_5 = 0 | X_0 = 0\}$, the probability of correct transmission through five stages.

$$\mathbf{2.2} \quad P_{00}^{(5)} = (1 - \alpha)^5 + 10(1 - \alpha)^3 \alpha^2 + 5(1 - \alpha) \alpha^4 = \frac{1}{2} [1 + (1 - 2\alpha)^5]$$

3.2.3 Let X_n denote the quality of the n th item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective." Suppose that X_n evolves as a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{matrix}.$$

What is the probability that the fourth item is defective given that the first item is defective?

Ans:
 Let $X_0, X_1, X_2, \dots$ be a Markov chain.
 $\Rightarrow$ the required is $\Pr\{X_3 = 1 | X_0 = 1\}$

$$\begin{aligned} P^2 &= \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix} \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix} \\ &= \begin{bmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} P^3 &= P \cdot P^2 \\ &= \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix} \begin{bmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{bmatrix} \\ &= \begin{bmatrix} 0.9737 & 0.0263 \\ 0.3152 & 0.6848 \end{bmatrix} \end{aligned}$$

$$\therefore \Pr\{X_3=1 | X_0=1\} = P_{11}^3 = 0.6848$$

is the probability that the fourth item is defective, given that the first item is defective. #

Similarly the probability that the fourth item is defective given that the first item is good is

$$\Pr\{X_3=1 | X_0=0\} = P_{01}^3 = 0.0263 = 2.63\%$$

Solution. Actually, we are asked to calculate $\mathbb{P}[X_3 = 1 | X_0 = 1] = ?$. So, we use iterations of the transition matrix:

$$P^2 = \begin{pmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{pmatrix} \begin{pmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{pmatrix} = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{pmatrix} \end{matrix}$$

and

$$P^3 = P \cdot P^2 = \begin{pmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{pmatrix} \begin{pmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{pmatrix} = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 0.9737 & 0.0263 \\ 0.3152 & 0.6848 \end{pmatrix} \end{matrix}$$

It follows that $\mathbb{P}[X_3 = 1 | X_0 = 1] = P_{11}^{(3)} = 0.6848$. ■

$$\mathbf{2.3} \quad P_{11}^{(3)} = .684772$$

3.2.4 Suppose X_n is a two-state Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{vmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{vmatrix} \end{matrix}.$$

Then, $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. Determine the transition probability matrix.

2.4

$$\begin{matrix} & \begin{matrix} (0,0) & (0,1) & (1,0) & (1,1) \end{matrix} \\ \begin{matrix} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{matrix} & \begin{vmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \end{vmatrix} \end{matrix}$$

3.2.5 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{vmatrix} \end{matrix}.$$

The Markov chain starts at time zero in state $X_0 = 0$. Let

$$T = \min\{n \geq 0; X_n = 2\}$$

be the first time that the process reaches state 2. Eventually, the process will reach and be absorbed into state 2. If in some experiment we observed such a process and noted that absorption had not yet taken place, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption had not yet taken place. Determine $\Pr\{X_3 = 0 | X_0, T > 3\}$.

Hint: The event $\{T > 3\}$ is exactly the same as the event $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$.

2.5 $\Pr\{X_3 = 0 | X_0 = 0, T > 3\} = P_{00}^{(3)} / (P_{00}^{(3)} + P_{01}^{(3)}) = .6652.$

3.3 Some Markov Chain Models

3.3.1 Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with

$$\Pr\{\xi_n = 0\} = 0.4, \quad \Pr\{\xi_n = 1\} = 0.3, \quad \Pr\{\xi_n = 2\} = 0.3,$$

and suppose $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end-of-period n .

Given $s=0, S=3$
 the possible values for X_n are
 3, 2, 1, 0 and -1 states
 no replenishment $\xrightarrow{\text{replenishment}}$ replenishment
 $\Pr\{\bar{X}_n = 0\} = 0.4, \Pr\{\bar{X}_n = 1\} = 0.3, \Pr\{\bar{X}_n = 2\} = 0.3$

Find the transition prob. MX

Ans:

$$P_{ij} = \begin{bmatrix} -1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 1 & 0 & 0.3 & 0.3 & 0.4 \\ 2 & 0.3 & 0.3 & 0.4 & 0 \\ 3 & 0 & 0.3 & 0.3 & 0.4 \end{bmatrix}$$

$$\left. \begin{aligned} P_{-1,-1} &= \Pr\{\bar{X}_{n+1} = S-j\}, i \leq s \\ &= \Pr\{\bar{X}_{n+1} = 4\} = 0 \\ P_{0,-1} &= \Pr\{\bar{X}_{n+1} = 4\} = 0 \\ P_{1,-1} &= \Pr\{\bar{X}_{n+1} = i-j\} \\ &= \Pr\{\bar{X}_{n+1} = 2\} = 0.3 \quad s < i \leq S \\ P_{2,-1} &= \Pr\{\bar{X}_{n+1} = 3\} = 0 \\ P_{3,-1} &= \Pr\{\bar{X}_{n+1} = 4\} = 0 \end{aligned} \right\} \left. \begin{aligned} P_{i,0} &= \Pr\{\bar{X}_{n+1} = S-j\}, i \leq s \\ &= \Pr\{\bar{X}_{n+1} = 3\} = 0 \\ P_{0,0} &= \Pr\{\bar{X}_{n+1} = 3\} = 0 \\ P_{1,0} &= \Pr\{\bar{X}_{n+1} = i-j\} \\ &= \Pr\{\bar{X}_{n+1} = 1\} = 0.3 \quad s < i \leq S \\ P_{2,0} &= \Pr\{\bar{X}_{n+1} = 2\} = 0.3 \\ P_{3,0} &= \Pr\{\bar{X}_{n+1} = 3\} = 0 \\ P_{3,3} &= \Pr\{\bar{X}_{n+1} = 0\} = 0.4 \end{aligned} \right\}$$

Solution. Here, the possible values of X_n are $S = 3, 2, 1, 0, -1$. So, invoking the inventory model, we obtain:

$$P_{ij} = \mathbb{P}\{X_{n+1} = j | X_n = i\} = \begin{cases} \mathbb{P}\{\xi_{n+1} = i - j\}, & \text{if } 0 < i \leq 3, \text{ (without replenishment),} \\ \mathbb{P}\{\xi_{n+1} = 3 - j\}, & \text{if } i \leq 0. \quad \text{(replenishment),} \end{cases}$$

Thus, the corresponding transition probability matrix is:

$$P = \begin{matrix} & \begin{matrix} -1 & 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.4 \end{pmatrix} \end{matrix}$$

3.3.3 Consider the inventory model of Section 3.3.1. Suppose that $S = 3$. Set up the corresponding transition probability matrix for the end-of-period inventory level X_n .

The transition probability matrix is given by

$$P = \begin{matrix} & \begin{matrix} -1 & 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{vmatrix} 0 & 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0 & 0.1 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 & 0 & 0 \\ 0 & 0.1 & 0.4 & 0.5 & 0 \\ 0 & 0 & 0.1 & 0.4 & 0.5 \end{vmatrix} \end{matrix}$$

where,

$$P_{ij} = \Pr\{X_{n+1} = j | X_n = i\} = \begin{cases} \Pr(\xi_{n+1} = 3 - j), & i \leq 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), & 0 < i \leq 3 & \text{without replenishment} \end{cases}$$

$$P\{\xi_n = 0\} = 0.5 \quad P\{\xi_n = 1\} = 0.4 \quad P\{\xi_n = 2\} = 0.1 \quad s=0 \quad X_n = 3, 2, 1, 0, -1 \quad P = \begin{cases} \xi_n = S - j & i \leq s \\ \xi_n = i - j & s < i \leq S \end{cases}$$

$$P\{\varepsilon_n = 0\} = 0.5, P\{\varepsilon_n = 1\} = 0.4, P\{\varepsilon_n = 2\} = 0.1 \quad x_n = \overset{S}{3}, 2, 1, \overset{S}{0}, -1$$

$$P_{-1,-1} = P\{x_{n+1} = -1 \mid x_n = -1\} = P\{\varepsilon_n = 3+1=4\} = P\{\varepsilon_n = 4\} = 0$$

$$P_{-1,0} = P\{x_{n+1} = 0 \mid x_n = -1\} = P\{\varepsilon_n = 3\} = 0$$

$$P_{-1,1} = P\{x_{n+1} = 1 \mid x_n = -1\} = P\{\varepsilon_n = 2\} = 0.1$$

$$P \begin{cases} \varepsilon_n = S-j & i \leq S \\ \varepsilon_n = i-j & S < i \leq S \end{cases}$$

$$P_{-1,2} = P\{x_{n+1} = 2 \mid x_n = -1\} = P\{\varepsilon_n = 1\} = 0.4$$

$$P_{-1,3} = P\{x_{n+1} = 3 \mid x_n = -1\} = P\{\varepsilon_n = 0\} = 0.5$$

$$P_{0,-1} = P\{x_{n+1} = -1 \mid x_n = 0\} = P\{\varepsilon_n = 4\} = 0$$

$$P_{0,0} = P\{x_{n+1} = 0 \mid x_n = 0\} = P\{\varepsilon_n = 3\} = 0$$

$$P_{0,1} = P\{x_{n+1} = 1 \mid x_n = 0\} = P\{\varepsilon_n = 2\} = 0.1$$

$$P_{0,2} = P\{x_{n+1} = 2 \mid x_n = 0\} = P\{\varepsilon_n = 1\} = 0.4$$

$$P_{0,3} = P\{x_{n+1} = 3 \mid x_n = 0\} = P\{\varepsilon_n = 0\} = 0.5$$

$$P_{1,-1} = P\{x_{n+1} = -1 \mid x_n = 1\} = P\{\varepsilon_n = 0\} = 0.1$$

$$P_{1,0} = P\{x_{n+1} = 0 \mid x_n = 1\} = P\{\varepsilon_n = 1\} = 0.4$$

$$P_{1,1} = P\{x_{n+1} = 1 \mid x_n = 1\} = P\{\varepsilon_n = 0\} = 0.5$$

$$P_{1,2} = P\{x_{n+1} = 2 \mid x_n = 1\} = P\{\varepsilon_n = -1\} = 0$$

$$P_{1,3} = P\{x_{n+1} = 3 \mid x_n = 1\} = P\{\varepsilon_n = -2\} = 0$$

$$P_{2-1} = P\{X_{n+1} = -1 \mid X_n = 2\} = P\{\xi_n = 3\} = 0$$

$$P_{20} = P\{X_{n+1} = 0 \mid X_n = 2\} = P\{\xi_n = 2\} = 0.1$$

$$P_{21} = P\{X_{n+1} = 1 \mid X_n = 2\} = P\{\xi_n = 1\} = 0.4$$

$$P_{22} = P\{X_{n+1} = 2 \mid X_n = 2\} = P\{\xi_n = 0\} = 0.5$$

$$P_{23} = P\{X_{n+1} = 3 \mid X_n = 2\} = P\{\xi_n = -1\} = 0$$

$$P_{3-1} = P\{X_{n+1} = -1 \mid X_n = 3\} = P\{\xi_n = 4\} = 0$$

$$P_{30} = P\{X_{n+1} = 0 \mid X_n = 3\} = P\{\xi_n = 3\} = 0$$

$$P_{31} = P\{X_{n+1} = 1 \mid X_n = 3\} = P\{\xi_n = 2\} = .1$$

$$P_{32} = P\{X_{n+1} = 2 \mid X_n = 3\} = P\{\xi_n = 1\} = 0.4$$

$$P_{33} = P\{X_{n+1} = 3 \mid X_n = 3\} = P\{\xi_n = 0\} = 0.5$$

3.3.4 Consider the inventory model of Section 3.3.1. Suppose that $S = 3$ and that the probability distribution for demand is $\Pr\{\xi = 0\} = 0.1$, $\Pr\{\xi = 1\} = 0.4$, $\Pr\{\xi = 2\} = 0.3$, and $\Pr\{\xi = 3\} = 0.2$. Set up the corresponding transition probability matrix for the end-of-period inventory level X_n .

The transition probability matrix is given by

$$P = \begin{matrix} & \begin{matrix} -2 & -1 & 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix} \end{matrix}$$

where,

$$P_{ij} = \Pr\{X_{n+1} = j | X_n = i\} = \begin{cases} \Pr(\xi_{n+1} = 3 - j), & i \leq 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), & 0 < i \leq 3 & \text{without replenishment} \end{cases}$$

$$\begin{aligned} P\{X_{n+1} = j | X_n = i\} &= P_{ij} & P_{ij} &= \begin{cases} P\{\xi_{n+1} = 3 - j\}, & i \leq 0 \\ P\{\xi_{n+1} = i - j\}, & 0 < i \leq 3 \end{cases} & S=3 & s=0 \\ P_{1,2} &= P\{\xi_{n+1} = 1-2\} = P\{\xi_{n+1} = -1\} = 0 \\ P_{1,1} &= P\{\xi_{n+1} = 1-1\} = P\{\xi_{n+1} = 0\} = 0.1 \\ P_{-1,-1} &= P\{\xi_{n+1} = 3 - (-1)\} = P\{\xi_{n+1} = 4\} = 0 \end{aligned}$$

$$P = \begin{matrix} & \begin{matrix} -2 & -1 & 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix} \end{matrix}$$

Problems

3.3.1 An urn contains six tags, of which three are red and three are green. Two tags are selected from the urn. If one tag is red and the other is green, then the selected tags are discarded and two blue tags are returned to the urn. Otherwise, the selected tags are resumed to the urn. This process repeats until the urn contains only blue tags. Let X_n denote the number of red tags in the urn after the n th draw, with $X_0 = 3$. (This is an elementary model of a chemical reaction in which red and green atoms combine to form a blue molecule.) Give the transition probability matrix.

$$\begin{array}{c}
 \mathbf{3.1} \qquad \qquad \qquad \begin{array}{ccccc}
 & 0 & 1 & 2 & 3 \\
 \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \parallel & \begin{array}{c} 1 \\ \frac{1}{15} \\ 0 \\ 0 \end{array} & \parallel & \begin{array}{c} 0 \\ \frac{14}{15} \\ \frac{8}{15} \\ 0 \end{array} & \parallel & \begin{array}{c} 0 \\ 0 \\ \frac{7}{15} \\ \frac{3}{5} \end{array} & \parallel & \begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{2}{5} \end{array}
 \end{array}
 \end{array}$$

3.3.2 Three fair coins are tossed, and we let X_1 denote the number of heads that appear. Those coins that were heads on the first trial (there were X_1 of them) we pick up and toss again, and now we let X_2 be the total number of tails, including those left from the first toss. We toss again all coins showing tails,

and let X_3 be the resulting total number of heads, including those left from the previous toss. We continue the process. The pattern is, count heads, toss heads, count tails, toss tails, count heads, toss heads, etc., and $X_0 = 3$. Then, $\{X_n\}$ is a Markov chain. What is the transition probability matrix?

$$\begin{array}{c}
 \mathbf{3.2} \qquad \qquad \qquad \begin{array}{ccccc}
 & 0 & 1 & 2 & 3 \\
 \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \parallel & \begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{1}{8} \end{array} & \parallel & \begin{array}{c} 0 \\ 0 \\ \frac{1}{4} \\ \frac{3}{8} \end{array} & \parallel & \begin{array}{c} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{8} \end{array} & \parallel & \begin{array}{c} 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \end{array}
 \end{array}
 \end{array}$$

3.3.3 Consider the inventory model of Section 3.3.1. Suppose that unfulfilled demand is not back ordered but is lost.

- Set up the corresponding transition probability matrix for the end-of-period inventory level X_n .
- Express the long run fraction of lost demand in terms of the demand distribution and limiting probabilities for the end-of-period inventory.

$$\mathbf{3.3 (a)} \quad \begin{array}{c} 0 \quad 1 \quad 2 \\ \mathbf{P} = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \left\| \begin{array}{ccc} .1 & .4 & .5 \\ .5 & .5 & 0 \\ .1 & .4 & .5 \end{array} \right\| \end{array}$$

$$\text{(b) Long run lost sales per period} = (.1) \pi_1 = .0444 \dots$$

3.3.4 Consider the queueing model of Section 3.4. Now, suppose that at most a single customer arrives during a single period, but that the service time of a customer is a random variable Z with the geometric probability distribution

$$\Pr\{Z = k\} = \alpha(1 - \alpha)^{k-1} \quad \text{for } k = 1, 2, \dots$$

Specify the transition probabilities for the Markov chain whose state is the number of customers waiting for service or being served at the start of each period. Assume that the probability that a customer arrives in a period is β and that no customer arrives with probability $1 - \beta$.

$$\mathbf{3.4} \quad \Pr\{\text{Customer completes service} \mid \text{In service}\} = \Pr\{Z = k \mid Z \geq k\} = \alpha.$$

$$P_{01} = \beta, \quad P_{00} = 1 - \beta, \quad P_{i,i+1} = \beta(1 - \alpha)$$

$$P_{i,i-1} = (1 - \beta)\alpha, \quad P_{ii} = (1 - \beta)(1 - \alpha) + \alpha\beta, \quad i \geq 1.$$

3.3.5 You are going to successively flip a quarter until the pattern HHT appears, that is, until you observe two successive heads followed by a tails. In order to calculate some properties of this game, you set up a Markov chain with the following states: $0, H, HH$, and HHT , where 0 represents the starting point, H represents a single observed head on the last flip, HH represents two successive heads on the last two flips, and HHT is the sequence that you are looking for. Observe that if you have just tossed a tails, followed by a heads, a next toss of a tails effectively starts you over again in your quest for the HHT sequence. Set up the transition probability matrix.

$$\begin{array}{c}
 \mathbf{3.5} \\
 \\
 \mathbf{P} = \begin{array}{c} 0 \\ H \\ HH \\ HHT \end{array} \begin{array}{c} 0 \\ H \\ HH \\ HHT \end{array} \begin{array}{c} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{array} \begin{array}{c} H \\ 0 \\ \frac{1}{2} \\ 0 \end{array} \begin{array}{c} HH \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{array} \begin{array}{c} HHT \\ 0 \\ \frac{1}{2} \\ 1 \end{array}
 \end{array}$$

3.3.6 Two teams, A and B, are to play a best of seven series of games. Suppose that the outcomes of successive games are independent, and each is won by A with probability p and won by B with probability $1 - p$. Let the state of the system be represented by the pair (a, b) , where a is the number of games won by A, and b is the number of games won by B. Specify the transition probability matrix. Note that $a + b \leq 7$ and that the entries end whenever $a = 4$ or $b = 4$.

$$\begin{array}{l}
 \mathbf{3.6} \quad P_{(a,b),(a,b+1)} = 1 - p \quad a \leq 3, b \leq 2 \\
 \quad \quad P_{(a,b),(a+1,b)} = p \quad a \leq 2, b \leq 3 \\
 \quad \quad P_{(3,b), A \text{ wins}} = p \quad b \leq 3 \\
 \quad \quad P_{(a,3), B \text{ wins}} = 1 - p, \quad a \leq 3 \\
 \quad \quad P_{A \text{ wins}, A \text{ wins}} = P_{B \text{ wins}, B \text{ wins}} = 1.
 \end{array}$$

3.3.7 A component in a system is placed into service, where it operates until its failure, whereupon it is replaced *at the end of the period* with a new component having statistically identical properties, and the process repeats. The probability that a component lasts for k periods is α_k , for $k = 1, 2, \dots$. Let X_n be the remaining life of the component in service *at the end-of-period* n . Then, $X_n = 0$ means that X_{n+1} will be the total operating life of the next component. Give the transition probabilities for the Markov chain $\{X_n\}$.

$$\begin{array}{l}
 \mathbf{3.7} \quad P_{0,k} = \alpha_k \quad \text{for } k = 1, 2, \dots \\
 \quad \quad P_{0,0} = 0 \\
 \quad \quad P_{k,k-1} = 1 \quad h \geq 1
 \end{array}$$

3.3.8 Two urns A and B contain a total of N balls. Assume that at time t , there were exactly k balls in A. At time $t + 1$, an urn is selected at random in proportion to its contents (i.e., A is chosen with probability k/N and B is chosen with probability $(N - k)/N$). Then, a ball is selected from A with probability p or from B with probability q and placed in the previously chosen urn. Determine the transition matrix for this Markov chain.

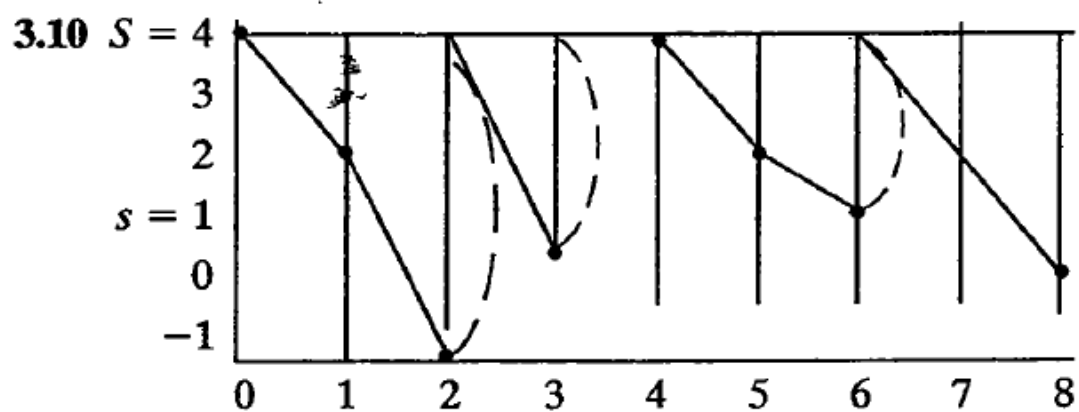
$$\begin{aligned}
3.8 \quad P_{k,k-1} &= q \left(\frac{k}{N} \right) \quad P_{k,k+1} = p \left(\frac{N-k}{N} \right) \\
P_{k,k} &= p \left(\frac{k}{N} \right) + q \left(\frac{N-k}{N} \right).
\end{aligned}$$

3.3.9 Suppose that two urns A and B contain a total of N balls. Assume that at time t , there are exactly k balls in A. At time $t + 1$, a ball and an urn are chosen with probability depending on the contents of the urn (i.e., a ball is chosen from A with probability k/N or from B with probability $(N - k)/N$). Then, the ball is placed into one of the urns, where urn A is chosen with probability k/N or urn B is chosen with probability $(N - k)/N$. Determine the transition matrix of the Markov chain with states represented by the contents of A.

$$\begin{aligned}
3.9 \quad P_{k,k-1} &= P_{k,k+1} = \left(\frac{k}{N} \right) \left(\frac{N-k}{N} \right) \\
P_{k,k} &= \left(\frac{k}{N} \right)^2 + \left(\frac{N-k}{N} \right)^2
\end{aligned}$$

3.3.10 Consider a discrete-time, periodic review inventory model and let ξ_n be the total demand in period n , and let X_n be the inventory quantity on hand at the end-of-period n . An (s, S) inventory policy is used: If the end-of-period stock is not greater than s , then a quantity is instantly procured to bring the level up to S . If the end-of-period stock exceeds s , then no replenishment takes place.

- (a) Suppose that $s = 1, S = 4$, and $X_0 = S = 4$. If the period demands turn out to be $\xi_1 = 2, \xi_2 = 3, \xi_3 = 4, \xi_4 = 0, \xi_5 = 2, \xi_6 = 1, \xi_7 = 2$, and $\xi_8 = 2$, what are the end-of-period stock levels X_n for periods $n = 1, 2, \dots, 8$?
- (b) Suppose $\xi_1, \xi_2, \dots$ are independent random variables where $\Pr\{\xi_n = 0\} = 0.1, \Pr\{\xi_n = 1\} = 0.3, \Pr\{\xi_n = 2\} = 0.3, \Pr\{\xi_n = 3\} = 0.2$, and $\Pr\{\xi_n = 4\} = 0.1$. Then, $X_0, X_1, \dots$ is a Markov chain. Determine P_{41} and P_{04} .



	ξ	=	2	3	4	0	2	1	2	2
(a)	n	=	1	2	3	4	5	6	7	8
	X_n	=	2	-1	0	4	2	1	2	0

(b) $P_{41} = Pr\{\xi = 3\} = .2$

$P_{04} = Pr\{\xi = 0\} = .1$

3.4 First Step Analysis

3.4.1 Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 1 & 0 & 0.7 & 0.2 & 0.1 \\ 2 & 0 & 0 & 0.9 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

pb 341 p105

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

trans. probs P_{ij}

Ans: $U_i = E[T | X_0 = i]$
 $i = 0, 1, 2$

Mean time to absorption

$$\Rightarrow U_0 = 1 + P_{00}U_0 + P_{01}U_1 + P_{02}U_2$$

$$U_1 = 1 + P_{10}U_0 + P_{11}U_1 + P_{12}U_2$$

$$U_2 = 1 + P_{20}U_0 + P_{21}U_1 + P_{22}U_2$$

$$\Rightarrow U_0 = 1 + 0.4U_0 + 0.3U_1 + 0.2U_2 \quad (1)$$

$$U_1 = 1 + 0.7U_1 + 0.2U_2 \quad (2)$$

$$U_2 = 1 + 0.9U_2 \quad (3)$$

$$(3) \Rightarrow U_2 = \frac{1}{0.1} = 10$$

$$(2) \Rightarrow U_1 = 1 + 0.7U_1 + 2$$

$$\therefore 0.3U_1 = 3$$

$$\therefore U_1 = \frac{3}{0.3} = 10$$

$\therefore$ By subs. values of U_1, U_2 in (1), we get

$$U_0 = 1 + 0.4U_0 + 3 + 2$$

$$0.6U_0 = 6$$

$$\therefore U_0 = 10 \Rightarrow U_0 = U_1 = U_2 = 10$$

الوقت المتوسط لوصول النظام للحالة 0 (Mean time to absorption)
 الى الحالة لخاصة 3

Solution. This four state Markov chain has a single absorbing state at $i = 3$. So, in a similar way as in (3.2.6), the law of total expectation can be applied to obtain similar equations related to the mean time to absorption depending on the starting state. Since, we have $v_3 = \mathbb{E}[T|X_0 = 3] = 0$, the first step analysis equations for $v_i = \mathbb{E}[T|X_0 = i]$, $i = 0, 1, 2$, are

$$\begin{aligned}v_0 &= 1 + (0.4)v_0 + (0.3)v_1 + (0.2)v_2 + (0.1)(0) \\v_1 &= 1 + (0)v_0 + (0.7)v_1 + (0.2)v_2 + (0.1)(0). \\v_2 &= 1 + (0)v_0 + (0)v_1 + (0.9)v_2 + (0.1)(0).\end{aligned}$$

Thus, multiplying by 10 and simplifying, we get the system

$$\begin{aligned}6v_0 - 3v_1 - 2v_2 &= 10 \\3v_1 - 2v_2 &= 10 \\v_2 &= 10,\end{aligned}$$

which results in $v_0 = v_1 = v_2 = 10$. Whence the mean time to reach state 3 starting from state 0 is equal to $v_0 = 10$. ■

3.4.2 Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

- (a) Starting in state 1, determine the probability that the Markov chain ends in state 0.
 (b) Determine the mean time to absorption.

2 pb 3.4.2 p. 105

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

abs → 0
non-abs → 1
abs → 2

a) $u = \Pr\{X_T = 0 | X_0 = 1\}$

$$u = P_{10} + P_{11}u$$

$$u = 0.1 + 0.6u$$

$$0.4u = 0.1 \Rightarrow u = \frac{1}{4}$$

$\therefore u = u_{10} = \frac{1}{4}$

b) $v = E[T | X_0 = 1]$

$$v = 1 + P_{11}v$$

$$v = 1 + 0.6v$$

$$0.4v = 1 \quad (\times 10)$$

$$\therefore v = \frac{10}{4} = 2.5 \text{ is the mean time of absorption}$$

$u_{10}?$

Solution. 1) As above, put $u = \mathbb{P}\{X_T = 0 | X_0 = 1\}$. Therefore, using (3.2.1), we obtain $u = (1)(0.1) + (u)(0.6) + (0)(0.3)$. Then $u = \frac{1}{4} = 0.25$.

2) Put $v = \mathbb{E}[T | X_0 = 1]$. So by (3.2.5) we have $v = 1 + (0.1)(0) + (v)(0.6) + (0)(0.3)$. Then $v = \frac{10}{4} = 2.5$. ■

Thus, we obtain the equation

$$u = \alpha + \beta u, \quad (3.2.1)$$

which gives

$$u = \frac{\alpha}{1 - \beta} = \frac{\alpha}{\alpha + \gamma}.$$

3.4.3 Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

- (a) Starting in state 1, determine the probability that the Markov chain ends in state 0.
 (b) Determine the mean time to absorption.

3.4.3 P^{100}

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

a) $u_1 = u_{10}$??

$$u_i = \Pr\{X_T = 0 \mid X_0 = i\}, i = 1, 2$$

$$\Rightarrow u_1 = p_{10} + p_{11} u_1 + p_{12} u_2$$

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 $X_0 = 1$

$$u_2 = p_{20} + p_{21} u_1 + p_{22} u_2$$

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 $X_0 = 2$

$$\Rightarrow \begin{cases} u_1 = 0.1 + 0.6 u_1 + 0.1 u_2 \\ u_2 = 0.2 + 0.3 u_1 + 0.4 u_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0.4 u_1 - 0.1 u_2 = 0.1 & \times 10 \\ 0.6 u_2 - 0.3 u_1 = 0.2 & \times 10 \end{cases}$$

$$\Rightarrow \begin{cases} 4 u_1 - u_2 = 1 & \textcircled{1} \\ 3 u_1 - 6 u_2 = -2 & \textcircled{2} \end{cases}$$

Solving ① and ②

$$\textcircled{1} \times 6 - \textcircled{2} \Rightarrow 21 u_1 = 8$$

$$\therefore u_1 = u_{10} = \frac{8}{21}$$

6) $\tau = E[T | X_0 = i], i = 1, 2$
is the mean time to absorption

$$\Rightarrow \begin{cases} \tau_1 = 1 + p_{11}\tau_1 + p_{12}\tau_2 \\ \tau_2 = 1 + p_{21}\tau_1 + p_{22}\tau_2 \end{cases}$$

$$\Rightarrow \begin{cases} \tau_1 = 1 + 0.6\tau_1 + 0.1\tau_2 \\ \tau_2 = 1 + 0.3\tau_1 + 0.4\tau_2 \end{cases}$$

$$\Rightarrow 0.4\tau_1 - 0.1\tau_2 = 1 \quad (1)$$

$$0.3\tau_1 - 0.6\tau_2 = -1 \quad (2)$$

$$(1) \times 6 - (2) \Rightarrow 2.1\tau_1 = 7$$

$$\therefore \tau_1 = \frac{7}{2.1} = \frac{70}{21} = \frac{10}{3}$$

$$\therefore \tau = \tau_0 = \frac{10}{3}$$

##

50 ??

3.4.3 Consider the Markov chain whose transition probability matrix is given by

for absorbed state $i=1,2$ non-absorbed state

(b) $V = E[T | X_0 = i]$

$V_{10} = V_1$ $V_{20} = V_2$

$V_1 = 1 + 0.6V_1 + 0.1V_2$

$V_2 = 1 + 0.3V_1 + 0.4V_2$

$V_1 = \frac{10}{3}$

$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$

$u_1 = u_{10} = P_{10} + P_{11}u_{10} + P_{12}u_{20} + P_{13}u_{30}$

$u_{30} = P\{X_T=0, X_0=3\}$

$u_2 = u_{20} = P\{X_T=0, X_0=2\}$

$= P_{20} + P_{21}u_{10} + P_{22}u_{20} + P_{23}u_{30}$

$u_1 = 0.1 + 0.6u_1 + 0.1u_2$

$u_2 = 0.2 + 0.3u_1 + 0.4u_2$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

$$u_1 = \frac{-1 + 0.1u_2}{-0.4}$$

$$u_2 = -2 + 0.3\left(\frac{-1 + 0.1u_2}{-0.4}\right) + 0.4u_2$$

$$60u_2 = 20 + 30\left[\frac{10 + 10u_2}{40}\right]$$

$$60u_2 = 20 + \frac{300}{40} + \frac{300u_2}{40}$$

$$60u_2 = 20 + 7.5 + 7.5u_2$$

$$52.5u_2 = 27.5$$

$$u_2 = 0.5238$$

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

absorption: 0, 3

non absorption: 1, 2

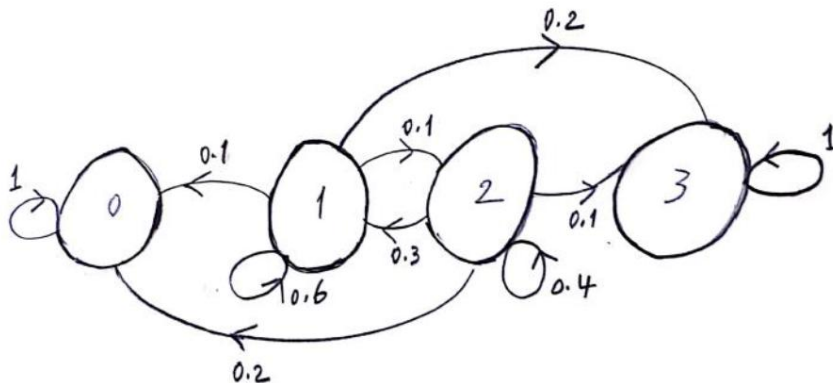
$$a) P\{X_1=0 | X_0=1\} = u_{10} = p_{10} + p_{11}u_{10} + p_{12}u_{20} + p_{13}u_{30} = 0.1 + 0.6u_{10} + 0.1u_{20} + 0$$

$$u_{20} = p_{20} + p_{21}u_{10} + p_{22}u_{20} + p_{23}u_{30} = 0.2 + 0.3u_{10} + 0.4u_{20} + 0 \Rightarrow \begin{cases} 0.4u_{10} - 0.1u_{20} = 0.1 \\ 0.6u_{10} - 0.3u_{20} = 0.2 \end{cases} \Rightarrow \begin{cases} 0.4u_{10} - \frac{1}{3}u_{20} = 0.1 \\ 0.2 + 0.3u_{10} = 0.2 \end{cases} \Rightarrow u_{20} = \frac{0.2 + 0.3u_{10}}{0.6} \Rightarrow u_{10} = \frac{8}{21}$$

$$b) E[\tau | X_0=1] = v_{10} = 1 + p_{11}v_{10} + p_{12}v_{20} = 1 + 0.6v_{10} + 0.1v_{20}$$

$$v_{20} = 1 + p_{21}v_{10} + p_{22}v_{20} = 1 + 0.3v_{10} + 0.4v_{20} \Rightarrow \begin{cases} 0.4v_{10} - 0.1v_{20} = 1 \\ v_{20} = 1 + 0.3v_{10} \end{cases} \Rightarrow \begin{cases} 0.4v_{10} - \frac{1}{6}v_{10} = 1 \\ v_{20} = \frac{10}{6} + \frac{1}{2}v_{10} \end{cases} \Rightarrow \frac{7}{20}v_{10} = \frac{7}{6} \Rightarrow v_{10} = \frac{10}{3}$$

(i) It's an absorbing Markov Chain.



Markov Chain Diagram

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{matrix} \right\| \end{matrix}$$

$$u_i = pr\{X_T = 0 | X_0 = i\} \quad \text{for } i=1,2,$$

$$\text{and } v_i = E[T | X_0 = i] \quad \text{for } i=1,2.$$

(ii)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$

$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$

$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \quad (1)$$

$$3u_1 - 6u_2 = -2 \quad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{8}{21} \quad \text{and} \quad u_2 = \frac{11}{21}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \frac{8}{21}$$

$$\approx 0.38$$

(iii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$

$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$

$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$4v_1 - v_2 = 10 \quad (1)$$

$$3v_1 - 6v_2 = -10 \quad (2)$$

Solving (1) and (2), we get

$$v_1 = v_2 = \frac{10}{3}$$

$$\therefore v_1 = v_{10} = \frac{10}{3} \\ \approx 3.3$$

3.4.6 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

Ans:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

$$u_i = pr\{X_T = 0 | X_0 = i\} \quad \text{for } i=1,2,$$

$$\text{and } v_i = E[T | X_0 = i] \quad \text{for } i=1,2.$$

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$

$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.4u_1 + 0.1u_2$$

$$u_2 = 0.2 + 0.1u_1 + 0.6u_2$$

$\Rightarrow$

$$6u_1 - u_2 = 1 \quad (1)$$

$$u_1 - 4u_2 = -2 \quad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{6}{23} \text{ and } u_2 = \frac{13}{23}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \frac{6}{23}$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$

$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.1v_2$$

$$v_2 = 1 + 0.1v_1 + 0.6v_2$$

$\Rightarrow$

$$6v_1 - v_2 = 10 \quad (1)$$

$$v_1 - 4v_2 = -10 \quad (2)$$

Solving (1) and (2), we get $v_2 = v_{20} = \frac{70}{23}$

$$v_1 = \frac{50}{23} \text{ and } v_2 = \frac{70}{23}$$

$\therefore v_1 = v_{10} = \frac{50}{23}$ is the mean time of absorption.

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

absorb: 0, 3

nicht absorb: 1, 2

$$a) P\{X_T=0 | X_0=1\} = u_1 = P_{10} + P_{11}u_{10} + P_{12}u_{20} = 0.1 + 0.4u_1 + 0.1u_2 \Rightarrow 0.6u_1 - 0.1u_2 = 0.1 \Rightarrow 0.6u_1 - \frac{1}{20} = \frac{1}{40}u_1 \Rightarrow 0.1$$

$$u_2 = P_{20} + P_{21}u_{10} + P_{22}u_{20} = 0.2 + 0.1u_1 + 0.6u_2 \Rightarrow u_2 = \frac{0.2+0.1u_1}{0.4} = \frac{1}{2} + \frac{1}{4}u_1 \Rightarrow \frac{23}{40}u_1 = \frac{3}{20} \Rightarrow u_1 = \frac{3}{20} \cdot \frac{40}{23} = \frac{6}{23}$$

$$b) E[T | X_0=1] = V_1 = 1 + P_{11}V_1 + P_{12}V_2 = 1 + 0.4V_1 + 0.1V_2 \Rightarrow \begin{cases} 0.6V_1 - 0.1V_2 = 1 \\ V_2 = \frac{10}{4} + \frac{1}{4}V_1 \end{cases} \Rightarrow 0.6V_1 - \frac{1}{4} \cdot \frac{1}{40}V_1 = 1 \Rightarrow \frac{23}{40}V_1 = \frac{5}{4} \Rightarrow V_1 = \frac{5}{4} \cdot \frac{40}{23} = \frac{50}{23}$$

$$V_{20} = 1 + P_{21}V_1 + P_{22}V_2 = 1 + 0.1V_1 + 0.6V_2 \Rightarrow V_2 = \frac{1+0.1V_1}{0.4}$$

$$\textcircled{a} u = P\{X_T=0 | X_0=1\} = u_{10}$$

$$u_{10} = P_{10} + P_{11}u_{10} + P_{12}u_{20} + \cancel{P_{13}u_{30}}$$

$$u_{10} = 0.1 + 0.4u_{10} + 0.1u_{20}$$

$$u_{20} = P_{20} + P_{21}u_{10} + P_{22}u_{20}$$

$$\textcircled{b} v = E[T | X_0=1] = V_1$$

$$V_{10} = 1 + P_{11}V_{10} + P_{12}V_{20}$$

$$V_{20} = 1 + P_{22}V_{20} + P_{21}V_{10}$$

Problems

3.4.1 Which will take fewer flips, on average: successively flipping a quarter until the pattern *HHT* appears, i.e., until you observe two successive heads followed by a tails; or successively flipping a quarter until the pattern *HTH* appears? Can you explain why these are different?

4.1 $0 = 0 \quad 1 = H \quad 2 = HH \quad 3 = HHT$

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\left. \begin{array}{l} v_0 = 1 + \frac{1}{2}v_0 + \frac{1}{2}v_1 \\ v_1 = 1 + \frac{1}{2}v_0 + \frac{1}{2}v_2 \\ v_2 = 1 + \frac{1}{2}v_2 \end{array} \right\} \begin{array}{l} v_0 = 8 \\ v_1 = 6 \\ v_2 = 2 \end{array}$$

$0 = 0 \quad 1 = H \quad 2 = HT \quad 3 = HTH$

$$\begin{array}{c|ccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\left. \begin{array}{l} v_0 = 1 + \frac{1}{2}v_0 + \frac{1}{2}v_1 \\ v_1 = 1 + \frac{1}{2}v_1 + \frac{1}{2}v_2 \\ v_2 = 1 + \frac{1}{2}v_0 \end{array} \right\} \begin{array}{l} v_0 = 10 \\ v_1 = 8 \\ v_2 = 6 \end{array}$$

3.4.2 A zero-seeking device operates as follows: If it is in state m at time n , then at time $n + 1$, its position is uniformly distributed over the states $0, 1, \dots, m - 1$. Find the expected time until the device first hits zero starting from state m .

Note: This is a highly simplified model for an algorithm that seeks a maximum over a finite set of points.

$$\begin{aligned}
 4.2 \quad v_1 &= 1 & v_1 &= 1 \\
 v_2 &= 1 + \frac{1}{2}v_1 & v_2 &= 1 + \frac{1}{2} \\
 v_3 &= 1 + \frac{1}{3}v_1 + \frac{1}{3}v_2 & v_3 &= 1 + \frac{1}{3} + \frac{1}{3}\left(\frac{3}{2}\right) = 1 + \frac{1}{2} + \frac{1}{3} \\
 &\vdots & & \\
 v_m &= 1 + \frac{1}{m}v_1 + \frac{1}{m}v_2 + \dots + \frac{1}{m}v_{m-1}
 \end{aligned}$$

Solve to get $v_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \sim \text{Log } m$

Note: $w_{jj} = 1$ and $w_{ij} = \frac{1}{j+1}$ for $i > j$.

3.4.3 A zero-seeking device operates as follows: If it is in state j at time n , then at time $n + 1$, its position is 0 with probability $1/j$, and its position is k (where k is one of the states $1, 2, \dots, j - 1$) with probability $2k/j^2$. Find the expected time until the device first hits zero starting from state m .

4.3 We will verify that $v_m = 2\left(\frac{m+1}{m}\right)\left(1 + \frac{1}{2} + \dots + \frac{1}{m}\right) - 3$ solves $v_m = 1 + \sum_{j=1}^{m-1} \frac{2j}{m^2}v_j$. Change variables to $V_k = kv_k$. To show: $V_k = 2(k+1)\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right) - 3k$ solves $V_m = m + \frac{2}{m}(V_1 + \dots + V_{m-1})$. Using the given V_k , use sums of numbers and interchange order as follows:

$$\begin{aligned}
 \sum_{k=1}^{m-1} V_k &= \sum_{k=1}^{m-1} \sum_{l=1}^k 2(k+1)\frac{1}{l} - 3 \sum_{k=1}^{m-1} k \\
 &= \sum_{l=1}^{m-1} 2 \sum_{k=l}^{m-1} (k+1) - \frac{3m(m-1)}{2} \\
 &= \sum_{l=1}^{m-1} 2 \left[\frac{m(m+1)}{2} - \frac{l(l+1)}{2} \right] - \frac{3m(m-1)}{2}.
 \end{aligned}$$

Then,

$$m + \frac{2}{m} \sum_{k=1}^{m-1} V_k = 2(m+1) \sum_{l=1}^{m-1} \frac{1}{l} - 3m + \frac{2(m+1)}{m} = V_m$$

as was to be shown.

As in Problem 4.2, $v_m \sim \log m$.

3.4.4 Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{bmatrix} \end{matrix}$$

Starting in state $X_0 = 1$, determine the probability that the process never visits state 2. Justify your answer.

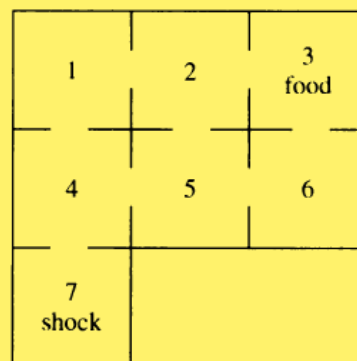
4.4 Change the matrix to

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ .1 & .2 & .5 & .2 \\ 0 & 0 & 1 & 0 \\ .2 & .2 & .3 & .3 \end{bmatrix} \end{matrix}$$

and calculate the probability that absorption takes place in State 0:

$$\left. \begin{aligned} u_{10} &= .1 + .2u_{10} + .2u_{30} \\ u_{30} &= .2 + .2u_{10} + .3u_{30} \end{aligned} \right\} \begin{aligned} u_{10} &= .2115 \\ u_{30} &= .3462 \end{aligned}$$

3.4.5 A white rat is put into compartment 4 of the maze shown here:



It moves through the compartments at random; i.e., if there are k ways to leave a compartment, it chooses each of these with probability $1/k$. What is the probability that it finds the food in compartment 3 before feeling the electric shock in compartment 7?

4.5

	1	2	3	4	5	6	7
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
2	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0
3	0	0	1	0	0	0	0
4	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	0	$\frac{1}{3}$
5	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0
6	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
7	0	0	0	0	0	0	1

$$u_{13} = \frac{7}{12} \quad u_{53} = \frac{2}{3}$$

$$u_{43} = \frac{5}{12}$$

$$u_{23} = \frac{3}{4} \quad u_{63} = \frac{5}{6}$$

3.4.6 Consider the Markov chain whose transition matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{vmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \end{matrix},$$

where $p + q = 1$. Determine the mean time to reach state 4 starting from state 0. That is, find $E[T|X_0 = 0]$, where $T = \min\{n \geq 0; X_n = 4\}$.

Hint: Let $v_i = E[T|X_0 = i]$ for $i = 0, 1, \dots, 4$. Establish equations for $v_0, v_1, \dots, v_4$ by using a first step analysis and the boundary condition $v_4 = 0$. Then, solve for v_0 .

$$\begin{aligned}
4.6 \quad v_0 &= 1 + qv_0 + pv_1 \\
v_1 &= 1 + qv_0 + pv_2 \quad , \quad q = 1 - p \\
v_2 &= 1 + qv_0 + pv_3 \\
v_3 &= 1 + qv_0
\end{aligned}$$

gives $v_0 = (1 + qv_0)(1 + p + p^2 + p^3) = (1 + p + p^2 + p^3) + v_0(1 - p^4)$
so

$$v_0 = \frac{1 + p + p^2 + p^3}{1 - (1 - p^4)} = \frac{1 - p^4}{p^4(1 - p)}.$$

3.4.7 Let X_n be a Markov chain with transition probabilities P_{ij} . We are given a “discount factor” β with $0 < \beta < 1$ and a cost function $c(i)$, and we wish to determine the total expected discounted cost starting from state i , defined by

$$h_i = E \left[\sum_{n=0}^{\infty} \beta^n c(X_n) | X_0 = i \right].$$

Using a first step analysis show that h_i satisfies the system of linear equations

$$h_i = c(i) + \beta \sum_j P_{ij} h_j \quad \text{for all states } i.$$

4.7 The stationary Markov transitions imply that $E \left[\sum_{n=1}^{\infty} \beta^n c(X_n) | X_0 = i, X_1 = j \right] = \beta h_j$ while $E[\beta^0 c(X_0) | X_0 = i] = c(i)$. Now use the law of total probability.

3.4.8 An urn contains five red and three green balls. The balls are chosen at random, one by one, from the urn. If a red ball is chosen, it is removed. Any green ball that is chosen is returned to the urn. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

4.8	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$\frac{1}{4}$	$\frac{3}{4}$	0	0	0	0
2	0	$\frac{2}{5}$	$\frac{3}{5}$	0	0	0
3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
4	0	0	0	$\frac{4}{7}$	$\frac{3}{7}$	0
5	0	0	0	0	$\frac{5}{8}$	$\frac{3}{8}$

X_n = # of red balls in use at n th draw.

$$v_5 = 4 + \frac{5}{2} + 2 + \frac{7}{4} + \frac{8}{5} = 11.85.$$

3.4.9 An urn contains five red and three yellow balls. The balls are chosen at random, one by one, from the urn. Each ball removed is replaced in the urn by a yellow ball. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

4.9	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$\frac{1}{8}$	$\frac{7}{8}$	0	0	0	0
2	0	$\frac{2}{8}$	$\frac{6}{8}$	0	0	0
3	0	0	$\frac{3}{8}$	$\frac{5}{8}$	0	0
4	0	0	0	$\frac{4}{8}$	$\frac{4}{8}$	0
5	0	0	0	0	$\frac{5}{8}$	$\frac{3}{8}$

X_n = # of red balls in use.

$$v_5 = \frac{8}{1} + \frac{8}{2} + \frac{8}{3} + \frac{8}{4} + \frac{8}{5} = 18\frac{4}{15} = 18.266....$$

3.4.10 You have five fair coins. You toss them all so that they randomly fall heads or tails. Those that fall tails in the first toss you pick up and toss again. You toss again those that show tails after the second toss, and so on, until all show heads. Let X be the number of coins involved in the *last* toss. Find $\Pr\{X = 1\}$.

4.10 Let X_n be the number of coins that fall tails in the n th toss, $X_0 = 5$. Make “1” an absorbing state.

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 3 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ 4 & \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & 0 \\ 5 & \frac{1}{32} & \frac{5}{32} & \frac{10}{32} & \frac{10}{32} & \frac{5}{32} & \frac{1}{32} \end{array}$$

$$U_{51} = .7235023041 \dots$$

3.4.11 An urn contains two red and two green balls. The balls are chosen at random, one by one, and removed from the urn. The selection process continues until all of the green balls have been removed from the urn. What is the probability that a single red ball is in the urn at the time that the last green ball is chosen?

4.11 Label the states (x, y) where $x = \#$ of red balls and $y = \#$ of green balls, “win” = $(1, 0)$, “lose” = $\{(0, 2), (2, 0)\}$.

$$\begin{array}{c|cccccc} & (2, 2) & (1, 2) & (2, 1) & (1, 1) & \text{win} & \text{lose} \\ \hline (2, 2) & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ (1, 2) & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ (2, 1) & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ (1, 1) & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \text{win} & 0 & 0 & 0 & 0 & 1 & 0 \\ \text{lose} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$U_{(2,2),\text{win}} = \frac{1}{3}.$$

3.4.12 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that when the process moves into state 2, it does so from state 1?

Hint: Let $T = \min\{n \geq 0; X_n = 2\}$, and let

$$z_i = \Pr\{X_{T-1} = 1 | X_0 = i\} \quad \text{for } i = 0, 1.$$

Establish and solve the first step equations

$$\begin{aligned} z_0 &= 0.3z_0 + 0.2z_1, \\ z_1 &= 0.4 + 0.5z_0 + 0.1z_1. \end{aligned}$$

4.12

$$\begin{matrix} & \begin{matrix} 0 & 1 & \text{win} & \text{lose} \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \text{win} \\ \text{lose} \end{matrix} & \begin{bmatrix} .3 & .2 & 0 & .5 \\ .5 & .1 & .4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} z_0 = U_{0, \text{win}} = \frac{8}{53}$$

3.4.13 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that the time $T = \min\{n \geq 0; X_n = 2\}$ is an odd number?

4.13 Use the matrix

$$\begin{array}{c|ccc|ccc}
 & 0 & 1 & 2 & 0' & 1' & 2' \\
 \hline
 0 & 0 & 0 & 0 & .3 & .2 & .5 \\
 1 & 0 & 0 & 0 & .5 & .1 & .4 \\
 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
 \hline
 0' & .3 & .2 & .5 & 0 & 0 & 0 \\
 1' & .5 & .1 & .4 & 0 & 0 & 0 \\
 2' & 0 & 0 & 0 & 0 & 0 & 1
 \end{array}$$

$$U_{0,2'} = .67669 \dots$$

3.4.14 A single die is rolled repeatedly. The game stops the first time that the sum of two successive rolls is either 5 or 7. What is the probability that the game stops at a sum of 5?

4.14 The long but sure way to solve the problem is to set up the 36×36 transition matrix for the Markov chain $Z_n = (X_{n-1}, X_n)$, making states (x, y) where $x + y = 5$ or $x + y = 7$ absorbing. However, all that is important prior to ending the game is the last roll, and by aggregating states where possible, we get the Markov transition matrix.

$$\begin{array}{c|ccccc}
 & 1 \text{ or } 2 & 3 \text{ or } 4 & 5 \text{ or } 6 & 5 & 7 \\
 \hline
 1 \text{ or } 2 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
 3 \text{ or } 4 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\
 5 \text{ or } 6 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} \\
 5 & 0 & 0 & 0 & 1 & 0 \\
 7 & 0 & 0 & 0 & 0 & 1
 \end{array}$$

$$U_{1,5} = \frac{22}{51} \quad U_{3,5} = \frac{21}{51} \quad U_{5,5} = \frac{16}{51}$$

$$\Pr\{5 \text{ before } 7\} = \frac{1}{3}(U_{1,5} + U_{3,5} + U_{5,5}) = \frac{59}{153}.$$

3.4.15 A simplified model for the spread of a rumor goes this way: There are $N = 5$ people in a group of friends, of which some have heard the rumor and the others have not. During any single period of time, two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friends is just as likely as between any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let X_n denote the number of friends who have heard the rumor at the end of the n th period.

Assuming that the process begins at time 0 with a single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

4.15

	1	2	3	4	5
1	.96	.04	0	0	0
2	0	.94	.06	0	0
3	0	0	.94	.06	0
4	0	0	0	.96	.04
5	0	0	0	0	1

$$v_1 = 133\frac{1}{3}$$

3.4.16 An urn contains five tags, of which three are red and two are green. A tag is randomly selected from the urn and replaced with a tag of the opposite color. This continues until only tags of a single color remain in the urn. Let X_n denote the number of red tags in the urn after the n th draw, with $X_0 = 3$. What is the probability that the game ends with the urn containing only red tags?

4.16

	0	1	2	3	4	5
0	1	0	0	0	0	0
1	.2	0	.8	0	0	0
2	0	.4	0	.6	0	0
3	0	0	.6	0	.4	0
4	0	0	0	.8	0	.2
5	0	0	0	0	0	1

$$U_{35} = \frac{17}{32}.$$

3.4.17 The *damage* X_n of a system subjected to wear is a Markov chain with the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

The system starts in state 0 and fails when it first reaches state 2. Let $T = \min\{n \geq 0; X_n = 2\}$ be the time of failure. Use a first step analysis to evaluate $\phi(s) = E[s^T]$ for a fixed number $0 < s < 1$. (This is called the *generating function* of T . See Section 3.9.)

4.17 Let $\varphi_i(s) = E[s^T | X_0 = i]$ for $i = 0, 1$

Then

$$\begin{aligned} \varphi_0(s) &= s[.7\varphi_0(s) + .3\varphi_1(s)] \\ \varphi_1(s) &= s[.6\varphi_1(s) + .4] \end{aligned}$$

which solves to give

$$\begin{aligned} \varphi_1(s) &= \frac{.4s}{1 - .6s} \\ \varphi_0(s) &= \left(\frac{.4s}{1 - .6s} \right) \left(\frac{.3s}{1 - .7s} \right) \end{aligned}$$

3.4.18 Time-dependent transition probabilities. A well-disciplined man, who smokes exactly one half of a cigar each day, buys a box containing N cigars. He cuts a cigar in half, smokes half, and returns the other half to the box. In general, on a day in which his cigar box contains w whole cigars and h half cigars, he will pick one of the $w + h$ smokes at random, each whole and half cigar being equally likely, and if it is a half cigar, he smokes it. If it is a whole cigar, he cuts it in half, smokes one piece, and returns the other to the box. What is the expected value of T , the day on which the last whole cigar is selected from the box?

Hint: Let X_n be the number of whole cigars in the box after the n th smoke. Then, X_n is a Markov chain whose transition probabilities vary with n . Define $v_n(w) = E[T|X_n = w]$. Use a first step analysis to develop a recursion for $v_n(w)$ and show that the solution is

$$v_n(w) = \frac{2Nw + n + 2w}{w + 1} - \sum_{k=1}^w \frac{1}{k},$$

whence

$$E[T] = v_0(N) = 2N - \sum_{k=1}^N \frac{1}{k}.$$

4.18 The transition probabilities are: If $X_n = w$, then $X_{n+1} = w$ w.pr. $\frac{h}{w+h}$ (and $h \rightarrow h-1$) or $X_{n+1} = w-1$ w.pr. $\frac{w}{w+h}$ (and $h \rightarrow h+1$) where $h = 2N - n - 2w$ is the number of half cigars in the box.

The recursion to verify is

$$v_n(w) = \frac{h}{w+h} v_{n+1}(w) + \frac{w}{w+h} v_{n+1}(w-1)$$

This is straight forward, and easiest if one first does the algebra to verify

$$\begin{aligned} v_{n+1}(w) &= v_n(w) + \frac{1}{1+w} \\ v_{n+1}(w-1) &= v_n(w) + \frac{n+2w-2N}{w(1+w)} \end{aligned}$$

3.4.19 Computer Challenge. Let N be a positive integer and let $Z_1, \dots, Z_N$ be independent random variables, each having the geometric distribution

$$\Pr\{Z = k\} = \left(\frac{1}{2}\right)^k, \quad \text{for } k = 1, 2, \dots$$

Since these are discrete random variables, the maximum among them may be unique, or there may be ties for the maximum. Let p_N be the probability that the maximum is unique. How does p_N behave when N is large? (Alternative formulation: You toss N dimes. Those that are heads you set aside; those that are tails you toss again. You repeat this until all of the coins are heads. Then, p_N is the probability that the last toss was of a single coin.)

$$4.19 \quad p_N = \frac{N}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left[1 - \left(\frac{1}{2}\right)^n\right]^{N-1}.$$

As a function of N , this does NOT converge but oscillates very slowly (cycles $\propto \log N$) and very slightly about the (Cesaro limit) $\frac{1}{2 \log 2}$.

3.5 Some Special Markov Chains

3.5.1 The probability of the thrower winning in the dice game called "craps" is $p = 0.4929$. Suppose Player A is the thrower and begins the game with \$5, and Player B, his opponent, begins with \$10. What is the probability that Player A goes bankrupt before Player B? Assume that the bet is \$1 per round.

Hint: Use equation (3.42).

pb 3.5.1 p.121 $i = \$5$ fortune for player A

Total amount $N = \$5 + \$10 = \$15$

$p = 0.4929 \Rightarrow q = 1 - p = 1 - 0.4929$
 $q = 0.5071$

$u_i = \text{pr} \{X_n \text{ reaches state 0 before state } N | X_0 = i\}$

$u_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}, \quad p \neq q$

$u_i = \frac{\left[\left(\frac{0.5071}{0.4929} \right)^5 - \left(\frac{0.5071}{0.4929} \right)^{15} \right]}{\left[1 - \left(\frac{0.5071}{0.4929} \right)^{15} \right]}$

$\therefore u_i = 0.71273$

Solution. Here, player A's initial fortune is $i = 5$ and the total of fortunes is $N = 5 + 10 = 15$, the craps is $p = 0.4929$ and $q = 1 - p = 1 - 0.4929 = 0.5071$. Thus by the formula, we have

$$u_5 = \mathbb{P} \{X_n = 0 | X_0 = 5\} = \frac{\left(\frac{0.5071}{0.4929} \right)^5 - \left(\frac{0.5071}{0.4929} \right)^{15}}{1 - \left(\frac{0.5071}{0.4929} \right)^{15}} \approx 0.71273 \blacksquare$$

3.5.2 Determine the gambler's ruin probability for Player A when both players begin with \$50, bet \$1 on each play, and where the win probability for Player A in each game is

(a) $p = 0.49292929$

(b) $p = 0.5029237$

(See Chapter 2, Section 2.2.)

What are the gambler's ruin probabilities when each player begins with \$500?

④

$$u_i = \frac{\left(\frac{0.50707071}{0.49292929} \right)^{500} - \left(\frac{0.50707071}{0.49292929} \right)^{1000}}{1 - \left(\frac{0.50707071}{0.49292929} \right)^{1000}}$$

$p = 0.49292929$

$i = 500$

$500 = n - i$

$n = 1000$

$$= \frac{1386596.552 - 1.92265 \times 10^{12}}{-1.92265 \times 10^{12}}$$

$$= 0.99999928$$

⑥ $u_i = \frac{2.87887 \times 10^{-3}}{0.999992}$

$= 0.0028789$

$$u_i = \Pr\{X_n \text{ reaches state 0 before state } N | X_0 = i\}$$

$$= \begin{cases} \frac{N-i}{N} & \text{when } p = q = \frac{1}{2}, \\ \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N} & \text{when } p \neq q. \end{cases} \quad (3.42)$$

Problems

3.5.1 As a special case of the successive maxima Markov chain whose transition probabilities are given in equation (3.34), consider the Markov chain whose transition probability matrix is given by

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & a_0 & a_1 & a_2 & a_3 \\ 1 & 0 & a_0 + a_1 & a_2 & a_3 \\ 2 & 0 & 0 & a_0 + a_1 + a_2 & a_3 \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

Starting in state 0, show that the mean time until absorption is $v_0 = 1/a_3$.

$$\mathbf{5.1} \quad v_0 = 1 + a_0 v_0 + a_1 v_1 + a_2 v_2$$

$$v_1 = 1 + (a_0 + a_1) v_1 + a_2 v_2$$

$$v_2 = 1 + (a_0 + a_1 + a_2) v_2$$

$$v_2 = \frac{1}{1 - (a_0 + a_1 + a_2)} = \frac{1}{a^3} = v_0 = v_1.$$

3.5.2 A component of a computer has an active life, measured in discrete units, that is a random variable T , where $\Pr\{T = k\} = a_k$ for $k = 1, 2, \dots$. Suppose one starts with a fresh component, and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n . Then, $\{X_n\}$ is a success runs Markov chain.

(a) Specify the probabilities p_i and q_i .

(b) A “planned replacement” policy calls for replacing the component upon its failure or upon its reaching age N , whichever occurs first. Specify the success runs probabilities p_i and q_i under the planned replacement policy.

$$\mathbf{5.2} \quad \begin{aligned} \text{(a)} \quad p_0 &= a_1, \quad q_0 = 1 - a_1 = (a_2 + a_3 + \dots) \\ p_k &= a_{k+1} / (a_{k+1} + a_{k+2} + \dots) \quad \text{for } k \geq 1. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad p_0 &= a_1, \quad q_0 = 1 - a_1 \\ p_k &= a_{k+1} / (a_{k+1} + a_{k+2} + \dots), \quad q_k = 1 - p_k \quad \text{for } 1 \leq k < N - 1. \\ p_{N-1} &= 1 \end{aligned}$$

The above assumes instantaneous replacement.

3.5.3 A Batch Processing Model. Customers arrive at a facility and wait there until K customers have accumulated. Upon the arrival of the K th customer, all are instantaneously served, and the process repeats. Let $\xi_0, \xi_1, \dots$ denote the arrivals in successive periods, assumed to be independent random variables whose distribution is given by

$$\Pr\{\xi_k = 0\} = \alpha, \quad \Pr\{\xi_k = 1\} = 1 - \alpha,$$

where $0 < \alpha < 1$. Let X_n denote the number of customers in the system at time n . Then, $\{X_n\}$ is a Markov chain on the states $0, 1, \dots, K - 1$. With $K = 3$, give the transition probability matrix for $\{X_n\}$. Be explicit about any assumptions you make.

5.3

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} \alpha & 1 - \alpha & 0 \\ 0 & \alpha & 1 - \alpha \\ 1 - \alpha & 0 & \alpha \end{vmatrix} \end{matrix}$$

3.5.4 Martha has a fair die with the usual six sides. She throws the die and records the number. She throws the die again and adds the second number to the first. She repeats this until the cumulative sum of all the tosses first exceeds 10. What is the probability that she stops at a cumulative sum of 13?

5.4 After reviewing (5.8), set up the matrix

	0	1	2	3	4	5	6	7	8	9	10	13	~13
0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0	0
1	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0
2	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0
3	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0
4	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0
5	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$
6	0	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{2}{6}$
7	0	0	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$
8	0	0	0	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{6}$
9	0	0	0	0	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$
10	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{5}{6}$
13	0	0	0	0	0	0	0	0	0	0	0	1	0
~13	0	0	0	0	0	0	0	0	0	0	0	0	1

$U_{0,13} = .1819$

3.5.5 Let $\{X_n\}$ be a random walk for which zero is an absorbing state and such that from a positive state, the process is equally likely to go up or down one unit. The transition probability matrix is given by (3.38) with $r_0 = 1$ and $p_i = q_i = \frac{1}{2}$ for $i \geq 1$. (a) Show that $\{X_n\}$ is a nonnegative martingale. (b) Use the maximal inequality in Chapter 2, (2.53) to limit the probability that the process ever gets as high as $N > 0$.

5.5 If $X_n = 0$, then $X_{n+1} = 0$ and $E[X_{n+1}|X_n = 0] = 0 = X_n$. If $X_n = i > 0$ then $X_{n+1} = i \pm 1$, each w. pr. $\frac{1}{2}$ so $E[X_{n+1}|X_n = i] = i = X_n$. So $\{X_n\}$ is a martingale. If $X_0 = i$, then $Pr\{\max X_n \geq N\} \leq \frac{E[X_0]}{N} = \frac{i}{N}$ by the maximal inequality. Of course, (5.13) asserts $Pr\{\max X_n \geq N\} = \frac{i}{N}$.

3.8 Branching Processes*

3.8.3 Suppose a parent has no offspring with probability $\frac{1}{2}$ and has two offspring with probability $\frac{1}{2}$. If a population of such individuals begins with a single parent and evolves as a branching process, determine u_n , the probability that the population is extinct by the n th generation, for $n = 1, 2, 3, 4, 5$.

Solution. The recursion (3.4.8) specializes to

$$u_n = \frac{1}{2} + \frac{1}{2}(u_{n-1})^2 = \frac{1 + (u_{n-1})^2}{2}.$$

Beginning with $u_0 = 0$, we successively compute the probabilities:

$$u_0 = 0, \quad u_3 = 0.695,$$

$$u_1 = 0.5, \quad u_4 = 0.742,$$

$$u_2 = 0.625, \quad u_5 = 0.775,$$

We see that the chances are $\frac{31}{40}$ that such a population will die out by the 5th generation. ■

Problems

3.8.1 Each adult individual in a population produces a fixed number M of offspring and then dies. A fixed number L of these remain at the location of the parent. These local offspring will either all grow to adulthood, which occurs with a fixed probability β , or all will die, which has probability $1 - \beta$. Local mortality is catastrophic in that it affects the entire local population. The remaining $N = M - L$ offspring disperse. Their successful growth to adulthood will occur statistically independently of one another, but at a lower probability $\alpha = p\beta$, where p may be thought of as the probability of successfully surviving the dispersal process. Define the random variable ξ to be the number of offspring of a single parent that survive to reach adulthood in the next generation. According to our assumptions, we may write ξ as

$$\xi = v_1 + v_2 + \cdots + v_N + (M - N)\Theta,$$

where $\Theta, v_1, v_2, \dots, v_N$ are independent with $\Pr\{v_k = 1\} = \alpha$, $\Pr\{v_k = 0\} = 1 - \alpha$, and with $\Pr\{\Theta = 1\} = \beta$ and $\Pr\{\Theta = 0\} = 1 - \beta$. Show that the mean number of offspring reaching adulthood is $E[\xi] = \alpha N + \beta(M - N)$, and since $\alpha < \beta$, the mean number of surviving offspring is maximized by dispersing none ($N = 0$). Show that the probability of having no offspring surviving to adulthood is

$$\Pr\{\xi = 0\} = (1 - \alpha)^N(1 - \beta)$$

and that this probability is made smallest by making N large.

8.1 No adults survive if a local catastrophe occurs, which has probability $1 - \beta$, and if independently, all N dispersed offspring fail to survive, which has probability $(1 - \alpha)^N$. Another example in which looking at mean values alone is misleading.

3.8.2 Let $Z = \sum_{n=0}^{\infty} X_n$ be the total family size in a branching process whose offspring distribution has a mean $\mu = E[\xi] < 1$. Assuming that $X_0 = 1$, show that $E[Z] = 1/(1 - \mu)$.

8.2 A first step analysis yields $E[Z] = 1 + \mu E[Z]$ whence $E[Z] = 1/(1 - \mu)$, $\mu < 1$.

3.8.3 Families in a certain society choose the number of children that they will have according to the following rule: If the first child is a girl, they have exactly one more child. If the first child is a boy, they continue to have children until the first girl, and then cease childbearing.

- (a) For $k = 0, 1, 2, \dots$, what is the probability that a particular family will have k children in total?
 (b) For $k = 0, 1, 2, \dots$, what is the probability that a particular family will have exactly k male children among their offspring?

8.3 Possible families:	GG	GB	BG	BBG	BBBG...
Probability:	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16} \dots$

Let N = Total children, X = Male children

(a) $Pr\{N = 2\} = \frac{3}{4}, \quad Pr\{N = k\} = \left(\frac{1}{2}\right)^k \quad \text{for } k \geq 3.$

(b) $Pr\{X = 1\} = \frac{1}{2}, \quad Pr\{X = 0\} = \frac{1}{4}, \quad Pr\{X = k\} = \left(\frac{1}{2}\right)^{k+1}, \quad k \geq 2.$

3.8.4 Let $\{X_n\}$ be a branching process with mean family size μ . Show that $Z_n = X_n/\mu^n$ is a nonnegative martingale. Interpret the maximal inequality as applied to $\{Z_n\}$.

8.4 $E[Z_{n+1}|Z_0, \dots, Z_n] = \frac{1}{\mu^{n+1}} E[\xi_1 + \dots + \xi_{X_n} | X_n = \mu^n Z_n] = \frac{1}{\mu^{n+1}} \mu X_n = Z_n$. Suppose $Z_0 = X_0 = 1$.

4 The Long Run Behavior of Markov Chains

4.1 Regular Transition Probability Matrices

4.1.1 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{vmatrix} \end{matrix}.$$

Determine the limiting distribution.

Solution. We aim to calculate the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2)$. First, observe that

$$\mathbf{P}^T = \begin{pmatrix} 0.7 & 0 & 0.5 \\ 0.2 & 0.6 & 0 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$$

So, referring to Equations (3.5.1)-(3.5.2), (in the matrix form (3.5.3)), we obtain

$$0.7\pi_0 + 0\pi_1 + 0.5\pi_2 = \pi_0, \quad (3.5.12)$$

$$0.2\pi_0 + 0.6\pi_1 + 0\pi_2 = \pi_1, \quad (3.5.13)$$

$$0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2 = \pi_2, \quad (3.5.14)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3.5.15)$$

We arbitrarily strike out (3.5.14) and simplify the remaining equations to get

$$3\pi_0 - 5\pi_2 = 0, \quad (3.5.16)$$

$$2\pi_0 - 4\pi_1 = 0, \quad (3.5.17)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3.5.18)$$

From (3.5.16) and (3.5.17), we get $\pi_1 = \frac{1}{2}\pi_0$, $\pi_2 = \frac{3}{5}\pi_0$. Substituting in (3.5.18), we get $\pi_0 = \frac{10}{21}$, $\pi_1 = \frac{5}{21}$, $\pi_2 = \frac{6}{21}$. ■

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$$\because \pi_j = \sum_{k=0}^2 \pi_k P_{kj}, \quad j = 0, 1, 2$$

$$\text{and } \sum_{k=0}^2 \pi_k = 1$$

$\Rightarrow$

$$\pi_0 = 0.7\pi_0 + 0.5\pi_2$$

$$\therefore \pi_2 = 0.6\pi_0 \quad (1)$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1$$

$$\therefore \pi_1 = 0.5\pi_0 \quad (2)$$

$$\because \pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

$\therefore$ by substituting (1) and (2) in (3)

we get $\pi_0 + 0.5\pi_0 + 0.6\pi_0 = 1$

$$\therefore \pi_0 = \frac{10}{21}$$

$$\therefore \pi_1 = 0.5\left(\frac{10}{21}\right) = \frac{5}{21}, \quad \pi_2 = 0.6\left(\frac{10}{21}\right) = \frac{6}{21}$$

$\therefore$ The limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (10/21, 5/21, 6/21)$

$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = 0.7\pi_0 + 0.5\pi_2$$

$$\pi_2 = \frac{0.3}{0.5}\pi_0 = \frac{3}{5}\pi_0 = \frac{2}{7}$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1, \quad \pi_2 = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2$$

$$\pi_1 = \frac{0.2}{0.4}\pi_0 = \frac{1}{2}\pi_0 \\ = \frac{5}{21}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\frac{10}{21}\pi_0 + \frac{5}{21}\pi_0 + \frac{6}{21}\pi_0 = 1$$

$$\frac{21}{21}\pi_0 = 1$$

$$\pi_0 = \frac{10}{21}$$

$$\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} = 0.7\pi_0 + 0.5\pi_2$$

$$\pi_1 = \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} = 0.2\pi_0 + 0.6\pi_1 \Rightarrow 0.4\pi_1 = 0.2\pi_0 \Rightarrow \pi_1 = \frac{1}{2}\pi_0 = 0.5\pi_0 = \frac{1}{2}\left(\frac{10}{21}\right) = \frac{5}{21}$$

$$\pi_2 = \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2 \Rightarrow 0.5\pi_2 = 0.1\pi_0 + 0.2\pi_0 \Rightarrow \pi_2 = \frac{2}{5}\pi_0 \Rightarrow \frac{2}{5}\left(\frac{10}{21}\right) = \frac{6}{21} = \frac{2}{7}$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \Rightarrow \pi_0 + \frac{1}{2}\pi_0 + \frac{2}{5}\pi_0 = 1 \Rightarrow \frac{21}{10}\pi_0 = 1 \Rightarrow \pi_0 = \frac{10}{21}$$

4.1.2 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{vmatrix} \end{matrix}.$$

Determine the limiting distribution.

Answer

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0 = 0.6\pi_0 + 0.3\pi_1 + 0.4\pi_2$$

$$\pi_1 = 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$4\pi_0 - 3\pi_1 - 4\pi_2 = 0 \quad (1)$$

$$3\pi_0 - 7\pi_1 + \pi_2 = 0 \quad (2)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

By solving equations using Cramer's rule, we get

$$\Delta = \begin{vmatrix} 4 & -3 & -4 \\ 3 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -66, \quad \Delta_0 = \begin{vmatrix} 0 & -3 & -4 \\ 0 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -31$$

$$\Delta_1 = \begin{vmatrix} 4 & 0 & -4 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -16, \quad \Delta_2 = \begin{vmatrix} 4 & -3 & 0 \\ 3 & -7 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -19$$

$$\therefore \pi_0 = \frac{\Delta_0}{\Delta} = \frac{31}{66}, \quad \pi_1 = \frac{\Delta_1}{\Delta} = \frac{16}{66}, \quad \pi_2 = \frac{\Delta_2}{\Delta} = \frac{19}{66}$$

$\therefore$ The limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (31/66, 16/66, 19/66)$

Solution. Same method of the latter example, yields $\pi_0 = \frac{31}{66}$, $\pi_1 = \frac{16}{66}$, $\pi_2 = \frac{19}{66}$. ■

Determine the limiting distribution. π_0, π_1, π_2

$$\begin{aligned} \pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = 0.6\pi_0 + 0.3\pi_1 + 0.4\pi_2 \\ \pi_1 &= \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21} = 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2 \\ \pi_2 &= \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2 \end{aligned} \Rightarrow \begin{cases} 0.4\pi_0 - 0.3\pi_1 - 0.4\pi_2 = 0 \\ 0.3\pi_0 - 0.7\pi_1 + 0.1\pi_2 = 0 \\ 0.1\pi_0 + 0.4\pi_1 - 0.5\pi_2 = 0 \end{cases} \Rightarrow \begin{cases} 0.4\pi_0 - 0.3\pi_1 - 0.4\pi_2 = 0 \\ 0.1\pi_0 + 0.4\pi_1 - 0.5\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\begin{vmatrix} 0.4 & -0.3 & -0.4 \\ 0.3 & -0.7 & 0.1 \\ 0.1 & 0.4 & -0.5 \end{vmatrix} \approx [0.16 + 0.15 - 0.04] - [-0.2 - 0.03 - 0.16] = 0.27 + 0.39 = 0.66 = \Delta$$

$$\Delta_0 = \begin{vmatrix} 0 & -0.3 & -0.4 \\ 0 & -0.7 & 0.1 \\ 0 & 0.4 & -0.5 \end{vmatrix} = 0.31 \quad \Delta_1 = \begin{vmatrix} 0.4 & 0 & -0.4 \\ 0.3 & 0 & 0.1 \\ 0.1 & 0 & -0.5 \end{vmatrix} = 0.16 \quad \Delta_2 = \begin{vmatrix} 0.4 & -0.3 & 0 \\ 0.3 & -0.7 & 0 \\ 0.1 & 0.4 & 0 \end{vmatrix} = 0.19$$

$$\pi_0 = \frac{\Delta_0}{\Delta} = \frac{31}{66} \quad \pi_1 = \frac{\Delta_1}{\Delta} = \frac{16}{66} \quad \pi_2 = \frac{\Delta_2}{\Delta} = \frac{19}{66}$$

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \end{matrix}$$

$$\pi_0 = \frac{-31}{-66} = 0.4697$$

$$\pi_1 = \frac{-16}{-66} = 0.2424$$

$$\pi_2 = \frac{-19}{-66} = 0.2878$$

Determine the limiting distribution.

$$\pi_0 = 0.6\pi_0 + 0.3\pi_1 + 0.4\pi_2$$

$$0.4\pi_2 - 0.3\pi_1 - 0.4\pi_0 = 0$$

$$\pi_1 = 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2$$

$$0.3\pi_0 - 0.7\pi_1 + 0.1\pi_2 = 0$$

$$\pi_2 = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2$$

$$0.1\pi_0 + 0.4\pi_1 - 0.5\pi_2 = 0$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\Delta = \begin{vmatrix} 4 & -3 & -4 \\ 3 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 3 & -7 \\ 1 & 1 \end{vmatrix} = -28 - 3 - 12 + 9 - 4 - 28 = -66$$

$$\Delta_0 = \begin{vmatrix} 0 & -3 & -4 \\ 0 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ 0 & -7 \\ 1 & 1 \end{vmatrix} = -3 - 28 = -31$$

$$\Delta_1 = \begin{vmatrix} 4 & 0 & -4 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 3 & 0 \\ 1 & 1 \end{vmatrix} = -12 - 4 = -16$$

$$\Delta_2 = \begin{vmatrix} 4 & -3 & 0 \\ 3 & -7 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 3 & -7 \\ 1 & 1 \end{vmatrix} = -28 + 9 = -19$$

4.1.3 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{matrix}.$$

What fraction of time, in the long run, does the process spend in state 1?

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{matrix}$$

$\downarrow \quad \downarrow \quad \downarrow$
 $\pi_0 \quad \pi_1 \quad \pi_2$

what fraction of time in the long run does the process spend in state 1? (π_1)??

Ans: $\pi_j = \sum_{k=0}^2 \pi_k P_{kj}$

at $j=0 \Rightarrow \pi_0 = 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2$
 $0.9\pi_0 = 0.2\pi_1 + 0.3\pi_2 \quad (\times 10)$
 $\therefore 9\pi_0 - 2\pi_1 - 3\pi_2 = 0 \quad (1)$

at $j=1 \Rightarrow \pi_1 = 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2$
 $0.1\pi_0 - 0.8\pi_1 + 0.3\pi_2 = 0 \quad (\times 10)$
 $\therefore \pi_0 - 8\pi_1 + 3\pi_2 = 0 \quad (2)$

$\therefore \pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$

Solving (1), (2) and (3)

Add (1), (2) $\Rightarrow 10\pi_0 - 10\pi_1 = 0 \Rightarrow \pi_0 = \pi_1$

$\therefore \pi_1 = \pi_0$

(1) + (3) $\times 3 \Rightarrow 12\pi_0 + \pi_1 = 3$

$\therefore 12\pi_1 + \pi_1 = 3 \Rightarrow 13\pi_1 = 3 \Rightarrow \pi_1 = \frac{3}{13}$

Solution. According to the above secondary interpretation of the limiting distribution, and using the same method as in the previous examples, we see that the long run fraction of time that the process spends in state 1 is $\pi_1 = \frac{3}{13}$. ■

What fraction of time, in the long run, does the process spend in state 1? π_1

$$\begin{aligned}\pi_0 &= \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} = 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} = 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\ \pi_2 &= \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1\end{aligned} \Rightarrow \begin{cases} 0.9\pi_0 - 0.2\pi_1 - 0.3\pi_2 = 0 \\ 0.1\pi_0 - 0.8\pi_1 + 0.3\pi_2 = 0 \\ 0.2\pi_0 + 0.6\pi_1 - 0.6\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} 0.9\pi_0 - 0.2\pi_1 - 0.3\pi_2 = 0 \\ 0.1\pi_0 - 0.8\pi_1 + 0.3\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Delta_0 = \begin{vmatrix} 0.9 & -0.2 & -0.3 \\ 0.1 & -0.8 & 0.3 \\ 1 & 1 & 1 \end{vmatrix} = -0.3 - 0.3 - 0.7 = -1.3$$

$$\Delta_1 = \begin{vmatrix} 0.9 & 0 & -0.3 \\ 0.1 & 0 & 0.3 \\ 1 & 1 & 1 \end{vmatrix} = -0.3$$

$$\Delta_2 = \begin{vmatrix} 0.9 & -0.2 & 0 \\ 0.1 & -0.8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -0.7$$

$$\pi_0 = \frac{3}{13}, \quad \pi_1 = \frac{3}{13}, \quad \pi_2 = \frac{7}{13}$$

What fraction of time, in the long run, does the process spend in state 1? π_1

$$\begin{aligned}\pi_0 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\ \pi_1 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\ \pi_2 &= 0.8\pi_0 + 0.6\pi_1 + 0.4\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1\end{aligned} \Rightarrow \begin{cases} 0.9\pi_0 - 0.2\pi_1 - 0.3\pi_2 = 0 \\ 0.1\pi_0 - 0.8\pi_1 + 0.3\pi_2 = 0 \\ 0.8\pi_0 + 0.6\pi_1 - 0.6\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -8 & 3 & 1 & -8 \\ 8 & 6 & -6 & 8 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 6 + 48 + 24 + 64 + 6 - 18 = 130$$

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 3 & 1 & 0 \\ 8 & 0 & -6 & 8 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 24 + 6 = 30$$

$$\pi_1 = \frac{30}{130} = \frac{3}{13}$$

4.1.4 A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{vmatrix} \end{matrix}.$$

Every period that the process spends in state 0 incurs a cost of \$2. Every period that the process spends in state 1 incurs a cost of \$5. Every period that the process spends in state 2 incurs a cost of \$3. What is the long run cost per period associated with this Markov chain?

Solution. First, we calculate the limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$. For, observe that

$$\mathbf{P}^T = \begin{pmatrix} 0.3 & 0.5 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.5 & 0.4 & 0.3 \end{pmatrix}$$

So, referring to Equations (3.5.1)-(3.5.2), (in the matrix form (3.5.3)), we obtain

$$0.3\pi_0 + 0.5\pi_1 + 0.5\pi_2 = \pi_0, \quad (3.5.19)$$

$$0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2 = \pi_1, \quad (3.5.20)$$

$$0.5\pi_0 + 0.4\pi_1 + 0.3\pi_2 = \pi_2, \quad (3.5.21)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3.5.22)$$

We arbitrarily strike out (3.5.21) and simplify the remaining equations to get

$$-7\pi_0 + 5\pi_1 + 5\pi_2 = 0, \quad (3.5.23)$$

$$2\pi_0 - 9\pi_1 + 2\pi_2 = 0, \quad (3.5.24)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3.5.25)$$

Multiplying (3.5.25) by -2 and adding the resulting equation to (3.5.24), we get $\pi_1 = \frac{2}{11}$.

Then, multiplying (3.5.25) by -5 and adding the resulting equation to (3.5.23), we get $\pi_0 = \frac{5}{12}$. Now, substituting the obtained values in (3.5.25), we obtain $\pi_2 = \frac{53}{132}$.

Next, invoking Formula (3.5.11), we see that the long run mean cost per period is equal to

$$\text{Mean cost} = \left(\frac{5}{12}\right)(2) + \left(\frac{2}{11}\right)(5) + \left(\frac{53}{132}\right)(3) = \frac{389}{132} = 2.95 \text{ \$}. \blacksquare$$

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \\ & \begin{matrix} \downarrow & \downarrow & \downarrow \\ \pi_0 & \pi_1 & \pi_2 \end{matrix} \end{matrix}$$

Find the long run cost per unit period ??

Ans:

* The long run mean cost per unit period

$$= \sum_{j=0}^2 \pi_j c_j$$

$$= \pi_0 c_0 + \pi_1 c_1 + \pi_2 c_2$$

$$= 2\pi_0 + 5\pi_1 + 3\pi_2$$

(I)

$$\pi_j = \sum_{k=0}^2 \pi_k P_{kj}$$

$$\text{at } j=0 \Rightarrow \pi_0 = 0.3\pi_0 + 0.5\pi_1 + 0.5\pi_2 \quad (1)$$

$$\text{at } j=1 \Rightarrow \pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2 \quad (2)$$

$$\text{S. } \pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

(1), (2) and (3) $\Rightarrow$

$$\begin{aligned} 7\pi_0 - 5\pi_1 - 5\pi_2 &= 0 \\ 2\pi_0 - 9\pi_1 + 2\pi_2 &= 0 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

(1)

(2)

(3)

3 Solving ①, ② and ③ using Cramer rule

$$\pi_0 = \frac{\Delta_0}{\Delta}, \pi_1 = \frac{\Delta_1}{\Delta}, \pi_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta = \begin{vmatrix} 7 & -5 & -5 \\ 2 & -9 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -55 - 24 + (-53) = -132$$

$$\Delta_0 = \begin{vmatrix} 0 & -5 & -5 \\ 0 & -9 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -55$$

$$\Delta_1 = \begin{vmatrix} 7 & 0 & -5 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -24$$

$$\Delta_2 = \begin{vmatrix} 7 & -5 & 0 \\ 2 & -9 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -53$$

$$\therefore \pi_0 = \frac{-55}{-132} = \frac{5}{12}, \pi_1 = \frac{-24}{-132} = \frac{2}{11}, \pi_2 = \frac{-53}{-132} = \frac{53}{132}$$

Subs. (II) in (I)
The mean cost per unit period

$$= 2\left(\frac{5}{12}\right) + 5\left(\frac{2}{11}\right) + 3\left(\frac{53}{132}\right)$$

$$= \frac{389}{132} \approx \$2.95$$

4.1.5 Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{vmatrix} 0.1 & 0.5 & 0 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} \end{matrix}$$

Determine the limiting distribution for the process.

Solution. Using the same method as in the above examples, we obtain $\pi_0 = \frac{10}{29}$, $\pi_1 = \frac{5}{29}$, $\pi_2 = \frac{5}{29}$, $\pi_3 = \frac{9}{29}$. ■

4.1.8 Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

Answer

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \quad (1)$$

$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \quad (2)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

By **solving** equations using Cramer's rule, we get

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 34, \quad \Delta_0 = \begin{vmatrix} 0 & -2 & -1 \\ 0 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 12$$

$$\Delta_1 = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 14, \quad \Delta_2 = \begin{vmatrix} 3 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 8$$

$$\therefore \pi_0 = \frac{\Delta_0}{\Delta} = \frac{6}{17}, \quad \pi_1 = \frac{\Delta_1}{\Delta} = \frac{7}{17}, \quad \pi_2 = \frac{\Delta_2}{\Delta} = \frac{4}{17}$$

$\therefore$ The limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (6/17, 7/17, 4/17)$

$\therefore$ In the long run, approximately 23.53% of families are upper class.

What fraction of families are upper class in the long run?

$$\begin{aligned} \pi_0 &= \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2 \Rightarrow 0.3\pi_0 - 0.2\pi_1 - 0.1\pi_2 = 0 \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 \Rightarrow 0.2\pi_0 - 0.4\pi_1 + 0.4\pi_2 = 0 \\ \pi_2 &= \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2 \Rightarrow 0.1\pi_0 + 0.2\pi_1 - 0.5\pi_2 = 0 \end{aligned}$$

$$\Rightarrow \begin{cases} 0.3\pi_0 - 0.2\pi_1 - 0.1\pi_2 = 0 \\ 0.1\pi_0 + 0.2\pi_1 - 0.5\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 0.3 & -0.2 & -0.1 \\ 0.1 & 0.2 & -0.5 \\ 1 & 1 & 1 \end{vmatrix} = 0.12 + 0.14 + 0.06 = 0.34$$

$$\Delta_2 = \begin{vmatrix} 0.3 & -0.2 & 0 \\ 0.1 & 0.2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0.06 + 0.02 = 0.08 \quad \pi_2 = \frac{8}{34} = \frac{4}{17}$$

Solution. Adopting the same notations as in Example 3.36, and proceeding in the same manner as in the solution of Example 3.42, we obtain $\pi_2 = \frac{4}{17}$. ■

Problems

4.1.1 Five balls are distributed between two urns, labeled A and B. Each period, an urn is selected at random, and if it is not empty, a ball from that urn is removed and placed into the other urn. In the long run what fraction of time is urn A empty?

1.1 Let X_n be the number of balls in Urn A. Then $\{X_n\}$ is a doubly stochastic Markov chain having 6 states, whence $\lim_{n \rightarrow \infty} Pr\{X_n = 0 | X_0 = i\} = \frac{1}{6}$.

4.1.2 Five balls are distributed between two urns, labeled A and B. Each period, one of the five balls is selected at random, and whichever urn it's in, it is moved to the other urn. In the long run, what fraction of time is urn A empty?

1.2 Let X_n be the number of balls in Urn A.

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$\pi_0 = \pi_5 = \frac{1}{32}, \quad \pi_1 = \pi_4 = \frac{5}{32}, \quad \pi_2 = \pi_3 = \frac{10}{32}.$$

$\pi_0 = \frac{1}{32}$

4.1.3 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \left\| \begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right\| \end{matrix},$$

where $\alpha_i \geq 0, i = 1, \dots, 6$, and $\alpha_1 + \dots + \alpha_6 = 1$. Determine the limiting probability of being in state 0.

1.3 The equations for the stationary distribution simplify to:

$$\pi_0 = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0 \quad (\sum \alpha_i = 1)$$

$$\pi_1 = (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0$$

$$\pi_2 = (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0$$

$$\pi_3 = (\alpha_4 + \alpha_5 + \alpha_6)\pi_0$$

$$\pi_4 = (\alpha_5 + \alpha_6)\pi_0$$

$$\pi_5 = \alpha_6\pi_0$$

$$1 = \left(\sum_{k=1}^6 k\alpha_k \right) \pi_0$$

$$\pi_0 = \frac{1}{\sum_{k=1}^6 k\alpha_k} = \frac{1}{\text{Mean of } \alpha \text{ distribution}}$$

4.1.4 A finite-state regular Markov chain has transition probability matrix $\mathbf{P} = \|P_{ij}\|$ and limiting distribution $\pi = \|\pi_i\|$. In the long run, what fraction of the *transitions* are from a prescribed state k to a prescribed state m ?

1.4 Formally, one can look at the Markov chain $Z_n = (X_n, X_{n+1})$ and ask for $\lim_{n \rightarrow \infty} Pr\{Z_n = (k, m)\} = \lim Pr\{X_n = k, X_{n+1} = m\} = \pi_k P_{km}$.

4.1.5 The four towns A, B, C, and D are connected by railroad lines as shown in the following diagram:

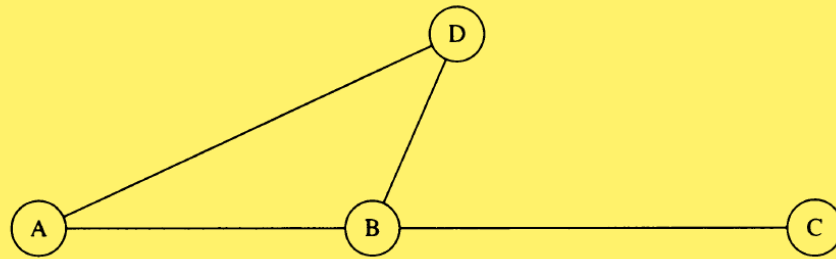


Figure 4.1 A graph whose nodes represent towns and whose arcs represent railroad lines.

Each day, in whichever town it is in, a train chooses one of the lines out of that town at random and traverses it to the next town, where the process repeats the next day. In the long run, what is the probability of finding the train in town D?

$$\begin{array}{c}
 \mathbf{1.5} \\
 \mathbf{P} = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{array}{c} \left\| \begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array} \right\| \end{array}
 \end{array}$$

$\pi_A = \frac{2}{8}; \quad \pi_B = \frac{3}{8}; \quad \pi_C = \frac{1}{8} \quad \boxed{\pi_D = \frac{2}{8}}$

Note: $\pi_{\text{NODE}} \propto \# \text{ arcs at NODE.}$

4.1.6 Determine the following limits in terms of the transition probability matrix $\mathbf{P} = \|P_{ij}\|$ and limiting distribution $\pi = \|\pi_j\|$ of a finite-state regular Markov chain $\{X_n\}$:

- (a) $\lim_{n \rightarrow \infty} \Pr\{X_{n+1} = j | X_0 = i\}.$
- (b) $\lim_{n \rightarrow \infty} \Pr\{X_n = k, X_{n+1} = j | X_0 = i\}.$
- (c) $\lim_{n \rightarrow \infty} \Pr\{X_{n-1} = k, X_n = j | X_0 = i\}.$

$$\begin{aligned}
 \mathbf{1.6} \quad & \lim_{n \rightarrow \infty} \Pr\{X_{n+1} = j | X_0 = i\} = \pi_j. \\
 & \lim_{n \rightarrow \infty} \Pr\{X_n = k, X_{n+1} = j | X_0 = i\} = \pi_k P_{kj}.
 \end{aligned}$$

4.1.7 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{vmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{vmatrix} \end{matrix}.$$

1.7 $\pi_0 = \frac{6}{19}; \quad \pi_1 = \frac{3}{19}; \quad \pi_2 = \frac{6}{19}; \quad \pi_3 = \frac{4}{19}.$

4.1.8 Show that the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{vmatrix} \end{matrix}$$

is regular and compute the limiting distribution.

1.8 P^8 has all positive entries.

$$\pi_0 = .2529 \quad \pi_1 = .2299 \quad \pi_2 = .3103 \quad \pi_3 = .1379 \quad \pi_4 = .0690$$

4.1.9 Determine the long run, or limiting, distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{matrix}.$$

1.9 $\pi_0 = \pi_1 = \frac{1}{10} \quad \pi_2 = \pi_3 = \frac{4}{10}.$

4.1.10 Consider a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & p_N \\ p_N & p_0 & p_1 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{bmatrix},$$

where $0 < p_0 < 1$ and $p_0 + p_1 + \cdots + p_N = 1$. Determine the limiting distribution.

1.10 The matrix is doubly stochastic, whence

$$\pi_k = \frac{1}{N+1} \quad \text{for } k = 0, 1, \dots, N.$$

4.1.11 Suppose that a production process changes state according to a Markov process whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0.3 & 0.5 & 0 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{bmatrix} \end{matrix}.$$

It is known that $\pi_1 = \frac{119}{379} = 0.3140$ and $\pi_2 = \frac{81}{379} = 0.2137$.

- Determine the limiting probabilities π_0 and π_3 .
- Suppose that states 0 and 1 are “In-Control” while states 2 and 3 are deemed “Out-of-Control.” In the long run, what fraction of time is the process Out-of-Control?
- In the long run, what *fraction of transitions* are from an In-Control state to an Out-of-Control state?

$$1.11 \text{ (a) } \pi_0 = \frac{117}{379} = .3087 \quad \pi_3 = \frac{62}{379} = .1636$$

$$\text{(b) } \pi_2 + \pi_3 = \frac{143}{379} = .3773.$$

$$\text{(c) } \pi_0(P_{02} + P_{03}) + \pi_1(P_{12} + P_{13}) = .1559$$

4.1.12 Let $\mathbf{P}$ be the transition probability matrix of a finite-state regular Markov chain, and let $\mathbf{\Pi}$ be the matrix whose rows are the stationary distribution π . Define $\mathbf{Q} = \mathbf{P} - \mathbf{\Pi}$.

(a) Show that $\mathbf{P}^n = \mathbf{\Pi} + \mathbf{Q}^n$.

(b) When

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

obtain an explicit expression for $\mathbf{Q}^n$ and then for $\mathbf{P}^n$.

1.12 (a) $\mathbf{\Pi P} = \mathbf{P\Pi} = \mathbf{\Pi} = \mathbf{\Pi}^2$. Then $\mathbf{Q}^2 = (\mathbf{P} - \mathbf{\Pi})(\mathbf{P} - \mathbf{\Pi}) = \mathbf{P}^2 - \mathbf{\Pi P} - \mathbf{P\Pi} + \mathbf{\Pi}^2 = \mathbf{P}^2 - \mathbf{\Pi}$. Similarly, $\mathbf{\Pi P}^n = \mathbf{P}^n \mathbf{\Pi} = \mathbf{\Pi}^{n+1} = \mathbf{\Pi}$ and $\mathbf{Q}^{n+1} = (\mathbf{P} - \mathbf{\Pi})(\mathbf{P}^n - \mathbf{\Pi}) = \mathbf{P}^{n+1} - \mathbf{\Pi}$.

$$\text{(b) } \mathbf{Q}^n = \frac{1}{2} \begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 & -\left(\frac{1}{2}\right)^n \\ 0 & 0 & 0 \\ -\left(\frac{1}{2}\right)^n & 0 & \left(\frac{1}{2}\right)^n \end{bmatrix}$$

4.1.13 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \end{matrix}$$

After a long period of time, you observe the chain and see that it is in state 1. What is the conditional probability that the previous state was state 2? That is, find

$$\lim_{n \rightarrow \infty} \Pr\{X_{n-1} = 2 | X_n = 1\}.$$

$$1.13 \quad \lim_{n \rightarrow \infty} \Pr\{X_{n-1} = 2 | X_n = 1\} = \frac{\pi_2 P_{21}}{\pi_1} = \frac{6 \times .2}{7} = .1714 \quad \pi_0 = \frac{11}{24} \quad \pi_1 = \frac{7}{24} \quad \pi_2 = \frac{6}{24}.$$

4.2 Examples

4.2.5 From purchase to purchase, a particular customer switches brands among products A, B, and C according to a Markov chain whose transition probability matrix is

$$P = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{matrix}$$

In the long run, what fraction of time does this customer purchase brand A?

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$$P = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{matrix}$$

$\downarrow \quad \downarrow \quad \downarrow$
 $\pi_0 \quad \pi_1 \quad \pi_2$

$$\pi_0 = \pi_A$$

??

Ans:

The limiting distⁿ is $\pi = (\pi_0, \pi_1, \pi_2)$

$$\pi_0 = 0.6\pi_0 + 0.1\pi_1 + 0.1\pi_2$$

$$0.4\pi_0 - 0.1\pi_1 - 0.1\pi_2 = 0 \quad (\times 10)$$

$$4\pi_0 - \pi_1 - \pi_2 = 0 \quad (1)$$

8

$$\pi_1 = 0.2\pi_0 + 0.7\pi_1 + 0.1\pi_2$$

$$0.2\pi_0 - 0.3\pi_1 + 0.1\pi_2 = 0 \quad (\times 10)$$

$$2\pi_0 - 3\pi_1 + \pi_2 = 0 \quad (2)$$

8

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

Solving (1), (2) and (3) by using Cramer rule

$$\Delta = \begin{vmatrix} 4 & -1 & -1 \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4 - 6 - 10 = -20$$

$$\Delta_0 = \begin{vmatrix} 0 & -1 & -1 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4, \quad \pi_0 = \frac{\Delta_0}{\Delta} = \frac{-4}{-20} = \frac{1}{5}$$

The fraction of time that the customer purchases brand A

$$\text{is } \pi_A = \pi_0 = \frac{1}{5} = 20\%$$

#

In the long run, what fraction of time does this customer purchase brand A? π_0

$$\begin{aligned}
 \pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = 0.6 \pi_0 + 0.1 \pi_1 + 0.1 \pi_2 \Rightarrow 0.4 \pi_0 - 0.1 \pi_1 - 0.1 \pi_2 = 0 \\
 \pi_1 &= \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21} = 0.2 \pi_0 + 0.7 \pi_1 + 0.1 \pi_2 \Rightarrow 0.2 \pi_0 - 0.3 \pi_1 - 0.1 \pi_2 = 0 \\
 \pi_2 &= \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} = 0.2 \pi_0 + 0.2 \pi_1 + 0.8 \pi_2 \Rightarrow 0.2 \pi_0 + 0.2 \pi_1 - 0.2 \pi_2 = 0
 \end{aligned}
 \quad \left\{ \begin{array}{l} 0.4 \pi_0 - 0.1 \pi_1 - 0.1 \pi_2 = 0 \\ 0.2 \pi_0 + 0.2 \pi_1 - 0.2 \pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{array} \right.$$

$$\begin{vmatrix} 0.4 & -0.1 & -0.1 \\ 0.2 & 0.2 & -0.2 \\ 1 & 1 & 1 \end{vmatrix} = 0.04 + 0.06 + 0.1 = 0.2 = \Delta \quad \Delta_0 = \begin{vmatrix} 0 & -0.1 & -0.1 \\ 0 & 0.2 & -0.2 \\ 1 & 1 & 1 \end{vmatrix} = 0.02 + 0.02 = 0.04 \quad \pi_0 = \frac{\Delta_0}{\Delta} = \frac{1}{5}$$

Problems

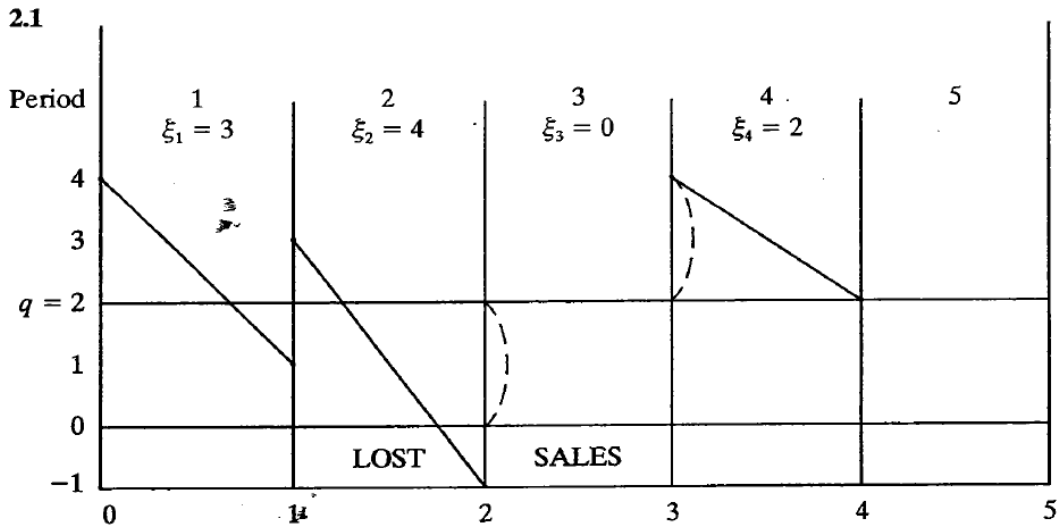
4.2.1 Consider a discrete-time periodic review inventory model (see Chapter 3, Section 3.3.1), and let ξ_n be the total demand in period n . Let X_n be the inventory quantity on hand at the end of period n . Instead of following an (s, S) policy, a (q, Q) policy will be used: If the stock level at the end of a period is less than or equal to $q = 2$ units, then $Q = 2$ additional units will be ordered and will be available at the beginning of the next period. Otherwise, no ordering will take place. This is a (q, Q) policy with $q = 2$ and $Q = 2$. Assume that demand that is not filled in a period is lost (no back ordering).

- (a) Suppose that $X_0 = 4$ and that the period demands turn out to be $\xi_1 = 3$, $\xi_2 = 4$, $\xi_3 = 0$, $\xi_4 = 2$. What are the end-of-period stock levels for periods $n = 1, 2, 3, 4$?
- (b) Suppose that $\xi_1, \xi_2, \dots$ are independent random variables, each having the probability distribution where

k	$=$	0	1	2	3	4
$\Pr\{\xi = k\}$	$=$	0.1	0.3	0.3	0.2	0.1

Then, $X_0, X_1, \dots$ is a Markov chain. Determine the transition probability distribution and the limiting distribution.

- (c) In the long run, during what fraction of periods are orders placed?



- (a) $X_0 = 4$ $X_1 = 1$ $X_2 = 0$ $X_3 = 2$ $X_4 = 2$

(b) $P =$

	0	1	2	3	4
0	.6	.3	.1	0	0
1	.3	.3	.3	.1	0
2	.1	.2	.3	.3	.1
3	.3	.3	.3	.1	0
4	.1	.2	.3	.3	.1

2.1 (c) $\pi_0 + \pi_1 + \pi_2 = .3559 + .2746 + .2288 = .8593$

4.2.2 A system consists of two components operating *in parallel*: The system functions if at least one of the components is operating. In any single period, if both components are operating at the beginning of the period, then each will fail, independently, during the period with probability α . When one component has failed, the remaining component fails during a period with a higher probability β . There is a single repair facility, and it takes two periods to repair a component.

- (a) Define an appropriate set of states for the system in the manner of the *Reliability and Redundancy* example and specify the transition probabilities in terms of α and β .
- (b) When $\alpha = 0.1$ and $\beta = 0.2$, in the long run what fraction of time is the system operating?

2.2 State is (x, y) where $x = \#$ machines operating and $y = \#$ days repair completed.

$$(a) \mathbf{P} = \begin{matrix} & \begin{matrix} (2,0) & (1,0) & (1,1) & (0,0) & (0,1) \end{matrix} \\ \begin{matrix} (2,0) \\ (1,0) \\ (1,1) \\ (0,0) \\ (0,1) \end{matrix} & \left\| \begin{array}{ccccc} (1-\alpha)^2 & 2\alpha(1-\alpha) & 0 & \alpha^2 & 0 \\ 0 & 0 & 1-\beta & 0 & \beta \\ (1-\beta) & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right\| \end{matrix}$$

$$(b) \pi_{(2,0)} + \pi_{(1,0)} + \pi_{(1,1)} = .6197 + .1840 + .1472 = .9508.$$

4.2.3 Suppose that a production process changes state according to a Markov process whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 0.2 & 0.2 & 0.4 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{array} \right\| \end{matrix}$$

- (a) Determine the limiting distribution for the process.
- (b) Suppose that states 0 and 1 are "In-Control," while states 2 and 3 are deemed "Out-of-Control." In the long run, what fraction of time is the process Out-of-Control?
- (c) In the long run, what fraction of transitions are from an In-Control state to an Out-of-Control state?

2.3 (a) $\pi_0 = .2549$; $\pi_1 = .2353$; $\pi_2 = .3529$; $\pi_3 = .1569$

(b) $\pi_2 + \pi_3 = .5098$

(c) $\pi_0(P_{02} + P_{03}) + \pi_1(P_{12} + P_{13}) = .2235.$

4.2.4 A component of a computer has an active life, measured in discrete units, that is a random variable ξ , where

k	$=$	1	2	3	4
$\Pr\{\xi = k\}$	$=$	0.1	0.3	0.2	0.4

Suppose that one starts with a fresh component, and each component is replaced by a new component upon failure. Let X_n be the *remaining life* of the component in service at the *end* of period n . When $X_n = 0$, a new item is placed into service at the *start* of the next period.

- (a) Set up the transition probability matrix for $\{X_n\}$.
- (b) By showing that the chain is regular and solving for the limiting distribution, determine the long run probability that the item in service at the end of a period has no remaining life and therefore will be replaced.
- (c) Relate this to the mean life of a component.

$$\begin{array}{c}
 \mathbf{2.4} \qquad \qquad \qquad \begin{array}{cccc}
 & 0 & 1 & 2 & 3 \\
 \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \parallel & \begin{array}{c} .1 \\ 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} .3 \\ 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} .2 \\ 0 \\ 0 \\ 1 \end{array} & \parallel & \begin{array}{c} .4 \\ 0 \\ 0 \\ 0 \end{array}
 \end{array}
 \end{array}$$

$$E[\xi] = 2.9$$

$$\pi_0 = \frac{10}{29} = \frac{1}{E[\xi]}; \quad \pi_1 = \frac{9}{29}; \quad \pi_2 = \frac{6}{29}; \quad \pi_3 = \frac{4}{29}.$$

4.2.5 Suppose that the weather on any day depends on the weather conditions during the previous 2 days. We form a Markov chain with the following states:

State (S, S) if it was sunny both today and yesterday,

State (S, C) if it was sunny yesterday but cloudy today,

State (C, S) if it was cloudy yesterday but sunny today,

State (C, C) if it was cloudy both today and yesterday,

and transition probability matrix

$$\mathbf{P} = \begin{array}{c} \text{Today's state} \\ \begin{array}{c} (S, S) \quad (S, C) \quad (C, S) \quad (C, C) \end{array} \\ \begin{array}{c} (S, S) \\ (S, C) \\ (C, S) \\ (C, C) \end{array} \left\| \begin{array}{cccc} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{array} \right\| \end{array}.$$

(a) Given that it is sunny on days 0 and 1, what is the probability it is sunny on day 5?

(b) In the long run, what fraction of days are sunny?

2.5 (a) $P_{(S,S),(S,S)}^{(4)} + P_{(S,S),(C,S)}^{(4)} = .3421 + .1368 = .4789.$

(b) $\pi_{(S,S)} + \pi_{(C,S)} = .25 + .15 = .40.$

$$\pi_0 = 0.7 \pi_0 + 0.5 \pi_2 \Rightarrow \pi_2 = \frac{3}{5} \pi_0$$

$$\pi_1 = 0.3 \pi_0 + 0.5 \pi_2 \Rightarrow \pi_1 = .3 \pi_0 + .5 \cdot \frac{3}{5} \pi_0 = 0.6 \pi_0$$

$$\pi_2 = 0.4 \pi_1 + 0.2 \pi_3$$

$$\pi_3 = 0.6 \pi_1 + 0.8 \pi_3 \Rightarrow \pi_3 = \frac{0.6 \times 0.6 \pi_0}{0.2} = \frac{9}{5} \pi_0$$

$$\pi_0 + 0.6 \pi_0 + \frac{3}{5} \pi_0 + \frac{9}{5} \pi_0 = 1$$

$$\pi_0 = \frac{1}{4} = 0.25$$

$$\text{Sunny days} = \pi_0 + \pi_1 = 0.25 + (0.6)(0.25) = 0.4$$

4.2.6 Consider a computer system that fails on a given day with probability p and remains “up” with probability $q = 1 - p$. Suppose the repair time is a random variable N having the probability mass function $p(k) = \beta(1 - \beta)^{k-1}$ for $k = 1, 2, \dots$, where $0 < \beta < 1$. Let $X_n = 1$ if the computer is operating on day n and

$X_1 = 0$ if not. Show that $\{X_n\}$ is a Markov chain with transition matrix

$$\begin{matrix} & 0 & 1 \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{vmatrix} \alpha & \beta \\ p & q \end{vmatrix} \end{matrix}$$

and $\alpha = 1 - \beta$. Determine the long run probability that the computer is operating in terms of α, β, p , and q .

2.6 To establish the Markov property and to get the transition probability matrix, use $\Pr\{N = n | N \geq n\} = \beta$.

$$\pi_1 = \frac{\beta}{p + \beta}.$$

4.2.7 Customers arrive for service and take their place in a waiting line. There is a single service facility, and a customer undergoing service at the beginning of a period will complete service and depart at the end of the period with probability β and will continue service into the next period with probability $\alpha = 1 - \beta$, and then the process repeats. This description implies that the service time η of an individual is a random variable with the geometric distribution,

$$\Pr\{\eta = k\} = \beta\alpha^{k-1} \quad \text{for } k = 1, 2, \dots,$$

and the service times of distinct customers are independent random variables.

At most a single customer can arrive during a period. We suppose that the actual number of arrivals during the n th period is a random variable ξ_n taking on the values 0 or 1 according to

$$\Pr\{\xi_n = 0\} = p$$

and

$$\Pr\{\xi_n = 1\} = q = 1 - p \quad \text{for } n = 0, 1, \dots$$

The state X_n of the system at the start of period n is defined to be the number of customers in the system, either waiting or being served. Then, $\{X_n\}$ is a Markov chain. Specify the following transition probabilities in terms of α, β, p , and q : $P_{00}, P_{01}, P_{02}, P_{10}, P_{11}$, and P_{12} . State any additional assumptions that you make.

$$\begin{aligned} \mathbf{2.7} \quad & P_{00} = p; \quad P_{0,1} = q = 1 - p; \quad P_{i,i+1} = \alpha q \\ & P_{ii} = \alpha p + \beta q; \quad P_{i,i-1} = \beta p \quad \text{for } i \geq 1. \end{aligned}$$

4.2.8 An airline reservation system has a single computer, which breaks down on any given day with probability p . It takes 2 days to restore a failed computer to normal service. Form a Markov chain by taking as states the pairs (x, y) , where x is the number of machines in operating condition at the end of a day and y is 1 if a day's labor has been expended on a machine, and 0 otherwise. The transition probability matrix is

$$\begin{array}{c}
 \text{To state} \\
 \hline
 \rightarrow (1, 0) \quad (0, 0) \quad (0, 1) \\
 \text{From state} \\
 \downarrow \\
 \begin{array}{c}
 (1, 0) \\
 \mathbf{P} = (0, 0) \\
 (0, 1)
 \end{array}
 \begin{array}{c}
 \left\| \begin{array}{ccc}
 q & p & 0 \\
 0 & 0 & 1 \\
 1 & 0 & 0
 \end{array} \right\|
 \end{array}
 \end{array}$$

Compute the system availability $\pi_{(1,0)}$ for $p = 0.01, 0.02, 0.05$, and 0.10 .

2.8 $\pi_{(1,0)} = \frac{1}{1 + 2p}.$

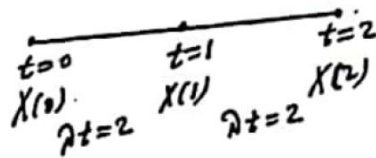
	$p = .01$	$.02$	$.05$	$.10$
$\pi_{(1,0)} =$	$.9804$	$.9615$	$.9091$	$.8333$

5 Poisson Processes

5.1 The Poisson Distribution and the Poisson Process

5.1.1 Defects occur along the length of a filament at a rate of $\lambda = 2$ per foot.

- (a) Calculate the probability that there are no defects in the first foot of the filament.
- (b) Calculate the conditional probability that there are no defects in the second foot of the filament, given that the first foot contained a single defect.



Solution.

1) Direct calculation gives $\mathbb{P}\{X(1) = 0\} = \frac{e^{-2(1)}(2(1))^0}{0!} = e^{-2}$.

2) We are asked to calculate $\mathbb{P}\{X(2) - X(1) = 0 | X(1) = 1\}$. We have

$$\mathbb{P}\{X(2) - X(1) = 0 | X(1) = 1\} = \frac{\mathbb{P}\{X(2) - X(1) = 0, X(1) - X(0) = 1\}}{\mathbb{P}\{X(1) = 1\}}.$$

By the independence of $X(2) - X(1)$ and $X(1) - X(0) = X(1)$, we get

$$\begin{aligned} \mathbb{P}\{X(2) - X(1) = 0 | X(1) = 1\} &= \frac{\mathbb{P}\{X(2) - X(1) = 0\} \mathbb{P}\{X(1) = 1\}}{\mathbb{P}\{X(1) = 1\}}, \\ &= \frac{e^{-2(1)}(2(1))^0}{0!} = e^{-2}. \blacksquare \end{aligned}$$

5.1.2 Let $p_k = \Pr\{X = k\}$ be the probability mass function corresponding to a Poisson distribution with parameter λ . Verify that $p_0 = \exp\{-\lambda\}$, and that p_k may be computed recursively by $p_k = (\lambda/k)p_{k-1}$.

Given $p_k = \Pr\{X = k\}$ for poisson (λ)

verify
$$\begin{cases} p_0 = e^{-\lambda} \\ p_k = \left(\frac{\lambda}{k}\right) p_{k-1} \end{cases} \quad \text{recurrence relation}$$

Ans: $\therefore p_k = \frac{e^{-\lambda} \lambda^k}{k!}, k = 0, 1, 2, \dots$

$\therefore p_0 = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda}, k = 0$

and $\therefore p_k = \frac{\lambda}{k} \cdot \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!}$

$\therefore p_k = \left(\frac{\lambda}{k}\right) p_{k-1}$

##

5.1.2 Let $p_k = \Pr\{X = k\}$ be the probability mass function corresponding to a Poisson distribution with parameter λ . Verify that $p_0 = \exp\{-\lambda\}$, and that p_k may be computed recursively by $p_k = (\lambda/k)p_{k-1}$.

$p_0 = e^{-\lambda} \because p_0 = \Pr\{X=0\} = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda}$

$p_k = \frac{e^{-\lambda} \lambda^k}{k!} = \frac{\lambda}{k} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} = \frac{\lambda}{k} p_{k-1}$

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β , respectively. Determine the conditional distribution of X , given that $N = X + Y = n$.

Ans: $X \sim \text{Poisson}(\alpha)$, $Y \sim \text{Poisson}(\beta)$
 $\therefore X+Y \sim \text{Poisson}(\alpha+\beta)$

$$\begin{aligned} \Pr\{X=k | N=n\} &= \Pr\{X=k | X+Y=n\} \\ &= \frac{\Pr\{X=k\} \cap \Pr\{X+Y=n\}}{\Pr\{X+Y=n\}} \\ &= \frac{\Pr\{X=k\} \cap \Pr\{Y=n-k\}}{\Pr\{X+Y=n\}} \\ &= \frac{e^{-\alpha} \alpha^k / k! \cdot e^{-\beta} \beta^{n-k} / (n-k)!}{e^{-(\alpha+\beta)} (\alpha+\beta)^n / n!} \\ &= \alpha^k \beta^{n-k} \left(\frac{1}{\alpha+\beta} \right)^n \frac{n!}{k! (n-k)!} \\ &= \binom{n}{k} \left(\frac{\alpha}{\alpha+\beta} \right)^k \left(\frac{\beta}{\alpha+\beta} \right)^{n-k} \\ \therefore \Pr\{X=k | N=n\} &= \binom{n}{k} p^k (1-p)^{n-k}, \quad p = \frac{\alpha}{\alpha+\beta} \end{aligned}$$

Note that $1-p = 1 - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$ which is Binomial dist'n

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β , respectively. Determine the conditional distribution of X , given that $N = X + Y = n$. $\text{Poisson} = \frac{e^{-\lambda} \lambda^x}{x!}$

$X \sim \text{Poisson}(\alpha)$ $Y \sim \text{Poisson}(\beta)$ $N = X+Y=n$ $X+Y \sim \text{Poisson}(\alpha+\beta)$

$$\begin{aligned} P(X=x | N=n) &= \Pr\{X=x | X+Y=n\} = \frac{\Pr\{X=x\} \cap \Pr\{X+Y=n\}}{\Pr\{X+Y=n\}} = \frac{\Pr\{X=x\} \cap \Pr\{Y=n-x\}}{\Pr\{X+Y=n\}} = \frac{\Pr\{X=x\} \cdot \Pr\{Y=n-x\}}{\Pr\{X+Y=n\}} \\ &= \frac{\frac{e^{-\alpha} \alpha^x}{x!} \cdot \frac{e^{-\beta} \beta^{n-x}}{(n-x)!}}{\frac{e^{-(\alpha+\beta)} (\alpha+\beta)^n}{n!}} = \frac{n!}{x! (n-x)!} \cdot \frac{\alpha^x \beta^{n-x}}{(\alpha+\beta)^n} \cdot \frac{(\alpha+\beta)^n}{(\alpha+\beta)^n} = \binom{n}{x} \left(\frac{\alpha}{\alpha+\beta} \right)^x \left(\frac{\beta}{\alpha+\beta} \right)^{n-x} = \binom{n}{x} p^x (1-p)^{n-x}, \text{ where } p = \frac{\alpha}{\alpha+\beta} \\ &= \text{Binomial distribution } (n, p) \end{aligned}$$

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β , respectively. Determine the conditional distribution of X , given that $N = X + Y = n$.

$$\text{Poisson} = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$X \sim \text{Poisson}(\alpha)$$

$$Y \sim \text{Poisson}(\beta)$$

$$X+Y \sim \text{Poisson}(\alpha + \beta)$$

$$P(X=x | N=n) = P(X=x | X+Y=n) = \frac{P(X=x) \cdot P(Y=n-x)}{P(X+Y=n)} = \frac{P(X=x) \cdot P(Y=n-x)}{P(X+Y=n)}$$

$$= \frac{\frac{e^{-\alpha} \alpha^x}{x!} \cdot \frac{e^{-\beta} \beta^{n-x}}{(n-x)!}}{\frac{e^{-(\alpha+\beta)} (\alpha+\beta)^n}{n!}} = \frac{n!}{x!(n-x)!} \cdot \frac{\alpha^x \beta^{n-x}}{(\alpha+\beta)^n} = \binom{n}{x} \left(\frac{\alpha}{\alpha+\beta} \right)^x \left(\frac{\beta}{\alpha+\beta} \right)^{n-x}$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \quad p = \frac{\alpha}{\alpha+\beta}$$

★ Binomial dist (n, p)

5.1.4 Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t .

(a) What is $\Pr\{X(t) = k\}$ for $k = 0, 1, \dots$?

(b) Consider fixed times $0 < s < t$. Determine the conditional probability $\Pr\{X(t) = n + k | X(s) = n\}$ and the expected value $E[X(t)X(s)]$.

$X(t)$ represents the # of customers that have arrived up to time t .

$$(a) \Pr\{X(t) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \dots$$

(b) The conditional probability
 $\Pr\{X(t) = n + k | X(s) = n\}$

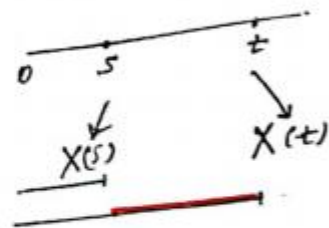
$$= \Pr\{X(t) - X(s) = k\}$$

$$= \Pr\{X(t-s) = k\}$$

$$= \frac{[\lambda(t-s)]^k e^{-\lambda(t-s)}}{k!}$$

and the expected value

$$\begin{aligned} E[X(t)X(s)] &= E[X(t)]E[X(s)] + E[X(s)] \\ &= \lambda t \cdot (\lambda s) + \lambda s \\ &= \lambda^2 ts + \lambda s \end{aligned}$$



$$a) \Pr\{X(t) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$b) \Pr\{X(t) = n + k | X(s) = n\} = \Pr\{X(t) - X(s) = k\} = \Pr\{X(t-s) = k\} = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}$$

$$\begin{aligned} E[X(t)X(s)] &= E[(X(t) - X(s) + X(s))(X(s) - X(0) + X(0))] = E[\underbrace{(X(t) - X(s))(X(s) - X(0))}_{\text{indep}} + X(s)(X(0))] = E[X(t) - X(s)] \cdot E[X(s) - X(0)] + E[X^2(s)] \\ &= E[X(t-s)] \cdot E[X(s)] + E[X(s)^2] = \lambda(t-s) \cdot \lambda s + \lambda s + \lambda^2 s^2 = \lambda s(\lambda(t-s) + 1 + \lambda s) = \lambda s(\lambda t - \lambda s + 1 + \lambda s) = \lambda s(\lambda t + 1) \end{aligned}$$

$$\text{Var}(X(s)) = E[X(s)^2] - (E[X(s)])^2 \Rightarrow E[X(s)^2] = \text{Var}(X(s)) + (E[X(s)])^2 = \lambda s + (\lambda s)^2$$

5.1.5 Suppose that a random variable X is distributed according to a Poisson distribution with parameter λ . The parameter λ is itself a random variable, exponentially distributed with density $f(\lambda) = \theta e^{-\theta\lambda}$ for $\lambda \geq 0$. Find the probability mass function for X .

* Pb 5.1.5 p. 229
 The conditional probability function of $X|\lambda$ is

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$
 and the prob. density fn of λ is

$$f(\lambda) = \theta e^{-\theta\lambda}$$
 The prob. fn of X is

$$f(x) = \int_0^{\infty} f(x|\lambda) f(\lambda) d\lambda$$

$$f(x) = \frac{\theta}{x!} \int_0^{\infty} \lambda^x e^{-(1+\theta)\lambda} d\lambda$$

7 let $(1+\theta)\lambda = u$
 $\Rightarrow \lambda = \frac{u}{1+\theta}, d\lambda = \frac{du}{1+\theta}$
 Then $f(x) = \frac{\theta}{x!} \int_0^{\infty} \left(\frac{u}{1+\theta}\right)^x e^{-u} \frac{du}{1+\theta}$

$$= \frac{\theta}{x!} \cdot \frac{1}{(1+\theta)^{x+1}} \int_0^{\infty} u^x e^{-u} du$$

$$= \frac{\theta}{x!} \cdot \frac{1}{(1+\theta)^{x+1}} \Gamma(x+1)$$

$$= \frac{\theta}{x!} \cdot \frac{1}{(1+\theta)^{x+1}} \cdot x!$$

$$\therefore f(x) = \frac{\theta}{(1+\theta)^{x+1}} = \frac{\theta}{1+\theta} \cdot \frac{1}{(1+\theta)^x}$$

$$\therefore f(x) = (1-p)p^x, x=0,1,\dots, p = \frac{1}{1+\theta}$$

 which is the prob. mass fn for X .

(Note: A circled note defines the Gamma function: $\int_0^{\infty} t^{\alpha} e^{-t} dt = \Gamma(\alpha+1) = \alpha!, \alpha=0,1,\dots$ Gamma function)

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad f(\lambda) = \theta e^{-\theta\lambda}$$

$$f(x) = \int_0^{\infty} f(x|\lambda) d\lambda = \int_0^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \theta e^{-\theta\lambda} d\lambda = \frac{\theta}{x!} \int_0^{\infty} \lambda^x e^{-(1+\theta)\lambda} d\lambda \quad u=(1+\theta)\lambda \quad du=(1+\theta)d\lambda \quad \lambda = \frac{u}{1+\theta}$$

$$\Rightarrow f(x) = \frac{\theta}{x!} \int_0^{\infty} \left(\frac{u}{1+\theta}\right)^x e^{-u} \frac{du}{1+\theta} = \frac{\theta}{x!} \cdot \frac{1}{(1+\theta)^{x+1}} \int_0^{\infty} u^x e^{-u} du = \frac{\theta}{(1+\theta)^{x+1}} \cdot \frac{1}{x!} \Gamma(x+1) = \frac{\theta}{(1+\theta)^{x+1}} \cdot \frac{1}{x!} (x!) = \frac{\theta}{(1+\theta)^{x+1}}$$

$$\Pr\{X = k\} = (1-p)p^k \quad \text{for } k = 0, 1, \dots \text{ where } p = 1/(1+\theta).$$

5.1.6 Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages per hour.

- (a) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?
 (b) What is the distribution of the time at which the first afternoon message arrives?

* pb 5.1.6 p. 229

(a) $X(t)$ represents the # messages that arrive at the telegraph office at any time.

$$\therefore \Pr\{X(s+t) - X(s) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \dots$$

$$\therefore \Pr\{X(12) - X(8) = 0\} = \frac{(3 \times 4)^0 e^{-3(4)}}{0!} = e^{-12}$$

$$= 6.1442 \times 10^{-6}$$

where $\lambda = 3$, $t = 12 - 8 = 4$ and $k = 0$.

(b) Consider T is the r.v. that represents the time at which the first afternoon message arrives.
 So, we can write

$$\Pr\{T > t\}$$

$$= \Pr\{\text{The first afternoon message arrives after } t \text{ units of time}\}$$

$$= \Pr\{X(t) - X(12) = 0\}$$

$$= \frac{[3(t-12)]^0 e^{-3(t-12)}}{0!}$$

$$= e^{-3(t-12)}, \text{ which is the survival/reliability function for } T.$$

$\xrightarrow{t}$
 12:00 P.M. 12:00 A.M.

Also, $\Pr\{T \leq t\} = 1 - \Pr\{T > t\}$

$$= 1 - e^{-3(t-12)}$$

$$= 1 - e^{-3x}, \quad x = t - 12$$

which is the cumulative distribution function for T .

$$\therefore T \sim \exp(3)$$

i.e. $T \sim \text{exponential dist.}^n$ with parameter equals 3.

$$a) \Pr\{X(12) - X(8) = 0\} = \Pr\{X(12-8) = 0\} = \Pr\{X(4) = 0\} = \frac{e^{-4(3)} (12)^0}{0!} = e^{-12}$$

$$b) T > t : \Pr\{T > t\} = \Pr\{X(t) - X(12) = 0\} = \Pr\{X(t-12) = 0\} = \frac{e^{-3(t-12)} (3(t-12))^0}{0!} = e^{-3(t-12)}$$

$$\Rightarrow \Pr\{T \leq t\} = 1 - e^{-3(t-12)} = 1 - e^{-3x}, \quad x = t - 12 \quad T \sim \exp(3)$$

5.1.7 Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t . Determine the following probabilities and conditional probabilities:

- (a) $\Pr\{X(1) = 2\}$.
- (b) $\Pr\{X(1) = 2 \text{ and } X(3) = 6\}$.
- (c) $\Pr\{X(1) = 2 | X(3) = 6\}$.
- (d) $\Pr\{X(3) = 6 | X(1) = 2\}$.

$X(t)$ is the # of customers (arrivals), $\lambda = 2$

$$\begin{array}{ccc} X(0) = 0 & \xrightarrow[2 \text{ Customers}]{\lambda t = 2(1) = 2} & X(1) = 2 \\ t = 0 & & t = 1 \end{array}$$

a) $\Pr\{X(1) = 2\}$

$$= \Pr\{X(1) - X(0) = 2\}$$

$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{2^2 e^{-2}}{2!} = 2 e^{-2}$$

b) $\Pr\{X(1) = 2 \text{ and } X(3) = 6\}$

$$\begin{array}{ccccc} X(0) = 0 & \xrightarrow[2 \text{ Cust.}]{\lambda t = 2(1) = 2} & X(1) = 2 & \xrightarrow[4 \text{ Cust.}]{\lambda t = 2(2) = 4} & X(3) = 6 \\ t = 0 & & t = 1 & & t = 3 \end{array}$$

$$= \Pr\{X(1) - X(0) = 2, X(3) - X(1) = 4\}$$

indep. r.v.s

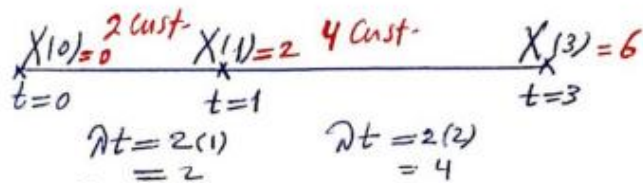
$$= \frac{2^2 e^{-2}}{2!} \cdot \frac{4^4 e^{-4}}{4!}$$

$$= 2 e^{-2} \cdot \frac{64}{6} e^{-4} = \frac{64}{3} e^{-6}$$

$\Pr(X, Y)$
 $= \Pr(X) \Pr(Y)$
 if X and Y are
 indep. r.v.s

$$d) \Pr \{X(3)=6 \mid X(1)=2\}$$

$$= \Pr \{X(3) - X(1) = 4 \mid X(1) - X(0) = 2\}$$



$$= \Pr \{X(3) - X(1) = 4\}$$

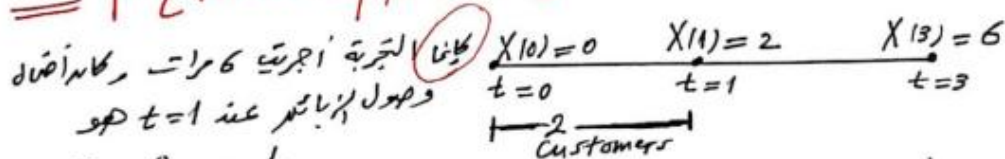
$$= \frac{4^4 e^{-4}}{4!}$$

$$= \frac{64}{6} e^{-4} = \frac{32}{3} e^{-4}$$

$X(3) - X(1)$
and $X(1) - X(0)$ are
2 indep. r.v.

Note that $\Pr(A|B) = \Pr(A)$
where A and B are
independent events.

$$c) \Pr \{X(1)=2 \mid X(3)=6\}$$



$$p = \frac{2}{6} = \frac{1}{3}$$

$$q = \frac{4}{6} = \frac{2}{3}$$

$$\Rightarrow X \sim \text{Bin}(6, \frac{1}{3})$$

$$X \sim \text{Bin}(6, \frac{1}{3})$$

$$\begin{aligned} \therefore \Pr \{X(1)=2 \mid X(3)=6\} &= \binom{n}{x} p^x q^{n-x} \\ &= \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4 \\ &= \frac{80}{243} \approx 0.3292 \end{aligned}$$

* حل آخر للفترة (c)

$$\begin{aligned}
 & \Pr\{X(1)=2 \mid X(3)=6\} \\
 &= \frac{\Pr\{X(1)=2 \text{ and } X(3)=6\}}{\Pr\{X(3)=6\}} \\
 &= \frac{(2e^{-2}/2!) \cdot (4^4 e^{-4}/4!)}{6^6 e^{-6}/6!} \\
 &= \frac{64/3 \cancel{e^6}}{64.8 \cancel{e^6}} \\
 &\approx 0.3292 \quad \#
 \end{aligned}$$

5.1.7 Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t . Determine the following probabilities and conditional probabilities:

$$\begin{aligned}
 & \Pr\{X(s+t) - X(s) = k\} \\
 &= \frac{(\lambda t)^k e^{-\lambda t}}{k!}
 \end{aligned}$$

- (a) $\Pr\{X(1) = 2\}$.
- (b) $\Pr\{X(1) = 2 \text{ and } X(3) = 6\}$.
- (c) $\Pr\{X(1) = 2 \mid X(3) = 6\}$.
- (d) $\Pr\{X(3) = 6 \mid X(1) = 2\}$.

$$a) \Pr\{X(1) = 2\} = \Pr\{X(1) - X(0) = 2\} = \frac{(2 \times 1)^2 \cdot e^{-2(1)}}{2!} = 2e^{-2}$$

$$b) \Pr\{X(1) = 2, X(3) = 6\} = \Pr\{X(1) - X(0) = 2, X(3) - X(1) = 4\} = \Pr\{X(1) - X(0) = 2\} \cdot \Pr\{X(3) - X(1) = 4\} = 2e^{-2} \cdot \frac{e^{-4} (2 \times 2)^4}{4!} = \frac{64}{3} e^{-6}$$

$$c) \Pr\{X(1) = 2 \mid X(3) = 6\} = \frac{\Pr\{X(1) = 2 \text{ and } X(3) = 6\}}{\Pr\{X(3) = 6\}} = \frac{\frac{64}{3} e^{-6}}{\frac{648}{10} e^{-6}} = \frac{64 \cdot 10}{3 \cdot 648} = \frac{640}{1944}$$

$$\Pr\{X(3) = 6\} = \Pr\{X(3) - X(0) = 6\} = \frac{(2 \times 3)^6 e^{-2(3)}}{6!} = \frac{6^6}{6!} e^{-6}$$

$$d) \Pr\{X(3) = 6 \mid X(1) = 2\} = \frac{\Pr\{X(3) = 6 \text{ and } X(1) = 2\}}{\Pr\{X(1) = 2\}} = \frac{\frac{64}{3} e^{-6}}{2 e^{-2}} = \frac{32}{3} e^{-4}$$

$$\underline{\underline{OY}} \Pr\{X(3) - X(1) = 4 \mid X(1) - X(0) = 2\} = \Pr\{X(3) - X(1) = 4\} = \frac{(2 \times 2)^4 e^{-4}}{4!} = \frac{32}{3} e^{-4}$$

$$(a) P\{X(3) = 0 | X(1) = 2\}.$$

$$(a) P\{X(1) = 2\} = P\{X(1) - X(0) = 2\} = \frac{e^{-2} 2^2}{2!} = 2 e^{-2}$$

$$(b) P\{X(1) = 2, X(3) = 6\} = P\{X(1) - X(0) = 2, X(3) - X(1) = 4\}$$

$$= P\{X(1) - X(0) = 2\} \cdot P\{X(3) - X(1) = 4\} = 2 e^{-2} \frac{e^{-4} 4^4}{4!} = \frac{64}{3} e^{-6}$$

$$(c) P\{X(1) = 2 | X(3) = 6\} = \frac{P\{X(1) = 2\} \cdot P\{X(3) = 6\}}{P\{X(3) = 6\}} = \frac{\frac{64}{3} e^{-6}}{64.8 e^{-6}} =$$

$$P\{X(3) = 6\} = P\{X(3) - X(0)\} = \frac{e^{-6} 6^6}{6!} = 64.8 e^{-6}$$

$$(d) P\{X(3) = 6 | X(1) = 2\} = \frac{P\{X(3) = 6 \text{ and } X(1) = 2\}}{P\{X(1) = 2\}} = \frac{\frac{64}{3} e^{-6}}{2 e^{-2}} = \frac{32}{3} e^{-4}$$

5.1.9 Let $\{X(t); t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

- (a) $E[X(2)]$.
- (b) $E[\{X(1)\}^2]$.
- (c) $E[X(1)X(2)]$.

For the Poisson process $\{X(t); t \geq 0\}$
 $E[X(t)] = \lambda t$ and $\text{Var}[X(t)] = \lambda t$

(a) $E[X(1)] = 2(1) = 2$,

$E[X(2)] = 2(2) = 4$, where $\lambda = 2$.

(b) To get $E[\{X(1)\}^2]$

$$\because \text{Var}[X(1)] = E[\{X(1)\}^2] - [E\{X(1)\}]^2$$

$$2 = E[\{X(1)\}^2] - 2^2$$

$$\therefore E[\{X(1)\}^2] = 2 + 4 = 6$$

* Another solution for (b)

$$E[\{X(1)\}^2] = E[X(1)]E[X(1)] + E[X(1)]$$

$$= 2(2) + 2 = 6$$

, See pb 5.1.4 p. 228

(c) $E[X(1)X(2)]$

$$= E[X(1)]E[X(2)] + E[X(1)]$$

$$= 2(4) + 2 = 10, \text{ See pb 5.1.4 p. 228}$$

5.1.9 Let $\{X(t); t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

- (a) $E[X(2)]$.
- (b) $E[\{X(1)\}^2]$.
- (c) $E[X(1)X(2)]$.

$$E[X(t)] = \lambda t$$

$$\text{Var}[X(t)] = \lambda t$$

a) $E[X(2)] = 2(2) = 4$

b) $E[X(1)^2] = \text{Var}[X(1)] + (E[X(1)])^2 = (1)(1) + (2)(1)^2 = 2 + 4 = 6$

c) $E[X(1)X(2)] = (2)^2(2)(1) + (2)(1) = 8 + 2 = 10$

$E[X(t)X(s)] = \lambda s(\lambda t + 1) = \lambda^2 ts + \lambda s, 0 \leq s \leq t$ from 5.1.4

5.1.9 Let $\{X(t); t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

(a) $E[X(2)]$.

$$E[X(t)] = \lambda t$$

(b) $E[(X(1))^2]$.

$$V[X(t)] = \lambda t$$

(c) $E[X(1)X(2)]$.

$$E[X(t)X(s)] = \lambda^2 ts + \lambda s$$

(a) $E[X(2)] = 2(2) = 4$

(b) $E[X(1)^2] = V(X(1)) + (E[X(1)])^2 = 1(2) + (2(1))^2 = 2 + 4 = 6$

(c) $E[X(1)X(2)] = \lambda s(\lambda t + 1) = 2(2)(2(1) + 1) = 12$

$$\lambda s(\lambda t + 1)$$

5.1.9 Let $\{X(t); t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

(a) $E[X(2)]$.

$$E[X(t)] = \lambda t$$

(b) $E[(X(1))^2]$.

$$V[X(t)] = \lambda t$$

(c) $E[X(1)X(2)]$.

a) $E[X(2)] = 2(2) = 4$

b) $E[X(1)^2] = V(X(1)) + (E[X(1)])^2 = 1(2) + (2(1))^2 = 2 + 4 = 6$

c) $E[X(1)X(2)] = (2)^2(2)(1) + (2)(1) = 8 + 2 = 10$

$E[X(t)X(s)] = \lambda s(\lambda t + 1) = \lambda^2 ts + \lambda s, 0 \leq s \leq t$ from 5.1.4

Problems

5.1.1 Let $\xi_1, \xi_2, \dots$ be independent random variables, each having an exponential distribution with parameter λ . Define a new random variable X as follows: If $\xi_1 > 1$, then $X = 0$; if $\xi_1 \leq 1$ but $\xi_1 + \xi_2 > 1$, then set $X = 1$; in general, set $X = k$ if

$$\xi_1 + \dots + \xi_k \leq 1 < \xi_1 + \dots + \xi_k + \xi_{k+1}.$$

Show that X has a Poisson distribution with parameter λ . (Thus, the method outlined can be used to simulate a Poisson distribution.)

Hint: $\xi_1 + \dots + \xi_k$ has a gamma density

$$f_k(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} \quad \text{for } x > 0.$$

Condition on $\xi_1 + \dots + \xi_k$ and use the law of total probability to show

$$\Pr\{X = k\} = \int_0^1 [1 - F(1-x)] f_k(x) dx,$$

where $F(x)$ is the exponential distribution function.

1.1 Following the hint:

$$\Pr\{X = k\} = \int_0^1 e^{-\lambda(1-x)} \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} dx = \frac{\lambda^k e^{-\lambda}}{(k-1)!} \int_0^1 x^{k-1} dx = \frac{\lambda^k e^{-\lambda}}{k!}$$

5.1.2 Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α , and that, independently, major defects are distributed over the cable according to a Poisson process of rate β . Let $X(t)$ be the number of defects, either major or minor, in the cable up to length t . Argue that $X(t)$ must be a Poisson process of rate $\alpha + \beta$.

1.2 The number of defects in any interval is Poisson distributed, from Theorem 1.1, and the numbers of defects in disjoint intervals are independent random variables because it's true for both major and minor types.

5.1.3 The *generating function* of a probability mass function $p_k = \Pr\{X = k\}$, for $k = 0, 1, \dots$, is defined by

$$g_X(s) = E[s^X] = \sum_{k=0}^{\infty} p_k s^k \quad \text{for } |s| < 1.$$

Show that the generating function for a Poisson random variable X with mean μ is given by

$$g_X(s) = e^{-\mu(1-s)}.$$

$$\mathbf{1.3} \quad g_X(s) = \sum_{k=0}^{\infty} \frac{\mu^k e^{-\mu}}{k!} s^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu s)^k}{k!} = e^{-\mu} e^{\mu s} = e^{-\mu(1-s)}, \quad |s| < 1.$$

5.1.4 (Continuation) Let X and Y be independent random variables, Poisson distributed with parameters α and β , respectively. Show that the generating function of their sum $N = X + Y$ is given by

$$g_N(s) = e^{-(\alpha+\beta)(1-s)}.$$

Hint: Verify and use the fact that the generating function of a sum of independent random variables is the product of their respective generating functions. See Chapter 3, Section 3.9.2.

$$\begin{aligned} \mathbf{1.4} \quad g_N(s) &= E[s^{X+Y}] = E[s^X s^Y] \\ &= E[s^X] E[s^Y] \quad (\text{using II, (1.10), (1.12)}). \\ &= g_X(s) g_Y(s). \end{aligned}$$

In the Poisson case (Problem 1.3)

$$g_N(s) = e^{-\alpha(1-s)} e^{-\beta(1-s)} = e^{-(\alpha+\beta)(1-s)}. \quad (\text{Poisson, } \theta = \alpha + \beta).$$

5.1.5 For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh . Let $p_k(h) = \Pr\{X(h) = k\}$ for $k = 0, 1, \dots$. Verify that

$$\lim_{h \rightarrow 0} \frac{1 - p_0(h)}{h} = \lambda, \quad \text{or } p_0(h) = 1 - \lambda h + o(h);$$

$$\lim_{h \rightarrow 0} \frac{p_1(h)}{h} = \lambda, \quad \text{or } p_1(h) = \lambda h + o(h);$$

$$\lim_{h \rightarrow 0} \frac{p_2(h)}{h} = 0, \quad \text{or } p_2(h) = o(h).$$

Here $o(h)$ stands for any remainder term of order less than h as $h \rightarrow 0$.

$$1.5 \text{ (a)} \quad \frac{1 - p_0(h)}{h} = \frac{1 - e^{-\lambda h}}{h} = \frac{\lambda h - \frac{1}{2}\lambda^2 h^2 + \frac{1}{3!}\lambda^3 h^3}{h}$$

$$= \lambda - \frac{1}{2}\lambda^2 h + \frac{1}{3!}\lambda^3 h^2 - \dots \rightarrow \lambda \text{ as } h \rightarrow 0.$$

$$(b) \quad \frac{p_1(h)}{h} = \frac{\lambda h e^{-\lambda h}}{h} = \lambda e^{-\lambda h} \rightarrow \lambda \text{ as } h \rightarrow 0.$$

$$(c) \quad \frac{p_2(h)}{h} = \frac{\frac{1}{2}\lambda^2 h^2 e^{-\lambda h}}{h} = \frac{1}{2}\lambda^2 h e^{-\lambda h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

5.1.6 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ . For $s, t > 0$, determine the conditional distribution of $X(t)$, given that $X(t+s) = n$.

$$1.6 \quad \Pr\{X(t) = k, X(t+s) = n\} = \Pr\{X(t) = k, X(t+s) - X(t) = n - k\}$$

$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} \frac{(\lambda s)^{n-k} e^{-\lambda s}}{(n-k)!}$$

$$\Pr\{X(t) = k | X(t+s) = n\} = \frac{\frac{(\lambda t)^k e^{-\lambda t}}{k!} \frac{(\lambda s)^{n-k} e^{-\lambda s}}{(n-k)!}}{\frac{[\lambda(t+s)]^n e^{-\lambda(t+s)}}{n!}}$$

$$= \binom{n}{k} \left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{n-k} \quad \left(\text{Binomial, } p = \frac{t}{t+s}\right)$$

5.1.7 Shocks occur to a system according to a Poisson process of rate λ . Suppose that the system survives each shock with probability α , independently of other shocks, so that its probability of surviving k shocks is α^k . What is the probability that the system is surviving at time t ?

$$1.7 \quad \Pr\{\text{Survive at time } t\} = \sum_{k=0}^{\infty} \Pr\{\text{Survive } k \text{ shocks}\} \Pr\{k \text{ shocks}\} = \sum_{k=0}^{\infty} \alpha^k \frac{e^{-\lambda t} (\lambda t)^k}{k!} = e^{-\lambda t(1-\alpha)}.$$

5.1.8 Find the probability $\Pr\{X(t) = 1, 3, 5, \dots\}$ that a Poisson process having rate λ is odd.

$$1.8 \quad e^{\lambda t} = 1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \frac{1}{3!}(\lambda t)^3 + \frac{1}{4!}(\lambda t)^4 + \dots$$

$$e^{-\lambda t} = 1 - \lambda t + \frac{1}{2}(\lambda t)^2 - \frac{1}{3!}(\lambda t)^3 + \frac{1}{4!}(\lambda t)^4 - \dots$$

$$\frac{1}{2}(e^{\lambda t} - e^{-\lambda t}) = \lambda t + \frac{1}{3!}(\lambda t)^3 + \frac{1}{5!}(\lambda t)^5 + \dots$$

$$Pr\{X(t) \text{ is odd}\} = e^{-\lambda t} \left[\frac{1}{2}(e^{\lambda t} - e^{-\lambda t}) \right] = \frac{1}{2}(1 - e^{-2\lambda t}).$$

5.1.9 Arrivals of passengers at a bus stop form a Poisson process $X(t)$ with rate $\lambda = 2$ per unit time. Assume that a bus departed at time $t = 0$ leaving no customers behind. Let T denote the arrival time of the next bus. Then, the number of passengers present when it arrives is $X(T)$. Suppose that the bus arrival time T is independent of the Poisson process and that T has the uniform probability density function

$$f_T(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

- (a) Determine the conditional moments $E[X(T)|T=t]$ and $E[\{X(T)\}^2|T=t]$.
 (b) Determine the mean $E[X(T)]$ and variance $\text{Var}[X(T)]$.

$$1.9 \quad (a) \quad E[X(T)|T=t] = \lambda t \quad E[X(T)^2|T=t] = \lambda t + \lambda^2 t^2.$$

$$(b) \quad E[X(T)] = \int_0^1 \lambda t dt = \frac{1}{2} \lambda = 1 \quad \text{when } \lambda = 2.$$

$$E[X(T)^2] = \int_0^1 (\lambda t + \lambda^2 t^2) dt = \frac{1}{2} \lambda + \frac{1}{3} \lambda^2.$$

$$\begin{aligned} \text{Var}[X(T)] &= E[X(T)^2] - E[X(T)]^2 = \frac{1}{2} \lambda + \frac{1}{3} \lambda^2 - \frac{1}{4} \lambda^2 \\ &= \frac{1}{2} \lambda + \frac{1}{12} \lambda^2 = \frac{4}{3} \quad \text{when } \lambda = 2. \end{aligned}$$

5.1.10 Customers arrive at a facility at random according to a Poisson process of rate λ . There is a waiting time cost of c per customer per unit time. The customers gather at the facility and are processed or dispatched in groups at fixed times $T, 2T, 3T, \dots$. There is a dispatch cost of K . The process is depicted in the following graph.

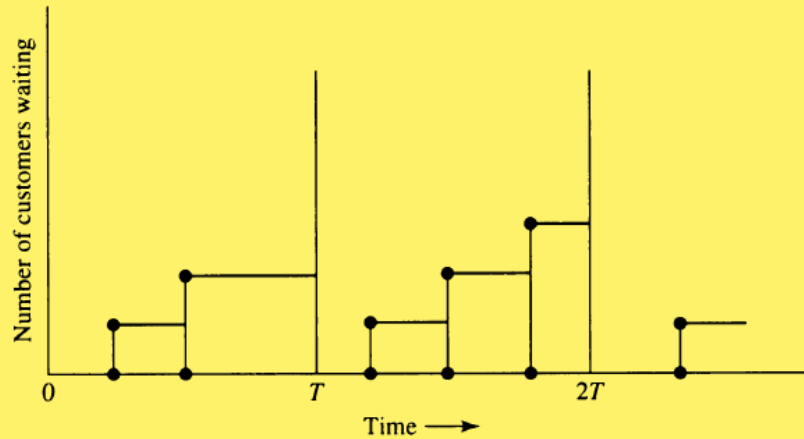


Figure 5.1 The number of customers in a dispatching system as a function of time.

- What is the total dispatch cost during the first cycle from time 0 to time T ?
- What is the mean total customer waiting cost during the first cycle?
- What is the mean total customer waiting + dispatch cost per unit time during the first cycle?
- What value of T minimizes this mean cost per unit time?

1.10 (a) K

$$(b) \quad c E \left[\int_0^T X(t) dt \right] = c \int_0^T \lambda t dt = \frac{1}{2} \lambda c T^2.$$

$$(c) \quad A.C. = \frac{1}{T} \left(K + \frac{1}{2} \lambda c T^2 \right).$$

$$(d) \quad \frac{d}{dT} A.C. = 0 \Rightarrow T^* = \sqrt{\frac{2K}{\lambda c}}.$$

Note: In (a), could argue for Dispatch Cost = $K \mathbf{1}\{X(T) > 0\}$ and $E[\text{Dispatch Cost}] = K(1 - e^{-\lambda T})$.

5.1.11 Assume that a device fails when a cumulative effect of k shocks occurs. If the shocks happen according to a Poisson process of parameter λ , what is the density function for the life T of the device?

1.11 The gamma density

$$f_k(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \quad x > 0$$

5.1.12 Consider the mixed Poisson process of Section 5.1.4, and suppose that the mixing parameter Θ has the exponential density $f(\theta) = e^{-\theta}$ for $\theta > 0$.

(a) Show that equation (5.3) becomes

$$\Pr\{X'(t) = j\} = \left(\frac{t}{1+t}\right)^j \left(\frac{1}{1+t}\right), \quad \text{for } j = 0, 1, \dots$$

(b) Show that

$$\Pr\{X'(t) = j, X'(t+s) = j+k\} = \binom{j+k}{j} t^j s^k \left(\frac{1}{1+s+t}\right)^{j+k+1},$$

so that $X'(t)$ and the increment $X'(t+s) - X'(t)$ are not independent random variables, in contrast to the simple Poisson process as defined in Section 5.1.2.

$$\begin{aligned} \mathbf{1.12} \quad (a) \quad \Pr\{X'(t) = k\} &= \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} e^{-\theta} d\theta \\ &= \frac{t^k}{k!} \int_0^\infty \theta^k e^{-(1+t)\theta} d\theta \\ &= \frac{1}{k!} \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right) \int_0^\infty x^k e^{-x} dx \\ &= \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right) \quad (\text{See I, (6.4)}). \end{aligned}$$

$$\begin{aligned} (b) \quad \Pr\{X'(t) = j, X'(t+s) = j+k\} &= \int_0^\infty \frac{(\theta t)^j e^{-\theta t}}{j!} \frac{(\theta s)^k e^{-\theta s}}{k!} e^{-\theta} d\theta \\ &= \binom{j+k}{j} \left(\frac{t}{1+s+t}\right)^j \left(\frac{s}{1+s+t}\right)^k \left(\frac{1}{1+s+t}\right). \end{aligned}$$

5.6 Compound and Marked Poisson Processes

5.6.2 Shocks occur to a system according to a Poisson process of intensity λ . Each shock causes some damage to the system, and these damages accumulate. Let $N(t)$ be the number of shocks up to time t , and let Y_i be the damage caused by the i th shock. Then

$$X(t) = Y_1 + \cdots + Y_{N(t)}$$

is the total damage up to time t . Determine the mean and variance of the total damage at time t when the individual shock damages are exponentially distributed with parameter θ .

Consider the compound Poisson process $Z(t) = \sum_{k=1}^{N(t)} Y_k$. If $\lambda > 0$ is the rate for the process $X(t)$ and $\mu = E[Y_1]$ and $\nu^2 = \text{Var}[Y_1]$ are the common mean and variance for $Y_1, Y_2, \dots$, then the moments of $Z(t)$ can be determined from the random sums formulas of II, Section 3.2 and are

$$E[Z(t)] = \lambda \mu t; \quad \text{Var}[Z(t)] = \lambda(\nu^2 + \mu^2)t. \quad (6.2)$$

$$E[Y_1] = \frac{1}{\theta}$$

$$\text{Var}[Y_1] = \frac{1}{\theta^2}$$

$$E[N(t)] = \lambda t$$

$$\text{Var}[N(t)] = \lambda t$$

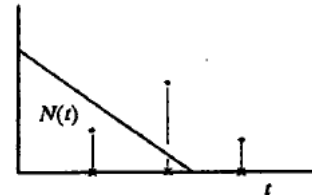
$$E[X(t)] = \frac{1}{\theta} \lambda t$$

$$\text{Var}[X(t)] = \frac{1}{\theta^2} \lambda t + \frac{1}{\theta^2} \lambda t$$

Problems

5.6.1 Suppose that points are distributed over the half line $[0, \infty)$ according to a Poisson process of rate λ . A sequence of independent and identically distributed nonnegative random variables $Y_1, Y_2, \dots$ is used to reposition the points so that a point formerly at location W_k is moved to the location $W_k + Y_k$. Completely describe the distribution of the relocated points.

6.1 Nonhomogeneous Poisson, intensity $\lambda(t) = \lambda G(t)$. To see this, let $N(t)$ be the number of points $\leq t$ in the relocated process = # points in Δ .



5.6.2 Suppose that particles are distributed on the surface of a circular region according to a spatial Poisson process of intensity λ particles per unit area. The polar coordinates of each point are determined, and each angular coordinate is shifted by a random amount, with the amounts shifted for distinct points being independent random variables following a fixed probability distribution. Show that at the end of the point movement process, the points are still Poisson distributed over the region.

6.2 The key observation is that, if Θ is uniform on $[0, 2\pi)$, and Y is independent with an arbitrary distribution, then $Y + \Theta \pmod{2\pi}$ is uniform.

5.6.3 Shocks occur to a system according to a Poisson process of intensity λ . Each shock causes some damage to the system, and these damages accumulate. Let $N(t)$ be the number of shocks up to time t , and let Y_i be the damage caused by the i th shock. Then

$$X(t) = Y_1 + \dots + Y_{N(t)}$$

is the total damage up to time t . Suppose that the system continues to operate as long as the total damage is strictly less than some critical value a , and fails in the contrary circumstance. Determine the mean time to system failure when the individual damages Y_i have a geometric distribution with $\Pr\{Y = k\} = p(1 - p)^k, k = 0, 1, \dots$

6.3 From the shock model of Section 6.1, modified for the discrete nature of the damage process, we have

$$\begin{aligned}
 E[T] &= \frac{1}{\lambda} \sum_{n=0}^{\infty} G^{(n)}(a-1) \\
 &= \frac{1}{\lambda} \left[1 + \sum_{n=1}^{\infty} \sum_{k=0}^{a-1} \binom{n+k-1}{k} p^n (1-p)^k \right] \quad (\text{See I, (3.6)}) \\
 &= \frac{1}{\lambda} \left[1 + \sum_{k=0}^{a-1} p(1-p)^k \left\{ \sum_{n=1}^{\infty} \binom{n-1+k}{n-1} p^{n-1} \right\} \right] \\
 &= \frac{1}{\lambda} \left[1 + \sum_{k=0}^{a-1} p(1-p)^k (1-p)^{-k-1} \right] \quad (\text{See I, (6.21)}) \\
 &= \frac{1}{\lambda} \left[1 + \frac{ap}{1-p} \right].
 \end{aligned}$$

5.6.4 Let $\{X(t); t \geq 0\}$ and $\{Y(t); t \geq 0\}$ be independent Poisson processes with respective parameters λ and μ . For a fixed integer a , let $T_a = \min\{t \geq 0; Y(t) = a\}$ be the random time that the Y process first reaches the value a . Determine $\Pr\{X(T_a) = k\}$ for $k = 0, 1, \dots$.

Hint: First consider $\xi = X(T_1)$ in the case in which $a = 1$. Then, ξ has a geometric distribution. Then, argue that $X(T_a)$ for general a has the same distribution as the sum of a independent ξ s and hence has a negative binomial distribution.

6.4 One can use the results of I, Section 5.2 or, since T_1 is exponentially distributed, μ

$$\begin{aligned}
 \Pr\{X(T_1) = k\} &= \int_0^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \mu e^{-\mu t} dt \\
 &= \left(\frac{\mu}{\lambda + \mu} \right) \left(\frac{\lambda}{\lambda + \mu} \right)^k, \quad k = 0, 1, \dots \quad (\text{See I, (6.4)}).
 \end{aligned}$$

T_a has a gamma density and a similar integration applies. See the last example in II, Section 4.

5.6.5. Let $\{X(t); t \geq 0\}$ and $\{Y(t); t \geq 0\}$ be independent Poisson processes with respective parameters λ and μ . Let $T = \min\{t \geq 0; Y(t) = 1\}$ be the random time of the first event in the Y process. Determine $\Pr\{X(T/2) = k\}$ for $k = 0, 1, \dots$.

6.5 $\frac{1}{2}T$ is exponentially distributed with rate parameter 2μ , whence

$$\begin{aligned}
 \Pr\left\{X\left(\frac{1}{2}T\right) = k\right\} &= \int_0^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} 2\mu e^{-2\mu t} dt \\
 &= \left(\frac{2\mu}{\lambda + 2\mu} \right) \left(\frac{\lambda}{\lambda + 2\mu} \right)^k, \quad k = 0, 1, \dots \quad (\text{See I, (6.4)})
 \end{aligned}$$

5.6.6 Let $W_1, W_2, \dots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ . A new point process is created as follows: Each point W_k is replaced by two new points located at $W_k + X_k$ and $W_k + Y_k$, where $X_1, Y_1, X_2, Y_2, \dots$ are independent and identically distributed nonnegative random variables, independent of the Poisson process. Describe the distribution of the resulting point process.

6.6 Instead of representing $W_k + X_k$ and $W_k + Y_k$ as distinct points on a single axis, write them as a single point $(W_k + X_k, W_k + Y_k)$ in the plane. The result is a nonhomogeneous Poisson point process in the positive quadrant with intensity

$$\lambda(x, y) = \lambda \int_0^{\min\{x, y\}} f(x-u)f(y-u)du$$

where $f(\cdot)$ is the common density for X, Y .

5.6.7 Let $\{N(t); t \geq 0\}$ be a Poisson process of intensity λ , and let $Y_1, Y_2, \dots$ be independent and identically distributed nonnegative random variables with

cumulative distribution function

$$G(y) = y^\alpha \quad \text{for } 0 < y < 1.$$

Determine $\Pr\{Z(t) > z | N(t) > 0\}$, where

$$Z(t) = \min\{Y_1, Y_2, \dots, Y_{N(t)}\}.$$

Describe the behavior for large t .

6.7 Refer to Exercise 6.3.

$$\Pr\{Z(t) > z | N(t) > 0\} = \frac{e^{-\lambda z^\alpha t} - e^{-\lambda t}}{1 - e^{-\lambda t}}; \quad 0 < z < 1.$$

Let $Y(t) = t^{1/\alpha} Z(t)$. For large t , we have

$$\begin{aligned} \Pr\{Y(t) > y | N(t) > 0\} &= \Pr\left\{Z(t) > \frac{y}{t^{1/\alpha}} \mid N(t) > 0\right\} \\ &= \frac{e^{-\lambda y^\alpha} - e^{-\lambda t}}{1 - e^{-\lambda t}} \xrightarrow{t \rightarrow \infty} e^{-\lambda y^\alpha} \quad (\text{Weibull}). \end{aligned}$$

5.6.8 Let $\{N(t); t \geq 0\}$ be a nonhomogeneous Poisson process of intensity $\lambda(t)$, $t > 0$, and let $Y_1, Y_2, \dots$ be independent and identically distributed nonnegative random variables with cumulative distribution function

$$G(y) = y^\alpha \quad \text{for } 0 < y < 1.$$

Suppose that the intensity process averages out in the sense that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda(u) du = \theta.$$

Let

$$Z(t) = \min \{Y_1, Y_2, \dots, Y_{N(t)}\}.$$

Determine

$$\lim_{t \rightarrow \infty} \Pr \left\{ t^{1/\alpha} Z(t) > z \right\}.$$

6.8 Following the remarks in Section 1.3 we may write $N(t) = M[\Lambda(t)]$ where $M(t)$ is a Poisson process of unit intensity.

$$\begin{aligned} \Pr\{Z(t) > z\} &= \Pr\left\{\min[Y_1, \dots, Y_{M[\Lambda(t)]}] > z\right\} \\ &= \exp\{-G(z)\Lambda(t)\} = \exp\{-z^\alpha \Lambda(t)\}. \\ \Pr\{t^{1/\alpha} Z(t) > z\} &= \Pr\left\{Z(t) > \frac{z}{t^{1/\alpha}}\right\} \\ &= \exp\left\{-z^\alpha \frac{\Lambda(t)}{t}\right\} \xrightarrow{t \rightarrow \infty} e^{-\theta z^\alpha}, \quad z > 0 \quad (\text{Weibull}) \end{aligned}$$

5.6.9 Let $W_1, W_2, \dots$ be the event times in a Poisson process of rate λ , and let $N(t) = N((0, t])$ be the number of points in the interval $(0, t]$. Evaluate

$$E \left[\sum_{k=1}^{N(t)} (W_k)^2 \right].$$

Note: $\sum_{k=1}^0 (W_k)^2 = 0$.

6.9 To carry the methods of this section a little further, write $N(dt) = N(t + dt) - N(t)$ so $N(dt) = 1$ if and only if $t < W_k \leq t + dt$ for some W_k . Then

$$\begin{aligned}\sum_{k=1}^{N(t)} (W_k)^2 &= \int_0^t x^2 N(dx) \quad \text{so} \\ E\left[\sum_{k=1}^{N(t)} (W_k)^2\right] &= E\left[\int_0^t x^2 N(dx)\right] = \int_0^t x^2 E[N(dx)] \\ &= \lambda \int_0^t x^2 dx = \frac{\lambda t^3}{3}.\end{aligned}$$

5.6.10 A Bidding Model Let $U_1, U_2, \dots$ be independent random variables, each uniformly distributed over the interval $(0, 1]$. These random variables represent successive bids on an asset that you are trying to sell, and that you must sell by time $t = 1$, when the asset becomes worthless. As a strategy, you adopt a secret number θ , and you will accept the first offer that is greater than θ . For example, you accept the second offer if $U_1 \leq \theta$ while $U_2 > \theta$. Suppose that the offers arrive according to a unit rate Poisson process ($\lambda = 1$).

- What is the probability that you sell the asset by time $t = 1$?
- What is the value for θ that maximizes your expected return? (You get nothing if you don't sell the asset by time $t = 1$.)
- To improve your return, you adopt a new strategy, which is to accept an offer at time t if it exceeds $\theta(t) = (1 - t)/(3 - t)$. What are your new chances of selling the asset, and what is your new expected return?

6.10 (a) $Pr\{\text{Sell asset}\} = 1 - e^{-(1-\theta)}.$

(b) Because the offers are uniformly distributed, the expected selling price, given that it is sold,

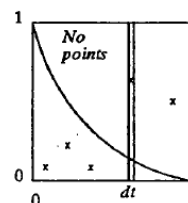
is $\left(\frac{1 + \theta}{2}\right).$

$$E[\text{Return}] = \frac{1 + \theta}{2} [1 - e^{-(1-\theta)}]$$

$$\theta^* = .2079 \quad \max_{\theta} E[\text{Return}] = .330425$$

(c) $Pr\{\text{Sell in } (t, t + dt)\} = e^{-\int_0^t [1 - \theta(u)] du} [1 - \theta(t)] dt.$

$$\begin{aligned}E[\text{Return}] &= \int_0^1 \left(\frac{1 + \theta(t)}{2}\right) \exp\left\{-\int_0^t [1 - \theta(u)] du\right\} [1 - \theta(t)] dt \\ &= \frac{2}{9} \int_0^1 (2 - t) dt = \frac{1}{3}, \text{ a miniscule improvement over (b).}\end{aligned}$$



6 Continuous Time Markov Chains

6.1 Pure Birth Processes

6.1.1 A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

Ans

at $n=0$ ③ $\Rightarrow P_0(t) = e^{-\lambda_0 t}, \lambda_0 = 1$

$\therefore P_0(t) = e^{-t}$ دالة الأضواء في الحالة 0

at $n=1$ ④ $\Rightarrow P_1(t) = \lambda_0 \left[\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right]$
 $\lambda_0 = 1, \lambda_1 = 3$

$P_1(t) = 1 \left[\frac{1}{3-1} e^{-t} + \frac{1}{1-3} e^{-3t} \right]$
 $\therefore P_1(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t} = \frac{1}{2} [e^{-t} - e^{-3t}]$ دالة الأضواء في الحالة 1

at $n=2$ ⑤ $\Rightarrow P_2(t) = \Pr\{X(t)=2 | X(0)=0\}$
 $= \lambda_0 \lambda_1 [B_{0,2} e^{-\lambda_0 t} + B_{1,2} e^{-\lambda_1 t} + B_{2,2} e^{-\lambda_2 t}]$ I

(5-2) $\Rightarrow B_{0,2} = \prod_{i=1}^2 \left(\frac{1}{\lambda_i - \lambda_0} \right)$

$B_{0,2} = \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} = \frac{1}{(3-1)(2-1)}$

$B_{0,2} = \frac{1}{2}$

$\lambda_0 = 1$
 $\lambda_1 = 3$
 $\lambda_2 = 2$

$$(5-1) \Rightarrow B_{1,2} = \prod_{i=0}^{2} \frac{1}{\lambda_i - \lambda_1} \quad i \neq 1$$

$$B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}$$

$$B_{1,2} = \frac{1}{(1-3)(2-3)} = \boxed{\frac{1}{2}}$$

$$\begin{aligned} \lambda_0 &= 1 \\ \lambda_1 &= 3 \\ \lambda_2 &= 2 \end{aligned}$$

$$(5-3) \Rightarrow B_{2,2} = \prod_{i=0}^{2} \frac{1}{\lambda_i - \lambda_2}$$

$$B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$B_{2,2} = \frac{1}{(1-2)(3-2)} = \boxed{-1}$$

Subs. $B_{2,2}$, $B_{1,2}$ and $B_{2,2}$ in (I), where

$\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, we get

$$P_2(t) = 3 \left[\frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t} - e^{-2t} \right]$$

Similarly, we can deduce that

$$P_3(t) = 6 \left[\frac{1}{8} e^{-t} + \frac{1}{4} e^{-3t} - \frac{1}{3} e^{-2t} - \frac{1}{24} e^{-5t} \right]$$

* Solve it as a homework.

Hint:

$$P_3(t) = \lambda_0 \lambda_1 \lambda_2 \left[B_{2,3} e^{\lambda_0 t} + B_{1,3} e^{\lambda_1 t} + B_{2,3} e^{\lambda_2 t} + B_{3,3} e^{\lambda_3 t} \right]$$

Solution.

$$P_0(t) = e^{-t}, \quad (\text{using (4.2.10)})$$

$$P_1(t) = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}, \quad (\text{using (4.2.11)})$$

$$P_2(t) = 3 \left(\frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-2t} \right), \quad (\text{using (4.2.12)-(4.2.15)})$$

$$P_3(t) = 6 \left(\frac{1}{8} e^{-t} + \frac{1}{4} e^{-3t} - \frac{1}{3} e^{-2t} - \frac{1}{24} e^{-5t} \right), \quad (\text{using (4.2.12)-(4.2.15)}) \blacksquare$$

6.1.5 Using equation (6.10), calculate the mean and variance for the Yule process where $X(0) = 1$.

$$\begin{aligned}
 P_n(t) &= p(1-p)^{n-1}, \quad p = e^{-\lambda t} \\
 &\Rightarrow \text{geometric dist'n} \\
 \therefore X(t) &\sim \text{geom}(p), \quad p = e^{-\lambda t} \\
 \text{Mean}[X(t)] &= 1/p = e^{\lambda t} \\
 \text{Variance}[X(t)] &= \frac{1-p}{p^2} \\
 &= \frac{1 - e^{-\lambda t}}{e^{-2\lambda t}} \quad \#
 \end{aligned}$$

Solution. Using Equation (4.2.16), we see that $P_n(t)$ is the geometric distribution with $p = e^{-\beta t}$, (i.e. $P_n(t) \sim \text{Geo}(e^{-\beta t})$). We know that for a geometric distribution $X \sim \text{Geo}(p)$, we have $\mathbb{E}[X] = \frac{1}{p}$ and $\text{Var}[X] = \frac{1-p}{p^2}$. Thus, since here we have $p = e^{-\beta t}$, we see that

$$\mathbb{E}[P_n(t)] = \frac{1}{e^{-\beta t}}, \text{ and } \text{Var}[P_n(t)] = \frac{1 - e^{-\beta t}}{e^{-2\beta t}}. \blacksquare$$

Problems

6.1.1 Let $X(t)$ be a Yule process that is observed at a random time U , where U is uniformly distributed over $[0, 1]$. Show that $\Pr\{X(U) = k\} = p^k / (\beta k)$ for $k = 1, 2, \dots$, with $p = 1 - e^{-\beta}$.

Hint: Integrate (6.10) over t between 0 and 1.

• pb(6.1.1) p.284

For Yule process

$$p_n(t) = e^{-\beta t} (1 - e^{-\beta t})^{n-1}, \quad n \geq 1$$

$\Pr\{X(U) = k\}$?? well

where U is a random time uniformly distributed from 0 to 1.

Ans: $\Pr\{X(U) = k\} = \int_0^1 e^{-\beta u} (1 - e^{-\beta u})^{k-1} du$

$$= \frac{1}{\beta} \int_0^1 (1 - e^{-\beta u})^{k-1} \cdot \beta e^{-\beta u} du$$

$$= \frac{1}{\beta} \left[\frac{(1 - e^{-\beta u})^k}{k} \right]_0^1$$

$$= \frac{1}{\beta k} [(1 - e^{-\beta})^k - 0]$$

$$\therefore \Pr\{X(U) = k\} = \frac{p^k}{\beta k}, \quad k = 1, 2, \dots$$

where $p = (1 - e^{-\beta})$

==

1.1 $\Pr\{X(U) = k\} = \int_0^1 e^{-\beta u} (1 - e^{-\beta u})^{k-1} du = \int_0^{1-e^{-\beta}} x^{k-1} \frac{dx}{\beta} = \frac{1}{\beta k} (1 - e^{-\beta})^k.$

6.1.2 A Yule process with immigration has birth parameters $\lambda_k = \alpha + k\beta$ for $k = 0, 1, 2, \dots$. Here, α represents the rate of immigration into the population, and β represents the individual birth rate. Supposing that $X(0) = 0$, determine $P_n(t)$ for $n = 0, 1, 2, \dots$.

1.2 $\lambda_k = \alpha + \beta k$

$$\begin{aligned}\lambda_0 \lambda_1 \cdots \lambda_{n-1} &= \beta^n \left(\frac{\alpha}{\beta}\right) \left(\frac{\alpha}{\beta} + 1\right) \cdots \left(\frac{\alpha}{\beta} + n - 1\right) \\ (\lambda_0 - \lambda_k) \cdots (\lambda_{k-1} - \lambda_k) &= \beta^k (-1)^k k! \\ (\lambda_n - \lambda_k) \cdots (\lambda_{k+1} - \lambda_k) &= \beta^{n-k} (n - k)! \\ P_n(t) &= \lambda_0 \cdots \lambda_{n-1} \sum_{k=0}^n B_{k,n} e^{-\lambda_k t} \\ &= \frac{\left(\frac{\alpha}{\beta}\right) \left(\frac{\alpha}{\beta} + 1\right) \cdots \left(\frac{\alpha}{\beta} + n - 1\right)}{n!} e^{-\alpha t} \sum_{k=0}^n \binom{n}{k} (-e^{-\beta t})^k \\ &= \binom{\frac{\alpha}{\beta} + n - 1}{n} e^{-\alpha t} (1 - e^{-\beta t})^n \quad \text{for } n = 0, 1, \dots\end{aligned}$$

Note: $\sum_{n=0}^{\infty} P_n(t) = 1$ using I, (6.18)–(6.20).

6.1.3 Consider a population comprising a fixed number N of individuals. Suppose that at time $t = 0$, there is exactly one *infected* individual and $N - 1$ *susceptible* individuals in the population. Once infected, an individual remains in that state forever. In any short time interval of length h , any given infected person will transmit the disease to any given susceptible person with probability $\alpha h + o(h)$. (The parameter α is the *individual infection rate*.) Let $X(t)$ denote the number of infected individuals in the population at time $t \geq 0$. Then, $X(t)$ is a pure birth process on the states $0, 1, \dots, N$. Specify the birth parameters.

1.3 The probabilistic rate of increase in the infected population is jointly proportional to the number who can give the disease, and the number available to catch it.

$$\lambda_k = \alpha k(N - k), \quad k = 0, 1, \dots, N.$$

6.1.4 A new product (a “Home Helicopter” to solve the commuting problem) is being introduced. The sales are expected to be determined by both media (newspaper and television) advertising and word-of-mouth advertising, wherein satisfied customers tell others about the product. Assume that media advertising creates new customers according to a Poisson process of rate $\alpha = 1$ customer per month. For the word-of-mouth advertising, assume that each purchaser of a Home Helicopter will generate sales to new customers at a rate of $\theta = 2$ customers per month. Let $X(t)$ be the total number of Home Helicopter customers up to time t .

- (a) Model $X(t)$ as a pure birth process by specifying the birth parameters λ_k , for $k = 0, 1, \dots$.
- (b) What is the probability that exactly two Home Helicopters are sold during the first month?

1.4 (a) $\lambda_k = \alpha + \theta k = 1 + 2k$.

$$(b) \quad P_2(t) = 2 \left[\frac{1}{8} e^{-t} - \frac{1}{4} e^{-3t} + \frac{1}{8} e^{-5t} \right]$$

$$P_2(1) = \frac{1}{4} e^{-1} - \frac{1}{2} e^{-3} + \frac{1}{4} e^{-5} = .0688.$$

6.1.5 Let W_k be the time to the k th birth in a pure birth process starting from $X(0) = 0$. Establish the equivalence

$$\Pr\{W_1 > t, W_2 > t + s\} = P_0(t)[P_0(s) + P_1(s)].$$

From this relation together with equation (6.7), determine the joint density for W_1 and W_2 , and then the joint density of $S_0 = W_1$ and $S_1 = W_2 - W_1$.

1.5 The two possibilities $X(w_2) = 0$ or $X(w_2) = 1$ give us

$$\begin{aligned}
 Pr\{W_1 > w_1, W_2 > w_2\} &= Pr\{X(w_1) = 0, X(w_2) = 0\} + Pr\{X(w_1) = 0, X(w_2) = 1\} \\
 &= P_0(w_1)P_0(w_2 - w_1) + P_0(w_1)P_1(w_2 - w_1) \\
 &= e^{-\lambda_0 w_2} + \lambda_0 e^{-\lambda_0 w_1} \left[\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0(w_2 - w_1)} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1(w_2 - w_1)} \right] \\
 &= \frac{\lambda_1}{\lambda_1 - \lambda_0} e^{-\lambda_0 w_2} + \frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 w_2} e^{-(\lambda_0 - \lambda_1)w_1} \\
 &= \int_{w_1}^{\infty} \int_{w_2}^{\infty} f_{w_1, w_2}(w'_1, w'_2) dw'_1, dw'_2,
 \end{aligned}$$

whence

$$f_{w_1, w_2}(w_1, w_2) = \frac{\partial}{\partial w_2} \frac{\partial}{\partial w_1} Pr\{W_1 > w_1, W_2 > w_2\} = \lambda_0 \lambda_1 e^{-\lambda_0 w_1} e^{-\lambda_1(w_2 - w_1)}$$

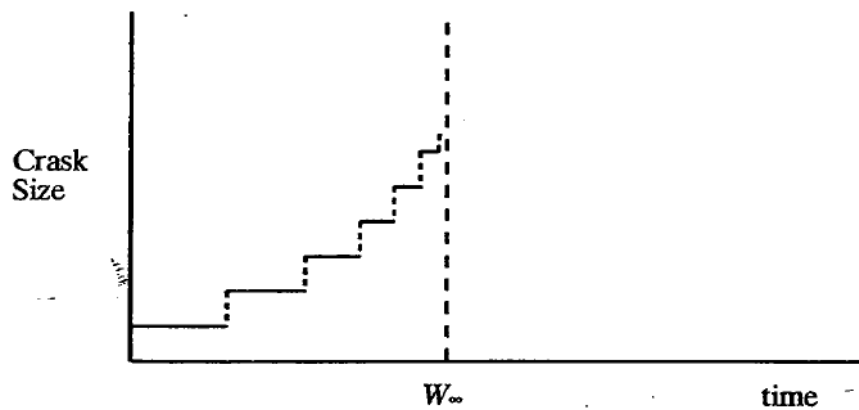
Setting $s_0 = w_1, s_1 = w_2 - w_1$ (Jacobian = 1) $f_{s_0, s_1}(s_0, s_1) = (\lambda_0 e^{-\lambda_0 s_0})(\lambda_1 e^{-\lambda_1 s_1})$.

6.1.6 A fatigue model for the growth of a crack in a discrete lattice proposes that the size of the crack evolves as a pure birth process with parameters

$$\lambda_k = (1 + k)^\rho \quad \text{for } k = 1, 2, \dots$$

The theory behind the model postulates that the growth rate of the crack is proportional to some power of the stress concentration at its ends and that this stress concentration is itself proportional to some power of $1 + k$, where k is the crack length. Use the sojourn time description to deduce that the mean time for the crack to grow to infinite length is finite when $\rho > 1$ and that, therefore, the failure time of the system is a well-defined and finite-valued random variable.

$$1.6 \quad E[S_k] = (1 + k)^{-\rho}; \quad E[W_\infty] = \sum_{k=1}^{\infty} (1 + k)^{-\rho} < \infty \text{ when } \rho > 1.$$



6.1.7 Let λ_0, λ_1 , and λ_2 be the parameters of the independent exponentially distributed random variables S_0, S_1 , and S_2 . Assume that no two of the parameters are equal.

(a) Verify that

$$\Pr\{S_0 > t\} = e^{-\lambda_0 t},$$

$$\Pr\{S_0 + S_1 > t\} = \frac{\lambda_1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 t},$$

and evaluate in similar terms

$$\Pr\{S_0 + S_1 + S_2 > t\}.$$

(b) Verify equation (6.8) in the case that $n = 2$ by evaluating

$$P_2(t) = \Pr\{X(t) = 2\} = \Pr\{S_0 + S_1 + S_2 > t\} - \Pr\{S_0 + S_1 > t\}.$$

1.7 (a) $\Pr\{S_0 \leq t\} = 1 - e^{-\lambda_0 t}$; $\Pr\{S_0 > t\} = e^{-\lambda_0 t}$.

$$\begin{aligned} \Pr\{S_0 + S_1 \leq t\} &= \int_0^t \left[1 - e^{-\lambda_1(t-x)}\right] \lambda_0 e^{-\lambda_0 x} dx = 1 - e^{-\lambda_0 t} - \left(\frac{\lambda_0}{\lambda_0 - \lambda_1}\right) e^{-\lambda_1 t} \left[1 - e^{-(\lambda_0 - \lambda_1)t}\right] \\ &= 1 - \frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} = 1 - \Pr\{S_0 + S_1 > t\}. \end{aligned}$$

$$\begin{aligned} \Pr\{S_0 + S_1 + S_2 \leq t\} &= \int_0^t \left\{1 - \frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1(t-x)} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0(t-x)}\right\} \lambda_2 e^{-\lambda_2 x} dx \\ &= 1 - \frac{\lambda_0 \lambda_2}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} - \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t} - \frac{\lambda_0 \lambda_1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 t}. \end{aligned}$$

$$\begin{aligned} \text{(b) } P_2(t) &= \frac{\lambda_0 \lambda_2}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t} + \frac{\lambda_0 \lambda_1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 t} \\ &\quad - \frac{\lambda_0}{(\lambda_0 - \lambda_1)} e^{-\lambda_1 t} - \frac{\lambda_1}{(\lambda_1 - \lambda_0)} e^{-\lambda_0 t} \\ &= \lambda_0 \lambda_1 \left[\frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t} + \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 t} \right]. \end{aligned}$$

6.1.8 Let $N(t)$ be a pure birth process for which

$$\Pr\{\text{an event happens in } (t, t+h) | N(t) \text{ is odd}\} = \alpha h + o(h),$$

$$\Pr\{\text{an event happens in } (t, t+h) | N(t) \text{ is even}\} = \beta h + o(h),$$

where $o(h)/h \rightarrow 0$ as $h \downarrow 0$. Take $N(0) = 0$. Find the following probabilities:

$$P_0(t) = \Pr\{N(t) \text{ is even}\}; \quad P_1(t) = \Pr\{N(t) \text{ is odd}\}.$$

Hint: Derive the differential equations

$$P'_0(t) = \alpha P_1(t) - \beta P_0(t) \quad \text{and} \quad P'_1(t) = -\alpha P_1(t) + \beta P_0(t)$$

and solve them by using $P_0(t) + P_1(t) = 1$.

1.8 Equations (1.2) become

$$P'_0(t) = -\beta P_0(t)$$

$$P'_n(t) = -\beta P_n(t) + \alpha P_{n-1}(t), \quad n = 2, 4, 6, \dots$$

$$P'_n(t) = -\alpha P_n(t) + \beta P_{n-1}(t), \quad n = 1, 3, 5, \dots$$

Seemmmg over $n = 0, 2, 4, \dots$ and $n = 1, 3, 5, \dots$ gives

$$P'_{\text{even}}(t) = \alpha P_{\text{odd}}(t) - \beta P_{\text{even}}(t) = \alpha[1 - P_{\text{even}}(t)] - \beta P_{\text{even}}(t)$$

$$P'_{\text{even}}(t) = \alpha - (\alpha + \beta)P_{\text{even}}(t).$$

Let $Q_{\text{even}}(t) = e^{(\alpha+\beta)t} P_{\text{even}}(t)$, then $Q'_{\text{even}}(t) = \alpha e^{(\alpha+\beta)t}$ and since $Q_{\text{even}}(0) = 1$, the solution is $Q_{\text{even}}(t) =$

$$\frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} e^{(\alpha+\beta)t} \text{ and } P_{\text{even}}(t) = e^{-(\alpha+\beta)t} Q_{\text{even}}(t) = \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} e^{-(\alpha+\beta)t}.$$

6.1.9 Under the conditions of Problem 6.8, determine $E[N(t)]$.

1.9 Equations (1.2) become

$$P'_0(t) = -\beta P_0(t)$$

$$P'_n(t) = -\beta P_n(t) + \alpha P_{n-1}(t), \quad n = 2, 4, 6, \dots$$

$$P'_n(t) = -\alpha P_n(t) + \beta P_{n-1}(t), \quad n = 1, 3, 5, \dots$$

Multiply the n th equation by n , sum, collect terms and simplify to get

$$M'(t) = \alpha P_{\text{odd}}(t) + \beta P_{\text{even}}(t) = \alpha + (\beta - \alpha)P_{\text{even}}(t),$$

and (See Problem 1.8 above)

$$M(t) = \frac{2\alpha\beta}{\alpha + \beta}t + \left(\frac{\beta - \alpha}{\beta + \alpha}\right)\left(\frac{\beta}{\alpha + \beta}\right)\left[1 - e^{-(\alpha+\beta)t}\right].$$

6.1.10 Consider a pure birth process on the states $0, 1, \dots, N$ for which $\lambda_k = (N - k)\lambda$ for $k = 0, 1, \dots, N$. Suppose that $X(0) = 0$. Determine $P_n(t) = \Pr\{X(t) = n\}$ for $n = 0, 1$, and 2 .

1.10 It is easily checked through (1.5) that $P_n(t) = \binom{N}{n} e^{-(N-n)\lambda} (1 - e^{-\lambda t})^n$, by working with $Q_n(t) = e^{\lambda t} P_n(t) = \binom{N}{n} (1 - e^{-\lambda t})^n$; $Q_0(t) \equiv 1$; $Q_n(t) = \lambda_{n-1} \int_0^t e^{-\lambda x} Q_{n-1}(x) dx$ (From (1.5)).

6.1.11 Beginning with $P_0(t) = e^{-\lambda_0 t}$ and using equation (6.5), calculate $P_1(t)$, $P_2(t)$, and $P_3(t)$ and verify that these probabilities conform with equation (6.7), assuming distinct birth parameters.

$$\mathbf{1.11} \quad P_1(t) = \lambda_0 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 x} e^{-\lambda_0 x} dx = \frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t}$$

$$\begin{aligned} P_2(t) &= \lambda_1 e^{-\lambda_2 t} \int_0^t \left[\frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 x} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 x} \right] e^{\lambda_2 x} dx \\ &= \lambda_0 \lambda_1 \left[\frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t} \right. \\ &\quad \left. + \left\{ \frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} \right\} e^{-\lambda_2 t} \right] \end{aligned}$$

and

$$\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$\begin{aligned} P_3(t) &= \lambda_0 \lambda_1 \lambda_2 e^{-\lambda_3 t} \int_0^t \left[\frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 x} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 x} \right. \\ &\quad \left. + \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 x} \right] e^{\lambda_3 x} dx \\ &= \lambda_0 \lambda_1 \lambda_2 \left[\frac{1}{(\lambda_3 - \lambda_0)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)} e^{-\lambda_0 t} + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_0 - \lambda_1)} e^{-\lambda_1 t} \right. \\ &\quad \left. + \frac{1}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)(\lambda_0 - \lambda_2)} e^{-\lambda_2 t} + \frac{1}{(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{-\lambda_3 t} \right] \end{aligned}$$

6.1.12 Verify that $P_2(t)$, as given by (6.8), satisfies (6.5) by following the calculations in the text that showed that $P_1(t)$ satisfies (6.5).

$$\begin{aligned}
\mathbf{1.12} \quad P_2(t) &= \lambda_1 e^{-\lambda_2 t} \int_0^t \left[\frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 x} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 x} \right] e^{\lambda_2 x} dx \\
&= \lambda_0 \lambda_1 e^{-\lambda_2 t} \left[\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} (1 - e^{-(\lambda_1 - \lambda_2)t}) + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} (1 - e^{-(\lambda_0 - \lambda_2)t}) \right] \\
&= \lambda_0 \lambda_1 \left[\frac{e^{-\lambda_0 t}}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} + \frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_0 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} \right]
\end{aligned}$$

Note $\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}.$

6.1.13 Using (6.5), derive $P_n(t)$ when all birth parameters are the same constant λ and show that

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, \dots$$

Thus, the postulates of Section 6.1.1 serve to define the Poisson processes.

Using the differential Eqs

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) \quad (1)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t), \quad n = 1, 2, \dots \quad (2)$$

where all birth parameters are the same constant λ ,

with initial condition $X(0) = 0$

show that $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \dots$
(Poisson process)

Ans: Let $X(t)$ represent the size of the population
initial condition $X(0) = 0$

$$X(0) = 0 \Rightarrow P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$X(0) = 0 \Rightarrow P_0(0) = 1, \quad P_n(0) = 0, \quad n = 1, 2, \dots$$

$$(1) \Rightarrow \frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

Separation of Variables

$$\frac{dP_0(t)}{P_0(t)} = -\lambda dt$$

and by Integration from 0 to t , we get

$$\int_0^t \frac{d\rho_0(u)}{\rho_0(u)} = -\lambda \int_0^t du$$

$$\therefore [\ln \rho_0(u)]_0^t = -\lambda t$$

$$\ln \rho_0(t) - \ln \rho_0(0) = -\lambda t, \quad \rho_0(0) = 1$$

$$\therefore \ln \rho_0(t) - \ln 1 = -\lambda t$$

$$\therefore \ln \rho_0(t) = -\lambda t \Rightarrow \boxed{\rho_0(t) = e^{-\lambda t}} \quad (3)$$

الحل هنا يعمل، والنتيجة تتطابق مع الحالة $t=0$ في الحالة 0.

$$(2) \Rightarrow \frac{d\rho_n(t)}{dt} = \lambda \rho_{n-1}(t) - \lambda \rho_n(t), \quad n=1, 2, \dots$$

$$\therefore \frac{d\rho_n(t)}{dt} + \lambda \rho_n(t) = \lambda \rho_{n-1}(t)$$

Multiply both sides by $e^{\lambda t}$

$$e^{\lambda t} \left[\frac{d\rho_n(t)}{dt} + \lambda \rho_n(t) \right] = \lambda \rho_{n-1}(t) e^{\lambda t}$$

$$\therefore \frac{d}{dt} [e^{\lambda t} \rho_n(t)] = \lambda \rho_{n-1}(t) e^{\lambda t}$$

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$$d[e^{\lambda t} \rho_n(t)] = \lambda \rho_{n-1}(t) e^{\lambda t} dt$$

$\therefore$ By integration from 0 to t

3

$$\int_0^t d[e^{\lambda x} p_n(x)] = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$[e^{\lambda x} p_n(x)]_0^t = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$e^{\lambda t} p_n(t) - \cancel{p_n(0)} = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$\therefore e^{\lambda t} p_n(t) = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx, n=1, 2, \dots$$

$$\therefore p_n(t) = \lambda e^{-\lambda t} \int_0^t p_{n-1}(x) e^{\lambda x} dx, n=1, 2, \dots$$

Recurrence Relation

تكرار العلاقة

(4)

at $n=1$

$$(4) \Rightarrow p_1(t) = \lambda e^{-\lambda t} \int_0^t p_0(x) e^{\lambda x} dx$$

$$(3) \Rightarrow p_0(x) = e^{-\lambda x}$$

$$\therefore p_1(t) = \lambda e^{-\lambda t} \int_0^t \cancel{e^{-\lambda x}} \cancel{e^{\lambda x}} dx$$

$$p_1(t) = \lambda e^{-\lambda t} \int_0^t dx = \lambda e^{-\lambda t} [x]_0^t$$

$$\therefore p_1(t) = \lambda t e^{-\lambda t}$$

(5)

4
at $n=2$

$$(4) \Rightarrow P_2(t) = \lambda e^{-\lambda t} \int_0^t P_1(x) e^{\lambda x} dx$$

$$(5) \Rightarrow P_1(x) = \lambda x e^{-\lambda x}$$

$$\therefore P_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x \cancel{e^{-\lambda x}} e^{\lambda x} dx$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \int_0^t x dx$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \left[\frac{x^2}{2} \right]_0^t$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \left(\frac{t^2}{2} \right)$$

$$\therefore P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$$

(6)

From (3), (5) and (6) we deduce that

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

, $n = 0, 1, 2, \dots$

==

Derive a general formula for $P_n(t)$ in case when all birth parameters are equal to the same constant λ .

Solution. Using the same method as in Equations (4.2.6) to (4.2.15), put $Q_n(t) = e^{\lambda t} P_n(t)$, $n = 0, 1, \dots$. By (4.2.5), we have

$$\boxed{P_0(t) = e^{-\lambda t}}.$$

Then, we see that

$$Q_0(t) = e^{\lambda t} P_0(t) = e^{\lambda t} e^{-\lambda t} = 1.$$

Also, as in (4.2.7), we see that

$$Q'_n(t) = e^{\lambda t} \lambda P_{n-1}(t) = \lambda Q_{n-1}(t).$$

It follows that $Q'_1(t) = \lambda Q_0(t) = \lambda$, whence $Q_1(t) = \lambda t$. But $Q_1(t) = e^{\lambda t} P_1(t)$, and therefore we obtain

$$\boxed{P_1(t) = \lambda t e^{-\lambda t}}.$$

Similarly, $Q'_2(t) = \lambda^2 t$, whence integrating to get $Q_2(t) = \frac{1}{2}(\lambda t)^2$. Likewise, we obtain $Q_3(t) = \frac{1}{3!}(\lambda t)^3$. Continuing in this way, we see that $Q_n(t) = \frac{1}{n!}(\lambda t)^n$.

It follows that $e^{\lambda t} P_n(t) = Q_n(t) = \frac{1}{n!}(\lambda t)^n$. Hence we obtain

$$\boxed{P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, \dots}.$$

This is just the formula of Poisson distribution with parameter λt . We conclude that the postulates of Definition 4.12 serve to define Poisson processes. ■

1.13 Let $Q_n(t) = e^{\lambda t} P_n(t)$. Then $Q_0(t) \equiv 1$ and (1.5) becomes $Q'_n(t) = \lambda Q_{n-1}(t)$ which solves to give $Q_1(t) = \lambda t$, $Q_2(t) = \frac{1}{2}(\lambda t)^2 \dots Q_n(t) = \frac{(\lambda t)^n}{n!}$ and $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$.

6.2 Pure Death Processes

6.2.1 A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 2$, and $\mu_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

Ans

For $N=3$ ① $\Rightarrow P_3(t) = e^{-\mu_3 t}$
 $P_3(t) = e^{-5t}$ (I)

For $n=2$ ② $\Rightarrow P_2(t) = \mu_3 [A_{2,2} e^{-\mu_2 t} + A_{3,2} e^{-\mu_3 t}]$



$$A_{2,2} = \frac{1}{\mu_3 - \mu_2}$$

$$A_{2,2} = \frac{1}{5-2} = \boxed{\frac{1}{3}}$$

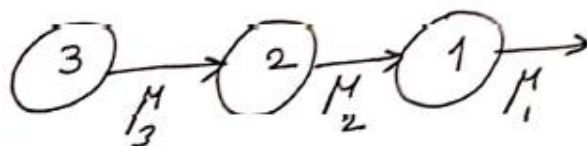
$$A_{3,2} = \frac{1}{\mu_2 - \mu_3} = \frac{1}{2-5} = \boxed{-\frac{1}{3}}$$

$$\therefore P_2(t) = 5 \left[\frac{1}{3} e^{-2t} - \frac{1}{3} e^{-5t} \right] \quad \text{(II)}$$

$$\begin{aligned} \mu_0 &= 0 \\ \mu_1 &= 3 \\ \mu_2 &= 2 \\ \mu_3 &= 5 \end{aligned}$$

For $n=1$

② $\Rightarrow P_1(t) = \mu_1 \mu_2 \mu_3 \left[A_{1,1} e^{-\mu_1 t} + A_{2,1} e^{-\mu_2 t} + A_{3,1} e^{-\mu_3 t} \right]$



$$A_{1,1} = \frac{1}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)}$$

$$= \frac{1}{(5-3)(2-3)}$$

$$A_{1,1} = \boxed{-\frac{1}{2}}$$

$$\begin{array}{l} \mu_0 = 0 \\ \mu_1 = 3 \\ \mu_2 = 2 \\ \mu_3 = 5 \end{array}$$

$$A_{2,1} = \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)}$$

$$A_{2,1} = \frac{1}{(5-2)(3-2)} = \boxed{\frac{1}{3}}$$

and $A_{3,1} = \frac{1}{(\mu_1 - \mu_3)(\mu_2 - \mu_3)}$

$$= \frac{1}{(3-5)(2-5)} = \boxed{\frac{1}{6}}$$

$$\therefore P_1(t) = 2(5) \left[-\frac{1}{2} e^{-3t} + \frac{1}{3} e^{-2t} + \frac{1}{6} e^{-5t} \right]$$

$$\therefore P_1(t) = 10 \left[\frac{1}{6} e^{-5t} + \frac{1}{3} e^{-2t} - \frac{1}{2} e^{-3t} \right] \quad (\text{III})$$

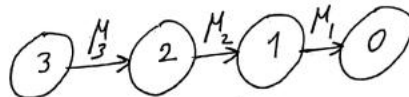
$$\therefore P_0(t) = 1 - [P_1(t) + P_2(t) + P_3(t)]$$

V

By substituting (I), (II), (III) in V

$$P_0(t) = 1 - \left[\left(1 - \frac{5}{3} + \frac{10}{6}\right) e^{-5t} + \left(\frac{5}{3} + \frac{10}{3}\right) e^{-2t} - \frac{10}{2} e^{-3t} \right]$$

$$\therefore P_0(t) = 1 + 5e^{-3t} - 5e^{-2t} - e^{-5t}$$



OR You can verify that:

$$P_0(t) = M_1 M_2 M_3 \left[A_{0,0} e^{-M_0 t} + A_{1,0} e^{-M_1 t} + A_{2,0} e^{-M_2 t} + A_{3,0} e^{-M_3 t} \right]$$

VI

where $A_{0,0} = \frac{1}{(M_3 - M_0)(M_2 - M_0)(M_1 - M_0)} = \boxed{\frac{1}{30}}$

$$A_{1,0} = \frac{1}{(M_3 - M_1)(M_2 - M_1)(M_0 - M_1)} = \boxed{\frac{1}{6}}$$

$$A_{2,0} = \frac{1}{(M_3 - M_2)(M_1 - M_2)(M_0 - M_2)} = \boxed{-\frac{1}{6}}$$

$$A_{3,0} = \frac{1}{(M_2 - M_3)(M_1 - M_3)(M_0 - M_3)} = \boxed{-\frac{1}{30}}$$

$$\therefore P_0(t) = 1 + 5e^{-3t} - 5e^{-2t} - e^{-5t}$$

Solution. Using Equation (4.2.18) with $N = 3$ and $\mu_3 = 5$, we get

$$P_3(t) = e^{-5t}$$

Now, using Equation (4.2.19), we see that $P_2(t) = \mu_3 (A_{2,2} e^{-\mu_2 t} + A_{3,2} e^{-\mu_3 t})$, where $A_{2,2}$, $A_{3,2}$ are given by Equation (4.2.20). So we obtain $A_{2,2} = \frac{1}{\mu_3 - \mu_2} = \frac{1}{3}$ and $A_{3,2} = \frac{1}{\mu_3 - \mu_2} = \frac{1}{3}$, which yields

$$P_2(t) = 5 \left(\frac{1}{3} e^{-2t} - \frac{1}{3} e^{-5t} \right)$$

Similarly, Equation (4.2.19) yields $P_1(t) = \mu_2 \mu_3 (A_{1,1} e^{-\mu_1 t} + A_{2,1} e^{-\mu_2 t} + A_{3,1} e^{-\mu_3 t})$, where $A_{1,1}$, $A_{2,1}$, $A_{3,1}$ are given by Equation (4.2.20). Substituting, we get $A_{1,1} = \frac{1}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)} = -\frac{1}{2}$, $A_{2,1} = \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} = \frac{1}{3}$ and $A_{3,1} = \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} = \frac{1}{6}$, which yields

$$P_1(t) = 10 \left(\frac{1}{6} e^{-5t} + \frac{1}{3} e^{-2t} - \frac{1}{2} e^{-3t} \right)$$

Finally, we use the unity-sum of probabilities to get

$$P_0(t) = 1 - P_1(t) - P_2(t) - P_3(t). \blacksquare$$

Problems

6.2.1 Let $X(t)$ be a pure death process starting from $X(0) = N$. Assume that the death parameters are $\mu_1, \mu_2, \dots, \mu_N$. Let T be an independent exponentially distributed random variable with parameter θ . Show that

$$\Pr\{X(T) = 0\} = \prod_{i=1}^N \frac{\mu_i}{\mu_i + \theta}.$$

2.1 Using the memoryless property (Section I, 5.2)

$$\Pr\{X(T) = 0\} = \Pr\{T > W_N\} = \prod_{i=1}^N \Pr\{T > W_i | T > W_{i-1}\} = \prod_{i=1}^N \left(\frac{\mu_i}{\mu_i + \theta} \right).$$

6.2.2 Let $X(t)$ be a pure death process with constant death rates $\mu_k = \theta$ for $k = 1, 2, \dots, N$. If $X(0) = N$, determine $P_n(t) = \Pr\{X(t) = n\}$ for $n = 0, 1, \dots, N$.

$$\mathbf{2.2} \quad P_N(t) = e^{-\theta t}$$

$$P_{N-1}(t) = \theta t e^{-\theta t}$$

$$P_n(t) = (\theta t)^{N-n} e^{-\theta t} / (N-n)! \quad \text{for } n = 1, 2, \dots, N$$

$$P_0(t) = 1 - \sum_{n=1}^N P_n(t).$$

6.2.3 A pure death process $X(t)$ with parameters $\mu_1, \mu_2, \dots$ starts at $X(0) = N$ and evolves until it reaches the absorbing state 0. Determine the mean area under the $X(t)$ trajectory.

Hint: This is $E[W_1 + W_2 + \dots + W_N]$.

2.3 Refer to Figure 2.1 to see that $\text{Area} = W_N + \dots + W_1$

$$\text{Hence } E[\text{Area}] = E[W_N] + \dots + E[W_1] = NE[S_N] + (N-1)E[S_{N-1}] + \dots + E[S_1] = \sum_{n=1}^N \frac{n}{\mu_n}.$$

- 6.2.4** A chemical solution contains N molecules of type A and M molecules of type B. An irreversible reaction occurs between type A and B molecules in which they bond to form a new compound AB. Suppose that in any small time interval of length h , any particular unbonded A molecule will react with any particular unbonded B molecule with probability $\theta h + o(h)$, where θ is a reaction rate. Let $X(t)$ denote the number of unbonded A molecules at time t .
- (a) Model $X(t)$ as a pure death process by specifying the parameters.
- (b) Assume that $N < M$ so that eventually all of the A molecules become bonded. Determine the mean time until this happens.

2.4 (a) $\mu_N = NM\theta$

$$\mu_{N-1} = (N-1)(M-1)\theta$$

$$\mu_k = k(M - (N - k))\theta, \quad k = 0, 1, \dots, N.$$

$$\begin{aligned}
 \text{(b) } E[W_N] &= \sum_{k=1}^N E[S_k] = \sum_{k=1}^N \frac{1}{k[M - (N - k)]\theta} \\
 &= \frac{1}{\theta} \sum_{k=1}^N \left\{ \frac{1}{(M - N)k} - \frac{1}{(M - N)(M - N + k)} \right\} \\
 &= \frac{1}{\theta(M - N)} \left\{ \sum_{k=1}^N \frac{1}{k} - \sum_{k=1}^N \frac{1}{M - N + k} \right\} \\
 &\sim \frac{1}{\theta(M - N)} \{ \log N - \log M + \log(M - N) \}.
 \end{aligned}$$

- 6.2.5** Consider a cable composed of fibers following the breakdown rule $K[l] = \sinh(l) = \frac{1}{2}(e^l - e^{-l})$ for $l \geq 0$. Show that the mean cable life is given by

$$\begin{aligned}
 E[W_N] &= \sum_{k=1}^N \{k \sinh(NL/k)\}^{-1} = \sum_{k=1}^N \left\{ \frac{k}{N} \sinh\left(\frac{L}{k/N}\right) \right\}^{-1} \left(\frac{1}{N}\right) \\
 &\approx \int_0^1 \{x \sinh(L/x)\}^{-1} dx.
 \end{aligned}$$

2.5 Breakdown rule is “exponential breakdown”.

$$\mu_k = kK \left[\frac{NL}{k} \right] = k \sinh \left[\frac{NL}{k} \right]$$

$$E[W_N] = \sum_{k=1}^N \frac{1}{k \sinh \left[\frac{NL}{k} \right]} = \sum_{k=1}^N \frac{1}{\left(\frac{k}{N} \right) \sinh \left(\frac{L}{R/N} \right)} \frac{1}{N} \cong \int_0^1 \frac{dx}{x \sinh \left(\frac{L}{x} \right)} \quad (\text{Riemann approx.})$$

6.2.6 Let T be the time to extinction in the linear death process with parameters $X(0) = N$ and α (see Section 6.2.1).

(a) Using the sojourn time viewpoint, show that

$$E[T] = \frac{1}{\alpha} \left[\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1} \right].$$

(b) Verify the result of (a) by using equation (6.15) in

$$E[T] = \int_0^{\infty} \Pr\{T > t\} dt = \int_0^{\infty} [1 - F_T(t)] dt.$$

Hint: Let $y = 1 - e^{-\alpha t}$.

2.6 (a) $E[T] = E[W_N] = \sum_{k=1}^N E[S_k] = \frac{1}{\alpha} \sum_{k=1}^N \frac{1}{k}.$

(b)
$$\begin{aligned} E[T] &= \int_0^{\infty} \Pr\{T > t\} dt = \int_0^{\infty} [1 - P_0(t)] dt = \int_0^{\infty} \left[1 - (1 - e^{-\alpha t})^N \right] dt = \frac{1}{\alpha} \int_0^1 [1 - y^N] \frac{dy}{1-y} \\ &= \frac{1}{\alpha} \int_0^1 [1 + y + y^2 + \cdots + y^{N-1}] dy = \frac{1}{\alpha} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right]. \end{aligned}$$

8 Brownian Motion and Related Processes

8.1 Brownian Motion and Gaussian Processes

8.1.1 Let $\{B(t); t \geq 0\}$ be a standard Brownian motion.

(a) Evaluate $\Pr\{B(4) \leq 3 | B(0) = 1\}$.

(b) Find the number c for which $\Pr\{B(9) > c | B(0) = 1\} = 0.10$.

Ans.:

$$\begin{aligned} \text{(a)} \quad \Pr\{B(4) \leq 3 | B(0) = 1\} &= \Phi\left(\frac{y-x}{\sqrt{t}}\right) \\ &\quad \text{, See Eq. (6)} \\ &= \Phi\left(\frac{3-1}{\sqrt{4}}\right) \end{aligned}$$

$$\therefore \Pr\{B(4) \leq 3 | B(0) = 1\} = \Phi(1) = 0.8413$$

From S.N.D. Table

$$\text{(b)} \quad \Pr\{B(9) > c | B(0) = 1\} = 0.10$$

$$\therefore 1 - \Pr\{B(9) \leq c | B(0) = 1\} = 0.10$$

$$\therefore \Pr\{B(9) \leq c | B(0) = 1\} = 0.90$$

$$\therefore \Phi\left(\frac{c-1}{3}\right) = 0.90$$

$$\therefore \frac{c-1}{3} = 1.282 \quad \text{by using S.N.D. Table}$$

$$\therefore c = 3(1.282) + 1 = 4.846$$

Solution.

(a) Taking $t = 4$, $s = 0$, $x = 1$, $y = 3$ in Formula (5.1.2), we obtain

$$\begin{aligned}\mathbb{P}\{B(4) \leq 3 | B(0) = 1\} &= \mathbb{P}\{B(0 + 4) \leq 3 | B(0) = 1\}, \\ &= \Phi\left(\frac{3-1}{\sqrt{4}}\right) = \Phi(1) = 0.8413.\end{aligned}$$

(b) Observe that

$$\begin{aligned}\mathbb{P}\{B(9) > c | B(0) = 1\} &= 1 - \mathbb{P}\{B(9) \leq c | B(0) = 1\} = 0.10 \\ &= 1 - \Phi\left(\frac{c-1}{\sqrt{9}}\right) = 0.10.\end{aligned}$$

It follows that

$$\Phi\left(\frac{c-1}{\sqrt{9}}\right) = 0.90.$$

From the table of the normal distribution, we infer that

$$\frac{c-1}{\sqrt{9}} = 1.282,$$

whence, we obtain the desired value:

$$c = 4.846. \quad \blacksquare$$

8.1.2 Let $\{B(t); t \geq 0\}$ be a standard Brownian motion and $c > 0$ a constant. Show that the process defined by $W(t) = cB(t/c^2)$ is a standard Brownian motion.

Ans:

$$\therefore \text{Cov}\{B(s), B(t)\} = s \text{ for } 0 \leq s \leq t$$

For standard Brownian motion ($\sigma^2=1$), we have

$$\text{Cov}\{B(s), B(t)\} = s \text{ for } 0 \leq s \leq t$$

$$\Rightarrow \text{Cov}\left\{B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right\} = \frac{s}{c^2},$$

Multiply both sides by c^2 we get $0 \leq \frac{s}{c^2} \leq \frac{t}{c^2}$

$$\therefore c^2 \text{Cov}\left\{B\left(\frac{s}{c^2}\right), B\left(\frac{t}{c^2}\right)\right\} = s$$

$$\therefore \text{Cov}\left\{cB\left(\frac{s}{c^2}\right), cB\left(\frac{t}{c^2}\right)\right\} = s$$

$$\therefore W(t) = cB(t/c^2)$$

$$\therefore \text{Cov}\{W(s), W(t)\} = s$$

, $0 \leq s \leq t$ (1)

Similarly, if $0 \leq t \leq s$, we can obtain

$$\text{Cov}\{W(s), W(t)\} = t \quad (2)$$

So, from (1) and (2), we conclude that

$$\text{Cov}\{W(s), W(t)\} = \min\{s, t\} \text{ for } s, t \geq 0$$

$\therefore W(t) = cB(t/c^2)$ is a standard Brownian motion.

—H—

Note

$$\begin{aligned} \text{Cov}(\alpha X, \alpha Y) \\ = \alpha^2 \text{Cov}(X, Y), \alpha \neq 0 \end{aligned}$$

8.1.5 Determine the covariance functions for the stochastic processes

(a) $U(t) = e^{-t} B(e^{2t})$, for $t \geq 0$.

(b) $V(t) = (1-t)B(t/(1-t))$, for $0 < t < 1$.

(c) $W(t) = tB(1/t)$, with $W(0) = 0$.

$B(t)$ is standard Brownian motion.

Ans:

(a)

$$\text{Cov}[U(s), U(t)]$$

$$= \text{Cov}[e^{-s} B(e^{2s}), e^{-t} B(e^{2t})]$$

$$= e^{-(s+t)} \text{Cov}[B(e^{2s}), B(e^{2t})]$$

$$= e^{-(s+t)} \min\{e^{2s}, e^{2t}\} \text{Cov prop for S.B.M.}$$

$$= e^{-(s+t)} e^{2s} = e^{s-t} \text{ if } s < t$$

$$= e^{-(s+t)} e^{2t} = e^{t-s} \text{ if } t < s$$

$$\text{Thus, Cov}[U(s), U(t)] = e^{-|s-t|} \quad \#$$

(b)

$$\text{Cov}[V(s), V(t)]$$

$$= \text{Cov}[(1-s)B(s/(1-s)), (1-t)B(t/(1-t))]$$

$$= (1-s)(1-t) \text{Cov}[B(s/(1-s)), B(t/(1-t))]$$

$$= (1-s)(1-t) \min\{s/(1-s), t/(1-t)\}$$

$$= (1-s)(1-t) \left(\frac{s}{1-s}\right) \text{ if } 0 < s < t < 1$$

$$= s(1-t), \text{ if } 0 < s < t < 1$$

$$= (1-s)(1-t) \left(\frac{t}{1-t}\right) \text{ if } 0 < t < s < 1$$

$$= t(1-s), \text{ if } 0 < t < s < 1 \quad \#$$

(c)

$$\text{Cov}[W(s), W(t)]$$

$$= \text{Cov}[sB(1/s), tB(1/t)]$$

$$= st \text{Cov}[B(1/s), B(1/t)]$$

$$\begin{aligned}
 \therefore \text{Cov}[W(s), W(t)] &= st \min\{1/s, 1/t\} \\
 &= st \left(\frac{1}{s}\right), \text{ if } \frac{1}{s} < \frac{1}{t} \\
 &= t, \quad 0 \leq t < s \\
 &= st \left(\frac{1}{t}\right), \text{ if } \frac{1}{t} < \frac{1}{s} \\
 &= s, \quad 0 \leq s < t \\
 &= \min\{s, t\} \quad \text{for } s, t \geq 0 \quad \#
 \end{aligned}$$

Problems

8.1.1 Consider the simple random walk

$$S_n = \xi_1 + \cdots + \xi_n, \quad S_0 = 0,$$

in which the summands are independent with $\Pr\{\xi = \pm 1\} = \frac{1}{2}$. In Chapter 3, Section 3.5.3, we showed that the mean time for the random walk to first reach $-a < 0$ or $b > 0$ is ab . Use this together with the invariance principle to show that $E[T] = ab$, where

$$T = T_{a,b} = \min\{t \geq 0; B(t) = -a \text{ or } B(t) = b\},$$

and $B(t)$ is standard Brownian motion.

Hint: The approximate Brownian motion (8.11) rescales the random walk in both time and space.

$$\begin{aligned} \mathbf{1.1} \text{ Let } T_n &= \min\{t \geq 0; B_n(t) \leq -a \text{ or } B_n(t) \geq b\} \\ &= \min\{t \geq 0; S_{[nt]} \leq -a\sqrt{n} \text{ or } S_{[nt]} \geq b\} \\ &= \frac{1}{n} \min\{k \geq 0; S_k \leq -a\sqrt{n} \text{ or } S_k \geq b\sqrt{n}\} \end{aligned}$$

$$\begin{aligned} E[T_n] &= \frac{1}{n} [a\sqrt{n}] [b\sqrt{n}] \quad (\text{III, Section 5.3}) \\ &\rightarrow ab \quad \text{as } n \rightarrow \infty. \end{aligned}$$

8.1.2 Evaluate $E[e^{\lambda B(t)}]$ for an arbitrary constant λ and standard Brownian motion $B(t)$.

Ans:

For $X(t) = e^{\lambda B(t)}$, $\ln X(t) = \lambda B(t)$
 $\Rightarrow \ln X(t) \sim N(0, \lambda^2 t)$ scale-invariance prop
 which is a Brownian motion
 with mean 0 and variance $\lambda^2 t$
 $\Rightarrow X(t) \sim \text{lognormal}(0, \lambda^2 t)$
 $\Rightarrow E[X(t)] = e^{(\mu + \sigma^2/2)t}$
 $= e^{0 + \frac{1}{2}\lambda^2 t}$
 $\therefore E[e^{\lambda B(t)}] = \exp\left[\frac{1}{2}\lambda^2 t\right] \quad \#$

if $B(t) \sim N(0, t)$
 then $\lambda B(t) \sim N(0, \lambda^2 t)$
 $B(t)$ is called scale-invariant

Also, we can find the Variance as
 $\text{Var}[X(t)] = e^{(2\mu + \sigma^2)t} (e^{\sigma^2 t} - 1)$
 $\therefore \text{Var}[e^{\lambda B(t)}] = e^{\lambda^2 t} [e^{\lambda^2 t} - 1] \quad \#$

Note:

The process of the form $X(t) = e^{\mu t + \sigma B(t)}$
 where $B(t)$ is a standard Brownian motion (i.e. $B(t) \sim N(0, t)$)
 is called a Geometric Brownian motion, also, it can
 be written as $X(t) = e^{\mu t + \sigma W(t)}$
 where $W(t)$ is referred to Wiener process ($W(t) \sim N(0, t)$).

Solution.

Observe that

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda\sqrt{t})^2} dx = e^{-\frac{1}{2}\lambda^2 t} \int_{-\infty}^{\infty} e^{\lambda\sqrt{t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

Making the variable change $y = \sqrt{t}x$ in the right-hand-side integral of the latter, we obtain

$$e^{-\frac{1}{2}\lambda^2 t} \int_{-\infty}^{\infty} e^{\lambda y} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}y^2/t} dy = 1.$$

On the other hand, referring to formula (5.1.3) of the density function, we obtain

$$\mathbb{E}[e^{\lambda B(t)}] = \int_{-\infty}^{\infty} e^{\lambda y} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}y^2/t} dy.$$

Combining the two latter formulas, we obtain

$$e^{-\frac{1}{2}\lambda^2 t} \mathbb{E}[e^{\lambda B(t)}] = 1$$

It follows that

$$\mathbb{E}[e^{\lambda B(t)}] = e^{\frac{1}{2}\lambda^2 t}. \blacksquare$$

1.2 Begin with $1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda\sqrt{t})^2} dx = e^{-\frac{1}{2}\lambda^2 t} \int_{-\infty}^{+\infty} e^{\lambda\sqrt{t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$
 $= e^{-\frac{1}{2}\lambda^2 t} \int_{-\infty}^{+\infty} e^{\lambda y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2/t} dy = e^{-\frac{1}{2}\lambda^2 t} \mathbb{E}[e^{\lambda B(t)}]$ so $\mathbb{E}[e^{\lambda B(t)}] = e^{\frac{1}{2}\lambda^2 t}.$

8.1.3 For a positive constant ϵ , show that

$$\Pr\left\{\frac{|B(t)|}{t} > \epsilon\right\} = 2\{1 - \Phi(\epsilon\sqrt{t})\}.$$

How does this behave when t is large ($t \rightarrow \infty$)? How does it behave when t is small ($t \approx 0$)?

1.3 $\Pr\left\{\frac{|B(t)|}{t} > \epsilon\right\} = \Pr\{|B(t)| > \epsilon t\} = 2\{1 - \Phi_t(\epsilon t)\} = 2\{1 - \Phi(\epsilon\sqrt{t})\} \quad (1.8)$

$$\Pr\left\{\frac{|B(t)|}{t} > \epsilon\right\} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{because } \Phi(\epsilon\sqrt{t}) \rightarrow 1$$

$$\Pr\left\{\frac{|B(t)|}{t} > \epsilon\right\} \rightarrow 1 \quad \text{as } t \rightarrow 0 \quad \text{because } \Phi(\epsilon\sqrt{t}) \rightarrow \frac{1}{2}.$$

8.1.4 Let $\alpha_1, \dots, \alpha_n$ be real constants. Argue that

$$\sum_{i=1}^n \alpha_i B(t_i)$$

is normally distributed with mean zero and variance

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \min \{t_i, t_j\}.$$

1.4 Every linear combination of jointly normal random variables is normally distributed.

$$E\left[\sum_{i=1}^n \alpha_i B(t_i)\right] = \sum \alpha_i E[B(t_i)] = 0$$

$$\begin{aligned} \text{Var}\left[\sum_{i=1}^n \alpha_i B(t_i)\right] &= E\left[\left\{\sum \alpha_i B(t_i)\right\}^2\right] = E\left[\left\{\sum_i \alpha_i B(t_i)\right\}\left\{\sum_j \alpha_j B(t_j)\right\}\right] \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j E[B(t_i)B(t_j)] = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \min\{t_i, t_j\}. \end{aligned}$$

8.1.5 Consider the simple random walk

$$S_n = \xi_1 + \cdots + \xi_n, \quad S_0 = 0,$$

in which the summands are independent with $\Pr\{\xi = \pm 1\} = \frac{1}{2}$. We are going to stop this random walk when it first drops a units below its maximum to date. Accordingly, let

$$M_n = \max_{0 \leq k \leq n} S_k, \quad Y_n = M_n - S_n, \quad \text{and} \\ \tau = \tau_a = \min\{n \geq 0; Y_n = a\}.$$

(a) Use a first step analysis to show that

$$\Pr\{M_\tau = 0\} = \frac{1}{1+a}.$$

(b) Why is $\Pr\{M_\tau \geq 2\} = \Pr\{M_\tau \geq 1\}^2$, and

$$\Pr\{M_\tau \geq k\} = \left(\frac{a}{1+a}\right)^k?$$

Identify the distribution of M_τ .

(c) Let $B(t)$ be standard Brownian motion, $M(t) = \max\{B(u); 0 \leq u \leq t\}$, $Y(t) = M(t) - B(t)$, and $\tau = \min\{t \geq 0; Y(t) = a\}$. Use the invariance principle to argue that $M(\tau)$ has an exponential distribution with mean a .

Note: τ is a popular strategy for timing the sale of a stock. It calls for keeping the stock as long as it is going up, but to sell it the first time that it drops a units from its best price to date. We have shown that $E[M(\tau)] = a$, whence $E[B(\tau)] = E[M(\tau)] - a = 0$, so that the strategy does not gain a profit, on average, in the Brownian motion model for stock prices.

1.5 (a) $\Pr\{M_\tau = 0\} = \Pr\{S_n \text{ drops to } -a < 0 \text{ before rising 1 unit}\} = \frac{1}{1+a}$ (Using III, 5.3).

(b) In order that $M_\tau \geq 2$, we must first have $M_\tau \geq 1$ followed by moving ahead to 2, starting afresh from $S' = 1$, before dropping a units below 1. The spatial homogeneity gives this second move

the same probability as $\Pr\{M(\tau) \geq 1\}$. Thus $\Pr\{M_\tau \geq 2\} = [\Pr\{M_\tau \geq 1\}]^2 = \left(\frac{a}{1+a}\right)^2$. The

argument repeats to give $\Pr\{M_\tau \geq k\} = \left(\frac{a}{1+a}\right)^k$.

(c) Divide the state space into increments of length $\frac{1}{n}$ and observe the Brownian motion as it

crosses lines $\frac{k}{n}$. That is, $\tau_1 = \min\left\{t \geq 0; B(t) = \frac{1}{n} \text{ or } B(t) = -\frac{1}{n}\right\}$, and if, for example, $B(\tau_1) =$

$\frac{1}{n}$, then let $\tau_2 = \min\left\{t \geq \tau_1; B(t) = \frac{2}{n} \text{ or } B(t) = \frac{0}{n}\right\}$, etc. $B(\tau_1), B(\tau_2), \dots$ has the same distribu-

tion as $\frac{1}{n}S_1, \frac{1}{n}S_2, \dots$. We apply part (b) to this approximating Brownian motion to see

$$\Pr\{M_n(\tau) > x\} = \left(\frac{na}{1+na}\right)^{nx} \rightarrow e^{-x/a}$$

As $n \rightarrow \infty$, the partially observed Brownian gets closer to the Brownian motion. We conclude

that $M(\tau)$ is exponentially distributed with mean a (Rate parameter $\frac{1}{a}$).

8.1.6 Manufacturers of crunchy munchies such as cheese crisps use compression testing machines to gauge product quality. The crisp, or whatever, is placed between opposing plates, which then move together. As the crisp is crunched, the force is measured as a function of the distance that the plates have moved. The output of the compression testing machine is a graph of force versus distance that is much like Figure 8.3. What aspects of the graph might be measures of product quality? Model the test as a row of tiny balloons between parallel plates. Each single balloon might follow a force–distance behavior of the form $\sigma = Ke(1 - q(e))$, where σ is the force, K is Young’s modulus, e is strain or distance, and $q(e)$ is a function that measures departures from Hooke’s law, to allow for soggy crisps. Each balloon obeys this relationship up until the random strain ζ at which it bursts. Determine the mean force as a function of strain. Use $F(x)$ for the cumulative distribution function of failure strain.

1.6 If $S_n(e)$ is the compressive force on n balloons at a strain of e , we have $E[S_n(e)] = nKe[1 - q(e)] \times [1 - F(e)]$. Because $\log E[S_n(e)] = \log nK + \log e - \log \frac{1}{[1 - q(e)][1 - F(e)]}$, a plot of $\log E[S_n(e)]$ versus $\log e$ would have an intercept $\log nK$ reflecting material volume and modulus. Any departures from linearity would reflect the failure strain distribution plus departures from Hooke’s law.

8.1.7 For $n = 0, 1, \dots$, show that (a) $B(n)$ and (b) $B(n)^2 - n$ are martingales (see Chapter 2, Section 2.5).

Ans:

(a) (1) $E[B(n)] = E[B(n)], n = 0, 1, \dots$
 $= 0 < \infty$, where $B(n) \sim N(0, n)$ المتوسط التباين

(2) $E[B(n+1) | B(0), \dots, B(n)]$
 $= E[B(n+1) - B(n) | B(0), \dots, B(n)] + B(n)$
 $= 0 + B(n)$
 $= B(n)$

$\therefore$ From (1) and (2), we obtain that $B(n), n = 0, 1, \dots$ is a martingale.

(b) (1) $E[B(n)^2 - n] = E[B(n)^2 - n], n = 0, 1, \dots$
 $= E[B(n)^2] - E(n)$
 $= \text{Var}[B(n)] + n - n$
 $= n - n = 0 < \infty$

(2) $E[B(n+1)^2 - (n+1) | B(n)^2 - n]$
 $= E[B(n+1)^2 - B(n)^2 - 1 + B(n)^2 - n | B(n)^2 - n]$
 $= \underbrace{B(n)^2 - n}_{[B(n)^2 - n]} + E[B(n+1)^2 - B(n)^2 - 1]$
 $= [B(n)^2 - n] + E[B(n+1)^2] - E[B(n)^2] - 1$
 $= [B(n)^2 - n] + \text{Var}[B(n+1)] - \text{Var}[B(n)] - 1$
 $= [B(n)^2 - n] + n+1 - n - 1$
 $= [B(n)^2 - n]$

$\text{Var}[B(n)] = E[B(n)^2] - E[B(n)]^2$
 $= E[B(n)^2]$

Remember:
 $E[g(X)|Y=y] = E[g(X)]$
 where X and Y are indep.
 $E[h(Y)|Y=y] = h(y)$

Thus, from (1) and (2), we get $[B(n)^2 - n], n = 0, 1, \dots$ is a martingale. $\#$

1.7 (a) $E[B(n+1) | B(0), \dots, B(n)] = E[B(n+1) - B(n) | B(0), \dots, B(n)] + B(n) = B(n).$

(b) $E[B(n+1)^2 - (n+1) | B(n)^2 - n] = E[\{B(n+1)^2 - B(n)^2 - 1\} + B(n)^2 - n | B(n)^2 - n]$
 $= B(n)^2 - n + E[B(n+1)^2 - B(n)^2] - 1 = B(n)^2 - n.$

8.1.8 Computer Challenge A problem of considerable contemporary importance is how to simulate a Brownian motion stochastic process. The invariance principle provides one possible approach. An infinite series expression that N. Wiener introduced may provide another approach. Let $Z_0, Z_1, \dots$ be a series of independent standard normal random variables. The infinite series

$$B(t) = \frac{t}{\sqrt{\pi}} Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{\sin mt}{m} Z_m, \quad 0 \leq t \leq 1,$$

is a standard Brownian motion for $0 \leq t \leq 1$. Try to simulate a Brownian motion stochastic process, at least approximately, by using finite sums of the form

$$B_N(t) = \frac{t}{\sqrt{\pi}} Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^N \frac{\sin mt}{m} Z_m, \quad 0 \leq t \leq 1.$$

If $B(t), 0 \leq t \leq 1$, is a standard Brownian motion on the interval $[0, 1]$, then $B'(t) = (1+t)B(1/(1+t)), 0 \leq t < \infty$, is a standard Brownian motion on the interval $[0, \infty)$. This suggests

$$B'_N(t) = (1+t)B_N\left(\frac{1}{1+t}\right), \quad 0 \leq t < \infty,$$

as an approximate standard Brownian motion. In what ways do these finite approximations behave like Brownian motion? Clearly, they are zero mean Gaussian processes. What is the covariance function, and how does it compare to that of Brownian motion? Do the gambler's ruin probabilities of (8.13) accurately describe their behavior? It is known that the squared variation of a Brownian motion stochastic process is not random, but constant:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left| B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right|^2 = 1.$$

This is a further consequence of the variance relation $E[(\Delta B)^2] = \Delta t$ (see the remark in Section 8.1.2). To what degree do the finite approximations meet this criterion?

1.8 The suggested approximations probably do well in mimicking Brownian motion in some tests, and poorly in others. Because each $B_N(t)$ is differentiable, $\Delta B_N(t)$ will be on the order of Δt , whereas $\Delta B(t)$ is on the order of $\sqrt{\Delta t}$.