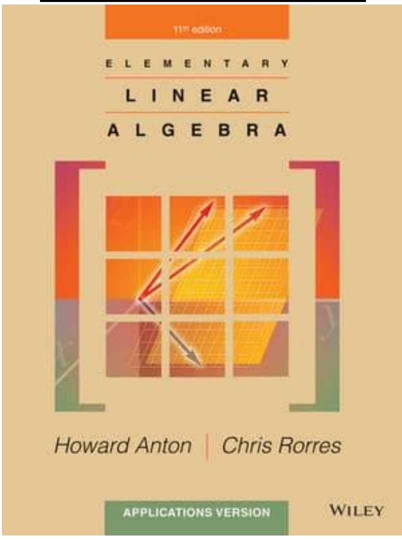
King Saud University
College of sciences
Department of Mathematics

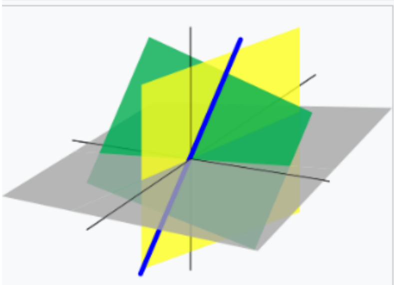
Math-244
Exercises (solved)

The exercises from :
Elementary Linear Algebra 11th Edition
by Howard Anton (Author)



Prepared by:
Lecturer: Fawaz bin Saud Alotaibi

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & n \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$



A line passing through the origin (blue, thick) in $\mathbf{R}^3$ is a linear subspace. It is the intersection of two planes (green and yellow).

"Page "means: in this file

1.2 Gaussian Elimination

In Exercises 5–8, solve the linear system by Gaussian elimination.

5.
$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 8 \\
-x_1 - 2x_2 + 3x_3 &= 1 \\
3x_1 - 7x_2 + 4x_3 &= 10
\end{aligned}
$$

Converting given equations into matrix form

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$R_2 \leftarrow R_2 + R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$R_3 \leftarrow R_3 - 3 \times R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$R_3 \leftarrow R_3 - 10 \times R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$x + y + 2z = 8 \rightarrow (1)$

$-y + 5z = 9 \rightarrow (2)$

$-52z = -104 \rightarrow (3)$

Now use back substitution method

From (3)

$-52z = -104$

$\Rightarrow z = \dfrac{-104}{-52} = 2$

From (2)

$-y + 5z = 9$

$\Rightarrow -y + 5(2) = 9$

$\Rightarrow -y + 10 = 9$

$\Rightarrow -y = 9 - 10$

$\Rightarrow -y = -1$

$\Rightarrow y = 1$

From (1)

$x + y + 2z = 8$

$\Rightarrow x + (1) + 2(2) = 8$

$\Rightarrow x + 5 = 8$

$\Rightarrow x = 8 - 5$

$\Rightarrow x = 3$

Solution using back substitution method.

$x = 3, y = 1$ and $z = 2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ (unique solution)}$$

7.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right)$$

$R_2 - 2R_1 \to R_2$ (multiply 1 row by 2 and subtract it from 2 row); $R_3 + 1R_1 \to R_3$ (multiply 1 row by 1 and add it to 3 row); $R_4 - 3R_1 \to R_4$ (multiply 1 row by 3 and subtract it from 4 row)

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right)$$

$R_2 / 3 \to R_2$ (divide the 2 row by 3)

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right)$$

$R_1 + 1R_2 \to R_1$ (multiply 2 row by 1 and add it to 1 row); $R_3 - 1R_2 \to R_3$ (multiply 2 row by 1 and subtract it from 3 row); $R_4 - 3R_2 \to R_4$ (multiply 2 row by 3 and subtract it from 4 row)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$Let\ w = t, z = s \ \Rightarrow y - 2z = 0 \ \Rightarrow y = 2s$$

$$x - w = -1 \ \Rightarrow x = t - 1$$

The system has infinitely many solutions:

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} t-1 \\ 2s \\ s \\ t \end{bmatrix}, \quad s, t \epsilon \mathbb{R}$$

▶ In Exercises 15–22, solve the given linear system by any method. ◀

15.
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution:
Total Equations are 3

$2x + y + 3z = 0 \rightarrow (1)$

$x + 2y + 0z = 0 \rightarrow (2)$

$0x + y + z = 0 \rightarrow (3)$

Converting given equations into matrix form

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 0.5 \times R_1$

$$= \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & 1.5 & -1.5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 0.6667 \times R_2$

$$= \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & 1.5 & -1.5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

i. e.

$2x + y + 3z = 0 \rightarrow (1)$

$1.5y - 1.5z = 0 \rightarrow (2)$

$2z = 0 \rightarrow (3)$

Now use back substitution method
From (3)
$2z = 0$

$\Rightarrow z = \frac{0}{2} = 0$

From (2)
$1.5y - 1.5z = 0$

$\Rightarrow 1.5y - 1.5(0) = 0$

$\Rightarrow 1.5y = 0$

$\Rightarrow y = \frac{0}{1.5} = 0$

From (1)
$2x + y + 3z = 0$

$\Rightarrow 2x + (0) + 3(0) = 0$

$\Rightarrow 2x = 0$

$\Rightarrow x = \frac{0}{2} = 0$

Solution using back substitution method.
$x = 0, y = 0$ and $z = 0$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ (unique solution)

19.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w \qquad - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

$$\left(\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right)$$

$R_1 \leftrightarrow R_2$ (interchange the 1 and 2 rows)

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right)$$

$R_3 - 2R_1 \rightarrow R_3$ (multiply 1 row by 2 and subtract it from 3 row); $R_4 + 2R_1 \rightarrow R_4$ (multiply 1 row by 2 and add it to 4 row)

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right)$$

$R_2 / 2 \rightarrow R_2$ (divide the 2 row by 2)

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right)$$

$R_3 - 3R_2 \rightarrow R_3$ (multiply 2 row by 3 and subtract it from 3 row); $R_4 - 1R_2 \rightarrow R_4$ (multiply 2 row by 1 and subtract it from 4 row)

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{array}\right)$$

$R_1 + 3R_3 \rightarrow R_1$ (multiply 3 row by 3 and add it to 1 row); $R_2 - 2R_3 \rightarrow R_2$ (multiply 3 row by 2 and subtract it from 2 row); $R_4 + 10R_3 \rightarrow R_4$ (multiply 3 row by 10 and add it to 4 row)

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$z = 0. Let \;\; y = s \;\Rightarrow x + y = 0 \;\Rightarrow x = -s$$

$$w - y = 0 \;\Rightarrow w = s$$

The system has infinitely many solutions:

$$X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ -s \\ s \\ 0 \end{bmatrix}, \quad s \epsilon \mathbb{R}$$

THEOREM 1.2.1 Free Variable Theorem for Homogeneous Systems

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

► In Exercises 25–26, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions. ◄

25.
$$\begin{aligned} x + 2y - \quad\quad 3z &= 4 \\ 3x - y + \quad\quad 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

25. For what a values does the system have one sol, infinitely many sols, or no sols. → for what a values.

No Solutions when $a = -4$; infinitely many solutions when $a = 4$; one solution for all values $a \neq -4$ and $a \neq 4$

$$\begin{aligned} x + 2y - \quad 3z &= 4 \\ 3x - y + \quad 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (a^2-14) & a+2 \end{bmatrix}$$

$-3R_1 + R_2 = R_2$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 4 & 1 & (a^2-14) & a+2 \end{bmatrix}$$

$-4R_1 + R_3 = R_3$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & (a^2-14)+12 & (a-14) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$-R_2 + R_3 = R_3$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & (a^2-16) & a-4 \end{bmatrix}$$

$-\frac{1}{7}R_2$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & (a-4)(a+4) & a-4 \end{bmatrix}$$

→

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & (a-4)(a+4) & a-4 \end{bmatrix}$$

If $a = \pm 4$

$+4 = a$

$(4-4)(4+4) = 4-4$ → $0=0$ Because there is no way to solve for the third unknown, the third unknown could be anything, so infinitely many sols

$-4 = a$

$(-8)(0) = -8$

$0 = -8$?

$5 = a$

$(1)(9) = 5-4$

$9 \neq 1$

$(a-4)(a+4) = a-4$

$\frac{(a-4)}{(a-4)}$ was this legal?

$a+4 = 1$

if $a = -3$

if $a > -3$ or $a < -3$

No Solutions

→ There's only one -3, so there is exactly one solution if $a = -3$

~~If a = 1~~ Summary:

If $a = \pm 4$, infinitely many sols

If $a > -3$ or $a < -3$, no sols. Or when $a = -4$

If $a = -3$, exactly one sol.

25. No solutions when $a = -4$;
infinitely many solutions when $a = 4$;
one solution for all values $a \neq -4$ and $a \neq 4$

▶ In Exercises 27–28, what condition, if any, must a, b, and c satisfy for the linear system to be consistent? ◀

27.
$$\begin{aligned} x + 3y - z &= a \\ x + y + 2z &= b \\ 2y - 3z &= c \end{aligned}$$

the matrix form of the above system is,

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

the Augmented matrix is, $\left[\begin{array}{ccc|c} 1 & 3 & -1 & a \\ 1 & 1 & 2 & b \\ 0 & 2 & -3 & c \end{array}\right]$

we convert it into row echelon form,

$$R_2 \to R_2 - R_1 \left[\begin{array}{ccc|c} 1 & 3 & -1 & a \\ 0 & -2 & 3 & b-a \\ 0 & 2 & -3 & c \end{array}\right]$$

$$R_3 \to R_3 + R_2 \left[\begin{array}{ccc|c} 1 & 3 & -1 & a \\ 0 & -2 & 3 & b-a \\ 0 & 0 & 0 & c+b-a \end{array}\right]$$

this is in row echelon form.

we have this system is consistent if $c + b - a = 0$
$\Rightarrow c = a - b.$

hence the given system is consistent for any value of a and b and for $c = a - b$.

Explanation:

to find consistency of the given system we convert the system into row echelon form.

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

Answer:

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are possible answers.

matrix that is in ***reduced row echelon form***. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***. (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

For finding the reduced row echelon form we have to perform row operation

Multiply each element of R_1 by $\frac{1}{2}$ to make the entry at $1, 1$ a 1.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

Perform the row operation $R_3 = R_3 - 3R_1$ to make the entry at $3, 1$ a 0.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -2 & -29 \\ 0 & \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

Multiply each element of R_2 by $-\frac{1}{2}$ to make the entry at $2, 2$ a 1.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{29}{2} \\ 0 & \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

Perform the row operation $R_3 = R_3 - \frac{5}{2}R_2$ to make the entry at $3, 2$ a 0.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{29}{2} \\ 0 & 0 & -\frac{143}{4} \end{bmatrix}$$

Multiply each element of R_3 by $-\frac{4}{143}$ to make the entry at $3, 3$ a 1.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{29}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Perform the row operation $R_2 = R_2 - \frac{29}{2}R_3$ to make the entry at $2, 3$ a 0.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Perform the row operation $R_1 = R_1 - \frac{3}{2}R_3$ to make the entry at $1, 3$ a 0.

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Perform the row operation $R_1 = R_1 - \frac{1}{2}R_2$ to make the entry at $1, 2$ a 0.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

37. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

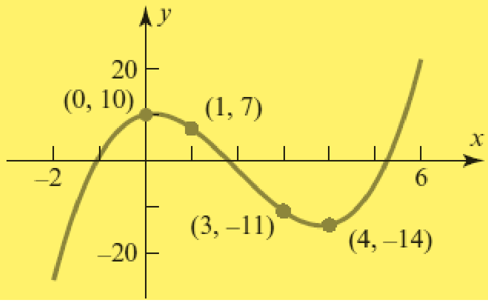


Figure Ex-37

Observe that, the equation is $y = ax^3 + bx^2 + cx + d$ and it passes through the points $(0, 10), (1, 7), (3, -11), (4, -14)$
Substitute $(0, 10)$ on the above equation.

$$10 = a(0)^3 + b(0)^2 + c(0) + d$$
$$d = 10$$

By substituting $d = 10$ on the equation $y = ax^3 + bx^2 + cx + d$ gives $y = ax^3 + bx^2 + cx + 10$.
Substitute $(1, 7)$ on the equation $y = ax^3 + bx^2 + cx + 10$.

$$7 = (1)^3 + b(1)^2 + c(1) + 10$$
$$a + b + c \quad 3$$

Substitute $(3, -11)$ on the equation $y = ax^3 + bx^2 + cx + 10$.

$$-11 = 27a + 9b + 3c + 10$$
$$9a + 3b + c = -7$$

Substitute $(4, -14)$ on the equation $y = ax^3 + bx^2 + cx + 10$.

$$-14 = (4)^3a + (4)^2b + 4c + 10$$
$$16a + 4b + c = -6$$

Subtract the above resulting first two equation and it gives $8a + 2b = -4, 4a + b = -2$
Subtract the above resulting next two equation and it gives $7a + b = 1$.

Substract the equations $4a + b = -2$ and $7a + b = 1$ gives the value which is $a = 1$.

By substituting $a = 1$ on $4a + b = -2$ gives $b = -6$.

By substituting $a = 1, b = -6$ on any one of the equations gives, $c = 2$.
Therefore, $a = 1, b = -6, c = 2$.

1.3 Matrices and Matrix Operations

In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

3. (a) $D + E$ (b) $D - E$ (c) $5A$

(d) $-7C$ (e) $2B - C$ (f) $4E - 2D$

(g) $-3(D + 2E)$ (h) $A - A$ (i) $\mathrm{tr}(D)$

(j) $\mathrm{tr}(D - 3E)$ (k) $4\,\mathrm{tr}(7B)$ (l) $\mathrm{tr}(A)$

a) $D + E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$

b) $D - E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

c) $$5A = 5\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$$

d) $$-7C = -7\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$$

e) $2B - C = 2[$ It is not possible because B is 2×2 matrix and C is 2×3 matrix add

f) $$4E - 2D = 4\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$$

g) $-3(D+2E)$

$$D+2E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + 2\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$$

$$-3(D+2E) = -3\begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

$$A - A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$tr(D) = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

trace = sum of diagonal elements

$$tr\, D = 1+0+4 = 5$$

j) $D - 3E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - 3\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} -17 & 2 & -7 \\ 2 & -3 & -5 \\ -9 & -1 & -5 \end{bmatrix}$

$tr(D-3E) = \text{trace of} \begin{bmatrix} -17 & 2 & -7 \\ 2 & -3 & -5 \\ -9 & -1 & -5 \end{bmatrix}$

$= -17 - 3 - 5$

$tr(D-3E) = -25$

k) $7B = 7\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 28 & -7 \\ 0 & 14 \end{bmatrix}$

$4\, tr(7B) = 4\, tr(7(B))$

$= 4\begin{bmatrix} 28 & -7 \\ 0 & 14 \end{bmatrix}$

$= 4[28+14]$

$= 4[42]$

$= 168$

l) $tr(A) = \text{trace of } [A]$ not possible

because it is not square matrix

3. (a) $\begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$

(e) Undefined (f) $\begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$ (g) $\begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$ (h) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) 5 (j) −25 (k) 168 (l) Undefined

In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

5. (a) AB (b) BA (c) $(3E)D$

(d) $(AB)C$ (e) $A(BC)$ (f) CC^T

(g) $(DA)^T$ (h) $(C^TB)A^T$ (i) $\operatorname{tr}(DD^T)$

(j) $\operatorname{tr}(4E^T - D)$ (k) $\operatorname{tr}(C^TA^T + 2E^T)$ (l) $\operatorname{tr}((EC^T)^TA)$

(a) Multiplying the matrices A and B , then

$$\mathrm{AB} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix} \qquad (1)$$

(b) Since B is a 2×2 matrix and A is a 3×2 matrix, matrix multiplication is not possible. Thus BA is undefined.

(c) First multiply the scalar 3 with the matrix E and then Multiplying the matrices $3\mathrm{E}$ and D ,

$$(3\mathrm{E})\mathrm{D} = \begin{bmatrix} 18 & 3 & 9 \\ -3 & 3 & 6 \\ 12 & 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$$

(d) Using (1) , and Multiplying the matrices AB and C ,

$$(\mathrm{AB})\mathrm{C} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(e) First multiply the matrices B and C and then Multiply the matrices A and BC

$$\mathrm{A}(\mathrm{BC}) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(f) Multiplying the matrices C and C^{T} ,

$$\mathrm{CC}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$$

(g) Multiplying the matrices D and A , and taking transpose,

$$\mathrm{DA} = \begin{bmatrix} 0 & 12 \\ -2 & 1 \\ 11 & 8 \end{bmatrix}$$

$$(\mathrm{DA})^{\mathrm{T}} = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$$

(h) First multiply the matrices $\mathrm{C^T}$ and B and then Multiply the matrices $\mathrm{C^TB}$ and $\mathrm{A^T}$,

$$(\mathrm{C^TB})\mathrm{A^T} = \begin{bmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$$

(i) Multiplying the matrices D and $\mathrm{D^T}$,

$$\mathrm{DD^T} = \begin{bmatrix} 30 & 1 & 21 \\ 1 & 2 & 1 \\ 21 & 1 & 29 \end{bmatrix}$$

Then $\mathrm{tr}\left(\mathrm{DD^T}\right) = 30 + 6 + 29$ implies $\mathrm{tr}\left(\mathrm{DD^T}\right) = 61$.

(j) Subtracting the matrices $4\mathrm{E^T}$ and D,

$$4\mathrm{E^T} - \mathrm{D} = \begin{bmatrix} 23 & -9 & 14 \\ 5 & 4 & 3 \\ 9 & 6 & 8 \end{bmatrix}$$

$\mathrm{tr}\left(4\mathrm{E^T} - \mathrm{D}\right) = 23 + 4 + 8$ implies $\mathrm{tr}\left(4\mathrm{E^T} - \mathrm{D}\right) = 35$.

(k) Multiply $\mathrm{C^T}$ and $\mathrm{A^T}$ and add the matrices $\mathrm{C^TA^T}$ with $2\mathrm{E^T}$,

$$\mathrm{C^TA^T} + 2\mathrm{E^T} = \begin{bmatrix} 15 & 3 & 12 \\ 14 & 0 & 7 \\ 12 & 12 & 13 \end{bmatrix}$$

Thus $\mathrm{tr}\left(\mathrm{C^TA^T} + 2\mathrm{E^T}\right) = 15 + 0 + 13$ $\mathrm{tr}\left(\mathrm{C^TA^T} + 2\mathrm{E^T}\right) = 28$.

(l) Multiplying the matrices $\left(\mathrm{EC^T}\right)^{\mathrm{T}}$ and A,

$$\left(\mathrm{EC^T}\right)^{\mathrm{T}}\mathrm{A} = \begin{bmatrix} 55 & 28 \\ 122 & 44 \end{bmatrix}$$

Thus $\mathrm{tr}\left(\left(\mathrm{EC^T}\right)^{\mathrm{T}}\mathrm{A}\right) = 55 + 44$ implies $\mathrm{tr}\left(\left(\mathrm{EC^T}\right)^{\mathrm{T}}\mathrm{A}\right) = 99$.

I-

To find the required answer, first we have to find C^T.

Here $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

Therefore $C^T = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T$

$\Rightarrow C^T = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$

$\text{Now, } EC^T = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$

Multiplying row of the first matrix vs column of the second matrix.

Therefore $EC^T = \begin{bmatrix} 16 & 34 \\ 7 & 8 \\ 14 & 28 \end{bmatrix}$

Now $(EC^T)^T = \begin{bmatrix} 16 & 34 \\ 7 & 8 \\ 14 & 28 \end{bmatrix}^T$

$\Rightarrow (EC^T)^T = \begin{bmatrix} 16 & 7 & 14 \\ 34 & 8 & 28 \end{bmatrix}$.

Now $(EC^T)^T A = \begin{bmatrix} 16 & 7 & 14 \\ 34 & 8 & 28 \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow (EC^T)^T A = \begin{bmatrix} 55 & 28 \\ 122 & 44 \end{bmatrix}$.

Explanation:

We know that The trace of a square matrix is the sum of the elements on its main diagonal (the diagonal that runs from the top left to the bottom right of the matrix).

Now in this case, $\operatorname{tr}\left((EC^T)^T A\right) = 55 + 44 = 99$.

5. **(a)** $\begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$ **(b)** Undefined **(c)** $\begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$ **(e)** $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(f) $\begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$ **(g)** $\begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$ **(h)** $\begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$ **(i)** 61 **(j)** 35 **(k)** 28 **(l)** 99

▶ In Exercises 15–16, find all values of k, if any, that satisfy the equation. ◀

15. $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

Multiply first two matrices,

In this case, the first matrix is 1×3 and the second matrix is 3×3 .

Multiply each row in the first matrix by each column in the second matrix.

$$[k \times 1 + 1 \times 1 + 1 \times 0 \quad k \times 1 + 1 \times 0 + 1 \times 2 \quad k \times 0 + 1 \times 2 + 1 \times -3]$$

Simplify each element of the matrix by multiplying out all the expressions.

$$[k+1 \quad k+2 \quad -1]$$

Now we have -

$$[k+1 \quad k+2 \quad -1] \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$$

Multiply each row in the first matrix by each column in the second matrix.

$$[(\mathrm{k}+1)\mathrm{k} + (\mathrm{k}+2) \times 1 - 1 \times 1] = 0$$

Simplify each element of the matrix by multiplying out all the expressions.

$$[\mathrm{k}^2 + 2\mathrm{k} + 1] = 0$$

Now

$$\mathrm{k}^2 + 2\mathrm{k} + 1 = 0$$

$$(\mathrm{k}+1)^2 = 0$$

$$(\mathrm{k}+1) = 0$$

$$\mathrm{k} = -1$$

Another solution of Dr. Wael Mustafa

scan me

▶ In Exercises 23–24, solve the matrix equation for a, b, c, and d. ◀

23. $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$

$a = 4$

$3 = d - 2c$ .... (1)

$-1 = d + 2c$ .... (2)

$a + b = -2 \Rightarrow 4 + b = -2 \Rightarrow b = -6$

Multiply Eq (2) by $(-1) \Rightarrow 1 = -d - 2c$ ... ... (3)

Add eq 1 with eq 3 $\Rightarrow 4 = -4c \Rightarrow c = -1$

In eq 1: $3 = d + 2 \Rightarrow d = 1$

1.4 Inverses; Algebraic Properties of Matrices

In Exercises 15–18, use the given information to find A.

15. $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

17. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

15) given that,

$$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \Rightarrow 7A = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1}$$

Now, let,

$$B = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad \det(B) = \begin{vmatrix} -3 & 7 \\ 1 & -2 \end{vmatrix} = (-3)(-2) - 7 = 6 - 7 = -1$$

Explanation:

We know that inverse of a matrix A = adj(A)/det(A) and adj(A)= transpose of the cofactor matrix of A.

So,

$$B^{-1} = \frac{1}{\det(B)} \text{adj}(B) = \frac{1}{-1} \begin{bmatrix} -2 & -1 \\ -7 & -3 \end{bmatrix}^{T} = -1 \begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

Now we have,

$$B^{-1} = 7A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} \frac{2}{7} & 1 \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$$

17-

Let $B = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

$$\begin{aligned} (I + 2A)^{-1} &= B \\ I + 2A &= B^{-1} \\ 2A &= B^{-1} - I \\ A &= \frac{1}{2}\left(B^{-1} - I\right). \end{aligned}$$

$$B^{-1} = \frac{1}{(-1)*5-2*4}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix}$$

$$A = \frac{1}{2}\left(\begin{bmatrix} \frac{-5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} \frac{-18}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{-12}{13} \end{bmatrix} = \begin{bmatrix} \frac{-9}{13} & \frac{1}{13} \\ \frac{2}{13} & \frac{-6}{13} \end{bmatrix}$$

▶ In Exercises 19–20, compute the following using the given matrix A.

(a) A^3 (b) A^{-3} (c) $A^2 - 2A + I$ ◀

19. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

a-

$$\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A} \cdot \mathbf{A}^2 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 41 & 15 \\ 30 & 11 \end{pmatrix}$$

b-

$$A^{-3} = (A^3)^{-1} = \begin{pmatrix} 41 & 15 \\ 30 & 11 \end{pmatrix}^{-1} = \frac{1}{41*11-15*30}\begin{pmatrix} 11 & -15 \\ -30 & 41 \end{pmatrix} =$$

$$\frac{1}{1}\begin{pmatrix} 11 & -15 \\ -30 & 41 \end{pmatrix} = \begin{pmatrix} 11 & -15 \\ -30 & 41 \end{pmatrix}$$

c-

$$A^2 - 2A + I = \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix} - 2\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 2 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 4 & 2 \end{pmatrix}$$

▶ In Exercises 21–22, compute $p(A)$ for the given matrix A and the following polynomials.

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$ ◀

21. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

a-

$$p(A) = A - 2I = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

b-

$$p(A) = 2A^2 - A + I$$

$$\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix}$$

$$p(A) = 2\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 19 & 7 \\ 14 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$$

c-

$$\mathbf{A}^3 = \mathbf{A} \cdot \mathbf{A}^2 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 11 & 4 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 41 & 15 \\ 30 & 11 \end{pmatrix}$$

$$p(A) = A^3 - 2A + I = \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix} - 2\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 35 & 13 \\ 26 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 13 \\ 26 & 10 \end{bmatrix}$$

In Exercises 23–24, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

23. Find all values of a, b, c, and d (if any) for which the matrices A and B commute.

First, recall that matrices A and B **commute** if

$$AB = BA$$

So let's find the products BA and AB.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

Setting corresponding entries equal, we get the following equations

$$\begin{aligned} c &= 0 \\ a &= d \\ 0 &= 0 \\ c &= 0 \end{aligned}$$

So, in order for matrices A and B to commute we must have

$$a = d, \;\; c = 0$$

Note that b can be any real number. We can write all possible values of a, b, c, d as parametric equations. Put $b = s$ and $d = t$. Then the set of possible values for a, b, c, d is given by parametric equations

$$a = t, \;\; b = s, \;\; c = 0, \;\; d = t$$

$$A = \begin{bmatrix} t & s \\ 0 & t \end{bmatrix}, \quad t, s \in \mathbb{R}$$

33. (a) Show that if a square matrix A satisfies the equation $A^2 + 2A + I = 0$, then A must be invertible. What is the inverse?

(b) Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.

a) If $A^2 + 2A + I = 0$ then

$$I = -A^2 - 2A$$
$$I = A(-A - 2I)$$

Hence, A is invertible and $A^{-1} = -A - 2I$

b)

$$p(A) = 0$$
$$a_n A^n + a_{n-1} A^{n-1} + \ldots + a_1 A + a_0 I = 0$$

Divide both sides by $a_0 \neq 0$:

$$\frac{a_n}{a_0} A^n + \frac{a_{n-1}}{a_0} A^{n-1} + \ldots + \frac{a_1}{a_0} A + I = 0$$
$$-\frac{a_n}{a_0} A^n - \frac{a_{n-1}}{a_0} A^{n-1} - \ldots - \frac{a_1}{a_0} A = I$$
$$A\left[-\frac{a_n}{a_0} A^{n-1} - \frac{a_{n-1}}{a_0} A^{n-2} - \ldots - \frac{a_1}{a_0}\right] = I$$

Thus A is invertible and:

$$A^{-1} = -\frac{a_n}{a_0} A^{n-1} - \frac{a_{n-1}}{a_0} A^{n-2} - \ldots - \frac{a_1}{a_0}$$

1.5 Elementary Matrices and a Method for Finding A^{-1}

▶ In Exercises 9–10, first use Theorem 1.4.5 and then use the inversion algorithm to find A^{-1}, if it exists. ◀

10. (a) $A = \begin{bmatrix} 1 & -5 \\ 3 & -16 \end{bmatrix}$ (b) $A = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

(a)

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 3 & -16 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_2 - 3R_1 \rightarrow R_2$ (multiply 1 row by 3 and subtract it from 2 row)

$$\left(\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right)$$

$R_2 / -1 \rightarrow R_2$ (divide the 2 row by -1)

$$\left(\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

$R_1 + 5R_2 \rightarrow R_1$ (multiply 2 row by 5 and add it to 1 row)

$$\left(\begin{array}{cc|cc} 1 & 0 & 16 & -5 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

Answer:

$$\mathbf{A}^{-1} = \begin{pmatrix} 16 & -5 \\ 3 & -1 \end{pmatrix}$$

(b)

Now finding inverse of the given matrix

$$\left[\begin{array}{cc|cc} 6 & 4 & 1 & 0 \\ -3 & -2 & 0 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 \div 6$

$$= \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{6} & 0 \\ -3 & -2 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 + 3 \times R_1$

$$= \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right]$$

The matrix is not invertible.

In Exercises 11–12, use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

12. (a) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

(a)

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_1 / \frac{1}{5} \rightarrow R_1$ (divide the 1 row by $\frac{1}{5}$)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right)$$

$R_2 - \frac{1}{5} R_1 \rightarrow R_2$ (multiply 1 row by $\frac{1}{5}$ and subtract it from 2 row); $R_3 - \frac{1}{5} R_1 \rightarrow R_3$ (multiply 1 row by $\frac{1}{5}$ and subtract it from 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right)$$

$R_2 \leftrightarrow R_3$ (interchange the 2 and 3 rows)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right)$$

$R_2 / -1 \rightarrow R_2$ (divide the 2 row by -1)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right)$$

$R_1 - 1\, R_2 \rightarrow R_1$ (multiply 2 row by 1 and subtract it from 1 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right)$$

$R_3 / \frac{1}{2} \rightarrow R_3$ (divide the 3 row by $\frac{1}{2}$)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right)$$

$R_1 + \frac{3}{2} R_3 \rightarrow R_1$ (multiply 3 row by $\frac{3}{2}$ and add it to 1 row); $R_2 + \frac{1}{2} R_3 \rightarrow R_2$ (multiply 3 row by $\frac{1}{2}$ and add it to 2 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right)$$

Answer:

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

(b)

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{21}{5} & -\frac{3}{5} & -\frac{3}{5} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_1 / \frac{1}{5} \to R_1$ (divide the 1 row by $\frac{1}{5}$)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ \frac{21}{5} & -\frac{3}{5} & -\frac{3}{5} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right)$$

$R_2 - \frac{21}{5} R_1 \to R_2$ (multiply 1 row by $\frac{21}{5}$ and subtract it from 2 row); $R_3 - \frac{1}{5} R_1 \to R_3$ (multiply 1 row by $\frac{1}{5}$ and subtract it from 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -\frac{24}{5} & \frac{39}{5} & -21 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right)$$

$R_2 / -\frac{24}{5} \to R_2$ (divide the 2 row by $-\frac{24}{5}$)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -\frac{13}{8} & \frac{35}{8} & -\frac{5}{24} & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right)$$

$R_1 - 1\,R_2 \to R_1$ (multiply 2 row by 1 and subtract it from 1 row); $R_3 + 1\,R_2 \to R_3$ (multiply 2 row by 1 and add it to 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{8} & \frac{5}{8} & \frac{5}{24} & 0 \\ 0 & 1 & -\frac{13}{8} & \frac{35}{8} & -\frac{5}{24} & 0 \\ 0 & 0 & -\frac{9}{8} & \frac{27}{8} & -\frac{5}{24} & 1 \end{array}\right)$$

$R_3 / -\frac{9}{8} \rightarrow R_3$ (divide the 3 row by $-\frac{9}{8}$)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{8} & \frac{5}{8} & \frac{5}{24} & 0 \\ 0 & 1 & -\frac{13}{8} & \frac{35}{8} & -\frac{5}{24} & 0 \\ 0 & 0 & 1 & -3 & \frac{5}{27} & -\frac{8}{9} \end{array}\right)$$

$R_1 + \frac{3}{8} R_3 \rightarrow R_1$ (multiply 3 row by $\frac{3}{8}$ and add it to 1 row); $R_2 + \frac{13}{8} R_3 \rightarrow R_2$ (multiply 3 row by $\frac{13}{8}$ and add it to 2 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{18} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{5}{54} & -\frac{13}{9} \\ 0 & 0 & 1 & -3 & \frac{5}{27} & -\frac{8}{9} \end{array}\right)$$

Answer:

$$\mathbf{A^{-1}} = \begin{pmatrix} -\frac{1}{2} & \frac{5}{18} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{5}{54} & -\frac{13}{9} \\ -3 & \frac{5}{27} & -\frac{8}{9} \end{pmatrix}$$

▶ In Exercises 13–18, use the inversion algorithm to find the inverse of the matrix (if the inverse exists). ◀

16. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$ **18.** $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

16-

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_2 - 1 R_1 \to R_2$ (multiply 1 row by 1 and subtract it from 2 row); $R_3 - 1 R_1 \to R_3$ (multiply 1 row by 1 and subtract it from 3 row); $R_4 - 1 R_1 \to R_4$ (multiply 1 row by 1 and subtract it from 4 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$R_2 / 3 \to R_2$ (divide the 2 row by 3)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$R_3 - 3 R_2 \to R_3$ (multiply 2 row by 3 and subtract it from 3 row); $R_4 - 3 R_2 \to R_4$ (multiply 2 row by 3 and subtract it from 4 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right)$$

$R_3 / 5 \to R_3$ (divide the 3 row by 5)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.2 & 0.2 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right)$$

$R_4 - 5R_3 \rightarrow R_4$ (multiply 3 row by 5 and subtract it from 4 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right)$$

$R_4 / 7 \rightarrow R_4$ (divide the 4 row by 7)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{array}\right)$$

Answer:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -0.2 & 0.2 & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

18-

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_1 \leftrightarrow R_2$ (interchange the 1 and 2 rows)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_4 - 2 R_1 \rightarrow R_4$ (multiply 1 row by 2 and subtract it from 4 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right)$$

$R_2 \leftrightarrow R_3$ (interchange the 2 and 3 rows)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right)$$

$R_2 / -1 \rightarrow R_2$ (divide the 2 row by -1)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right)$$

$R_4 - 1 R_2 \rightarrow R_4$ (multiply 2 row by 1 and subtract it from 4 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 8 & -5 & 0 & -2 & 1 & 1 \end{array}\right)$$

$R_2 + 3\,R_3 \rightarrow R_2$ (multiply 3 row by 3 and add it to 2 row); $R_4 - 8\,R_3 \rightarrow R_4$ (multiply 3 row by 8 and subtract it from 4 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1.5 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -4 & -2 & 1 & 1 \end{array}\right)$$

$R_4 / -5 \rightarrow R_4$ (divide the 4 row by -5)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1.5 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.8 & 0.4 & -0.2 & -0.2 \end{array}\right)$$

$R_1 - 1\,R_4 \rightarrow R_1$ (multiply 4 row by 1 and subtract it from 1 row)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -0.8 & 0.6 & 0.2 & 0.2 \\ 0 & 1 & 0 & 0 & 1.5 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.8 & 0.4 & -0.2 & -0.2 \end{array}\right)$$

Answer:

$$A^{-1} = \begin{pmatrix} -0.8 & 0.6 & 0.2 & 0.2 \\ 1.5 & 0 & -1 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0.8 & 0.4 & -0.2 & -0.2 \end{pmatrix}$$

▶ In Exercises 27–28, show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A, and then use that result to find a matrix C such that $CA = B$. ◀

28. $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

Applying row operations of matrix A to obtain B ,

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} (\mathrm{R_2} = \mathrm{R_2} - 2\mathrm{R_1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ -5 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; (\mathrm{R_3} - 2\mathrm{R_1})$$

$$\begin{bmatrix} 2 & 1 & 0 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \; (\mathrm{R_1} = \mathrm{R_1} - 4\mathrm{R_3})$$

$$\begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means that the matrix B can be written as:

$$\begin{aligned} \mathrm{B} &= \mathrm{E_3}.\,(\mathrm{E_2}.\,(\mathrm{E_1}.\,\mathrm{A})) \\ &= \mathrm{E_3}.\,\mathrm{E_2}.\,\mathrm{E_1}.\,\mathrm{A} \end{aligned}$$

Now,

$$\mathrm{CA} = \mathrm{B}$$

$$\begin{aligned} \mathrm{C} &= \mathrm{E_3}.\,\mathrm{E_2}.\,\mathrm{E_1} \\ &= \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 0 & -4 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 0 & -4 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \end{aligned}$$

1.6 More on Linear Systems and Invertible Matrices

► In Exercises 1–8, solve the system by inverting the coefficient matrix and using Theorem 1.6.2. ◄

3.
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

3) Given :-
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

Here $AX = B$

where $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$

$\because AX = B$

$\Rightarrow$ Pre-multiply both sides by A^{-1}

$\Rightarrow A^{-1}AX = A^{-1}B$

$\Rightarrow \boxed{X = A^{-1}B}$ $\quad (\because A^{-1}A = AA^{-1} = I)$

Now, $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

a_{ij} = element of i^{th} row and j^{th} column of A.

C_{ij} = co-factor of element a_{ij}

$= (-1)^{i+j}$ [Determinant of sub-matrix left on deleting i^{th} row & j^{th} column]

$$a_{11} = 1 \Rightarrow C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

$$a_{12} = 3 \Rightarrow C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

$$a_{13} = 1 \Rightarrow C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2$$

$$a_{21} = 2 \Rightarrow C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0$$

$$a_{22} = 2 \Rightarrow C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$a_{23} = 1 \Rightarrow C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3$$

$$a_{31} = 2 \Rightarrow C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$a_{32} = 3 \Rightarrow C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$a_{33} = 1 \Rightarrow C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -4$$

$\therefore$ Matrix of Co-factors $= \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 1 & -4 \end{bmatrix}$

Transpose of co-factor matrix $= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$

And $\det.(A) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

$= 1(2-3) - 3(2-2) + 1(6-4)$

$= -1 - 0 + 2$

$= \boxed{1}$

$\therefore$ $A^{-1} = \dfrac{\text{Cofactor matrix transpose}}{\det.(A)}$

$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$

$\therefore$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$

$\therefore$ $x_1 = -1$, $x_2 = 4$, $x_3 = -7$ (ans)

In Exercises 13–17, determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

15.
$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

Convert to matrix form:

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{bmatrix}$$

Now we need to achieve row echelon form by performing row operations:

- Row operation $-4R_1 + R_2$ and $3R_1 + R_3$

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{bmatrix}$$

- Row operation $R_2 + R_3$

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{bmatrix}$$

In order for this matrix system to be consistent, we must have a solution. This only occurs if we do **not** have any rows that are like this:

$$\begin{bmatrix} 0 & 0 & 0 & \text{non-zero number} \end{bmatrix}$$

because $0x_1 + 0x_2 + 0x_3 \neq$ non-zero number for any x_1, x_2, x_3.

Therefore, from row 3 of the row echelon matrix above, we *must* have

$$-b_1 + b_2 + b_3 = 0.$$

Thus, the values of b_1, b_2, b_3 must satisfy the condition of

$$b_1 = b_2 + b_3$$

16.

$$\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$$

16) given the system of equations are

$x_1 - 2x_2 - x_3 = b_1$

$-4x_1 + 5x_2 + 2x_3 = b_2$

$-4x_1 + 7x_2 + 4x_3 = b_3$

The Argumented matrix form $\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right]$

Apply row reduction enchelon form

$R_2 = R_2 + 4R_1 \quad \text{and} \quad R_3 = R_3 + 4R_1$

$$\approx \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & b_2 + 4b_1 \\ 0 & -1 & 0 & b_3 + 4b_1 \end{array}\right]$$

$R_3 = 3R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -3 & b_2 + 4b_1 \\ 0 & 0 & 2 & 3b_3 + 8b_1 - b_2 \end{array}\right]$$

In the above matrix, the system is consistent. Hence the system has a solution for all values of b_1, b_2, and b_3.

1.7 Diagonal, Triangular, and Symmetric Matrices

▶ In Exercises 7–10, find A^2, A^{-2}, and A^{-k} (where k is any integer) by inspection. ◀

9. $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

$$\mathrm{A} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\mathrm{A}^2 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

$$\mathrm{A}^{-2} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\mathrm{A}^{-\mathrm{k}} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-k} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-k} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-k} \end{bmatrix}$$

In Exercises 13–14, compute the indicated quantity.

13. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39}$

Consider the matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39}$$

Apply the diagonal property on the given matrix and rewrite the matrix as,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39} = \begin{bmatrix} 1^{39} & 0 \\ 0 & (-1)^{39} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Explanation:

Property of diagonal matrix,

If $\mathrm{A} = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$ is a diagonal matrix, then A^{n} ,

$\begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}^{\mathrm{n}} = \begin{bmatrix} d_{11}^n & 0 \\ 0 & d_{22}^n \end{bmatrix}$, where n is any integer.

Exponential property,

$(-1)^{\mathrm{odd}\nu\mathrm{mber}} = -1$

CHAPTER 1: Systems of Linear Equations and Matrices

Useful links:

https://atozmath.com/CONM/GaussEli.aspx?q=GEBS2&q1=2%602x-y-3z%3d0%3b-x%2b2y-3z%3d0%3bx%2by%2b4z%3d0%60GEBS2%60%601.25%60false&dm=D&dp=4&do=1#PrevPart

https://onlinemschool.com/math/assistance/equation/gaus/

https://atozmath.com/LinearEqn_HK.aspx?q=1&m=US

2.1 Determinants by Cofactor Expansion

► In Exercises 5–8, evaluate the determinant of the given matrix. If the matrix is invertible, use Equation (2) to find its inverse. ◄

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$ **8.** $\begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$

Let,

5. $A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$, $|A| = \begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 3\times 4 - (-2)\times 5 = 12 + 10 = 22$

As $|A| \neq 0$, then matrix A is invertiable.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{11} & -\frac{5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$$

note: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2x2 matrix, then

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

6. Let, $B = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$, $|B| = \begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = 4\times 2 - 8\times 1 = 0$

As $\det(B) = 0$, then B is not invertible, therefore B^{-1} is not exist.

7. Let $C = \begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$, $|C| = \begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = (-5)\times(-2) - (-7)\times(7)$

$= 10 + 49$

$= 59 \neq 0$

As ~~det~~ $\det(C) \neq 0$, then C is invertible.

$$C^{-1} = \frac{1}{|C|}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \frac{1}{59}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{59} & \frac{-7}{59} \\ \frac{7}{59} & -\frac{5}{59} \end{bmatrix}$$

8. Let $D = \begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$, then $|D| = \begin{vmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{vmatrix}$

$= \sqrt{2}\times\sqrt{3} - 4\times\sqrt{6}$

$= \sqrt{6} - 4\sqrt{6}$

$= -3\sqrt{6} \neq 0$

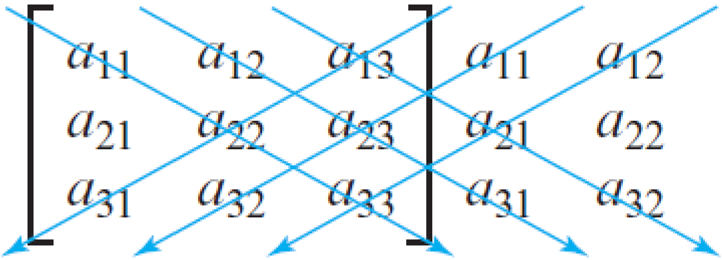
As $\det(D) \neq 0$, then D is invertible,

$$D^{-1} = \frac{1}{|D|}\begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ -4 & \sqrt{2} \end{bmatrix} = \frac{1}{3\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ -4 & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3\sqrt{2}} & -\frac{1}{3} \\ -\frac{4}{3\sqrt{6}} & \frac{1}{3\sqrt{3}} \end{bmatrix}$$

▶ In Exercises 9–14, use the arrow technique to evaluate the determinant. ◀

$$\textbf{11.}\quad \begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$$



(11).

Consider the following determinant:

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}\begin{bmatrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{bmatrix} = (-2)(5)(2) + (1)(-7)(1) + (4)(3)(6) - (4)(5)(1) - (-2)(-7)(6) - (1)(3)(2)$$

$$-20 - 7 + 72 - 20 - 84 - 6$$

$$= -65$$

In Exercises 15–18, find all values of λ for which $\det(A) = 0$.

15. $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$

Calculate the determinant of matrix A which is

$det(A) = (\lambda - 2)(\lambda + 4) - (-5)(1) = \lambda^2 + 2\lambda - 8 + 5 = \lambda^2 + 2\lambda - 3$

Set the determinant equal to zero to find the values of λ that make the determinant zero: $\lambda^2 + 2\lambda - 3 = 0$

Factor the quadratic equation to find the values of λ: $(\lambda + 3)(\lambda - 1) = 0$

Solve for λ to get $\lambda = -3$ and $\lambda = 1$.
Hence, the values of λ for which $det(A) = 0$ in Exercise 15 are $\lambda = -3$ and $\lambda = 1$.

In Exercises 21–26, evaluate det(A) by a cofactor expansion along a row or column of your choice.

21. $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$	**25.** $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of*** A, and the sums themselves are called ***cofactor expansions of*** A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (7)$$

[cofactor expansion along the jth column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (8)$$

[cofactor expansion along the ith row]

21-

Solution:

$$|A| = \begin{vmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{vmatrix}$$

$$= -3 \times \begin{vmatrix} 5 & 1 \\ 0 & 5 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} + 7 \times \begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix}$$

$$= -3 \times (5 \times 5 - 1 \times 0) + 0 \times (2 \times 5 - 1 \times (-1)) + 7 \times (2 \times 0 - 5 \times (-1))$$

$$= -3 \times (25 + 0) + 0 \times (10 + 1) + 7 \times (0 + 5)$$

$$= -3 \times (25) + 0 \times (11) + 7 \times (5)$$

$$= -75 + 0 + 35$$

$$= -40$$

Note: Choosing the second column is faster in calculations

25-

Solution:

$$|A| = \begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix}$$

$$= 3 \times \begin{vmatrix} 2 & 0 & -2 \\ 1 & -3 & 0 \\ 10 & 3 & 2 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & 0 & -2 \\ 4 & -3 & 0 \\ 2 & 3 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 2 & -2 \\ 4 & 1 & 0 \\ 2 & 10 & 2 \end{vmatrix} - 5 \times \begin{vmatrix} 2 & 2 & 0 \\ 4 & 1 & -3 \\ 2 & 10 & 3 \end{vmatrix}$$

$$= 3 \times (-78) - 3 \times (-48) + 0 - 5 \times (30)$$

$$= -234 + 144 + 0 - 150$$

$$= -240$$

Note: Choosing the third column is faster in calculations

2.2 Evaluating Determinants by Row Reduction

▶ In Exercises 9–14, evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion. ◀

14. $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

$R_2 \to R_2 - 5R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{bmatrix}$$

$R_4 \to R_4 - 12R_2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{bmatrix}$$

$R_4 \to R_4 + 36R_3$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{bmatrix}$$

Expand along Column-1 :

$$(1)(-1)^{1+1}\begin{vmatrix} 1 & -9 & -2 \\ 0 & -3 & -1 \\ 0 & 0 & -13 \end{vmatrix} + (0)(-1)^{2+1}\begin{vmatrix} -2 & 3 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -13 \end{vmatrix} +$$

$$(0)\,(-1)^{3+1}\begin{vmatrix} -2 & 3 & 1 \\ 1 & -9 & -2 \\ 0 & 0 & -13 \end{vmatrix} + (0)\,(-1)^{4+1}\begin{vmatrix} -2 & 3 & 1 \\ 1 & -9 & -2 \\ 0 & -3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -9 & -2 \\ 0 & -3 & -1 \\ 0 & 0 & -13 \end{vmatrix} + 0 + 0 + 0$$

Expand along column 1:

$$= (1)\,(-1)^{1+1}\begin{vmatrix} -3 & -1 \\ 0 & -13 \end{vmatrix} + (0)\,(-1)^{2+1}\begin{vmatrix} -9 & -2 \\ 0 & -13 \end{vmatrix} + (0)\,(-1)^{3+1}\begin{vmatrix} -9 & -2 \\ -3 & -1 \end{vmatrix}$$

$$= (-3)\times(-13) + 0\times(-1) + 0 + 0$$

$$\left(\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\right)$$

$$= \boxed{39}$$

Hence, determinant of Given matrix is $\boxed{39}$ Answer

In Exercises 15–22, evaluate the determinant, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

20. $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$

22. $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$

20-

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} \xrightarrow{-3R_{13}} \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} \xrightarrow{\frac{1}{2}R_2}$$

$$2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(-6) = -12$$

(22) $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 2 \times 0 = \boxed{0}$ Answer

► In Exercises 25–28, confirm the identities without evaluating the determinants directly. ◄

28. $$\begin{vmatrix} a_1 & b_1 + ta_1 & c_1 + rb_1 + sa_1 \\ a_2 & b_2 + ta_2 & c_2 + rb_2 + sa_2 \\ a_3 & b_3 + ta_3 & c_3 + rb_3 + sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

For question 28, the given matrices are equal without directly evaluating the determinants because the second matrix is obtained by adding multiples of the first column to the second and third columns

▶ In Exercises 29–30, show that $\det(A) = 0$ without directly evaluating the determinant. ◀

30. $A = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$

Proceed with row operations.

Explanation:

Add the first row to each of the subsequent rows i.e. apply a row operation $R_1 \to R_1 + R_2 + R_3 + R_4 + R_5$.

$$\det A = \det \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$$

$$= 0$$

2.3 Properties of Determinants; Cramer's Rule

▶ In Exercises 15–18, find the values of k for which the matrix A is invertible. ◀

18. $A = \begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$

Expanding matrix A through row 2 :

$$\begin{aligned}
|\mathrm{A}| &= \begin{vmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{vmatrix} \\
&= (-\mathrm{k})\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + (-\mathrm{k})\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \\
&= (-\mathrm{k})(2-0) + 1(1-0) + (-\mathrm{k})(2-0) \\
&= (-\mathrm{k})(2) + 1(1) + (-\mathrm{k})(2) \\
&= -2\mathrm{k} + 1 - 2\mathrm{k} \\
&= -4\mathrm{k} + 1
\end{aligned}$$

Now A will be invertible if $|\mathrm{A}| \neq 0$

Now we will find the values of k at which the matrix will not be invertible.

$\Rightarrow |\mathrm{A}| = 0$

$\Rightarrow -4\mathrm{k} + 1 = 0$

$\Rightarrow -4\mathrm{k} = -1$

$\Rightarrow \mathrm{k} = \frac{-1}{-4} = \frac{1}{4}$

So at $\mathrm{k} = \frac{1}{4}$, matrix A will not be invertible.

In Exercises 19–23, decide whether the matrix is invertible, and if so, use the adjoint method to find its inverse.

20. $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} + 3 \times \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix}$$

$$= 2 \times (3 \times (-4) - 2 \times 0) + 0 \times (0 \times (-4) - 2 \times (-2)) + 3 \times (0 \times 0 - 3 \times (-2))$$

$$= 2 \times (-12 + 0) + 0 \times (0 + 4) + 3 \times (0 + 6)$$

$$= 2 \times (-12) + 0 \times (4) + 3 \times (6)$$

$$= -24 + 0 + 18$$

$$= -6$$

$$Adj(A) = \text{Adj} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 0 & -4 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(3 \times (-4) - 2 \times 0) & -(0 \times (-4) - 2 \times (-2)) & +(0 \times 0 - 3 \times (-2)) \\ -(0 \times (-4) - 3 \times 0) & +(2 \times (-4) - 3 \times (-2)) & -(2 \times 0 - 0 \times (-2)) \\ +(0 \times 2 - 3 \times 3) & -(2 \times 2 - 3 \times 0) & +(2 \times 3 - 0 \times 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-12+0) & -(0+4) & +(0+6) \\ -(0+0) & +(-8+6) & -(0+0) \\ +(0-9) & -(4+0) & +(6+0) \end{bmatrix}^T$$

$$= \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}^T$$

$$= \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} \times Adj(A)$

$$= \frac{1}{-6} \times \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1.5 \\ 0.6667 & 0.3333 & 0.6667 \\ -1 & 0 & -1 \end{bmatrix}$$

We can find $|A|$ from matrix C directly by the following Def:

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (7)$$

[cofactor expansion along the *j*th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (8)$$

[cofactor expansion along the *i*th row]

In Exercises 24–29, solve by Cramer's rule, where it applies.

26.
$$\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$

$$\Delta = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = -55$$

Show detailed calculation of the determinant

$$\Delta_1 = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 144$$

Show detailed calculation of the determinant

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 61$$

Show detailed calculation of the determinant

$$\Delta_3 = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = -230$$

Show detailed calculation of the determinant

$$x = \frac{\Delta_1}{\Delta} = \frac{144}{-55} = -\frac{144}{55}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{61}{-55} = -\frac{61}{55}$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-230}{-55} = \frac{46}{11}$$

More details:

Use Cramer's Rule to find the values of x, y, z.

$$\frac{x}{D_x} = \frac{-y}{D_y} = \frac{z}{D_z} = \frac{-1}{D}$$

$$D_x = \begin{vmatrix} -4 & 1 & -6 \\ -1 & 2 & 1 \\ 2 & -3 & 20 \end{vmatrix}$$

$$= -4 \times \begin{vmatrix} 2 & 1 \\ -3 & 20 \end{vmatrix} -1 \times \begin{vmatrix} -1 & 1 \\ 2 & 20 \end{vmatrix} -6 \times \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix}$$

$= -4 \times (2 \times 20 - 1 \times (-3)) - 1 \times (-1 \times 20 - 1 \times 2) - 6 \times (-1 \times (-3) - 2 \times 2)$

$= -4 \times (40 + 3) - 1 \times (-20 - 2) - 6 \times (3 - 4)$

$= -4 \times (43) - 1 \times (-22) - 6 \times (-1)$

$= -172 + 22 + 6$

$= -144$

$$D_y = \begin{vmatrix} 1 & 1 & -6 \\ 4 & 2 & 1 \\ 2 & -3 & 20 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} 2 & 1 \\ -3 & 20 \end{vmatrix} -1 \times \begin{vmatrix} 4 & 1 \\ 2 & 20 \end{vmatrix} -6 \times \begin{vmatrix} 4 & 2 \\ 2 & -3 \end{vmatrix}$$

$= 1 \times (2 \times 20 - 1 \times (-3)) - 1 \times (4 \times 20 - 1 \times 2) - 6 \times (4 \times (-3) - 2 \times 2)$

$= 1 \times (40 + 3) - 1 \times (80 - 2) - 6 \times (-12 - 4)$

$= 1 \times (43) - 1 \times (78) - 6 \times (-16)$

$= 43 - 78 + 96$

$= 61$

$$D_z = \begin{vmatrix} 1 & -4 & -6 \\ 4 & -1 & 1 \\ 2 & 2 & 20 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} -1 & 1 \\ 2 & 20 \end{vmatrix} + 4 \times \begin{vmatrix} 4 & 1 \\ 2 & 20 \end{vmatrix} - 6 \times \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= 1 \times (-1 \times 20 - 1 \times 2) + 4 \times (4 \times 20 - 1 \times 2) - 6 \times (4 \times 2 - (-1) \times 2)$$

$$= 1 \times (-20 - 2) + 4 \times (80 - 2) - 6 \times (8 + 2)$$

$$= 1 \times (-22) + 4 \times (78) - 6 \times (10)$$

$$= -22 + 312 - 60$$

$$= 230$$

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} + 4 \times \begin{vmatrix} 4 & 2 \\ 2 & -3 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= 1 \times (-1 \times (-3) - 2 \times 2) + 4 \times (4 \times (-3) - 2 \times 2) + 1 \times (4 \times 2 - (-1) \times 2)$$

$$= 1 \times (3 - 4) + 4 \times (-12 - 4) + 1 \times (8 + 2)$$

$$= 1 \times (-1) + 4 \times (-16) + 1 \times (10)$$

$$= -1 - 64 + 10$$

$$= -55$$

$$\frac{x}{D_x} = \frac{-y}{D_y} = \frac{z}{D_z} = \frac{-1}{D}$$

$$\therefore \frac{x}{-144} = \frac{-y}{61} = \frac{z}{230} = \frac{-1}{-55}$$

$$\therefore \frac{x}{-144} = \frac{-1}{-55}, \frac{-y}{61} = \frac{-1}{-55}, \frac{z}{230} = \frac{-1}{-55}$$

$$\therefore x = \frac{144}{-55}, y = \frac{61}{-55}, z = \frac{-230}{-55}$$

CHAPTER 2: Determinants

Useful links:

https://onlinemschool.com/math/assistance/matrix/determinant/

https://atozmath.com/matrix.aspx?q=det

CHAPTER 4: General Vector Spaces

4.1 Real Vector Spaces

In Exercises 3–12, determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail.

7. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

Given definition of scalar multiplication as

$k(x, y, z) = (k^2x, k^2y, k^2z)$

Lets verify distributive axiom : We know In a vector space , if k_1 and k_2 are two scalars and $\mathbf{u}$ is an element in vector space then $(k_1 + k_2)u = k_1u + k_2u.$

Here

$$\begin{aligned}(k_1 + k_2)u &= (k_1 + k_2)(x, y, z) \\ &= \left((k_1 + k_2)^2x, (k_1 + k_2)^2y, (k_1 + k_2)^2z\right) \\ &= ((k_1^2 + k_2^2 + 2k_1k_2)x, (k_1^2 + k_2^2 + 2k_1k_2)y, (k_1^2 + k_2^2 + 2k_1k_2)z) \\ &= ((k_1^2x + k_2^2x + 2k_1k_2x), (k_1^2y + k_2^2y + 2k_1k_2y), (k_1^2z + k_2^2z + 2k_1k_2z)) \\ &= (k_1^2x, k_1^2y, k_1^2z) + (k_2^2x, k_2^2y, k_2^2z) + (2k_1k_2x, 2k_1k_2y, 2k_1k_2z) \\ &= k_1(x, y, z) + k_2(x, y, z) + (2k_1k_2x, 2k_1k_2y, 2k_1k_2z) \\ &= k_1u + k_2u + (2k_1k_2x, 2k_1k_2y, 2k_1k_2z)\end{aligned}$$

$$(k_1 + k_2)u \neq k_1u + k_2u$$

Since, it does not follow the distributive property, hence given definition is not a vector space.

11. The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

Let $u = (1, y), v = (1, y'), w = (1, y'')$ be in V and k is scalar.

Axiom 1.

Let $u = (1, y), v = (1, y')$ be in V then

$u + v = (1, y) + (1, y')$ according to (1)

$= (1, y + y')$

If u and v are in V, then $u + v$ is in V. V is closed under addition.

Axiom 2.

Check $u + v = v + u$

$u + v = (1, y) + (1, y')$ according to (1)

$= (1, y + y')$

$= (1, y' + y)$

$= (1, y') + (1, y)$

$= v + u$

Axiom 2 is satisfied.

Axiom 3. Associativity

if $u, v, w \in V$ then

$u + (v + w) = (1, y) + ((1, y') + (1, y''))$

$= (1, y) + (1, y' + y'')$

$= (1, y + y' + y'')$

$= (1, (y + y') + y'')$

$= (1, y + y') + (1, y'')$

$= ((1, y) + (1, y')) + (1, y'')$

$= (u + v) + w$

Associativity is satisfied.

Axiom 4.

Let $0 = (1, 0)$ be zero vector in V ,then

$u + 0 = (1, y) + (1, 0)$

$= (1, y)$

$= u$

Axiom 4 satisfied.

Axiom 5.

Let $u = (1, y)$ and $v = (1, y')$ be in V such that $u + v = 0$

$u + v = (1, y) + (1, y')$

$\Rightarrow (1, 0) = (1, y + y')$

$\Rightarrow y + y' = 0$

$\Rightarrow y = -y'$

So, $v = (1, -y')$ is the inverse of $u \in V$

Axiom 6.

Let k be scalar in $\mathbb{R}$, then ku also be in $\mathbb{R}$.

$ku = k(1, y)$

$= (1, ky)$ according to (1)

If $u \in V$ and k is scalar then $ku \in V$

Axiom 7:

Let k is scalar, show that $k(u + v) = ku + kv$

$k(u + v) = k((1, y) + (1, y'))$ according to (1)

$= k(1, y + y')$

$= (1, k(y + y'))$

$= (1, ky + ky')$

$= (1, ky) + (1, ky')$ according to (2)

$= k(1, y) + k(1, y')$

$= ku + kv$

Axiom 7 is satisfied.

Axiom 8:

Let k,m are scalars, show that $(k + m)u = ku + km$

$(k + m)u = (k + m)(1, y)$ according to (2)

$= (1, (k + m)y)$

$= (1, ky + my)$

$= (1, ky) + (1, my)$

$= k(1, y) + m(1, y)$

$= ku + mu$

Axiom 8 is satisfied.

Axiom9:

Let k,m be scalars, show that $\mathbf{k(mu) = (km)u}$

$$\begin{aligned}\mathbf{k(mu)} &= \mathbf{k(m(1,y))} \\ &= \mathbf{k(1,my)} \quad \text{according to (2)} \\ &= \mathbf{(1,kmy)} \\ &= \mathbf{(km)(1,y)} \\ &= \mathbf{(km)u}\end{aligned}$$

Axiom 9 is satisfied.

Axiom 10:

To show that $\mathbf{1u = u}$ then

$$\begin{aligned}\mathbf{1u} &= \mathbf{1(1,y)} \\ &= \mathbf{(1,1y)} \\ &= \mathbf{(1,y)} \\ &= \mathbf{u}\end{aligned}$$

Axiom 10 is satisfied.

Therefore, V is the vector space.

CHAPTER 4: General Vector Spaces

4.2 Subspaces

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3.

(a) All vectors of the form $(a, 0, 0)$.

(b) All vectors of the form $(a, 1, 1)$.

(c) All vectors of the form (a, b, c), where $b = a + c$.

(d) All vectors of the form (a, b, c), where $b = a + c + 1$.

(e) All vectors of the form $(a, b, 0)$.

Let's go through each part to determine whether it satisfies the conditions of Theorem 4.2.1:

(a) All vectors of the form (*a*,0,0).

To check if this set is a subspace, we need to verify conditions (a) and (b) of the theorem.

- (a) If $\mathbf{u} = (a_1, 0, 0)$ and $\mathbf{v} = (a_2, 0, 0)$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2, 0, 0)$. This vector is still of the form (*a*,0,0), so (a) holds.
- (b) If *k* is any scalar and $\mathbf{u} = (a, 0, 0)$, then $k\mathbf{u} = (ka, 0, 0)$. This vector is still of the form (*a*,0,0), so (b) holds.

Therefore, the set of all vectors of the form (*a*,0,0) is a subspace of *R*3.

(b) All vectors of the form (*a*,1,1).

- (a) If $\mathbf{u} = (a_1, 1, 1)$ and $\mathbf{v} = (a_2, 1, 1)$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2, 2, 2)$. This vector is not necessarily of the form (*a*,1,1), so (a) does not hold.

Therefore, the set of all vectors of the form (*a*,1,1) is not a subspace of $\mathbf{R}^3$.

(c) All vectors of the form (*a*,*b*,*c*), where $b = a + c$.

- (a) If $\mathbf{u} = (a_1, b_1, c_1)$ and $\mathbf{v} = (a_2, b_2, c_2)$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$. Since $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$, we have $b_1 + b_2 = (a_1 + a_2) + (c_1 + c_2)$, and this satisfies the condition $b = a + c$. Therefore, (a) holds.
- (b) If *k* is any scalar and $\mathbf{u} = (a, b, c)$, then $k\mathbf{u} = (ka, kb, kc)$. Since $b = a + c$, we have $kb = ka + kc$, and this satisfies the condition $b = a + c$. Therefore, (b) holds.

Therefore, the set of all vectors of the form (a, b, c), where $b = a + c$, is a subspace of $\mathbf{R}^3$.

(d) All vectors of the form (*a*,*b*,*c*), where $b = a + c + 1$.

- (a) If $\mathbf{u} = (a_1, b_1, c_1)$ and $\mathbf{v} = (a_2, b_2, c_2)$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$. Since $b_1 = a_1 + c_1 + 1$ and $b_2 = a_2 + c_2 + 1$, we have $b_1 + b_2 = (a_1 + a_2) + (c_1 + c_2) + 2$, and this does not satisfy the condition *b*=*a*+*c*+1. Therefore, (a) does not hold.

Therefore, the set of all vectors of the form (*a*,*b*,*c*), where $b = a + c + 1$, is not a subspace of $\mathbf{R}^3$.

Another solution:

a) Let S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$
Then S is a non-empty subset of $\mathbb{R}^3$, since $(0, 0, 0) \in S$
Let $u = (x_1, 0, 0),\ v = (x_2, 0, 0) \in S$; Then x_1, x_2 are real.
Let $a, b \in \mathbb{R}$. Then $au + bv = a(x_1, 0, 0) + b(x_2, 0, 0) = (ax_1 + bx_2, 0, 0) \in S$, since $ax_1 + bx_2 \in \mathbb{R}$.
This proves that S is a subspace of $\mathbb{R}^3$

Step 3

b)
Let S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 1\}$
Then S is a non-empty subset of $\mathbb{R}^3$, since $(1, 1, 1) \in S$. But S does not contain the null vector $(0, 0, 0)$.
S is not a subspace of $\mathbb{R}^3$, since every subspace W of a vector space V must contain the null vector θ of V.

Step 4

c)
Let S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = x + z\}$
Then S is a non-empty subset of $\mathbb{R}^3$, since $(0, 0, 0) \in S$
Let $u = (x_1, y_1, z_1),\ v = (x_2, y_2, z_2) \in S$; Then x_i, y_i, z_i are real and $y_1 = x_1 + z_1, y_2 = x_2 + z_2$
Let $a, b \in \mathbb{R}$. Then $au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \in S$, [
$ax_1 + bx_2 + az_1 + bz_2 = a(x_1 + z_1) + b(x_2 + z_2) = ay_1 + by_2$]
This proves that S is a subspace of $\mathbb{R}^3$

Step 5

d)
Let S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = x + z + 1\}$
Then S is a non-empty subset of $\mathbb{R}^3$, since $(1, 3, 1) \in S$. But S does not contain the null vector $(0, 0, 0)$.
S is not a subspace of $\mathbb{R}^3$, since every subspace W of a vector space V must contain the null vector θ of V.

3. Use Theorem 4.2.1 to determine which of the following are subspaces of P_3.

(a) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

(b) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

(c) All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

(d) All polynomials of the form $a_0 + a_1x$, where a_0 and a_1 are real numbers.

Let **V be** a vector space over a field **F** and suppose **T** is a subset of vectors contained in **V.** Then, T is a subspace of V if it satisfies the following properties:-

a.)T contains the zero vector i.e.$0 \in \mathrm{T}$
b.)if $\mathrm{a, b} \in \mathrm{F} \Rightarrow \mathrm{au} + \mathrm{bv} \in \mathrm{T}$, forall $\mathrm{u, v} \in \mathrm{T}$.

a.)Let $\mathrm{a, b} \in \mathrm{R}$(we suppose the field is R here).Let T be subset here,clearly,$0 \in \mathrm{T}$.
$\mathrm{Let u} = a_0 + a_1x + a_2x^2 + a_3x^3, v = b_0 + b_1x + b_2x^2 + b_3x^3 \in \mathrm{T}$.
$\Rightarrow a_0 = b_0 = 0 \ldots (1)$
$\Rightarrow au + bv = a(a_0 + a_1x + a_2x^2 + a_3x^3) + b(b_0 + b_1x + b_2x^2 + b_3x^3)$
$\Rightarrow au + bv = aa_0 + bb_0 + a(a_1x + a_2x^2 + a_3x^3) + b(b_1x + b_2x^2 + b_3x^3)$
$\Rightarrow aa_0 + bb_0 = 0$ from(1)
$\Rightarrow au + bv = 0 + aa_1x + aa_2x^2 + aa_3x^3 + bb_1x + bb_2x^2 + bb_3x^3)$
$\Rightarrow au + bv = 0 + (aa_1 + bb_1)x + (aa_2 + bb_2)x^2 + (aa_3 + bb_3)x^3 \in \mathrm{T}$
So, T is a subspace here.

Step 2

b.)Let $\mathrm{a, b} \in \mathrm{R}$(we suppose the field is R here).Let T be subset here,clearly,$0 \in \mathrm{T}$.
$\mathrm{Let u} = a_0 + a_1x + a_2x^2 + a_3x^3, v = b_0 + b_1x + b_2x^2 + b_3x^3 \in \mathrm{T}$.
$\Rightarrow a_0 + a_1 + a_2 + a_3 = b_0 + b_1 + b_2 + b_3 = 0$
$\Rightarrow a(a_0 + a_1 + a_2 + a_3) + b(b_0 + b_1 + b_2 + b_3) = 0$
$\Rightarrow aa_0 + bb_0 + aa_1 + bb_1 + aa_2 + bb_2 + aa_3 + bb_3 = 0 \ldots\ldots (2)$
$\Rightarrow au + bv = a(a_0 + a_1x + a_2x^2 + a_3x^3) + b(b_0 + b_1x + b_2x^2 + b_3x^3)$
$\Rightarrow au + bv = (aa_0 + bb_0) + (aa_1 + bb_1)x + (aa_2 + bb_2)x^2 + (aa_3 + bb_3)x^3$
$\Rightarrow au + bv \in \mathrm{T}$. from(2)

So, T is a subspace here.

c.)Let $\mathrm{a, b} \in \mathrm{R}$(we suppose the field is R here).Let T be subset here,clearly,$\mathbf{0} \in \mathrm{T}$.
we have, $\sqrt{2}, 1 \in \mathrm{R}$ and $\mathrm{x}, 0 \in \mathrm{T}$, BUT $\sqrt{2}\mathrm{x} + 1.0 = \sqrt{2}\mathrm{x} \notin \mathrm{T}$,since $\sqrt{2}$ is not a rational number.

So, T is not a subspace here.

d.)Let $\mathrm{a, b} \in \mathrm{R}$(we suppose the field is R here).Let T be subset here,clearly,$\mathbf{0} \in \mathrm{T}.\mathrm{Let} \mathrm{u} = \mathrm{a}_0 + \mathrm{a}_1\mathrm{x}, \mathrm{v} = \mathrm{b}_0 + \mathrm{b}_1\mathrm{x} \in \mathrm{T}$.
$\Rightarrow \mathrm{a}_0, \mathrm{a}_1, \mathrm{b}_0, \mathrm{b}_1 \in \mathrm{R}$
$\Rightarrow (\mathrm{aa}_0 + \mathrm{bb}_0), (\mathrm{aa}_1 + \mathrm{bb}_1) \in \mathrm{R} \ldots\ldots (3).$
$\Rightarrow \mathrm{au} + \mathrm{bv} = \mathrm{a}(\mathrm{a}_0 + \mathrm{a}_1\mathrm{x}) + \mathrm{b}(\mathrm{b}_0 + \mathrm{b}_1\mathrm{x})$
$\Rightarrow \mathrm{au} + \mathrm{bv} = (\mathrm{aa}_0 + \mathrm{bb}_0) + (\mathrm{aa}_1 + \mathrm{bb}_1)\mathrm{x}$
$\Rightarrow \mathrm{au} + \mathrm{bv} \in \mathrm{T}$. from(3)

So, T is a subspace here.

7. Which of the following are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

(a) $(2, 2, 2)$ (b) $(0, 4, 5)$ (c) $(0, 0, 0)$

(a) Suppose that the given vector $w = (2, 2, 2)$.
Then, the linear combination should be written as,
$(2, 2, 2) = a(0, -2, 2) + b(1, 3, -1)$

Explanation:

If two vectors are linear combinations of other then it can be written as $\vec{w} = a\vec{u} + b\vec{v}$

Step 2

On comparing both sides the system of equation should be,
$0 + b = 2$ ...(1)
$-2a + 3b = 2$ ...(2)
$2a - b = 2$ ...(3)
The solution for the system of equations should be $a = 2$ and $b = 2$.
Thus, $(2, 2, 2)$ should be written as a linear combination of the given vectors.
(b) Suppose that the given vector $w = (0, 4, 5)$.
Then, the linear combination should be written as,
$(0, 4, 5) = a(0, -2, 2) + b(1, 3, -1)$
On comparing both sides the system of equation should be,
$0 + b = 0$ ...(1)
$-2a + 3b = 4$ ...(2)
$2a - b = 5$ ...(3)
There will no unique solution for the system of equations.
Thus, $(0, 4, 5)$ should not be written as a linear combination of the given vectors.
(c) Suppose that the given vector $w = (0, 0, 0)$.
Then, the linear combination should be written as,
$(0, 0, 0) = a(0, -2, 2) + b(1, 3, -1)$
On comparing both sides the system of equation should be,
$0 + b = 0$ ...(1)
$-2a + 3b = 0$ ...(2)
$2a - b = 0$ ...(3)
The solution for the system of equations should be $a = 0$ and $b = 0$.
Thus, $(0, 0, 0)$ should be written as a linear combination of the given vectors.

8. Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

(a) $(-9, -7, -15)$ (b) $(6, 11, 6)$ (c) $(0, 0, 0)$

a)Let, $au + bv + cw = (-9, -7, -15)$

The augmented matrix = $\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$

Apply elementary row operation

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

$R_1 = \frac{R_1}{2}$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

$R_2 = R_2 - R_1$, $R_3 = R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$R_2 = -\frac{2}{3}R_2$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{4}{3} \end{array}\right]$$

$R_3 = -\frac{3}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$R_2 = R_2 + \frac{1}{3}R_3$, $R_1 = R_1 - \frac{3}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$R_1 = R_1 - \frac{1}{2}R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This gives, a=-2,b=1,c=-2

Then, $-2u + v - 2w = (-9, -7, -15)$

b)Let, $au + bv + cw = (6, 11, 6)$

The augmented matrix = $\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$

Apply elementary row operation

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

$R_1 = \frac{R_1}{2}$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 3 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

$R_2 = R_2 - R_1$, $R_3 = R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & -\frac{3}{2} & \frac{1}{2} & 8 \\ 0 & 1 & -1 & -6 \end{array}\right]$$

$R_2 = -\frac{2}{3}R_2$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 1 & -1 & -6 \end{array}\right]$$

$R_3 = R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 0 & -\frac{2}{3} & -\frac{2}{3} \end{array}\right]$$

$R_3 = -\frac{3}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 1 & -\frac{1}{3} & -\frac{16}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_2 = R_2 + \frac{1}{3}R_3$, $R_1 = R_1 - \frac{3}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 = R_1 - \frac{1}{2}R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This gives, a=4,b=-5, c=1

Then. $4u - 5v + w = (6.11.6)$

c)Let, $\mathbf{au} + \mathbf{bv} + \mathbf{cw} = (0, 0, 0)$

The augmented matrix = $\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right]$

Apply elementary row operation

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \text{RREF=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This gives, a=0,b=0,c=0

$0.\,\mathbf{u} + 0.\,\mathbf{v} + 0.\,\mathbf{w} = (0, 0, 0)$

c)

$$(0, 0, 0) = \alpha_1(2, 1, 4) + \alpha_2(1, -1, 3) + \alpha_3(3, 2, 5)$$

In parts $a)$ and $b)$ we saw that coefficients matrix of correspondent linear system is regular, which means vectors u, v and w are independent, which implies that there is only trivial solution for this vector equation:

$$\Longrightarrow \quad \alpha_1 = 0$$
$$\Longrightarrow \quad \alpha_2 = 0$$
$$\Longrightarrow \quad \alpha_3 = 0$$

9. Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

The given matrices are

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

Let $aA + bB + cC = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$$

$$\begin{bmatrix} 4a+b & -b+2c \\ -2a+2b+c & -2a+3b+4c \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 4a+b &= d_1 \\ -b+2c &= d_2 \\ -2a+2b+c &= d_3 \\ -2a+3b+4c &= d_4 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & d_1 \\ 0 & -1 & 2 & d_2 \\ -2 & 2 & 1 & d_3 \\ -2 & 3 & 4 & d_4 \end{array}\right]$$

The combined augmented matrix for $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is

$$\left[\begin{array}{ccc|ccc} 4 & 1 & 0 & 6 & 0 & -1 \\ 0 & -1 & 2 & -8 & 0 & 5 \\ -2 & 2 & 1 & -1 & 0 & 7 \\ -2 & 3 & 4 & -8 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & \frac{1}{4} & 0 & \frac{3}{2} & 0 & -\frac{1}{4} \\ 0 & -1 & 2 & -8 & 0 & 5 \\ -2 & 2 & 1 & -1 & 0 & 7 \\ -2 & 3 & 4 & -8 & 0 & 1 \end{array}\right] R_1 \to \frac{1}{4}R_1$$

DATE / /20

PAGE NO

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & -2 & 8 & 0 & -5 \\ 0 & \frac{5}{2} & 1 & 2 & 0 & \frac{13}{2} \\ 0 & \frac{7}{2} & 4 & -5 & 0 & \frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_2 \to -R_2 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{4} \\ 0 & 1 & -2 & 8 & 0 & -5 \\ 0 & 0 & 6 & -18 & 0 & 19 \\ 0 & 0 & 11 & -33 & 0 & 18 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{1}{4}R_2 \\ R_3 \to R_3 - \frac{5}{2}R_2 \\ R_4 \to R_4 - \frac{7}{2}R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{3}{4} \\ 0 & 1 & -2 & 8 & 0 & -5 \\ 0 & 0 & 1 & -3 & 0 & \frac{19}{6} \\ 0 & 0 & 11 & -33 & 0 & 18 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & 2 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & -3 & 0 & \frac{19}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{-101}{6} \end{array}\right]$$

- For $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$, the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of the coefficient matrix = Rank of the augmented matrix = 3

∴ The system is consistent and has a unique solution

$a = -2,\ b = 2,\ c = -3$

$\therefore \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a Linear Combination of A, B, C

and $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = -2A + 2B - 3C$

- For $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow a=0, b=0, c=0$$

$\therefore \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a linear combination of A, B, C

and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$

- For $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 15/6 \\ 0 & 0 & 0 & \frac{-101}{6} \end{array}\right]$$

Rank of the coefficient matrix = 3

Rank of the augmented matrix = 4

Since rank of the coefficient matrix $\neq$ rank of the augmented matrix

$\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is not a linear combination of A, B, C

10. In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

(a) $-9 - 7x - 15x^2$ (b) $6 + 11x + 6x^2$

(c) 0 (d) $7 + 8x + 9x^2$

(a) Here the given vector is $-9-7x-15x^2$.

Let, $-9-7x-15x^2 = ap_1 + bp_2 + cp_3$,

where a, b, c are some scalar.

$\therefore -9-7x-15x^2 = a(2+x+4x^2) + b(1-x+3x^2) + c(3+2x+5x^2)$

$\Rightarrow -9-7x-15x^2 = (2a+b+3c) + (a-b+2c)x + (4a+3b+5c)x^2$

Comparing the coefficient of the like terms, we have

$2a+b+3c = -9$

$a-b+2c = -7$

$4a+3b+5c = -15$.

Solving for a, b, c

we have, $a = -2, b = 1$ and $c = -2$.

Therefore we have

$-9-7x-15x^2 = -2(2+x+4x^2) + 1(1-x+3x^2) - 2(3+2x+5x^2)$

(b) Here the given vector is $6+11x+6x^2$.

Let, $6+11x+6x^2 = ap_1 + bp_2 + cp_3$,

where a, b, c are some scalar.

$\therefore 6+11x+6x^2 = a(2+x+4x^2) + b(1-x+3x^2) + c(3+2x+5x^2)$

$\Rightarrow 6+11x+6x^2 = (2a+b+3c) + (a-b+2c)x + (4a+3b+5c)x^2$

Comparing the coefficient of the like terms, we have

$2a+b+3c = 6$

$a-b+2c = 11$

$4a+3b+5c = 6$.

Solving for a, b, c

we have, $a = 4, b = -5$ and $c = 1$.

Therefore we have

$6+11x+6x^2 = 4(2+x+4x^2) - 5(1-x+3x^2) + 1(3+2x+5x^2)$

(c) Here the given vector is 0 .
Let , $0 = ap_1 + bp_2 + cp_3$,
where a, b, c are some scalar.
$\therefore 0 = a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) \Rightarrow 0 = (2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2$
Comparing the coefficient of the like terms, we have
$2a + b + 3c = 0$
$a - b + 2c = 0$
$4a + 3b + 5c = 0$.
Solving for a, b, c
we have, $a = b = c = 0$.
Therefore we have
$0 = 0(2 + x + 4x^2) + 0(1 - x + 3x^2) + 0(3 + 2x + 5x^2)$.

(d) Here the given vector is $7 + 8x + 9x^2$.
Let , $7 + 8x + 9x^2 = ap_1 + bp_2 + cp_3$,
where a, b, c are some scalar.
$\therefore 7 + 8x + 9x^2 = a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2)$
$\Rightarrow 7 + 8x + 9x^2 = (2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2$
Comparing the coefficient of the like terms, we have
$2a + b + 3c = 7$
$a - b + 2c = 8$
$4a + 3b + 5c = 9$.
Solving for a, b, c
we have, $a = 0, b = -2 \quad \text{and} \quad c = 3$.
Therefore we have
$7 + 8x + 9x^2 = 0(2 + x + 4x^2) - 2(1 - x + 3x^2) + 3(3 + 2x + 5x^2)$

11. In each part, determine whether the vectors span R^3.

(a) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1)$

(b) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

For the first two, it's very simple. A vector spans R3 if every vector u in R3 can be written as a linear combination of the vectors in the set S.

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + \ldots c_n \vec{u}_n = \vec{v}$$

Where v is a vector understood to be any vector in R3.

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

For a square matrix, we need only take its determinant to find out if it spans R3. Expand by cofactors and minors on the top row:

$$\begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \left(2\right)\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - \left(0\right)\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \left(0\right)\begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix}$$

Easy enough.

$$= 2(0*1 - 1*3) + 0 + 0 = -6$$

Since the determinant is nonzero, the vectors span R3.

Second one.

$$\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = \left(2\right)\begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} - \left(4\right)\begin{vmatrix} -1 & -1 \\ 3 & 8 \end{vmatrix} + \left(8\right)\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (2)(10) - (4)(-5) + (8)(-5) = 0$$

Since the determinant = 0, the set does not span R3.

12. Suppose that $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, and $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(2, 3, -7, 3)$ (b) $(0, 0, 0, 0)$

(c) $(1, 1, 1, 1)$ (d) $(-4, 6, -13, 4)$

To check which of the following vectors are in the span of $\{v_1, v_2, v_3\}$.

12) (a)

$$\begin{bmatrix}2\\3\\-7\\3\end{bmatrix} = a\begin{bmatrix}2\\1\\0\\3\end{bmatrix} + b\begin{bmatrix}3\\-1\\5\\2\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

Simplify each term.

$$\begin{bmatrix}2\\3\\-7\\3\end{bmatrix} = \begin{bmatrix}2a\\a\\0\\3a\end{bmatrix} + \begin{bmatrix}3b\\-b\\5b\\2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}2\\3\\-7\\3\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\0+5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add 0 and 5b.

$$\begin{bmatrix}2\\3\\-7\\3\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}2\\3\\-7\\3\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b+0\\5b+2c\\3a+2b+c\end{bmatrix}$$

Add $a-b$ and 0.

$$\begin{bmatrix}2\\3\\-7\\3\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b\\5b+2c\\3a+2b+c\end{bmatrix}$$

Write as a linear system of equations.

$2 = 2a + 3b - c$
$3 = a - b$
$-7 = 5b + 2c$
$3 = 3a + 2b + c$

Solve the system of equations.

$b = -1, c = -1, a = 2$

$(2, 3, -7, 3)$ is in the span of $\{v_1, v_2, v_3\}$.

(b)

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = a\begin{bmatrix}2\\1\\0\\3\end{bmatrix} + b\begin{bmatrix}3\\-1\\5\\2\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

Simplify each term.

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2a\\a\\0\\3a\end{bmatrix} + \begin{bmatrix}3b\\-b\\5b\\2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\0+5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add 0 and 5b.

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b+0\\5b+2c\\3a+2b+c\end{bmatrix}$$

Add $a-b$ and 0.

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b\\5b+2c\\3a+2b+c\end{bmatrix}$$

Write as a linear system of equations.

$0 = 2a + 3b - c$
$0 = a - b$
$0 = 5b + 2c$
$0 = 3a + 2b + c$

Solve the system of equations.

$a = 0, c = 0, b = 0$

$(0, 0, 0, 0)$ is in the span of $\{v_1, v_2, v_3\}$.

(c)

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = a\begin{bmatrix}2\\1\\0\\3\end{bmatrix} + b\begin{bmatrix}3\\-1\\5\\2\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

Simplify each term.

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2a\\a\\0\\3a\end{bmatrix} + \begin{bmatrix}3b\\-b\\5b\\2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\0+5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add 0 and 5b.

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b+0\\5b+2c\\3a+2b+c\end{bmatrix}$$

Add a − b and 0.

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b\\5b+2c\\3a+2b+c\end{bmatrix}$$

Write as a linear system of equations.

$1 = 2a + 3b - c$
$1 = a - b$
$1 = 5b + 2c$
$1 = 3a + 2b + c$

Solve the system of equations.

$\mathbf{No solution}$

$(1, 1, 1, 1)$ is not in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(d)

$$\begin{bmatrix}-4\\6\\-13\\4\end{bmatrix} = a\begin{bmatrix}2\\1\\0\\3\end{bmatrix} + b\begin{bmatrix}3\\-1\\5\\2\end{bmatrix} + c\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

Simplify each term.

$$\begin{bmatrix}-4\\6\\-13\\4\end{bmatrix} = \begin{bmatrix}2a\\a\\0\\3a\end{bmatrix} + \begin{bmatrix}3b\\-b\\5b\\2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}-4\\6\\-13\\4\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\0+5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add 0 and 5b.

$$\begin{bmatrix}-4\\6\\-13\\4\end{bmatrix} = \begin{bmatrix}2a+3b\\a-b\\5b\\3a+2b\end{bmatrix} + \begin{bmatrix}-c\\0\\2c\\c\end{bmatrix}$$

Add the corresponding elements.

$$\begin{bmatrix}-4\\6\\-13\\4\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b+0\\5b+2c\\3a+2b+c\end{bmatrix}$$

Add a − b and 0.

$$\begin{bmatrix}-4\\6\\-13\\4\end{bmatrix} = \begin{bmatrix}2a+3b-c\\a-b\\5b+2c\\3a+2b+c\end{bmatrix}$$

Write as a linear system of equations.

$-4 = 2a + 3b - c$
$6 = a - b$
$-13 = 5b + 2c$
$4 = 3a + 2b + c$

Solve the system of equations.

$b = -3, c = 1, a = 3$

$(-4, 6, -13, 4)$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

13. Determine whether the following polynomials span P_2

$$\mathbf{p}_1 = 1 - x + 2x^2, \quad \mathbf{p}_2 = 3 + x,$$
$$\mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

To determine whether the given polynomials span (P_2) (the vector space of polynomials of degree at most 2), we need to check if any polynomial in (P_2) can be expressed as a linear combination of these polynomials.

Let's denote the polynomials as follows

$[p_1(x) = 1 - x + 2x^2]$ and $[p_2(x) = 3 + x]$

$[p_3(x) = 5 - x + 4x^2]$ and $[p_4(x) = -2 - 2x + 2x^2]$

Now, we need to find coefficients (a, b,) and (c) such that

$$[f(x) = a \cdot p_1(x) + b \cdot p_2(x) + c \cdot p_3(x) + d \cdot p_4(x)]$$

Substituting the expressions of the given polynomials

$$[f(x) = a(1 - x + 2x^2) + b(3 + x) + c(5 - x + 4x^2) + d(-2 - 2x + 2x^2)]$$

$$[= (a - d) + (b - c - 2d) \cdot x + (2a + 4c + 2d) \cdot x^2]$$

Now, equating the coefficients of f(x) and the expression above

For constant term

a - d = c - 2d = 0

a = d

c = 2d

For coefficient of (x)

b - c - 2d = 0

b = c + 2d

b = 3d

For coefficient of (x^2)

$2a + 4c + 2d = 0$

$2d + 8d + 2d = 0$

$12d = 0$

d = 0

From $d = 0$, we get $a = 0 \quad \text{and} \quad c = 0.$

Therefore, $b = 3d = 0.$

14. Let $\mathbf{f} = \cos^2 x$ and $\mathbf{g} = \sin^2 x$. Which of the following lie in the space spanned by **f** and **g**?

(a) $\cos 2x$ (b) $3 + x^2$ (c) 1 (d) $\sin x$ (e) 0

Let

$$f = \cos^2 x \quad \text{and} \quad g = \sin^2 x$$

Consider the identity

$$\cos^2 x - \sin^2 x = \cos 2x$$

Then

$$\cos 2x \in \text{span}\{\cos x, \sin x\}$$

Now we know that

$$\sin^2 x + \cos^2 x = 1$$

So

$$1 \in \text{span}\{\sin^2 x, \cos^2 x\}$$

Also note that

$$0\sin^2 x + 0\cos^2 x = 0$$
$$\Rightarrow 0 \in \text{span}\{f, g\}$$

Let if possible

$$\sin x \in \text{span}\{f, g\}$$

Then there exist

$$a, b \in \mathbb{R} \quad \text{and}$$
$$a\sin^2 x + b\cos^2 x = \sin x$$

For x=0

$$b = 0$$

For

$$x = \frac{\pi}{2}$$
$$\Rightarrow a = 1$$
$$\text{and} \quad \text{for } x = -\frac{\pi}{2}$$
$$\Rightarrow a = -1$$

Then a has two different values. Since this is impossible therefore

$$\sin x \not\in \text{span}\{f, g\}$$

Next we suppose that

$$3 + x^2 \in \text{span}\{f, g\}$$
$$\Rightarrow a\cos^2 x + b\sin^2 x = 3 + x^2$$

Then

$$a\cos^2 x + b - b\cos^2 x = 3 + x^2$$
$$\Rightarrow (a - b)\cos^2 x + b = 3 + x^2$$

Without loss of generality we assume that $a \geq b$
Then maximum value of left hand side is

$$a - b + b = a$$

and the maximum value of RHS doesn't exist because

$$\lim_{x \to \infty} 3 + x^2 = \infty$$

So LHS and RHS cannot be equal. Then our assumption was wrong and

$$3 + x^2 \not\in \text{span}\{f, g\}$$

22. Let $\mathbf{v}_1 = (1, 6, 4)$, $\mathbf{v}_2 = (2, 4, -1)$, $\mathbf{v}_3 = (-1, 2, 5)$, and $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 8, 9)$. Use Theorem 4.2.6 to show that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

THEOREM 4.2.6 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then*

$$span\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} = span\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

Given that, $\mathbf{v}_1 = (1,6,4), \mathbf{v}_2 = (2,4,-1), \mathbf{v}_3 = (-1,2,5)$ and $\mathbf{w}_1 = (1,-2,-5), \mathbf{w}_2 = (0,8,9)$.
To prove $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, we need to show that each vector from one set can be written as linear combination of vectors from other set.
Set, $\mathbf{v}_1 = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2$ where α_1, α_2 are constants to be determined.
So, $(1,6,4) = \alpha_1(1,-2,-5) + \alpha_2(0,8,9)$
$\Rightarrow (1,6,4) = (\alpha_1, -2\alpha_1, -5\alpha_1) + (0, 8\alpha_2, 9\alpha_2)$
$\Rightarrow (1,6,4) = (\alpha_1, -2\alpha_1 + 8\alpha_2, -5\alpha_1 + 9\alpha_2)$
Equating each components, we get
$1 = \alpha_1$
$6 = -2\alpha_1 + 8\alpha_2$
$4 = -5\alpha_1 + 9\alpha_2$
Substituting $\alpha_1 = 1$ in second and third equation, we get
$\alpha_2 = 1$
$\alpha_2 = 1$
Thus, $\mathbf{v}_1 = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2$
$\Rightarrow \mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_2$ $[\because \alpha_1 = 1, \alpha_2 = 1]$

Set, $\mathbf{v}_2 = \beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2$
$\Rightarrow (2,4,-1) = \beta_1(1,-2,-5) + \beta_2(0,8,9)$
$\Rightarrow (2,4,-1) = (\beta_1, -2\beta_1, -5\beta_1) + (0, 8\beta_2, 9\beta_2)$
$\Rightarrow (2,4,-1) = (\beta_1, -2\beta_1 + 8\beta_2, -5\beta_1 + 9\beta_2)$
Equating each components, we get
$2 = \beta_1$
$4 = -2\beta_1 + 8\beta_2$
$-1 = -5\beta_1 + 9\beta_2$
Substituting $\beta_1 = 2$ in second and third equations, we get
$\beta_2 = 1$
$\beta_2 = 1$
Thus, $\mathbf{v}_2 = \beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2$
$\Rightarrow \mathbf{v}_2 = 2\mathbf{w}_1 + \mathbf{w}_2$ $[\because \beta_1 = 2, \beta_2 = 1]$

Set, $v_3 = \gamma_1 w_1 + \gamma_2 w_2$
$\Rightarrow (-1,2,5) = \gamma_1(1,-2,-5) + \gamma_2(0,8,9)$
$\Rightarrow (-1,2,5) = (\gamma_1,-2\gamma_1,-5\gamma_1) + (0,8\gamma_2,9\gamma_2)$
$\Rightarrow (-1,2,5) = (\gamma_1,-2\gamma_1+8\gamma_2,-5\gamma_1+9\gamma_2)$
Equating each components, we get
$-1 = \gamma_1$
$2 = -2\gamma_1 + 8\gamma_2$
$5 = -5\gamma_1 + 9\gamma_2$
Substituting $\gamma_1 = -1$ in second and third equations, we get
$\gamma_2 = 0$
$\gamma_2 = 0$
Thus, $v_3 = \gamma_1 w_1 + \gamma_2 w_2$
$\Rightarrow v_3 = -w_1 \qquad [\because \gamma_1 = -1, \gamma_2 = 0]$

Explanation:

Suppose $u_1, u_2, \ldots\ldots, u_m$ are any vectors in a vector space V. Any vector of the form $a_1u_1 + a_2u_2 + \ldots\ldots\ldots + a_mu_m$ where the a_i are scalars, is called a linear combination of $u_1, u_2, \ldots\ldots, u_m$. The collection of all such linear combinations, denoted by $\text{span}\{u_1, u_2, \ldots\ldots, u_m\}$, is called the linear span of $u_1, u_2, \ldots\ldots, u_m$.

So, $v_1 = w_1 + w_2$
$v_2 = 2w_1 + w_2$
$v_3 = -w_1 = -w_1 + 0.\, w_2$
Thus, v_1, v_2, v_3 can be expressed as the linear combinations of w_1, w_2.
Also, $w_1 = -v_3 \qquad [\because v_3 = -w_1]$
$\Rightarrow w_1 = 0.\, v_1 + 0.\, v_2 + (-1).\, v_3$
And, $w_2 = v_1 - w_1 \qquad [\because v_1 = w_1 + w_2]$
$\Rightarrow w_2 = v_1 - (-v_3) \qquad [\because w_1 = -v_3]$
$\Rightarrow w_2 = v_1 + v_3$
$\Rightarrow w_2 = v_1 + 0.\, v_2 + v_3$
Hence, $w_1 = 0.\, v_1 + 0.\, v_2 + (-1).\, v_3$ and
$w_2 = v_1 + 0.\, v_2 + v_3$
Thus, w_1, w_2 can be expressed as the linear combinations of v_1, v_2, v_3.

Explanation:

If $S = \{v_1, v_2, \ldots\ldots\ldots, v_r\}$ and $S' = \{w_1, w_2, \ldots\ldots, w_k\}$ are non-empty sets of vectors in a vector space V, then $\text{span}\{v_1, v_2, \ldots\ldots, v_r\} = \text{span}\{w_1, w_2, \ldots\ldots, w_k\}$ if and only if each vector in S is a linear combination of those in S' and each vector in S' is a linear combination of those in S.

4.3 Linear Independence

2. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^3.

(a) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

(b) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

a-

We know that the vectors $\{u_1, u_2, \ldots, u_n\}$ is said to be linearly independent if $c_1u_1 + c_2u_2 + \ldots + c_nu_n = 0$ then $c_1 = c_2 = \ldots = c_n = 0$.

(a) Given that $u_1 = (-3, 0, 4), u_2 = (5, -1, 2), u_3 = (1, 1, 3)$.
Let us consider:

$$c_1u_1 + c_2u_2 + c_3u_3 = \vec{0}$$
$$\Rightarrow c_1(-3, 0, 4) + C_2(5, -1, 2) + c_3(1, 1, 3) = \vec{0}$$
$$\Rightarrow (-3c_1 + 5c_2 + c_3, -c_2 + c_3, 4c_1 + 2c_2 + 3c_3) = (0, 0, 0)$$

So we get,

$$-3c_1 + 5c_2 + c_3 = 0 \ldots (1)$$
$$-c_2 + c_3 = 0 \ldots (2)$$
$$4c_1 + 2c_2 + 3c_3 = 0 \ldots (3)$$

From equation (2):

$c_2 = c_3$

From equation (1):

$$-3c_1 + 5c_2 + c_3 = 0$$
$$\Rightarrow -3c_1 + 5c_2 + c_2 = 0, \text{ since } c_2 = c_3$$
$$\Rightarrow -3c_1 + 6c_2 = 0$$
$$\Rightarrow c_1 = 2c_2$$

From equation (3):

$$4c_1 + 2c_2 + 3c_3 = 0$$
$$\Rightarrow 8c_2 + 2c_2 + 3c_2 = 0, \text{since } c_1 = 2c_2 \text{ and } c_3 = c_2$$
$$\Rightarrow 13c_2 = 0$$
$$\Rightarrow c_2 = 0$$

Since $c_2 = 0$ therefore $c_1 = 2c_2 = 0$ and $c_3 = c_2 = 0$.
Hence, $(-3, 0, 4), (5, -1, 2), (1, 1, 3)$ are linearly independent.

The determinant of the vector is needed to be determined and the vector system is checked that linearly dependent or independent.

The vector matrix is

$$A = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

The determinant of the above matrix is determined as

$$|A| = -3(-3-2) - 5(0-4) + 1(0+4)$$
$$= 39$$

The determinant of the matrix is not equal to zero which means the vectors are linearly independent.

b-

THEOREM 4.3.3 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.*

$$r = 4 \,, n = 3 \,, r > n \,,$$

so by this theorem, these vectors are linearly dependent.

3. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

(a) $(3, 8, 7, -3)$, $(1, 5, 3, -1)$, $(2, -1, 2, 6)$, $(4, 2, 6, 4)$

(b) $(3, 0, -3, 6)$, $(0, 2, 3, 1)$, $(0, -2, -2, 0)$, $(-2, 1, 2, 1)$

Solution (a)
Given vectors are

$$(3,8,7,-3),(1,5,3,-1),(2,-1,2,6),(4,2,6,4)$$

Let a,b,c,d are the scalars such as

$$\mathrm{a}(3,8,7,-3)+\mathrm{b}(1,5,3,-1)+\mathrm{c}(2,-1,2,6)+\mathrm{d}(4,2,6,4)=(0,0,0,0) \qquad (1)$$
$$\Rightarrow 3\mathrm{a}+\mathrm{b}+2\mathrm{c}+4\mathrm{d}=0 \qquad (2)$$
$$8\mathrm{a}+5\mathrm{b}-\mathrm{c}+2\mathrm{d}=0 \qquad (3)$$
$$7\mathrm{a}+3\mathrm{b}+2\mathrm{c}+6\mathrm{d}=0 \qquad (4)$$
$$-3\mathrm{a}-\mathrm{b}+6\mathrm{c}+4\mathrm{d}=0 \qquad (5)$$

Solving above equations we get
a=-d, b=d, c=-d.
For particular, let d=1. Then a=-1,b=1,c=-1.
This shows that the vectors are linearly dependent.

(b) Given vectors are

$$(3,0,-3,6),(0,2,3,1),(0,-2,-2,0),(-2,1,2,1)$$

Let a,b,c,d are scalars such that

$$\mathrm{a}(3,0,-3,6)+\mathrm{b}(0,2,3,1)+\mathrm{c}(0,-2,-2,0)+\mathrm{d}(-2,1,2,1)=(0,0,0,0)$$

Then we get

$$3\mathrm{a}-2\mathrm{d}=0 \qquad (1)$$
$$2\mathrm{b}-2\mathrm{c}+\mathrm{d}=0 \qquad (2)$$
$$-3\mathrm{a}+3\mathrm{b}-2\mathrm{c}+2\mathrm{d}=0 \qquad (3)$$
$$6\mathrm{a}+\mathrm{b}+\mathrm{d}=0 \qquad (4)$$

Solving above equations we get
a=b=c=d=0
This shows that the given vectors are linearly independent.

4. In each part, determine whether the vectors are linearly independent or are linearly dependent in P_2.

(a) $2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$

(b) $1 + 3x + 3x^2,\ x + 4x^2,\ 5 + 6x + 3x^2,\ 7 + 2x - x^2$

a) we have the given vectors

$2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$
Let us consider

$a(2 - x + 4x^2) + b(3 + 6x + 2x^2) + c(2 + 10x - 4x^2) = 0$ for a, b, c reals.
$\Rightarrow (2a + 3b + 2c) + x(-a + 6b + 10c) + x^2(4a + 2b - 4c) = 0$

computing each power of x, we get
$2a + 3b + 2c = 0$
$-a + 6b + 10c = 0$
$4a + 2b - 4c = 0$

solving we get

$a = 0, b = 0 \quad \text{and} \quad c = 0$
Thus the vectors $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$ are linearly independent.

b) we have the given vectors

$1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$
Let us consider

$a(1 + 3x + 3x^2) + b(x + 4x^2) + c(5 + 6x + 3x^2) + d(7 + 2x - x^2) = 0$ for a, b, c, d reals.
$\Rightarrow (a + 5c + 7d) + x(3a + b + 6c + 2d) + x^2(3a + 4b + 3c - d) = 0$

computing each power of x, we get
$a + 5c + 7d = 0$
$3a + b + 6c + 2d = 0$
$3a + 4b + 3c - d = 0$

solving we get
Solve the equation for c.
$c = -\frac{a}{5} - \frac{7d}{5}$
$3a + b + 6c + 2d = 0$
$3a + 4b + 3c - d = 0$

Solve the equation for b.
$b = -\frac{9a}{5} + \frac{32d}{5}$
$c = -\frac{a}{5} - \frac{7d}{5}$
$3a + 4b + 3c - d = 0$

Solve the equation for d.
$d = \frac{4a}{17}$
$b = -\frac{9a}{5} + \frac{32d}{5}$
$c = -\frac{a}{5} - \frac{7d}{5}$

Simplify the right side.
$b = -\frac{5a}{17}$
$d = \frac{4a}{17}$
$c = -\frac{a}{5} - \frac{7d}{5}$

Simplify the right side.
$c = -\frac{9a}{17}$
$b = -\frac{5a}{17}$
$d = \frac{4a}{17}$

Thus the scalars are not all zero. Hence the vectors $1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$ are linearly dependent.

5. In each part, determine whether the matrices are linearly independent or dependent.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}

a) For the matrices

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \in M_{22}$$

Linear independence

$$c_1\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + c_2\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Expanding the equation

$$\begin{bmatrix} c_1 + c_2 & 2c_2 + c_3 \\ c_1 + 2c_2 + 2c_3 & 2c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the following system of equations:

$$c_1 + c_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots eq1$$
$$2c_2 + c_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots eq2$$
$$c_1 + 2c_2 + 2c_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots eq3$$
$$2c_1 + c_2 + c_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots eq4$$

from equation (1)

$$c_1 = = -c_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots eq5$$

put the value it in equation (2,3)

$$-c_2 + 2c_2 + 2c_3 = 0$$
$$c_2 + 2c_3 = 0$$
$$c_2 = -2c_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots eq6$$

put this in equation (2)

$$2(-2c_3) + c_3 = 0$$
$$-3c_3 = 0$$
$$c_3 = 0$$

from(6)

$$c_2 = 0$$

from(5)

$$c_1 = 0$$

vectors are linearly dependent

b-

b)

Let us consider the relation

$$b_1\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+b_2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}+b_3\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}b_1&0&b_2\\0&b_3&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

Comparing both side we get

$b_1=0$

$b_2=0$

and $b_3=0$

Thus $b_1\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+b_2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}+b_3\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$

implies $b_1=b_2=b_3=0$

so, $\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$ and $\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}$ are linearly independent in M_{23}

10) a) The given vectors are.

$v_1=(1,2,3,4)$, $v_2=(0,1,0,-1)$ and $v_3=(1,3,3,3)$

Let us consider the relation

$$c_1v_1+c_2v_2+c_3v_3=0$$

$$\Rightarrow c_1(1,2,3,4)+c_2(0,1,0,-1)+c_3(1,3,3,3)=(0,0,0,0)$$

$$\Rightarrow (c_1+c_3,\ 2c_1+c_2+3c_3,\ 3c_1+3c_3,\ 4c_1-c_2+3c_3)=(0,0,0,0)$$

Comparing both side we get

$c_1+c_3=0$ --- (5)

$2c_1+c_2+3c_3=0$ --- (6)

$3c_1+3c_3=0$ --- (7)

and $4c_1-c_2+3c_3=0$ --- (8)

From (5) we get $c_1=-c_3$

Put, $c_1 = -c_3$ in (6) we get

$$-2c_3 + c_2 + 3c_3 = 0 \Rightarrow c_2 + c_3 = 0$$
$$\Rightarrow c_2 = -c_3$$

Let, $c_3 = k$, then $c_1 = -k$ and $c_2 = -k$

Thus for $k \neq 0$ c_1, c_2, c_3 not equal to zero.

So $v_1 = (1,2,3,4)$, $v_2 = (0,1,0,-1)$ and $v_3 = (1,3,3,3)$ are linearly dependent set in $\mathbb{R}^4$.

b) Let, $v_1 = d_1 v_2 + d_2 v_3$ --- (9)

$$\Rightarrow (1,2,3,4) = d_1(0,1,0,-1) + d_2(1,3,3,3)$$
$$\Rightarrow (1,2,3,4) = (d_2, d_1 + 3d_2, 3d_2, -d_1 + 3d_2)$$

Comparing both side we get.

$d_2 = 1$, $d_1 + 3d_2 = 2$ --- (10)

$3d_2 = 3$ --- (11)

and $-d_1 + 3d_2 = 4$ --- (12)

From (11) we get $\underline{d_2 = 1}$

put $d_2 = 1$ in (10) we get.

$d_1 + 3 = 2 \Rightarrow \underline{d_1 = -1}$

From (9) we get

$v_1 = -v_2 + v_3$ --- (13) $[\because d_1 = -1, d_2 = 1]$

Thus v_1 is a linear combination of v_2 and v_3

From (13) we get

$v_2 = -v_1 + v_3$

Thus v_2 is a linear combination of v_1 and v_3

From (13) we get

$v_3 = v_1 + v_2$

Thus v_3 is a linear combination of v_1 and v_2.

6. Determine all values of k for which the following matrices are linearly independent in M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Let $c_1\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + c_2\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c_3\begin{bmatrix} 3 & 0 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Then $c_1 - c_2 + 3c_3 = 0$

$c_1 + kc_2 + 2c_3 = 0$

$kc_1 + c_2 + 14c_3 = 0$

This is a homogeneous system of three equations in c_1, c_2, c_3.

The coefficient determinant of the system is $\begin{vmatrix} 1 & -1 & 3 \\ 1 & k & 2 \\ k & 1 & 14 \end{vmatrix}$.

Now, $\begin{vmatrix} 1 & -1 & 3 \\ 1 & k & 2 \\ k & 1 & 14 \end{vmatrix} = 1(14k - 2) + 1(14 - 2k) + 3(1 - k^2)$

$= -3k^2 + 12k + 15$

$= -3(k - 5)(k + 1)$

The given matrices are linearly independent if

$\begin{vmatrix} 1 & -1 & 3 \\ 1 & k & 2 \\ k & 1 & 14 \end{vmatrix} \neq 0$

$i.e.,\ \text{ if } \ -3(k-5)(k+1) \neq 0$

$i.e.,\ \text{ if } \ k \neq -1, 5$

Hence the given matrices are linearly independent for all values of k except -1 and $5.$

7. In each part, determine whether the three vectors lie in a plane in R^3.

(a) $\mathbf{v}_1 = (2, -2, 0)$, $\mathbf{v}_2 = (6, 1, 4)$, $\mathbf{v}_3 = (2, 0, -4)$

(b) $\mathbf{v}_1 = (-6, 7, 2)$, $\mathbf{v}_2 = (3, 2, 4)$, $\mathbf{v}_3 = (4, -1, 2)$

- Three vectors in R^3 are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two (Figure 4.3.4).

Explanation:

We know that If there are three vectors in a 3d-space and their scalar triple product is zero, then these three vectors are coplanar.

So ,

(a) We have given the vectors $\mathbf{v1} = (2, -2, 0), \mathbf{v2} = (6, 1, 4), \mathbf{v3} = (2, 0, -4)$

Taking the scalar tripple product,

$$\Rightarrow \begin{vmatrix} 2 & -2 & 0 \\ 6 & 1 & 4 \\ 2 & 0 & -4 \end{vmatrix} = 2 \times (-4 - 0) - -2 \times (-24 - 8) + 0$$ =-8-64=-72

Hence the Scalar tripple product is not equal to zero. **So the given points are not lie in same plane.**

(b) We have given the vectors $\mathbf{v1} = (-6, 7, 2), \mathbf{v2} = (3, 2, 4), \mathbf{v3} = (4, -1, 2)$

Taking scalar tripple product,

$$\begin{vmatrix} -6 & 7 & 2 \\ 3 & 2 & 4 \\ 4 & -1 & 2 \end{vmatrix} = -6 \times (4 + 4) - 7 \times (6 - 16) + 2 \times (-3 - 8) = -48 + 70 - 22 = 0$$

Here the scalar tripple product is zero. **Therefore the given vectors lie in same plane.**

9. (a) Show that the three vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$, and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

a. The vectors can be written as:

$$k_1(0,3,1,-1) + k_2(6,0,5,1) + k_3(4,-7,1,3) = (0,0,0,0)$$

The augmented matrix is given by: $\left[\begin{array}{ccc|c} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$.

Applying $R_1 \leftrightarrow R_3$,

$$= \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 3 & 0 & -7 & 0 \\ 0 & 6 & 4 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 - 3R_1$ and $R_4 \rightarrow R_4 + R_1$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -15 & -10 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 6 & 4 & 0 \end{array}\right]$$

Applying $R_2 \rightarrow -\frac{1}{5}R_2$ and $R_4 \rightarrow R_4 - R_3$,

$$= \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_3 \rightarrow -2R_2 + R_3$,

$$= \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equations become:

$$3k_2 + 2k_3 = 0$$

$$k_2 = -\frac{2}{3}k_3$$

And,

$$k_1 + 5k_2 + k_3 = 0$$

$$k_1 - \frac{10}{3}k_3 + k_3 = 0$$

$$k_1 = \frac{7}{3}k_3$$

Since the solution depends upon values of k_3, therefore, the solution is non-trivial.

The vectors form a linearly dependent set in $\mathbb{R}^4$.

b.

Expressing $\mathbf{v}_1$ as LC of $\mathbf{v}_2$ and $\mathbf{v}_3$,

$$k_1 v_1 = -k_2 v_2 - k_3 v_3$$

$$v_1 = -\frac{k_2}{k_1}v_2 - \frac{k_3}{k_1}v_3$$

$$= -\frac{-\frac{2}{3}k_3}{\frac{7}{3}k_3}v_2 - \frac{k_3}{\frac{7}{3}k_3}v_3$$

$$= \frac{2}{7}v_2 - \frac{3}{7}v_3$$

Expressing $\mathbf{v}_2$ as LC of $\mathbf{v}_1$ and $\mathbf{v}_3$,

$$v_2 = -\frac{k_1}{k_2}v_1 - \frac{k_3}{k_2}v_3$$

$$= -\frac{\frac{7}{3}k_3}{-\frac{2}{3}k_3}v_1 - \frac{k_3}{-\frac{2}{3}k_3}v_3$$

$$= \frac{7}{2}v_1 + \frac{3}{2}v_3$$

Expressing $\mathbf{v}_3$ as LC of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$v_3 = -\frac{k_1}{k_3}v_1 - \frac{k_2}{k_3}v_2$$

$$= -\frac{\frac{7}{3}k_3}{k_3}v_1 - \frac{-\frac{2}{3}k_3}{k_3}v_2$$

$$= -\frac{7}{3}v_1 + \frac{2}{3}v_2$$

Explanation:

Use the relation $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ to evaluate the linear combinations.

11. For which real values of λ do the following vectors form a linearly dependent set in R^3?

$$\mathbf{v}_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right), \quad \mathbf{v}_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right), \quad \mathbf{v}_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

It has been given the vectors

$$\mathbf{v}_1 = \left(\lambda, \frac{1}{2}, \frac{1}{2}\right)$$
$$\mathbf{v}_2 = \left(\frac{1}{2}, \lambda, \frac{1}{2}\right)$$
$$\mathbf{v}_3 = \left(\frac{1}{2}, \frac{1}{2}, \lambda\right)$$

Explanation:

Since we know that determinant of the linearly dependent vectors is 0

Then we must have

$$\det\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = 0$$

Step 2

Now we have obtained that

$$\det\begin{bmatrix} \lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \lambda \end{bmatrix} = 0$$

$$\lambda\begin{vmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{vmatrix} + \frac{1}{2}\begin{vmatrix} \frac{1}{2} & \lambda \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = 0$$

$$\lambda\left(\lambda^2 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{\lambda}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{\lambda}{2}\right) = 0$$

$$\lambda^3 - \frac{3\lambda}{4} + \frac{1}{4} = 0$$

$$4\lambda^3 - 3\lambda + 1 = 0$$

$$4\lambda^2(\lambda + 1) - 4\lambda(\lambda + 1) + (\lambda + 1) = 0$$

$$(\lambda + 1)(4\lambda^2 - 4\lambda + 1) = 0$$

$$(\lambda + 1)(2\lambda - 1)^2 = 0$$

$$\lambda = -1, \frac{1}{2}$$

4.4 Coordinates and Basis

2. Use the method of Example 3 to show that the following set of vectors forms a basis for R^3.

$$\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

Let $A = [u_1 \quad u_2 \quad u_3]$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix}$$

$|A| = 3(40 - 24) - 2(8 + 16) + 1(6 + 20)$

$|A| = 3(16) - 2(24) + 26$

$|A| = 26 \neq 0$

Hence given set of vectors is linearly independent.

And we know that any linearly independent set containing three vectors forms a basis of R^3 .

Hence set of vectors $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$ forms basis of R^3.

4. Show that the following polynomials form a basis for P_3.

$$1+x, \quad 1-x, \quad 1-x^2, \quad 1-x^3$$

$\{1+x, 1-x, 1-x^2, 1-x^3\}$.

$a(1+x) + b(1-x) + c(1-x^2) + d(1-x^3) = 0$

$(a+b+c+d) + (a-b)x + (-c)x^2 - dx^3 = 0.$

$\Rightarrow \quad -c = 0 \Rightarrow c = 0$

$-d = 0 \Rightarrow d = 0$

$a-b = 0$

$a+b = 0$

$\Rightarrow a = 0$ and $b = 0$.

Hence $\{1+x, 1-x, 1-x^2, 1-x^3\}$ is linearly independent set and dimension is 4.

Hence It is a basis for P_3.

5. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix}3 & 6\\3 & -6\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -8\\-12 & -4\end{bmatrix}, \begin{bmatrix}1 & 0\\-1 & 2\end{bmatrix}$$

Explanation:

We know that if V be a vector space of dimension n over a field F then any linearly independent set of n vectors of V is a basis of V

5) Here we take the given matrices as a set of S that is

$$S = \left\{ \begin{bmatrix}3 & 6\\3 & -6\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -8\\-12 & -4\end{bmatrix}, \begin{bmatrix}1 & 0\\-1 & 2\end{bmatrix} \right\}$$

Since we know that dimension of a matrix $M_{m\times n}$ is mn so, here dimension of M_{22} will be $(2\times 2) = 4$

Also we see that S contains 4 matrix as vectors so it is enough to show that S is linearly independent and then it is clearly dimension of S=4

Now we take the relation for some scalar c_1, c_2, c_3, c_4 as

$$c_1\begin{bmatrix}3 & 6\\3 & -6\end{bmatrix} + c_2\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} + c_3\begin{bmatrix}0 & -8\\-12 & -4\end{bmatrix} + c_4\begin{bmatrix}1 & 0\\-1 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3c_1 & 6c_1\\3c_1 & -6c_1\end{bmatrix} + \begin{bmatrix}0 & -c_2\\-c_2 & 0\end{bmatrix} + \begin{bmatrix}0 & -8c_3\\-12c_3 & -4c_3\end{bmatrix} + \begin{bmatrix}c_4 & 0\\-c_4 & 2c_4\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3c_1 & 6c_1-c_2\\3c_1-c_2 & -6c_1\end{bmatrix} + \begin{bmatrix}0 & -8c_3\\-12c_3 & -4c_3\end{bmatrix} + \begin{bmatrix}c_4 & 0\\-c_4 & 2c_4\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3c_1+c_4 & 6c_1-c_2-8c_3\\3c_1-c_2-12c_3-c_4 & -6c_1-4c_3+2c_4\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Equating each elements from both side we get,

$3c_1 + c_4 = 0$ ......(1) , $6c_1 - c_2 - 8c_3 = 0$ ....(2)

$3c_1 - c_2 - 12c_3 - c_4 = 0$ ....(3) and $-6c_1 - 4c_3 + 2c_4 = 0$ ....(4)

Now from (1) , $c_4 = -3c_1$ putting this into (3) we have ,

$3c_1 - c_2 - 12c_3 + 3c_1 = 0 \Rightarrow 6c_1 - c_2 - 12c_3 = 0$ .....(5)

subtract (5) from (2) we get,

$6c_1 - c_2 - 8c_3 - 6c_1 + c_2 + 12c_3 = 0 \Rightarrow 4c_3 = 0 \Rightarrow c_3 = 0$

so, from (4) we get, $-6c_1 + 2(-3c_1) = 0 \Rightarrow -12c_1 = 0 \Rightarrow c_1 = 0$ this shows that $c_4 = -3c_1 = 0$

and then from (2) we get, $c_2 = 0$

This shows that S is linearly independent as $c_1 = c_2 = c_3 = c_4 = 0$

and then dimension of S = 4

Thus, by the above explanation we can say that S is a basis for M_{22}

7. In each part, show that the set of vectors is not a basis for R^3.

(a) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

(b) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

We have given that the set

1) $\{\mathbf{u}_1 = (2, -3, 1), \mathbf{u}_2 = (4, 1, 1), \mathbf{u}_3 = (0, -7, 1)\}$

To show given set is not basis for $\mathbf{R}^3$

any set is basis for vector space V if it is linearily independent and spans V

Therefore, to determine linearily independent

Here, the vector $\mathbf{u}_2$ is linear combination of $\mathbf{u}_1$ and $\mathbf{u}_3$

i.e. $(4, 1, 1) = 2(2, -3, 1) - (0, -7, 1)$

i.e. $\mathbf{u}_2 = 2\mathbf{u}_1 - \mathbf{u}_3$

Therefore, given set is linearily dependent

Hence it is not basis

2) we have given that the set

$\{\mathbf{u}_1 = (1, 6, 4), \mathbf{u}_2 = (2, 4, -1), \mathbf{u}_3 = (-1, 2, 5)\}$

Here, the vector $\mathbf{u}_2$ is also linear combination of other two vectors $\mathbf{u}_1$ and $\mathbf{u}_3$

i.e.

$(2, 4, -1) = (1, 6, 4) - (-1, 2, 5)$

i.e. $\mathbf{u}_2 = \mathbf{u}_1 - \mathbf{u}_3$

Therefore, given set is linearily dependent

Hence it is not basis for $\mathbb{R}^3$

8. Show that the following vectors do not form a basis for P_2.

$$1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$$

Let $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ be the vectors where $\mathrm{v}_1 = 1 - 3\mathrm{x} + 2\mathrm{x}^2$, $\mathrm{v}_2 = 1 + \mathrm{x} + 4\mathrm{x}^2$ and $\mathrm{v}_3 = 1 - 7\mathrm{x}$.

Step 2

Now, we can write
$1 - 7\mathrm{x} = 2(1 - 3\mathrm{x} + 2\mathrm{x}^2) - (1 + \mathrm{x} + 4\mathrm{x}^2)$
$\Rightarrow \mathrm{v}_3 = 2\mathrm{v}_1 - \mathrm{v}_2$
Since the vector v_3 is a linear combination of the vectors v_1 and v_2 , the set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ is not linearly independent and therefore the vectors $1 - 3\mathrm{x} + 2\mathrm{x}^2$, $1 + \mathrm{x} + 4\mathrm{x}^2$ and $1 - 7\mathrm{x}$ do not form a basis for P_2 . [Proved]

9. Show that the following matrices do not form a basis for M_{22}.

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}2 & -2\\3 & 2\end{bmatrix}, \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}$$

Solution:

The matrices are, $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}2 & -2\\3 & 2\end{bmatrix}, \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}$.

The matrices are linearly independent.

$$a\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + b\begin{bmatrix}2 & -2\\3 & 2\end{bmatrix} + c\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

On simplifying,

$$\begin{bmatrix}a+2b+c & -2b-c-d\\a+3b+c+d & a+2b+d\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

The equations are,

$a+2b+c=0$

$-2b-c-d=0$

$a+3b+c+d=0$

$a+2b+d=0$

The augmented matrix is $\left[\begin{array}{cccc|c}1 & 2 & 1 & 0 & 0\\0 & -2 & -1 & -1 & 0\\1 & 3 & 1 & 1 & 0\\1 & 2 & 0 & 1 & 0\end{array}\right]$

$R_3 = R_3 - R_1$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 2 & 1 & 0 & 0\\0 & -2 & -1 & -1 & 0\\0 & 1 & 0 & 1 & 0\\1 & 2 & 0 & 1 & 0\end{array}\right]$$

$R_4 = R_4 - R_1$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 2 & 1 & 0 & 0\\0 & -2 & -1 & -1 & 0\\0 & 1 & 0 & 1 & 0\\0 & 0 & -1 & 1 & 0\end{array}\right]$$

$R_3 = R_3 + \frac{R_2}{2}$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 2 & 1 & 0 & 0\\0 & -2 & -1 & -1 & 0\\0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0\\0 & 0 & -1 & 1 & 0\end{array}\right]$$

On simplifying,

$R_4 = R_4 - 2R_3$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 2 & 1 & 0 & 0\\0 & -2 & -1 & -1 & 0\\0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

Here, take $d = x$

On substituting,

$c = \frac{0-\left(\frac{1}{2}\right)^{x}}{-\frac{1}{2}} = x$

$b = \frac{0-(-1)x-(-1)(x)}{-2} = -x$

$a = 0 - 0(x) - (1)x - 2(-x) = x$

$\Rightarrow a = x, b = x, c = x, d = x$

Therefore, the matrices are linearly dependent.

So, they do not form a basis for M_{22}.

10. Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$, $\mathbf{v}_3 = \cos 2x$.

(a) Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for V.

(b) Find a basis for V.

Let V be the space spanned by

$\mathrm{v_1 = cos^2x, v_2 = sin^2x \quad and \quad v_3 = cos\,2x}$

a) Observe that $\mathrm{v_1 - v_2 - v_3 = cos^2x - sin^2x - cos\,2x}$

$= \mathrm{cos^2x - sin^2x - (cos^2x - sin^2x)}$.

$= \mathrm{cos^2x - sin^2x - cos^2x + sin^2x}$

$= 0$

for all $\mathrm{x} \in (-\infty, \infty)$ which gives us that the set $\mathrm{S = \{v_1. v_2, v_3\}}$ is a linearly dependent set on V and therefore S is not a basis for V.

b) Consider the set $\mathrm{S' = \{v_1, v_2\}}$

Take scalars a and b such that $\mathrm{acos^2x + bsin^2x = 0}$ for all $\mathrm{x} \in (-\infty, \infty)$.

Taking x = 0, we have a = 0 and taking x = $\frac{\pi}{2}$ we have b = 0.

Hence, the set $\mathrm{S'}$ is a linearly independent set in V.

Consider a vector f(x) = $\mathrm{acos\,2^x{+}bsin^2x + cos\,2x \in V}$

Since cos2x = $\mathrm{cos^2x - sin^2x}$ for all $\mathrm{x} \in (-\infty, \infty)$, we have

$\mathrm{f(x) = acos^2x + bsin^2x + c(cos^2x - sin^2x)}$

$\mathrm{= acos^2x + bsin^2x + ccos^2x - csin^2x}$

$\mathrm{= (a + c)cos^2x + (b - c)sin^2x}$

which gives us that f(x) $\in \mathrm{span(S')}$ and therefore $\mathrm{S'}$ spans V .

Hence, $\mathrm{S'}$ is a linearly independent set of V which spans V and therefore it is a basis for V.

4.5 Dimension

► In Exercises 1–6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space. ◄

5.
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

$x_1 - 3x_2 + x_3 = 0$

$2x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 9x_2 + 3x_3 = 0$

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced row echelon form is,

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 - 3x_2 + x_3 = 0$

Take, $x_2 = s$, $x_3 = t$

Then, $x_1 - 3s + t = 0$

$x_1 = 3s - t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s - t \\ s \\ t \end{bmatrix}$$

$$= s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension $= 2$

Remark It can be shown that for any homogeneous linear system, the method of the last example *always* produces a basis for the solution space of the system. We omit the formal proof.

7. In each part, find a basis for the given subspace of R^3, and state its dimension.

(a) The plane $3x - 2y + 5z = 0$.

7. To find a basis for the given subspaces of $\mathbb{R}^3$

(a) W_1 : The plane $3x - 2y + 5z = 0$

Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W_1$

$$3x - 2y + 5z = 0$$
$$\Rightarrow 3x = 2y - 5z$$
$$\Rightarrow x = \frac{2}{3}y - \frac{5}{3}z$$

Putting $y = 3s, z = 5t$

$$\Rightarrow x = 2s - 5t$$

$$\therefore \vec{v} = \begin{bmatrix} 2s - 5t \\ 3s \\ 3t \end{bmatrix} = s\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$$

$\therefore$ A basis for W_1 is $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \right\}$

Explanation:

The dimension of a subspace is equal to the number of vectors in its basis.

$\therefore$ The dimension of $W_1 = \mathbf{2}$

8. In each part, find a basis for the given subspace of R^4, and state its dimension.

(a) All vectors of the form $(a, b, c, 0)$.

(b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

(c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

A)

Let subspace $\mathrm{A} = \{(\mathrm{a}, \mathrm{b}, \mathrm{c}, 0)\}$

$\Rightarrow \mathrm{A} = \{\mathrm{a}(1,0,0,0) + \mathrm{b}(0,1,0,0) + \mathrm{c}(0,0,1,0)\}$

this is subspace spanned by these vectors

$\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$

Therefore, dimension of A is 3.

Explanation:

Since, number of unknows $(\mathrm{a}, \mathrm{b}, \mathrm{c})$

Step 2

B)

Let $\mathrm{B} = \{(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) : \mathrm{d} = \mathrm{a} + \mathrm{b}, \mathrm{c} = \mathrm{a} - \mathrm{b}\}$

$\Rightarrow \mathrm{d} = \mathrm{a} + \mathrm{b}, \mathrm{c} = \mathrm{a} - \mathrm{b}$

$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) = (\mathrm{a}, \mathrm{b}, \mathrm{a} - \mathrm{b}, \mathrm{a} + \mathrm{b})$

$= \mathrm{a}(1,0,1,1) + \mathrm{b}(0,1,-1,1)$

subspace spanned by $\{(1,0,1,1),(0,1,-1,1)\}$

dim of B $=$ number of unknows $(\mathrm{a}, \mathrm{b}) = 2.$

C)

Let $\mathrm{C} = \{(, \mathrm{b}, \mathrm{c}, \mathrm{d}) : \mathrm{a} = \mathrm{b} = \mathrm{c} = \mathrm{d}\}$

$\Rightarrow (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) = (\mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{a}) = \mathrm{a}(1,1,1,1)$

$\Rightarrow$ Subspace spanned by $\{(1,1,1,1)\}$

and $\dim(\mathrm{C}) = 1$

Explanation:

Since, $\dim$ of c = number of unknowns $(\mathrm{a}) = 1.$

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.

(a) $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$

(b) $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

THEOREM 4.5.5 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(*a*) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

#(a)

We have given that,

$v_1 = (-1, 2, 3)$
$v_2 = (1, -2, -2)$

let,

$v_3 = (a, b, c)$

we need to find a vector in standard basis of R^3,so that

when added to the set $\{v_1, v_2\}$it form a basis for R^3.

Now , we form an augmented matrix as,

$$\begin{bmatrix} -1 & 1 & a \\ 2 & -2 & b \\ 3 & -2 & c \end{bmatrix}$$

We now reduce this augmented matrix to row reduce echelon form by using elementary row operation

Multiply each element of R_1 by -1 to make the entry at $1,1$ a 1.

$$\begin{bmatrix} 1 & -1 & -a \\ 2 & -2 & b \\ 3 & -2 & c \end{bmatrix}$$

Perform the row operation $R_2 = R_2 - 2R_1$ to make the entry at $2,1$ a 0.

$$\begin{bmatrix} 1 & -1 & -a \\ 0 & 0 & b+2a \\ 3 & -2 & c \end{bmatrix}$$

Perform the row operation $R_3 = R_3 - 3R_1$ to make the entry at $3,1$ a 0.

$$\begin{bmatrix} 1 & -1 & -a \\ 0 & 0 & b+2a \\ 0 & 1 & c+3a \end{bmatrix}$$

Swap R_3 with R_2 to put a nonzero entry at $2,2$.

$$\begin{bmatrix} 1 & -1 & -a \\ 0 & 1 & c+3a \\ 0 & 0 & b+2a \end{bmatrix}$$

So these vectors are linearly independent if and only if,

$$b + 2a \neq 0$$

$$b \neq -2a$$

so let,

a=0

b=1

c=0

Then we get,

$$v_3 = (0, 1, 0)$$

So now ,

$\{v_1, v_2, v_3\}$ will form a basis for R^3.

So we get,

$$v_3 = (0, 1, 0)$$

#2)

We have given that,

$v_1 = (1, -1, 0)$
$v_2 = (3, 1, -2)$

Now let,

$v_3 = (a, b, c)$

we form an augmented Matrix as,

$$\begin{bmatrix} 1 & 3 & a \\ -1 & 1 & b \\ 0 & -2 & c \end{bmatrix}$$

We now reduce this augmented matrix to row reduce echelon form by using elementary row operations.

Perform the row operation $R_2 = R_2 + R_1$ to make the entry at $2, 1$ a 0.

$$\begin{bmatrix} 1 & 3 & a \\ 0 & 4 & b+a \\ 0 & -2 & c \end{bmatrix}$$

Multiply each element of R_2 by $\frac{1}{4}$ to make the entry at $2, 2$ a 1.

$$\begin{bmatrix} 1 & 3 & a \\ 0 & 1 & \frac{b+a}{4} \\ 0 & -2 & c \end{bmatrix}$$

Perform the row operation $R_3 = R_3 + 2R_2$ to make the entry at $3, 2$ a 0.

$$\begin{bmatrix} 1 & 3 & a \\ 0 & 1 & \frac{b+a}{4} \\ 0 & 0 & \frac{2c+b+a}{2} \end{bmatrix}$$

So these vectors are linearly independent if and only if,

$$\frac{2c+b+a}{2} \neq 0$$

So we get,

$$2c+b+a \neq 0$$

We can choose ,

a=1

b=0

c=0

So ,

$$v_3 = (1,0,0)$$

Hence,

$$\{v_1, v_2, v_3\} \quad \text{is a basis of } R^3.$$

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

1. Linear Independence:

$$\begin{aligned} c_1u_1 + c_2u_2 + c_3u_3 &= 0 \\ c_1v_1 + c_2(v_1 + v_2) + c_3(v_1 + v_2 + v_3) &= 0 \\ (c_1 + c_2 + c_3)v_1 + (c_2 + c_3)v_2 + c_3v_3 &= 0 \end{aligned}$$

Since $\{v_1, v_2, v_3\}$ is a basis of V so it is linearly independent.
Hence we have

$$\begin{aligned} (c_1 + c_2 + c_3) &= 0 \\ c_2 + c_3) &= 0 \\ c_3 &= 0 \end{aligned}$$

Putting $c_3 = 0$ in equation 2 we have $c_2 = 0$
putting $c_2 = c_3 = 0$ in equation 1 we have $c_1 = 0$
Hence $c_1 = c_2 = c_3 = 0$
Therefore, $\{u_1, u_2, u_3\}$ is linearly independent and hence a basis.

15. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^3.

Given vectors :
$v_1 = (1, -2, 3)$
$v_2 = (0, 5, -3)$

These vectors are linearly independent.

Motive: We want to enlarge $\{v_1, v_2\}$ to a basis for R^3.

Explanation:

To enlarge the set $\{v_1, v_2\}$ to a basis for R^3, we need to add one more vector that is linearly independent of $\{v_1, v_2\}$.

Let's find such a vector:

Let's call the new vector $v_3 = (a, b, c)$. We want v_3 to be linearly independent of v_1 and v_2. This means that the determinant of the matrix formed by these vectors should be nonzero.

So,

$$\begin{vmatrix} 1 & -2 & 3 \\ 0 & 5 & -3 \\ a & b & c \end{vmatrix} \neq 0$$

$\Rightarrow 1(5c + 3b) - 0(-2c - 3b) + a(6 - 15) \neq 0$

$\Rightarrow -9a + 3b + 5c \neq 0$

Now, we can choose any values for a, b, and c that satisfy this condition.
Let's arbitrarily choose $a = 1, b = 0 \quad \text{and} \quad c = 1$
so, $v_3 = (1, 0, 1)$

Now, $\{v_1, v_2, v_3\}$ forms a basis for R^3 because it is a set of three linearly independent vectors in R^3.

16. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4.

Step 1

The given vectors are $\mathbf{v}_1 = (1, 0, 0, 0) \quad \textbf{and} \quad \mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent vectors.

Consider the vectors in matrix form $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

here pivot 1 is in the first and second column

Step 2

therefore we add a third and fourth elements of the standard basis in the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ in order to obtain a basis of $\mathbb{R}^4$

Step 1

The third vector, $\mathbf{v3}$, can be $(0, 0, 1, 0)$

Explanation:

This vector is linearly independent from $\mathbf{v1}$ and $\mathbf{v2}$ because it has a non-zero entry in a position where both $\mathbf{v1}$ and $\mathbf{v2}$ have zeros.

Step 2

The fourth vector, $\mathbf{v4}$, can be $(0, 0, 0, 1)$

Explanation:

This vector is linearly independent from $\mathbf{v1}$, $\mathbf{v2}$, and $\mathbf{v3}$ because it has a non-zero entry in a position where all the other vectors have zeros.

17. Find a basis for the subspace of R^3 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (2, 0, 1), \quad \mathbf{v}_4 = (0, 0, -1)$$

Step 1

Given vectors:
$\mathbf{v}_1 = (1,0,0),\ \mathbf{v}_2 = (1,0,1),\ \mathbf{v}_3 = (2,0,1),\ \mathbf{v}_4 = (0,0,-1)$
The aim is to find the basis.

First, write the vectors in matrix form.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Swap R_3 with R_2 to put a nonzero entry at $2, 2$.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Perform the row operation $\mathrm{R}_1 = \mathrm{R}_1 - \mathrm{R}_2$ to make the entry at $1, 2$ a 0.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here rank of the matrix is 2 and only two pivot elements are there.
So, the $\mathbf{basis} = \{\mathbf{v}_1, \mathbf{v}_2\}$

Explanation:

Get the basis by using the reduced row echelon form of the matrix.

The basis is $\{v_1 = (1,0,0), v_2 = (1,0,1)\}$

4.6 Change of Basis

3. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

(a) Find the transition matrix B to B'.

(b) Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

and use (12) to compute $[\mathbf{w}]_{B'}$.

(c) Check your work by computing $[\mathbf{w}]_{B'}$ directly.

$P_{B' \to B} = \left[[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B \right]$	$P_{B \to B'} = \left[[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \cdots \mid [\mathbf{u}_n]_{B'} \right]$

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}]$$

(a).

The given basis is $\mathrm{B} = \{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\}$

$$\mathrm{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathrm{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathrm{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The given basis is $\mathrm{B}' = \{\mathrm{u}'_1, \mathrm{u}'_2, \mathrm{u}'_3\}$

$$\mathrm{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \mathrm{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathrm{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Now we find the transition matrix($\mathrm{T_m}$) from B to B' by this

$[\mathrm{B}'|\mathrm{B}] = [\mathrm{I}|\mathrm{T_m}]$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$$

By row reduce echelon form we get

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

Therefore the transition matrix from B to B' is

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix}$$

(b).

The given basis is $B = \{u_1, u_2, u_3\}$

$$u_1 = \begin{bmatrix}2\\1\\1\end{bmatrix}, u_2 = \begin{bmatrix}2\\-1\\1\end{bmatrix}, u_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

We have to find the corresponding coordinate vector $[W]_\beta$. $W = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$ I claim that the coordinate vector entries x_1, x_2, x_3 Satisfies thr following criterion.

Then,

$$x_1\begin{bmatrix}2\\1\\1\end{bmatrix} + x_2\begin{bmatrix}2\\-1\\1\end{bmatrix} + x_3\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c}2 & 2 & 1 & -5\\1 & -1 & 2 & 8\\1 & 1 & 1 & -5\end{array}\right] \approx \left[\begin{array}{ccc|c}2 & 2 & 1 & -5\\0 & 4 & -3 & -21\\0 & 0 & -1 & 5\end{array}\right] \begin{cases}(R_3 = -2\times R_3 + R_1)\\ R_2 = -2\times R_2 + R_1\end{cases}$$

$\Rightarrow x_3 = -5$

$\Rightarrow 4x_2 - 3x_3 = -21 \Rightarrow x_2 = -9$

$\Rightarrow 2x_1 + 2x_2 + 1x_3 = 5 \Rightarrow x_1 = 9$

Therefore the coordinate vectors are $[W]_\beta = (9, -9, -5)$

(c).

The given basis is $B' = \{u'_1, u'_2, u'_3\}$

$$u'_1 = \begin{bmatrix}3\\1\\-5\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\\-3\end{bmatrix}, u_3 = \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

We have to find the corresponding coordinate vector $[W]_{\beta'}$. $W = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$ I claim that the coordinate vector entries x_1, x_2, x_3 Satisfies thr following criterion.

Then,

$$x'_1\begin{bmatrix}3\\1\\-5\end{bmatrix} + x'_2\begin{bmatrix}1\\1\\-3\end{bmatrix} + x'_3\begin{bmatrix}-1\\0\\2\end{bmatrix} = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}3 & 1 & -1 & -5\\1 & 1 & 0 & 8\\-5 & -3 & 2 & -5\end{array}\right] \approx \left[\begin{array}{ccc|c}3 & 1 & -1 & -5\\0 & 2 & 1 & 29\\0 & -4 & 1 & -40\end{array}\right] \begin{cases}(R_3 = 3R_3 + 5R_1)\\ R_2 = 3R_2 - R_1\end{cases} \approx \left[\begin{array}{ccc|c}3 & 1 & -1 & -5\\0 & 2 & 1 & 29\\0 & 0 & 3 & 18\end{array}\right] R_3 = 2R_2 + R_3$$

$\Rightarrow 3x'_3 = 18 \Rightarrow x'_3 = 6$

$\Rightarrow 2x'_2 + x'_3 = 29 \Rightarrow x'_2 = \frac{23}{2}$

$\Rightarrow 3x'_1 + x'_2 - x'_3 = -5 \Rightarrow x'_1 = -\frac{7}{2}$

Therefore the coordinate vectors are $[W]_{\beta'} = \left(-\frac{7}{2}, \frac{23}{2}, 6\right)$

(c).

Now we check the coordinate vector of $[W]_{\beta'}$ directly

That,

$$3x'_1 + x'_2 - x'_3 = -5$$
$$x'_1 + x'_2 - 0x'_3 = 8$$
$$-5x'_1 - 3x'_2 + 2x'_3 = -5$$

Now we check the coordinate vectors are $[W]_{\beta'} = \left(-\frac{7}{2}, \frac{23}{2}, 6\right)$

Here, $x'_1 = -\frac{15}{2}, x'_2 = \frac{23}{2}, x'_3 = 6$

Therefore we clearly see that the coordinate vectors are satisfies these three equations.

Explanation:

Simply Put the value of variable on left hand side and get R. H. S. And satisfies this.

12. If B_1, B_2, and B_3 are bases for R^2, and if

$$P_{B_1 \to B_2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad P_{B_2 \to B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}$$

then $P_{B_3 \to B_1} =$ _________.

$$P_{B_1 \to B_2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \qquad P_{B_2 \to B_2} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}$$

To find $P_{B_3 \to B_1}$

$$\left[\begin{array}{cc|cc} 3 & 1 & 7 & 2 \\ 5 & 2 & 4 & -1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 5 & 2 & 4 & -1 \\ 3 & 1 & 7 & 2 \end{bmatrix}$$

$R_2 \to R_2 - \frac{3}{5} R_1$

$$\begin{bmatrix} 5 & 2 & 4 & -1 \\ 0 & -\frac{1}{5} & \frac{23}{5} & \frac{13}{5} \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 5 & 2 & 4 & -1 \\ 0 & -1/5 & \frac{23}{5} & \frac{13}{5} \end{array}\right]$$

$R_2 \to R_2 \times (-5)$

$$\left[\begin{array}{cc|cc} 5 & 2 & 4 & -1 \\ 0 & +1 & -23 & -13 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 5 & 0 & 50 & 25 \\ 0 & 1 & -23 & -13 \end{bmatrix}$$

$R_1 \to R_1 \div (5)$

$$\left[\begin{array}{cc|cc} 1 & 0 & 10 & 5 \\ 0 & 1 & -23 & -13 \end{array}\right]$$

Hence

$$P_{B_3 \to B_1} = \begin{bmatrix} 10 & 5 \\ -23 & -13 \end{bmatrix}$$ Ans.

13. If P is the transition matrix from a basis B' to a basis B, and Q is the transition matrix from B to a basis C, what is the transition matrix from B' to C? What is the transition matrix from C to B'?

Given: If **P** is the transition matrix from a basis B' to a basis **B** , and **Q** is the transition matrix from **B** to a basis **C** , what is the transition matrix from B' to **C** ? What is the transition matrix from **C** to B' ?

Solution:

Let's denote the following:

- P is the transition matrix from basis B' to basis B.
- Q is the transition matrix from basis B to basis C.

Explanation:

We need to find:

1. The transition matrix from B' to C.
2. The transition matrix from C to B'.

1. Transition Matrix from B' to C

To find the transition matrix from B' to C, we need to combine the transition from B' to B (given by P) and the transition from B to C (given by Q).

The transition matrix from B' to C is obtained by multiplying Q and P :

$$R = QP$$

where R is the transition matrix from B' to C.

2. Transition Matrix from C to B'

To find the transition matrix from C to B', we need to consider the inverse of the transition matrices in reverse order.

First, let's find the inverse of Q which transitions from C to B :

$$Q^{-1}$$

Next, find the inverse of P which transitions from B to B' :

$$P^{-1}$$

The transition matrix from C to B' is then the product of these inverses:

$$S = P^{-1}Q^{-1}$$

where S is the transition matrix from C to B'.

4.7 Row Space, Column Space, and Null Space

13. (a) Use the methods of Examples 6 and 7 to find bases for the row space and column space of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

(b) Use the method of Example 9 to find a basis for the row space of A that consists entirely of row vectors of A.

THEOREM 4.7.4 *Elementary row operations do not change the row space of a matrix.*

THEOREM 4.7.6 *If A and B are row equivalent matrices, then:*

(*a*) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(*b*) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

The given matrix is $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$

Lets find the row echelon form

Performing the row operation $R_2 = R_2 + 2R_1$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

Performing the row operation $R_3 = R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

Performing the row operation $R_4 = R_4 + 3R_1$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 6 & 1 & 0 \end{bmatrix}$$

Performing the row operation $R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 6 & 1 & 0 \end{bmatrix}$$

Performing the row operation $R_4 = R_4 - 2R_2$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Performing the row operation $R_4 = R_4 - R_3$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Performing the row operation $R_1 = R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The column space is a space spanned by the columns of the initial matrix that correspond to the pivot columns of the reduced matrix.

Thus, the column space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (answer)

b) Now basis for the row space is the span of row space

Hence basis for the row space= $\mathbf{span} \left(\begin{bmatrix} 1 \\ 0 \\ 11 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$ (answer)

b-Solution We will transpose *A*, thereby converting the row space of *A* into the column space of A^T; then we will find a basis for the column space of A^T ; and then we will transpose again to convert column vectors back to row vectors.

Transposing *A* yields

$$A^T = \begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$$

and then reducing this matrix to row echelon form we obtain

Row echelon form
Given matrix

$$\begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$$

$R_2 \leftarrow R_2 + 2 \times R_1$

$$= \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 5 \times R_1$

$$= \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$$

$R_5 \leftarrow R_5 - 3 \times R_1$

$$= \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 3 \times R_2$

$$= \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After interchanging rows $R_3 \leftrightarrow R_4$

$$= \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and third columns contain the leading 1's, so the corresponding column vectors in A^T form a basis for the column space of A^T ; these are

$$c_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 0 \\ 3 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 5 \\ -7 \\ 0 \\ -6 \end{bmatrix}, c_3 = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

r1 = [1 -2 5 0 3], **r**2 = [-2 5 -7 0 -6] , **r**3 =[-1 3 -2 1 -3]

for the row space of *A*

▶ In Exercises 14–15, find a basis for the subspace of R^4 that is spanned by the given vectors. ◀

15. $(1, 1, 0, 0)$, $(0, 0, 1, 1)$, $(-2, 0, 2, 2)$, $(0, -3, 0, 3)$

Consider the matrix A wit given vectors as the columns.

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Find the row echelon form

Perform the row operation $\mathrm{R}_2 = \mathrm{R}_2 - \mathrm{R}_1$ to make the entry at $2, 1$ a 0.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Swap R_3 with R_2 to put a nonzero entry at $2, 2$.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Perform the row operation $\mathrm{R}_4 = \mathrm{R}_4 - \mathrm{R}_2$ to make the entry at $4, 2$ a 0.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Multiply each element of R_3 by $\frac{1}{2}$ to make the entry at $3, 3$ a 1.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Multiply each element of R_4 by $\frac{1}{3}$ to make the entry at $4, 4$ a 1.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The leading $1'$s are in all four columns.

Hence, all four columns are pivot columns.

Therefore, the basis for subspace $\mathbb{R}^4$ which is spanned by the given vectors is the set of all given four vectors.

i.e. a basis is $\{(1, 1, 0, 0), (0, 0, 1, 1), (-2, 0, 2, 2), (0, -3, 0, 3)\}$.

► In Exericses 16–17, find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors. ◄

16. $\mathbf{v}_1 = (1, 0, 1, 1)$, $\mathbf{v}_2 = (-3, 3, 7, 1)$, $\mathbf{v}_3 = (-1, 3, 9, 3)$, $\mathbf{v}_4 = (-5, 3, 5, -1)$

16)

Given vectors:

$\mathbf{v}_1 = (1,0,1,1)$, $\mathbf{v}_2 = (-3,3,7,1)$, $\mathbf{v}_3 = (-1,3,9,3)$, $\mathbf{v}_4 = (-5,3,5,-1)$

We will put these vectors into a matrix and row reduce to find the basis.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -3 & 3 & 7 & 1 \\ -1 & 3 & 9 & 3 \\ -5 & 3 & 5 & -1 \end{bmatrix}$$

Perform the row operation $R_2 = R_2 + 3R_1$.

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ -1 & 3 & 9 & 3 \\ -5 & 3 & 5 & -1 \end{bmatrix}$$

Perform the row operation $R_3 = R_3 + R_1$.

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \\ -5 & 3 & 5 & -1 \end{bmatrix}$$

Perform the row operation $R_4 = R_4 + 5R_1$.

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \end{bmatrix}$$

Multiply each element of R_2 by $\frac{1}{3}$.

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \end{bmatrix}$$

Perform the row operation $R_3 = R_3 - 3R_2$.

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 10 & 4 \end{bmatrix}$$

Perform the row operation $R_4 = R_4 - 3R_2$.

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the third and fourth vectors are linearly dependent on the first and second vectors. So, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the space spanned by the given vectors.

Now, let's express $\mathbf{v}_3$ and $\mathbf{v}_4$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

For $\mathbf{v}_3$:

$\mathbf{v}_3 = (-1,3,9,3) = 2(1,0,1,1) + 1(-3,3,7,1)$

For $\mathbf{v}_4$:

$\mathbf{v}_4 = (-5,3,5,-1) = -2(1,0,1,1) + 1(-3,3,7,1)$

17. $\mathbf{v}_1 = (1, -1, 5, 2)$, $\mathbf{v}_2 = (-2, 3, 1, 0)$, $\mathbf{v}_3 = (4, -5, 9, 4)$, $\mathbf{v}_4 = (0, 4, 2, -3)$, $\mathbf{v}_5 = (-7, 18, 2, -8)$

17)

Given: $\mathbf{v}_1 = (1, -1, 5, 2)$, $\mathbf{v}_2 = (-2, 3, 1, 0)$, $\mathbf{v}_3 = (4, -5, 9, 4)$, $\mathbf{v}_4 = (0, 4, 2, -3)$, $\mathbf{v}_5 = (-7, 18, 2, -8)$ We will put these vectors into a matrix and row reduce to find the basis.

$$\begin{bmatrix} 1 & -1 & 5 & 2 \\ -2 & 3 & 1 & 0 \\ 4 & -5 & 9 & 4 \\ 0 & 4 & 2 & -3 \\ -7 & 18 & 2 & -8 \end{bmatrix}$$

Perform the row operation $R_2 = R_2 + 2R_1$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 4 & -5 & 9 & 4 \\ 0 & 4 & 2 & -3 \\ -7 & 18 & 2 & -8 \end{bmatrix}$$

Perform the row operation $R_3 = R_3 - 4R_1$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & -1 & -11 & -4 \\ 0 & 4 & 2 & -3 \\ -7 & 18 & 2 & -8 \end{bmatrix}$$

Perform the row operation $R_5 = R_5 + 7R_1$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & -1 & -11 & -4 \\ 0 & 4 & 2 & -3 \\ 0 & 11 & 37 & 6 \end{bmatrix}$$

Perform the row operation $R_3 = R_3 + R_2$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & -3 \\ 0 & 11 & 37 & 6 \end{bmatrix}$$

Perform the row operation $R_4 = R_4 - 4R_2$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -42 & -19 \\ 0 & 11 & 37 & 6 \end{bmatrix}$$

Perform the row operation $R_5 = R_5 - 11R_2$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -42 & -19 \\ 0 & 0 & -84 & -38 \end{bmatrix}$$

Swap R_4 with R_3 to put a nonzero entry at $3, 3$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & 0 & -42 & -19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -84 & -38 \end{bmatrix}$$

Multiply each element of $\mathbf{R}_3$ by $-\frac{1}{42}$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & 0 & 1 & \frac{19}{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -84 & -38 \end{bmatrix}$$

Perform the row operation $\mathbf{R}_5 = \mathbf{R}_5 + 84\mathbf{R}_3$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 11 & 4 \\ 0 & 0 & 1 & \frac{19}{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Perform the row operation $\mathbf{R}_2 = \mathbf{R}_2 - 11\mathbf{R}_3$.

$$= \begin{bmatrix} 1 & -1 & 5 & 2 \\ 0 & 1 & 0 & -\frac{41}{42} \\ 0 & 0 & 1 & \frac{19}{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Perform the row operation $\mathbf{R}_1 = \mathbf{R}_1 - 5\mathbf{R}_3$.

$$= \begin{bmatrix} 1 & -1 & 0 & -\frac{11}{42} \\ 0 & 1 & 0 & -\frac{41}{42} \\ 0 & 0 & 1 & \frac{19}{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Perform the row operation $\mathbf{R}_1 = \mathbf{R}_1 + \mathbf{R}_2$.

$$= \begin{bmatrix} 1 & 0 & 0 & -\frac{26}{21} \\ 0 & 1 & 0 & -\frac{41}{42} \\ 0 & 0 & 1 & \frac{19}{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the first two vectors form a basis for the space spanned by the given vectors

Now, let's express $\mathbf{v}_3$ and $\mathbf{v}_5$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$:

For $\mathbf{v}_4$:

$\mathbf{v}_3 = (4, -5, 9, 4) = 2(1, -1, 5, 2) - 1(-2, 3, 1, 0) + 0(0, 4, 2, -3)$

For $\mathbf{v}_5$:

$\mathbf{v}_5 = (-7, 18, 2, -8) = -1(1, -1, 5, 2) + 3(-2, 3, 1, 0) + 2(0, 4, 2, -3)$

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

▶ In Exercises 1–2, find the rank and nullity of the matrix A by reducing it to row echelon form. ◀

1. (a) $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

THEOREM 4.8.2 Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$rank(A) + nullity(A) = n$$

1) a) Apply the row operation on the matrix as follows:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \leftarrow R_2 - 2R_1, R_3 \leftarrow R_3 - 3R_1, R_4 \leftarrow R_4 - 4R_1$$

Since the row echelon form of A contain 1 nonzero row so $\text{rank}(A) = 1$ and $\text{nullity}(A) = 4 - 1 = 3$ (since A has 4 columns) [$\text{nullity}(A) + \text{rank}(A) = 4$].

b) Apply the row operation on the matrix as follows:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} R_2 \leftarrow R_2 + 3R_1, R_3 \leftarrow R_3 - 2 \times R_1$$

$$= \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_2 \leftarrow \frac{R_2}{5}, R_3 \leftarrow R_3 - R_2$$

7. In each part, find the largest possible value for the rank of A and the smallest possible value for the nullity of A.

(a) A is 4×4 (b) A is 3×5 (c) A is 5×3

$$\text{rank}(A) \leq \min(m, n)$$

in which $\min(m, n)$ is the minimum of m and n.

$\mathbf{A}$ is 4×4

For a 4×4 matrix $\mathbf{A}$, the largest possible value for the rank of $\mathbf{A}$ is 4, which means all four rows or columns are linearly independent.

Explanation:

We know that:

The rank of a matrix is the number of linearly independent rows or columns. So, the largest possible value for the rank of an $\mathbf{m} \times \mathbf{n}$ matrix is $\min(\mathbf{m}, \mathbf{n})$.

The smallest possible value for the nullity of $\mathbf{A}$ is 0, indicating that the null space is empty.

$\mathbf{A}$ is 3×5

For a 3×5 matrix $\mathbf{A}$, the largest possible value for the rank of $\mathbf{A}$ is $\min(3, 5) = 3$, which means 3 rows or columns are linearly independent.

Now, According to the Rank-Nullity Theorem,

$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) =$ number of columns in $\mathbf{A}$.

smallest possible value for the nullity of the matrix $\mathbf{A} = 5 - \text{rank}(\mathbf{A})$

$$= 5 - 3$$

$$= 2$$

$\mathbf{A}$ is 5×3

For a 5×3 matrix $\mathbf{A}$, the largest possible value for the rank of $\mathbf{A}$ is $\min(5, 3) = 3$, which means 3 rows or columns are linearly independent.

Now, According to the Rank-Nullity Theorem,

$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) =$ number of columns in $\mathbf{A}$.

smallest possible value for the nullity of the matrix $\mathbf{A} = 3 - \text{rank}(\mathbf{A})$

$$= 3 - 3$$

$$= 0$$

15. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

Solution: Given matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

We know that rank of matrix is common dimension of row space and column space of matrix .

Let us suppose $\mathrm{r} = 2, \mathrm{s} = 1$ then matrix is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$ Let's apply some operations,

$R_4 \to R_4 - \frac{3}{2} R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $R_3 \to \frac{R_3}{4}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now $R_3 \to R_3 - \frac{R_2}{2}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{2}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Rank is 2.

Rank can not be 1.

If $\mathrm{r} = 2$ and $\mathrm{s} = 1$ then

$\mathrm{rank} = 2$

19. (a) If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ______. Why?

(b) If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ______. Why?

(c) If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ______. Why?

(d) If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ______. Why?

The proof of Theorem 4.8.1 shows that the rank of A can be interpreted as the number of leading 1's in any row echelon form of A.

THEOREM 4.8.3 *If A is an $m \times n$ matrix, then*

(a) rank(A) = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.

(b) nullity(A) = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

a)

Answer is 3

Explanation:

Reason:- There are 3 row in given matrix A and number of rows is less than number of columns

b)

Answer is 5

Reason:- There are 5 columns in given matrix A

c)

Answer is 3

Reason:- There are at most three columns can be povot columns

d)

Answer is 3

Reason:- There are 3 columns in given matrix A

21. Let A be a 5×7 matrix with rank 4.

(a) What is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

(b) Is $A\mathbf{x} = \mathbf{b}$ consistent for all vectors $\mathbf{b}$ in R^5? Explain.

(a) The dimension of the solution space of $\mathbf{Ax} = \mathbf{0}$ is equal to the difference between the number of columns in matrix $\mathbf{A}$ and its rank.
Dimension of solution space = Number of columns - Rank

In this case, $\mathbf{A}$ is a 5×7 matrix with rank 4.
Dimension of solution space = Number of columns - Rank
Dimension of solution space$= 7 - 4 = 3$
So, the dimension of the solution space of $\mathbf{Ax} = \mathbf{0}$ is 3.

THEOREM 4.8.9 *Let A be an $m \times n$ matrix.*

(*a*) (***Overdetermined Case***). *If $m > n$, then the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in R^n.*

(*b*) (***Underdetermined Case***). *If $m < n$, then for each vector $\mathbf{b}$ in R^m the linear system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.*

(b) No, there is not a solution for $\mathbf{Ax} = \mathbf{b}$ for all vectors $\mathbf{b}$ in $\mathbf{R5}$. In fact, since the matrix $\mathbf{A}$ has rank 4, it means that its row space (the space spanned by the rows of $\mathbf{A}$) is of dimension 4.
The row space of $\mathbf{A}$ is a subspace of $\mathbf{R}^7$ (since $\mathbf{A}$ is a 5×7 matrix), and its dimension is 4.
This means that the equation $\mathbf{Ax} = \mathbf{b}$ can be consistent (i.e., have a solution) only if $\mathbf{b}$ lies in the row space of $\mathbf{A}$, which is a subspace of $\mathbf{R}^7$ with dimension 4.
Therefore, there are many vectors in $\mathbf{R}^5$ that are not in the row space of $\mathbf{A}$, and for those vectors, the equation $\mathbf{Ax} = \mathbf{b}$ does not have a solution.

6.1 Inner Products

In Exercises 17–18, find $\|\mathbf{u}\|$ and $d(\mathbf{u}, \mathbf{v})$ relative to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + 3u_2v_2$ on R^2.

18. $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$

DEFINITION 2 If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle}$$

A vector of norm 1 is called a ***unit vector***.

For the given vector $\mathrm{u} = (-1, 2)$:

$$||\mathrm{u}|| = \sqrt{2(-1)(-1) + 3(2)(2)}$$

$$||\mathrm{u}|| = \sqrt{2 + 12} = \sqrt{14}.$$

(ii) Distance $(\mathrm{d}(\mathrm{u}, \mathrm{v}))$:

The distance between two vectors u and v in this inner product space is given by the norm of their difference.

$$\mathrm{d}(\mathrm{u}, \mathrm{v}) = ||\mathrm{u} - \mathrm{v}||$$

For the given vectors $\mathrm{u} = (-1, 2)$ and $\mathrm{v} = (2, 5)$:

$$\mathrm{u} - \mathrm{v} = (-1 - 2, 2 - 5) = (-3, -3)$$

$$\mathrm{d}(\mathrm{u}, \mathrm{v}) = ||-3 \quad -3||$$

$$\mathrm{d}(\mathrm{u}, \mathrm{v}) = \sqrt{2(-3)(-3) + 3(-3)(-3)}$$

$$\mathrm{d}(\mathrm{u}, \mathrm{v}) = \sqrt{18 + 27} = \sqrt{45} = 3\sqrt{5}.$$

In Exercises 19–20, find $\|\mathbf{p}\|$ and $d(\mathbf{p}, \mathbf{q})$ relative to the standard inner product on P_2.

20. $\mathbf{p} = -5 + 2x + x^2, \quad \mathbf{q} = 3 + 2x - 4x^2$

As one know that, $\|p\| = \sqrt{< p, p >}$ and $d(p, q) = \|p - q\|$

For any $p = a + bx + cx^2$ and $q = d + ex + fx^2$, standard inner product can be given by,

$$< p, q > = ad + be + cf$$

For given p ,

$$\begin{aligned} < p, p > &= (-5)^2 + (2)^2 + (1) \\ &= 25 + 4 + 1 \\ &= 29 \end{aligned}$$

Therefore, $\|p\| = \sqrt{< p, p >} = \sqrt{29}$.

Moreover,

$$\begin{aligned} p - q &= \left(-5 + 2x + x^2\right) - \left(3 + 2x - 4x^2\right) \\ &= -8 + 5x^2 \end{aligned}$$

Therefore, to find $d(p, q)$, let first find $< p - q, p - q >$:

$$\begin{aligned} < p - q, p - q > &= (-8)^2 + (5)^2 \\ &= 64 + 25 \\ &= 89 \end{aligned}$$

So, $d(p, q)$ can be calculated as,

$$\begin{aligned} d(p - q) &= \|p - q\| \\ &= \sqrt{< p - q, p - q >} \\ &= \sqrt{89} \end{aligned}$$

In Exercises 27–28, suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in an inner product space such that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2, \quad \langle \mathbf{v}, \mathbf{w} \rangle = -6, \quad \langle \mathbf{u}, \mathbf{w} \rangle = -3$$
$$\|\mathbf{u}\| = 1, \quad \|\mathbf{v}\| = 2, \quad \|\mathbf{w}\| = 7$$

Evaluate the given expression.

27. (a) $\langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle$ (b) $\|\mathbf{u} + \mathbf{v}\|$

To evaluate $\langle 2v - w, 3u + 2w \rangle$ use the distributive property of the dot product and then substitute the given values.

$$= \langle 2v - w, 3u + 2w \rangle$$
$$= \langle 2v, 3u + 2w \rangle - \langle w, 3u + 2w \rangle$$
$$= \langle 2v, 3u \rangle + \langle 2v, 2w \rangle - \langle w, 3u \rangle - \langle w, 2w \rangle$$
$$= 6\langle v, u \rangle + 4\langle v, w \rangle - 3\langle w, u \rangle - 2\langle w, w \rangle$$

Substitute the values then

$$= 6\langle u, v \rangle + 4\langle v, w \rangle - 3\langle u, w \rangle - 2||w||^2$$
$$= 6 \times 2 + 4 \times (-6) - 3 \times (-3) - 2 \times 7^2$$
$$= 12 - 24 + 9 - 98$$
$$= 21 - 122$$
$$\langle 2v - w, 3u + 2w \rangle = -101$$

(b) Find $||u + v||$.

Explanation:

To evaluate $||u + v||$, use the definition of the norm and substitute the given values

$$||u + v|| = \sqrt{\langle u + v, u + v \rangle}$$
$$= \sqrt{\langle u, u + v \rangle + \langle v, u + v \rangle}$$
$$= \sqrt{\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle}$$
$$= \sqrt{||u||^2 + \langle u, v \rangle + \langle u, v \rangle + ||v||^2}$$
$$= \sqrt{1 + 2 + 2 + 4} =$$
$$= \sqrt{9}$$
$$[||u + v|| = 3$$

▶ In Exercises 33–34, let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Show that the expression does *not* define an inner product on R^3, and list all inner product axioms that fail to hold. ◀

33. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$

34. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$

Let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$.

Explanation:

The inner product $\langle\ ,\ \rangle$ satisfies the properties :

(1) **Linearity** : $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

(2) **Symmetry** : $\langle u, v \rangle = \langle v, u \rangle$

(3) **Positive definite** : $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$

33. $\langle u, v \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$ ......(i)

(1) **Linearity** : Let $a, b \in R$. For any three vectors $u, v, w \in R^3$,

$au + bv = a(u_1, u_2, u_3) + b(v_1, v_2, v_3) = (au_1 + bv_1, au_2 + bv_2, au_3, bv_3)$

$\Rightarrow \langle au + bv, w \rangle = (au_1 + bv_1)^2 w_1^2 + (au_2 + bv_2)^2 w_2^2 + (au_3 + bv_3)^2 w_3^2$

and

$a\langle u, w \rangle +\ b\langle v, w \rangle = a\left(u_1^2 w_1^2 + u_2^2 w_2^2 + u_3^2 w_3^2\right) + b\left(v_1^2 w_1^2 + v_2^2 w_2^2 + v_3^2 w_3^2\right)$

$\Rightarrow a\langle u, w \rangle +\ b\langle v, w \rangle = \left(au_1^2 + bv_1^2\right) w_1^2 + \left(au_2^2 + bv_2^2\right) w_2^2 + \left(au_3^2 + bv_3^2\right) w_3^2$

Therefore, $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$ if

$(au_1 + bv_1)^2 w_1^2 + (au_2 + bv_2)^2 w_2^2 + (au_3 + bv_3)^2 w_3^2 = \left(au_1^2 + bv_1^2\right) w_1^2 + \left(au_2^2 + bv_2^2\right) w_2^2 + \left(au_3^2 + bv_3^2\right) w_3^2$

i.e. if $(au_i + bv_i)^2 = au_i^2 + bv_i^2$, where $i = 1, 2, 3$

which is not true in general.

Hence (i) doesn't define an inner product on R^3.

(2) **Symmetry** :

$\langle u, v \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 \ \Rightarrow \langle u, v \rangle = v_1^2 u_1^2 + v_2^2 u_2^2 + v_3^2 u_3^2$

$\Rightarrow \langle u, v \rangle = \langle v, u \rangle$

(i) satisfies the symmetry property.

(3) **Positive definite** :

$\langle u, u \rangle = u_1^2 u_1^2 + u_2^2 u_2^2 + u_3^2 u_3^2$

$\Rightarrow \ \langle u, u \rangle = 2u_1^2 + 2u_2^2 + 2u_3^2 \geq 0$ ($\therefore$ sum of square of real numbers is always non-negative)

Also $\langle u, u \rangle = 0 \Leftrightarrow 2u_1^2 + 2u_2^2 + 2u_3^2 = 0$

$\Leftrightarrow u_1^2 + u_2^2 + u_3^2 = 0 \Leftrightarrow u_1 = 0, u_2 = 0, u_3 = 0$

$\Leftrightarrow u = 0$

Hence, $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$

Therefore, (i) satisfies the positive definite property.

34. $\langle u, v\rangle = u_1v_1 - u_2v_2 + u_3v_3$ ....(ii)

(1) **Linearity** : Let $a, b \in R$. For any three vectors $u, v, w \in R^3$,

$\Rightarrow \langle au + bv, w\rangle = (au_1 + bv_1)w_1 - (au_2 + bv_2)w_2 + (au_3 + bv_3)w_3$

$\Rightarrow \langle au + bv, w\rangle = a(u_1w_1 - u_2w_2 + u_3w_3) + b(v_1w_1 - v_2w_2 + v_3w_3)$

$\Rightarrow \langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$

Hence, (ii) satisfies the linear property.

(2) **Symmetry** :

$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3$

$\Rightarrow \langle u, v\rangle = v_1u_1 - v_2u_2 + v_3u_3$

$\Rightarrow \langle u, v\rangle = \langle v, u\rangle$

Hence, (ii) satisfies the symmetric property.

(3) Positive definite :

$\langle u, u\rangle = u_1u_1 - u_2u_2 + u_3u_3$

$\Rightarrow \langle u, u\rangle = u_1^2 - u_2^2 + u_3^2$

Hence, $\langle u, u\rangle$ need not to be non-negative.

For example : Take $u = (1, 2, 1)$

$\Rightarrow \langle u, u\rangle = 1^2 - 2^2 + 1^2 = 1 - 4 + 1 = -2 < 0$

Hence, (ii) doesn't satisfies the positive definite property.

6.2 Angle and Orthogonality in Inner Product Spaces

▶ In Exercises 3–4, find the cosine of the angle between the vectors with respect to the standard inner product on P_2. ◀

3. $\mathbf{p} = -1 + 5x + 2x^2, \ \mathbf{q} = 2 + 4x - 9x^2$

4. $\mathbf{p} = x - x^2, \ \mathbf{q} = 7 + 3x + 3x^2$

the average inner product on p2.

$3. \mathrm{p} = 1 + 5\mathrm{x} + 2\mathrm{x}2, \mathrm{q} = 2 + 4\mathrm{x} - 9\mathrm{x}2$

The fact that

$$\sin(\theta) = \frac{< \mathrm{p}, \mathrm{q} >}{||\mathrm{p}||. ||\mathrm{q}||}$$

$$< \mathrm{p}, \mathrm{q} > = < (-1 + 5\mathrm{x} + 2\mathrm{x}^2), (2 + 4\mathrm{x} + 9\mathrm{x}^2)$$

$$= (-1\mathrm{x}2) + (5\mathrm{x}4) + (-18)$$

$$= -2 + 20 - 18$$

$$< \mathrm{p}, \mathrm{q} > = 0$$

$$||\mathrm{p}|| = \sqrt{< \mathrm{p}, \mathrm{p} >}$$

$$= \sqrt{1 + 25 + 4}$$

$$= \sqrt{30}$$

$$||\mathrm{q}|| = \sqrt{< \mathrm{q}, \mathrm{q} >}$$

$$= \sqrt{4 + 16 + 81}$$

$$= \sqrt{101}$$

$$\therefore \cos\theta = \frac{0}{\sqrt{30. \sqrt{101}}}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1} 0$$

$$\theta = 9^0$$

the average inner product on p2.

$4. \mathrm{p} = \mathrm{x} - \mathrm{x}2, \mathrm{q} = 7 + 3\mathrm{x} + 3\mathrm{x}2$

$$\cos\theta = \frac{< \mathrm{p}, \mathrm{q} >}{||\mathrm{p}||. ||\mathrm{q}||}$$

$$= \frac{< \mathrm{x} - \mathrm{x}^2, 7 + 3\mathrm{x} + 3\mathrm{x}^2 >}{\sqrt{< \mathrm{x} - \mathrm{x}^2, \mathrm{x} - \mathrm{x}^2 >}, \sqrt{7 + 3\mathrm{x} + 3\mathrm{x}^2, 7 + 3\mathrm{x} + 3\mathrm{x}^2}}$$

$$= \frac{0 + 3 - 3}{\sqrt{2}. \sqrt{67}}$$

$$= 0$$

$$\theta = 9^0$$

In Exercises 7–8, determine whether the vectors are orthogonal with respect to the Euclidean inner product.

7. (a) $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (4, 2, -1)$

(b) $\mathbf{u} = (-2, -2, -2)$, $\mathbf{v} = (1, 1, 1)$

(c) $\mathbf{u} = (a, b)$, $\mathbf{v} = (-b, a)$

DEFINITION 1 Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V called ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Two vectors u and v are orthogonal if inner product of vector u.v =0

Que A)

$$u = (-1, 3, 2) \quad , v = (4, 2, -1)$$

$$\begin{aligned} u.v &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= (-1)(4) + (3)(2) + (2)(-1) \\ &= -4 + 6 - 2 \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

Since the dot product is equal to zero the vectors are orthogonal

Que B)

$$u = (-2, -2, -2) \quad \text{and} \quad v = (1, 1, 1)$$

$$\begin{aligned} u.v &= (-2)(1) + (-2)(1) + (-2)(1) \\ &= -2 - 2 - 2 \\ &= -6 \end{aligned}$$

Dot product is not equal zero
Therefore the vectors are not orthogonal

Que C)

$$u = (a, b) \quad \text{and} \quad v = (-b, a)$$

$$\begin{aligned} u.v &= u_1v_1 + u_2v_2 \\ &= (a)(-b) + (b)(a) \\ &= -ab + ba \\ &= 0 \end{aligned}$$

Since the dot product is equal to zero
The vectors are orthogonal

▶ In Exercises 11–12, show that the matrices are orthogonal with respect to the standard inner product on M_{22}. ◀

12. $U = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$

To show that two matrices are orthogonal concerning the standard inner product on $\mathrm{M_{22}}$, we need to verify if their inner product is zero.

The standard inner product of two matrices A and B is defined as the sum of the products of their corresponding entries.

Let's calculate the standard inner product of U and V:

$$\mathrm{U} = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix}$$
$$\mathrm{V} = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$$

The standard inner product (U, V) is given by:

$$(\mathrm{U}, \mathrm{V}) = \mathrm{U}[0][0] \times \mathrm{V}[0][0] + \mathrm{U}[0][1] \times \mathrm{V}[0][1] + \mathrm{U}[1][0] \times \mathrm{V}[1][0] + \mathrm{U}[1][1] \times \mathrm{V}[1][1]$$

Substituting the values:

$$(\mathrm{U}, \mathrm{V}) = (5 \times 1) + (-1 \times 3) + (2 \times -1) + (-2 \times 0)$$
$$(\mathrm{U}, \mathrm{V}) = 5 - 3 - 2 + 0$$
$$(\mathrm{U}, \mathrm{V}) = 0$$

Since the standard inner product of U and V is 0, we can conclude that the matrices U and V are orthogonal with respect to the standard inner product on $\mathrm{M_{22}}$.

15. If the vectors $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (2, -4)$ are orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2$, what must be true of the weights w_1 and w_2?

Given vectors $\mathrm{u} = (1, 2)$ and $\mathrm{v} = (2, -4)$, determine the weights w_1 and w_2 in the weighted Euclidean inner product formula
$\langle \mathrm{u}, \mathrm{v} \rangle = \mathrm{w}_1 \mathrm{u}_1 \mathrm{v}_1 + \mathrm{w}_2 \mathrm{u}_2 \mathrm{v}_2$
such that u and v are orthogonal.

Substitute the Vector Components: Substitute the components of u and v into the inner product formula:
$\langle \mathrm{u}, \mathrm{v} \rangle = \mathrm{w}_1 \cdot 1 \cdot 2 + \mathrm{w}_2 \cdot 2 \cdot (-4).$

Simplify the Expression:
$\langle \mathrm{u}, \mathrm{v} \rangle = 2\mathrm{w}_1 - 8\mathrm{w}_2.$

Set the Inner Product to Zero for Orthogonality: For u and v to be orthogonal, $\langle \mathrm{u}, \mathrm{v} \rangle = 0$. Therefore, we set the equation to zero:
$2\mathrm{w}_1 - 8\mathrm{w}_2 = 0.$

Solve for the Relationship Between w_1 and w_2 :
$2\mathrm{w}_1 = 8\mathrm{w}_2 \Rightarrow \mathrm{w}_1 = 4\mathrm{w}_2.$
This equation represents the relationship between w_1 and w_2 for the vectors to be orthogonal.

16. Let R^4 have the Euclidean inner product. Find two unit vectors that are orthogonal to all three of the vectors $\mathbf{u} = (2, 1, -4, 0)$, $\mathbf{v} = (-1, -1, 2, 2)$, and $\mathbf{w} = (3, 2, 5, 4)$.

①

given vectors are

$u = (2,1,-4,0)$, $v = (-1,-1,2,2)$, $w = (3,2,5,4)$

Let (a,b,c,d) be the vector that is orthogonal to u, v, w.

$(a,b,c)\cdot(2,1,-4,0) = 0 \Rightarrow 2a+b-4c=0$

$(a,b,c)\cdot(-1,-1,2,2) = 0 \Rightarrow -a-b+2c+2d=0$

$(a,b,c)\cdot(3,2,5,4) = 0 \Rightarrow 3a+2b+5c+4d=0$

$$\therefore \begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \quad \text{———} (*)$$

Now to solve above system, reduce the coef^n matrix to rref.

$$R_1 \to \frac{R_1}{2} \quad \begin{bmatrix} 1 & 1/2 & -2 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \quad \begin{bmatrix} 1 & \frac{1}{2} & -2 & 0 \\ 0 & -\frac{1}{2} & 0 & 2 \\ 0 & \frac{1}{2} & -10 & 4 \end{bmatrix}$$

$$R_2 \to -2R_2 \quad \begin{bmatrix} 1 & \frac{1}{2} & -2 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & \frac{1}{2} & -10 & 4 \end{bmatrix}$$

$R_1 \longrightarrow R_1 - \frac{R_2}{2}$

$R_3 \longrightarrow R_3 - \frac{R_2}{2}$

$$\begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -1 & 6 \end{bmatrix}$$

$R_3 \longrightarrow -R_3$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \longrightarrow R_1 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

(*) is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$a - 10d = 0$

$b - 4d = 0$

$c - 6d = 0$

$$m = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 6 \\ 1 \end{bmatrix}, \quad d = 1$$

$$n = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ 12 \\ 2 \end{bmatrix}, \quad d = 2$$

$m = \begin{bmatrix} 10 \\ 4 \\ 6 \\ 1 \end{bmatrix}$, $n = \begin{bmatrix} 20 \\ 8 \\ 12 \\ 2 \end{bmatrix}$ are two vectors, orthogonal to the given vectors

so required unit vectors are

$$\frac{1}{\sqrt{100+16+36+1}} \begin{bmatrix} 10 \\ 4 \\ 6 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{400+64+144+4}} \begin{bmatrix} 20 \\ 8 \\ 12 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 10/\sqrt{153} \\ 4/\sqrt{153} \\ 6/\sqrt{153} \\ 1/\sqrt{153} \end{bmatrix}, \quad \begin{bmatrix} 20/\sqrt{612} \\ 8/\sqrt{612} \\ 12/\sqrt{612} \\ 2/\sqrt{612} \end{bmatrix}$$

17. Do there exist scalars k and l such that the vectors

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2?

Polynomials are $\mathrm{p_1 = 2 + kx + 6x^2}$, $\mathrm{p_2 = l + 5x + 3x^2}$, $\mathrm{p_3 = 1 + 2x + 3x^2}$

If vectors are mutually orthogonal then inner product of any 2 vectors is 0.

Inner product of $\mathrm{a_1 + a_2x + .. a_nx^{n-1}}$ and $\mathrm{b_1 + b_2x + .. b_nx^{n-1}}$ is $[a_1 \quad a_2 \ldots a_n]^{\mathrm{T}} \cdot [b_1 \quad b_2 \ldots b_n]^{\mathrm{T}}$

Inner product $\langle \mathrm{p_1, p_2} \rangle = [2 \quad k \quad 6]^{\mathrm{T}} \cdot [l \quad 5 \quad 3]^{\mathrm{T}}$

$\langle \mathrm{p_1, p_2} \rangle = \mathrm{2l + 5k + 3 \times 6}$
$\langle \mathrm{p_1, p_2} \rangle = \mathrm{2l + 5k + 18}$

Equating dot product to 0

$\mathrm{2l + 5k + 18 = 0}$
$\mathrm{2l = -5k - 18}$
$\mathrm{l = \frac{-5k-18}{2}}$
$\mathrm{l = -\frac{5}{2}k - \frac{18}{2}}$
$\mathrm{l = -\frac{5}{2}k - 9}$

Inner product $\langle \mathrm{p_1, p_3} \rangle = [2 \quad k \quad 6]^{\mathrm{T}} \cdot [1 \quad 2 \quad 3]^{\mathrm{T}}$

$\langle \mathrm{p_1, p_3} \rangle = \mathrm{2 \times 1 + 2k + 3 \times 6}$
$\langle \mathrm{p_1, p_3} \rangle = \mathrm{2k + 18 + 2}$
$\langle \mathrm{p_1, p_3} \rangle = \mathrm{2k + 20}$

Equating dot product to 0

$\mathrm{2k + 20 = 0}$
$\mathrm{2k = -20}$
$\mathrm{k = -\frac{20}{2}}$
$\mathrm{k = -10}$

For $\mathrm{k = -10}$, $\mathrm{l = -\frac{5}{2}k - 9 = -\frac{5}{2} \times -10 - 9}$
$\mathrm{l = 5 \times 5 - 9 = 25 - 9}$
$\mathrm{l = 16}$

Inner product $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = [l \quad 5 \quad 3]^{\mathrm{T}} \cdot [1 \quad 2 \quad 3]^{\mathrm{T}}$

$\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = \mathrm{l} + 5 \times 2 + 3 \times 3$
$\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = \mathrm{l} + 10 + 9$
$\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = \mathrm{l} + 19$

Equating dot product to 0

$\mathrm{l} + 19 = 0$

Substitute $\mathrm{l} = 16$ in above equation

$16 + 19 = 0$
$35 = 0$

But $35 \neq 0$ so above equation is false.

It means for $\mathrm{l} = 16$, $\mathbf{k} = -10$: all vectors are not mutually orthogonal.

This means for any value of $\mathrm{l}, \mathbf{k}$ all 3 vectors are not mutually orthogonal.

So there exist no scalars $\mathrm{l}, \mathbf{k}$ such that vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are mutually orthogonal.

46. Use the Cauchy–Schwarz inequality to prove that for all real values of a, b, and θ,

$$(a\cos\theta + b\sin\theta)^2 \leq a^2 + b^2$$

Solution:

of Dr. Wael Mustafa

scan me

6.3 Gram–Schmidt Process; *QR*-Decomposition

2. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^3.

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

(c) $(1, 0, 0)$, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(0, 0, 1)$

(d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

2.

a) Given

$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

we have to check whether set of vectors orthogonal or orthonormal or not.

Let $u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$v = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

$w = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$\therefore u \cdot v = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$

$u w = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2} + 0 + \frac{1}{2} = 0$

$v \cdot w = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}} \neq 0$

So, the given set of vectors not orthogonal also not orthonormal.

b) Given

$\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right); \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right); \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

we have to check whether set of vectors orthogonal or orthonormal or not.

Let $u = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$v = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

$w = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\therefore u \cdot v = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0$

$u w = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$

$v w = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} + \frac{2}{3} - \frac{4}{3} = 0$

& $\|u\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$

$\|v\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$

$\|w\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$

So, the given set of vectors are orthonormal set.

c) Given

$(1,0,0), (0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}), (0,0,1)$

we have to check whether the set of vectors orthogonal or orthonormal or not.

Let $u=(1,0,0)$

$v=(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

$w=(0,0,1)$

$\therefore u.v=(1,0,0).(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=0+0+0=0$

$u.w=(1,0,0).(0,0,1)=0+0+0=0$

$v.w=(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})(0,0,1)=0+0+\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\neq 0$

So, the given set of vectors neither orthogonal nor orthonormal.

d) Given

$(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}), (\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$

we have to check whether the set of vectors orthogonal or orthonormal or not

Let $u=(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}})$

$v=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$

$\therefore u.v=(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}).(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)=\frac{1}{\sqrt{12}}-\frac{1}{\sqrt{12}}+0=0$

& $\|u\|=\sqrt{(\frac{1}{\sqrt{6}})^2+(\frac{1}{\sqrt{6}})^2+(-\frac{2}{\sqrt{6}})^2}=\sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}}=1$

& $\|v\|=\sqrt{(\frac{1}{\sqrt{2}})^2+(-\frac{1}{\sqrt{2}})^2+0^2}=\sqrt{\frac{1}{2}+\frac{1}{2}}=1$

So, the given set of vectors from orthonormal set

7. Verify that the vectors

$$\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right), \quad \mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right), \quad \mathbf{v}_3 = (0, 0, 1)$$

form an orthonormal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (1, -2, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

THEOREM 6.3.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.*

THEOREM 6.3.2

(*a*) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if* **u** *is any vector in V, then*

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \qquad (3)$$

(*b*) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if* **u** *is any vector in V, then*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \qquad (4)$$

7. $v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $v_3 = (0,0,1)$

clearly $v_1 \cdot v_2 = \frac{-12}{25} + \frac{12}{25} = 0$

$v_1 \cdot v_3 = 0$ & $v_2 \cdot v_3 = 0$.

Also, $\|v_1\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

$\|v_2\| = 1$, $\|v_3\| = 1$.

So, v_1, v_2, v_3 ~~forms~~ form a orthonormal basis of $\mathbb{R}^3$.

Now, $u = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4/5 \\ 3/5 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

where $c_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3/5 \\ 4/5 \\ 0 \end{pmatrix} = -\frac{3}{5} - \frac{8}{5} = \frac{-11}{5}$

$c_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4/5 \\ 3/5 \\ 0 \end{pmatrix} = \frac{4}{5} - \frac{6}{5} = -\frac{2}{5}$

$c_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2$

In Exercises 29–30, let R^3 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

29. $\mathbf{u}_1 = (1, 1, 1), \ \mathbf{u}_2 = (-1, 1, 0), \ \mathbf{u}_3 = (1, 2, 1)$

$$v_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_2 = u_2 - \frac{< u_2, v_1 >}{||v_1||^2} v_1$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{-1+1+0}{1+1+1} v_1$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_3 = u_3 - \frac{< u_3, v_1 >}{||v_1||^2} v_1 - \frac{< u_3, v_2 >}{||v_2||^2} v_2$$

$$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \left(\frac{1+2+1}{1+1+1}\right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-1+2}{1+1} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{pmatrix}$$

now find all three norms

$$||v_1||^2 = 1 + 1 + 1 = 3$$
$$||v_2||^2 = 1 + 1 = 2$$
$$||v_3||^2 = \frac{1}{36} + \frac{1}{36} + \frac{1}{9} = \frac{1}{6}$$

orthonormal vectors be

$$\left\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||} \right\}$$

$$\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \right\}$$

32. Let R^3 have the Euclidean inner product. Find an orthonormal basis for the subspace spanned by (0, 1, 2), (−1, 0, 1), (−1, 1, 3).

Solution :, Given that the subspace spanned by (0,1,2), (−1,0,1) and (−1,1,3)

i.e. $x_1 = (0,1,2)$,

$x_2 = (-1,0,1)$, $x_3 = (-1,1,3)$

Let $\boxed{y_1 = x_1 = (0,1,2)}$

And $y_2 = x_2 - \dfrac{\langle x_2, y_1\rangle}{\|y_1\|^2} y_1$

$$y_2 = (-1,0,1) - \frac{\langle(-1,0,1),(0,1,2)\rangle}{\|(0,1,2)\|^2}(0,1,2)$$

$$y_2 = (-1,0,1) - \frac{(0+0+2)}{(0^2+1^2+2^2)}(0,1,2)$$

$$y_2 = (-1,0,1) - \frac{2}{5}(0,1,2)$$

$$y_2 = (-1,0,1) - \left(0, \frac{2}{5}, \frac{4}{5}\right)$$

$$y_2 = \left(-1-0,\ 0-\frac{2}{5},\ 1-\frac{4}{5}\right)$$

$$\boxed{y_2 = \left(-1, -2/5, \frac{1}{5}\right)}$$

And

$$y_3 = x_3 - \frac{\langle x_3, y_1\rangle}{\|y_1\|^2} y_1 - \frac{\langle x_3, y_2\rangle}{\|y_2\|^2} y_2$$

$$v_3 = (-1, 1, 3) - \frac{\langle (-1, 1, 3), (0, 1, 2) \rangle}{0^2 + 1^2 + 2^2} \cdot (0, 1, 2)$$

$$- \frac{\langle (-1, 1, 3) \; (-1, \frac{-2}{5}, \frac{1}{5}) \rangle}{1^2 + (\frac{-2}{5})^2 + (\frac{1}{5})^2} \cdot (-1, \frac{-2}{5}, \frac{1}{5})$$

$$= (-1, 1, 3) - \frac{0 + 1 + 6}{5} \cdot (0, 1, 2) - \frac{(1 - \frac{2}{5} + \frac{3}{5})}{1 + \frac{4}{25} + \frac{1}{25}} (-1, \frac{-2}{5}, \frac{1}{5})$$

$$= (-1,1,3) - \frac{7}{5}(0,1,2) - \frac{6/5}{6/5}\left(-1,-\frac{2}{5},\frac{1}{5}\right)$$

$$= (-1,1,3) + \left(0,-\frac{7}{5},-\frac{14}{5}\right) + \left(1,\frac{2}{5},-\frac{1}{5}\right)$$

$$= (0,0,0)$$

i.e. $\boxed{y_3 = (0,0,0)}$

This is a linearly dependent vector. So this vector not belong the given basis vector.

the set $\{y_1, y_2\} = \{(0,1,2), (-1,-2/5,\frac{1}{5})\}$ be the set of orthogonal basis.

Now,

$$z_1 = \frac{y_1}{\|y_1\|}$$

$$z_1 = \frac{(0,1,2)}{\sqrt{0^2+1^2+2^2}}$$

$$z_1 = \frac{(0,1,2)}{\sqrt{5}}$$

$$\boxed{z_1 = \frac{1}{\sqrt{5}}(0,1,2)}$$

And $z_2 = \frac{y_2}{\|y_2\|}$

$$z_2 = \frac{(-1,-2/5,1/5)}{\sqrt{(-1)^2+(\frac{-2}{5})^2+(\frac{1}{5})^2}}$$

$$z_2 = \frac{(-1, -2/5, 1/5)}{\sqrt{1 + \frac{4}{25} + \frac{1}{25}}}$$

$$z_2 = \frac{(-1, -2/5, 1/5)}{\sqrt{\frac{30}{25}}}$$

$$\boxed{z_2 = \sqrt{\frac{5}{6}}\,(-1, -2/5, 1/5)}$$

Hence the set $\{z_1, z_2\}$ be the orthonormal set of vector basis.

Answer

37. Let R^3 have the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$$

Use the Gram–Schmidt process to transform $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (1, 0, 0)$ into an orthonormal basis.

Given inner product on R^3 as:

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$$

where $\vec{u} = (u_1, u_2, u_3) \quad \text{and} \quad \vec{v} = (v_1, v_2, v_3)$

Using the Gram-Schmidt process to transform the following vectors into an orthonormal basis.

$\vec{u}_1 = (1, 1, 1), \vec{u}_2 = (1, 1, 0), \vec{u}_3 = (1, 0, 0)$.

Here

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 = (1, 1, 1) \\ \Rightarrow ||\vec{v}_1|| &= \langle \vec{u}_1, \vec{u}_1 \rangle^{1/2} \\ &= (1 + 2 + 3)^{1/2} \\ &= \sqrt{6} \end{aligned}$$

Explanation:

Because $||\vec{u}|| = \langle \vec{u}, \vec{u} \rangle^{1/2}$.

$$\begin{aligned} \Rightarrow \vec{w}_1 &= \frac{1}{||\vec{v}_1||} \vec{v}_1 \\ &= \frac{1}{\sqrt{6}} (1, 1, 1) \\ &= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \end{aligned}$$

To find $\vec{v}_2$.

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1$$

To find $\langle \vec{u}_2, \vec{v}_1 \rangle$.

$$\begin{aligned} \langle \vec{u}_2, \vec{v}_1 \rangle &= 1 + 2 \\ &= 3 \end{aligned}$$

and

$$\langle \vec{v}_1, \vec{v}_1 \rangle = 6$$

Hence,

$$\begin{aligned} \vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 \\ &= (1,1,0) - \frac{3}{6}(1,1,1) \\ &= (1,1,0) - \frac{1}{2}(1,1,1) \\ &= \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned} \therefore ||\vec{v}_2|| &= \left(\frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} \right)^{1/2} \\ &= \sqrt{\frac{1}{4} + \frac{2}{4} + \frac{3}{4}} \\ &= \sqrt{\frac{6}{4}} \\ &= \frac{\sqrt{6}}{2} \end{aligned}$$

$$\begin{aligned} \therefore \vec{w}_2 &= \frac{1}{||\vec{v}_2||} \vec{v}_2 \\ &= \frac{2}{\sqrt{6}} \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \\ &= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \end{aligned}$$

To find $\vec{v}_3$.

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{v}_1, \vec{u}_3 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{v}_2, \vec{u}_3 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2$$

$v_1 = (1,1,1), v_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), u_3 = (1,0,0)$

$$\langle \vec{v}_1, \vec{u}_3 \rangle = 1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = 6$$

$$\langle \vec{v}_2, \vec{u}_3 \rangle = \frac{1}{2}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{3}{2}$$

$$\begin{aligned}
\therefore \vec{v}_3 &= (1,0,0) - \frac{1}{6}(1,1,1) - \frac{1/2}{3/2}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \\
&= (1,0,0) - \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) - \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}\right) \\
&= \left(1 - \frac{1}{6} - \frac{1}{6}, -\frac{1}{6} - \frac{1}{6}, -\frac{1}{6} + \frac{1}{6}\right) \\
&= \left(1 - \frac{1}{3}, -\frac{1}{3}, 0\right) \\
&= \left(\frac{2}{3}, -\frac{1}{3}, 0\right)
\end{aligned}$$

$$\begin{aligned}
||\vec{v}_3|| &= \langle \vec{v}_3, \vec{v}_3 \rangle^{1/2} \\
&= \left(\frac{4}{9} + 2 \times \frac{1}{9}\right)^{1/2} \\
&= \frac{\sqrt{6}}{3}
\end{aligned}$$

$$\begin{aligned}
\therefore \vec{w}_3 &= \frac{1}{||\vec{v}_3||}\vec{v}_3 \\
&= \frac{3}{\sqrt{6}}\left(\frac{2}{3}, -\frac{1}{3}, 0\right) \\
&= \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right)
\end{aligned}$$

Hence the orthonormal basis is:

$$\left\{\vec{w}_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \vec{w}_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), \vec{w}_3 = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right)\right\}$$

Let $\mathbb{R}^3$ have the inner product $< u, v >= u_1v_1+2u_2v_2+3u_3v_3$. Use Gram-Schmidt to transform

$$u_1 = (1, 1, 1) , \quad u_2 = (1, 1, 0) , \quad u_3 = (1, 0, 0) ,$$

into an orthonormal basis with respect to this inner product.
We start with $v_1 = \frac{u_1}{\sqrt{<u_1,u_1>}}$ or $v_1 = \frac{1}{\sqrt{6}}(1, 1, 1)$. Now

$$\begin{aligned} v_2 &= \frac{u_2- < u_2, v_1 > v_1}{||u_2- < u_2, v_1 > v_1||} , \\ &= \frac{(1,1,0) - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}{||u_2- < u_2, v_1 > v_1||} , \\ &= \frac{(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})}{\sqrt{< (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}) >}} , \\ &= \frac{1}{\sqrt{6}}(1, 1, -1) . \end{aligned}$$

Finally,

$$\begin{aligned} v_3 &= \frac{u_3- < u_3, v_2 > v_2- < u_3, v_1 > v_1}{||u_3- < u_3, v_2 > v_2- < u_3, v_1 > v_1||} . \\ &= \frac{(1,0,0) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}{||u_3- < u_3, v_2 > v_2- < u_3, v_1 > v_1||} , \\ &= \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right) . \end{aligned}$$

8.1 General Linear Transformations

▶ In Exercises 1–2, suppose that T is a mapping whose domain is the vector space M_{22}. In each part, determine whether T is a linear transformation, and if so, find its kernel. ◀

1. (a) $T(A) = A^2$

(c) $T(A) = A + A^T$

1 (a) Consider, $A, B \in M_{22}$.
Now,

$$\begin{aligned} T(A+B) &= (A+B)^2 \\ &= A^2 + B^2 + 2AB \\ &= A^2 + B^2 + AB + BA \end{aligned}$$

Now, $T(A) = A^2$ and $T(B) = B^2$.
Since, $T(A+B) \neq T(A) + T(B)$.
Therefore, $T(A)$ is not a linear transformation.

Explanation:

The formula for $(a+b)^2 = a^2 + b^2 + 2ab$.

c) Consider, $A, B \in M_{22}$.
Now,

$$\begin{aligned} T(A+B) &= (A+B) + (A+B)^T \\ &= A + B + B^T + A^T \\ &= (A + A^T) + (B + B^T) \\ &= T(A) + T(B) \end{aligned}$$

Now,

$$\begin{aligned} T(aA) &= aA + (aA)^T \\ &= a(A + A^T) \\ &= aT(A) \end{aligned}$$

Since, $T(A+B) = T(A) + T(B)$ and $T(aA) = aT(A)$.
Therefore, $T(A)$ is a linear transformation.
Now, the kernel is given by

$$\begin{aligned} \ker(T) &= \{A \in M_{22} : T(A) = 0\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a = 0, b + c = 0, 2d = 0 \right\} \\ &= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\} \end{aligned}$$

10. Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(T)$?

(a) x^2 (b) 0 (c) $1 + x$ (d) $-x$

(10)

T is a linear transformation from the vector space P_2 to the vector space P_3 defined by $T(p(x)) = xp(x)$.

To find the kernel of T, we need to find all polynomials $p(x)$ in P_2 such that $T(p(x)) = xp(x) = 0$.

Since the zero polynomial is the only polynomial that satisfies this condition, the kernel of T consists only of the zero polynomial.

So, only the zero polynomial is in $\ker(T)$.

11. Let $T : P_2 \to P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$?

(a) $x + x^2$ (b) $1 + x$ (c) $3 - x^2$ (d) $-x$

(11)

T is a linear transformation from the vector space P_2 to the vector space P_3 defined by $T(p(x)) = xp(x)$.

To find the range of T, we need to find all polynomials $q(x)$ in P_3 such that there exists a polynomial $p(x)$ in P_2 with $T(p(x)) = xp(x) = q(x)$.

Since any polynomial $q(x)$ in P_3 can be written as $q(x) = xp(x)$ for some polynomial $p(x)$ in P_2, it follows that the range of T is the entire vector space P_3.

So, all polynomials in P_3 are in $R(T)$.

Out of the given options, only $(a) \quad x + x^2 \quad \text{and} \quad (d) \quad -x$ **are in** $R(T)$**, since they are both polynomials in** P_3.

13. In each part, use the given information to find the nullity of the linear transformation T.

(a) $T: R^5 \to P_5$ has rank 3.

(b) $T: P_4 \to P_3$ has rank 1.

(c) The range of $T: M_{mn} \to R^3$ is R^3.

(d) $T: M_{22} \to M_{22}$ has rank 3.

13. (a) $T: \mathbb{R}^5 \longrightarrow P_5$ has rank 3

then By rank-nullity theorem

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(\mathbb{R}^5)$$

$$\Rightarrow \text{nullity}(T) = 5-3 = 2$$

(b) $T: P_4 \to P_3$ has rank 1

then By rank-nullity theorem

$$\text{Rank}(T) + \text{nullity}(T) = \dim(P_4)$$

$$\Rightarrow \text{nullity}(T) = \dim(P_4) - \text{Rank}(T)$$

$$= 5-1 = 4$$

(c) The range of $T: M_{mn} \to \mathbb{R}^3$ is $\mathbb{R}^3$

$\Rightarrow$ dimension of Range of T is 3

$\Rightarrow$ $\text{Rank}(T) = \text{dimension of Range space of } T$

$= 3$

$$\Rightarrow \text{nullity}(T) = \dim(M_{mn}) - \text{Rank}(T)$$

$$= m.n - 3$$

(d) $T: M_{22} \to M_{22}$ has rank 3

$$\Rightarrow \text{nullity}(T) = \dim(M_{22}) - \text{Rank}(T)$$

$$= 4-3$$

$$= 1.$$

14. In each part, use the given information to find the rank of the linear transformation T.

(a) $T: R^7 \to M_{32}$ has nullity 2.

(b) $T: P_3 \to R$ has nullity 1.

(c) The null space of $T: P_5 \to P_5$ is P_5.

(d) $T: P_n \to M_{mn}$ has nullity 3.

14)

a)Given $\mathrm{T}: \mathbb{R}^7 \to \mathrm{M}_{32}$ $\mathrm{nullity(T)} = 2$

By RNT

$\mathrm{rank(T)} + \mathrm{nullity(T)} = \dim(\mathbb{R}^7)$

$\mathrm{rank(T)} + 2 = 7$

$\Rightarrow \mathrm{rank(T)} = 7 - 2 = 5$

$\Longrightarrow \mathrm{rank(T)} = 5$

b)$\mathrm{T}: \mathrm{P}_3 \to \mathbb{R}$ $\mathrm{nullity(T)} = 1$

By RNT

$\mathrm{rank(T)} + \mathrm{nullity(T)} = \dim(\mathrm{P}_3)$

$\Rightarrow \mathrm{rank(T)} + 1 = 4$

$\Rightarrow \mathrm{rank(T)} = 4 - 1 = 3$

$\Rightarrow \mathrm{rank(T)} = 3$

c)$\mathrm{T}: \mathrm{P}_5 \to \mathrm{P}_5$ nullspace is $\mathrm{P}_5 \Rightarrow \dim(\mathrm{P}_5) = 6$

By RNT

$\mathrm{rank(T)} + \mathrm{nullity(T)} = \dim(\mathrm{P}_5)$

$\mathrm{rank(T)} + 6 = 6$

$\Rightarrow \mathrm{rank(T)} = 6 - 6$

$\Rightarrow \mathrm{rank(T)} = 0$

d)$\mathrm{T}: \mathrm{P}_n \to \mathrm{M}_{mn}$ $\mathrm{nullity(T)} = 3$

By RNT

$\mathrm{rank(T)} + \mathrm{nullity(T)} = \dim(\mathrm{P}_n)$

$\mathrm{rank(T)} + 3 = n + 1$

$\Rightarrow \mathrm{rank(T)} = n + 1 - 3 = n - 2$

$\Longrightarrow \mathrm{rank(T)} = n - 2$

15. Let $T: M_{22} \rightarrow M_{22}$ be the dilation operator with factor $k = 3$.

(a) Find $T\left(\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}\right)$.

(b) Find the rank and nullity of T.

EXAMPLE 4 Dilation and Contraction Operators

If V is a vector space and k is any scalar, then the mapping $T: V \rightarrow V$ given by $T(\mathbf{x}) = k\mathbf{x}$ is a linear operator on V, for if c is any scalar and if $\mathbf{u}$ and $\mathbf{v}$ are any vectors in V, then

$$T(c\mathbf{u}) = k(c\mathbf{u}) = c(k\mathbf{u}) = cT(\mathbf{u})$$
$$T(\mathbf{u} + \mathbf{v}) = k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

If $0 < k < 1$, then T is called the ***contraction*** of V with factor k, and if $k > 1$, it is called the ***dilation*** of V with factor k.

THEOREM 8.1.4 Dimension Theorem for Linear Transformations

If $T: V \rightarrow W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and

$$\mathit{rank}(T) + \mathit{nullity}(T) = \dim(V) \tag{7}$$

In the special case where A is an $m \times n$ matrix and $T_A: R^n \rightarrow R^m$ is multiplication by A, the kernel of T_A is the null space of A, and the range of T_A is the column space of A. Thus, it follows from Theorem 8.1.4 that

$$\operatorname{rank}(T_A) + \operatorname{nullity}(T_A) = n$$

Q) Let $T : M_{22} \to M_{22}$ be the dilation operator with factor $k = 3$

i) Find $T\left(\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}\right)$

ii) Find Rank & Nullity of T.

Soln) Here $T : M_{22} \to M_{22}$ the dilation operator with factor $k = 3$ is given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

i) $$T\left(\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}\right) = \begin{bmatrix} 3\cdot 1 & 3\cdot 2 \\ 3(-4) & 3\cdot 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -12 & 9 \end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}\right) = \begin{bmatrix} 3 & 6 \\ -12 & 9 \end{bmatrix}$$

ii) We find Rank & Nullity of T as follow
let us determine the transformation matrix T, so

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So transformation matrix T is

$$T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\boxed{\text{Rank}\,T = 4}$ and By Rank - Nullity theorem

$$\text{Rank}\,T + \text{Nullity}\,T = \dim(M_{22})$$

$$\text{Nullity}\,T = 4 - 4$$

$$\boxed{\text{Nullity}\,T = 0}$$

Thus we obtained Rank & Nullity of T.

16. Let $T : P_2 \to P_2$ be the contraction operator with factor $k = 1/4$.

(a) Find $T(1 + 4x + 8x^2)$.

(b) Find the rank and nullity of T.

EXAMPLE 4 Dilation and Contraction Operators

If V is a vector space and k is any scalar, then the mapping $T : V \to V$ given by $T(\mathbf{x}) = k\mathbf{x}$ is a linear operator on V, for if c is any scalar and if $\mathbf{u}$ and $\mathbf{v}$ are any vectors in V, then

$$T(c\mathbf{u}) = k(c\mathbf{u}) = c(k\mathbf{u}) = cT(\mathbf{u})$$
$$T(\mathbf{u} + \mathbf{v}) = k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

If $0 < k < 1$, then T is called the ***contraction*** of V with factor k, and if $k > 1$, it is called the ***dilation*** of V with factor k.

The operator $\mathrm{T} : \mathrm{P_2} \to \mathrm{P_2}$ is the contraction operator with factor $\mathrm{k} = \frac{1}{4}$

i.e, $\mathrm{T(p(x))} = \frac{1}{4}\mathrm{p(x)}$ for all $\mathrm{p(x)} \in \mathrm{P_2}$

(a) Therefore, $\mathrm{T}(1 + 4x + 8x^2)$

$= \frac{1}{4}(1 + 4x + 8x^2)$

$= \frac{1}{4} + x + 2x^2$

(b) Now, $\mathrm{Ker(T)} = \{\mathrm{p(x)} \in \mathrm{P_2} : \mathrm{T(p(x))} = 0\}$

$= \{\mathrm{p(x)} \in \mathrm{P_2} : \frac{1}{4}\mathrm{p(x)} = 0\}$

$= \{\mathrm{p(x)} \in \mathrm{P_2} : \mathrm{p(x)} = 0\}$

$= \{0\}$

Hence, $\dim(\mathrm{Ker(T)}) = 0$

$\therefore$ Nullity $(\mathrm{T}) = 0$

*** Let, $\mathrm{q(x)} \in \mathrm{P_2}$ (co-domain) be arbitrary

Consider $\mathrm{p(x)} = 4\mathrm{q(x)}$

Then $\mathrm{p(x)} \in \mathrm{P_2}$ (domain) and $\mathrm{T(p(x))} = \frac{1}{4}\mathrm{p(x)} = \mathrm{q(x)}$

Hence, T is surjective

$\therefore$ Range $(\mathrm{T}) = \mathrm{P_2}$

$\therefore \mathrm{Rank(T)} = \dim(\mathrm{P_2}) = 3$

Hence, $\mathrm{Rank(T)} = 3$

19. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, 0)$, and let $T: R^2 \to R^2$ be the linear operator for which

$$T(\mathbf{v}_1) = (1, -2) \quad \text{and} \quad T(\mathbf{v}_2) = (-4, 1)$$

Find a formula for $T(x_1, x_2)$, and use that formula to find $T(5, -3)$.

Let us consider basis for $\mathbb{R}^2$ given by:

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and let us consider linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which:

$$T(v_1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad T(v_2) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Let us first find formula for $T(x_1, x_2)$:

$$\begin{aligned} (x_1, x_2) &= av_1 + bv_2 \\ &= a \begin{bmatrix} 1 \\ -2 \end{bmatrix} + b \begin{bmatrix} -4 \\ 1 \end{bmatrix} \\ &\Leftrightarrow \begin{cases} x_1 = a + b \\ x_2 = a \end{cases} \Rightarrow \underline{\underline{a = x_2 \text{ and } b = x_1 - x_2}} \end{aligned}$$

which gives us:

$$(x_1, x_2) = x_2 v_1 + (x_1 - x_2) v_2$$

Therefore, we have:

$$\begin{aligned} T(x_1, x_2) &= T(x_2 v_1 + (x_1 - x_2) v_2) \\ &\Leftrightarrow x_2 T(v_1) + (x_1 - x_2) T(v_2) \\ &= x_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (x_1 - x_2) \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4x_1 + 5x_2 \\ x_1 - 3x_2 \end{bmatrix} \end{aligned}$$

Which gives us:

$$T(5, -3) = \begin{bmatrix} -35 \\ 14 \end{bmatrix}$$

20. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$, and let $T: R^2 \to R^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = (-1, 2, 0) \quad \text{and} \quad T(\mathbf{v}_2) = (0, -3, 5)$$

Find a formula for $T(x_1, x_2)$, and use that formula to find $T(2, -3)$.

Let us consider basis for $\mathbb{R}^2$ given by:

$$S = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

and let us consider linear operator $T: \mathbb{R}^2 \to \mathbb{R}^3$ for which:

$$T(v_1) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad T(v_2) = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$$

Let us first find formula for $T(x_1, x_2)$:

$$\begin{aligned} (x_1, x_2) &= av_1 + bv_2 \\ &= a\begin{bmatrix} -2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &\Leftrightarrow \begin{cases} x_1 = -2a + b \\ x_2 = a + 3b \end{cases} \quad \Rightarrow \quad \underline{\underline{a = \frac{1}{7}(-3x_1 + x_2) \text{ and } b = \frac{1}{7}(x_1 + 2x_2)}} \end{aligned}$$

which gives us:

$$(x_1, x_2) = \frac{1}{7}(-3x_1 + x_2)v_1 + \frac{1}{7}(x_1 + 2x_2)v_2$$

Therefore, we have:

$$\begin{aligned} T(x_1, x_2) &= T\left(\frac{1}{7}(-3x_1 + x_2)v_1 + \frac{1}{7}(x_1 + 2x_2)v_2\right) \\ &\Leftrightarrow \frac{1}{7}(-3x_1 + x_2)T(v_1) + \frac{1}{7}(x_1 + 2x_2)T(v_2) \\ &= \frac{1}{7}(-3x_1 + x_2)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{7}(x_1 + 2x_2)\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3x_1 - x_2}{7} \\ \frac{-9x_1 - 4x_2}{7} \\ \frac{5x_1 + 10x_2}{7} \end{bmatrix} \end{aligned}$$

23. Let $T : P_3 \to P_2$ be the mapping defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = 5a_0 + a_3x^2$$

(a) Show that T is linear.

(b) Find a basis for the kernel of T.

(c) Find a basis for the range of T.

(a) Showing that T is linear:

To show that a mapping T is linear, we need to demonstrate that it satisfies two properties:

Explanation:

1. **Additivity:** For any vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain of T, we should have $\mathrm{T}(\mathbf{u}+\mathbf{v}) = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$.
2. **Scalar Multiplication:** For any vector $\mathbf{u}$ in the domain of T and any scalar c, $\mathrm{T}(\mathrm{cu}) = \mathrm{cT}(\mathrm{u})$.

et's check each property for the given mapping T:

Let $\mathrm{u} = a_0 + a_1x + a_2x^2 + a_3x^3$ and $\mathrm{v} = b_0 + b_1x + b_2x^2 + b_3x^3$, where a_i and b_i are constants.

1. **Additivity:** $\mathrm{T}(\mathrm{u}+\mathrm{v}) = \mathrm{T}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3\right)$

$= \mathrm{T}\left((a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3\right)$

$= 5(a_0+b_0) + (a_3+b_3)x_2$

$= 5a_0 + a_3x^2 + 5b_0 + b_3x^2$

$= \mathrm{T}\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) + \mathrm{T}\left(b_0 + b_1x + b_2x^2 + b_3x^3\right)$

$= \mathrm{T}(\mathrm{u}) + \mathrm{T}(\mathrm{v})$

2. **Scalar Multiplication:** Let c be a scalar.

$\mathrm{T}(\mathrm{cu}) = \mathrm{T}\left(c\left(a_0 + a_1x + a_2x^2 + a_3x^3\right)\right)$

$= \mathrm{T}\left(ca_0 + ca_1x + ca_2x^2 + ca_3x^3\right)$

$= 5ca_0 + ca_3x^2$

$= c5a_0 + ca_3x^2$

$= c\mathrm{T}\left(a_0 + a_1x + a_2x^2 + a_3x^3\right)$

$= c\mathrm{T}(\mathrm{u})$

Since T satisfies both properties of linearity, it is indeed a linear mapping.

(b) Finding a basis for the kernel of T:

Explanation:

The kernel (also called the null space) of a linear transformation T is the set of vectors in the domain that map to the zero vector in the codomain.

In this case, we want to find the vectors $\mathbf{u} = a_0 + a_1x + a_2x^2 + a_3x^3$ that satisfy $\mathrm{T}(\mathbf{u}) = 0$.

From the definition of T, we have $5a_0 + a_3x^2 = 0$.

This implies that $a_0 = 0$ and $a_3 = 0$, since there is no $\mathbf{x}$ term in the right-hand side.

So, the vectors in the kernel of T are of the form $a_1x + a_2x^2$, where a_1 and a_2 are arbitrary constants.

A basis for the kernel can be chosen as $\{x, x^2\}$, as these two vectors are linearly independent and span the kernel.

(c) Finding a basis for the range of T:

Explanation:

The range of a linear transformation T is the set of all possible vectors that can be obtained by applying T to vectors in the domain

From the definition of T, we see that the range of T is spanned by $5a_0 + a_3x^2$.
This means that any polynomial of the form $5a_0 + a_3x^2$ can be obtained in the range.
Since the range is the space of polynomials in P_2, a basis for the range can be chosen as $\{5, x^2\}$.
These two vectors are linearly independent and span the range of T.

24. Let $T: P_2 \to P_2$ be the mapping defined by

$$T(a_0 + a_1x + a_2x^2) = 3a_0 + a_1x + (a_0 + a_1)x^2$$

(a) Show that T is linear.

(b) Find a basis for the kernel of T.

(c) Find a basis for the range of T.

Given $T: P_2 \to P_2$ is defined by $T(a_0 + a_1x + a_2x^2) = 3a_0 + a_1x + (a_0+a_1)x^2$

(i) Let $A = a_0 + a_1x + a_2x^2$ and $B = b_0 + b_1x + b_2x^2 \in P_2$.

Then, $T(A) = 3a_0 + a_1x + (a_0+a_1)x^2$ and $T(B) = 3b_0 + b_1x + (b_0+b_1)x^2$.

Now, $T(A+B) = 3(a_0+b_0) + (a_1+b_1)x + [(a_0+b_0) + (a_1+b_1)]x^2$

$= [3a_0 + a_1x + (a_0+a_1)x^2] + [3b_0 + b_1x + (b_0+b_1)x^2]$

$= T(A) + T(B)$.

Thus, $T(A+B) = T(A) + T(B)$ for all $A, B \in P_2$.

Again, let $c \in \mathbb{R}$.

Then, $T(cA) = T(ca_0 + ca_1x + ca_2x^2)$

$= 3(ca_0) + (ca_1)x + (ca_0 + ca_1)x^2$

$= c[3a_0 + a_1x + (a_0+a_1)x^2]$

$= c \cdot T(A)$

Thus, $T(cA) = c \cdot T(A)$ for all $c \in \mathbb{R}$ and all $A \in P_2$.

Therefore, T is linear.

(ii) $\text{Ker } T = \{a_0 + a_1x + a_2x^2 \in P_2 : T(a_0 + a_1x + a_2x^2) = 0\}$.

Now, $T(a_0 + a_1x + a_2x^2) = 0$ gives,

$3a_0 + a_1x + (a_0+a_1)x^2 = 0$

Then, $T(cA) = T(ca_0 + ca_1x + ca_2x^2)$

$= 3(ca_0) + (ca_1)x + (ca_0 + ca_1)x^2$

$= c[3a_0 + a_1x + (a_0 + a_1)x^2]$

$= c \cdot T(A)$

Thus, $T(cA) = c \cdot T(A)$ for all $c \in \mathbb{R}$ and all $A \in P_2$.

Therefore, T is linear.

(ii) $\text{Ker } T = \{a_0 + a_1x + a_2x^2 \in P_2 : T(a_0 + a_1x + a_2x^2) = 0\}$.

Now, $T(a_0 + a_1x + a_2x^2) = 0$ gives,

$3a_0 + a_1x + (a_0 + a_1)x^2 = 0$

i.e., $3a_0 = 0$, $a_1 = 0$, $a_0 + a_1 = 0$

i.e., $a_0 = 0$, $a_1 = 0$

Let $a_2 = k$.

Then, $(a_0, a_1, a_2) = (0, 0, k) = k(0, 0, 1)$ where k is real.

Therefore, the required basis for the Kernel of T is $\{(0, 0, 1)\}$.

(iii) Here, $T(1) = T(1 + 0\cdot x + 0\cdot x^2)$

$$= 3\cdot 1 + 0\cdot x + (1+0)x^2$$

$$= 3 + x^2$$

i.e., $T(1,0,0) = (3,0,1)$

And, $T(x) = T(0 + 1\cdot x + 0\cdot x^2) = 3\cdot 0 + 1\cdot x + (0+1)x^2$

$$= 0 + x + x^2$$

i.e., $T(0,1,0) = (0,1,1)$

And, $T(x^2) = T(0 + 0\cdot x + 1\cdot x^2) = 3\cdot 0 + 0\cdot x + (0+0)x^2 = 0$

i.e., $T(0,0,1) = (0,0,0)$.

Image of T is the linear span of the vectors $(3,0,1), (0,1,1)$.

Therefore, $\text{Im}(T) = \{(3,0,1), (0,1,1)\}$.

Hence, the basis for the range of T is $= \{(3,0,1), (0,1,1)\}$.

31. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V, and let $T: V \to R^3$ be a linear transformation for which

$$T(\mathbf{v}_1) = (1, -1, 2), \quad T(\mathbf{v}_2) = (0, 3, 2),$$
$$T(\mathbf{v}_3) = (-3, 1, 2)$$

Find $T(2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3)$.

$T(2v_1-3v_2+4v_3)=2T(v_1)-3T(v_2)+4T(v_3)$

$=2(1,-1,2)-3(0,3,2)+4(-3,1,2)=(2-12,-2-9+4,4-6+8)=(-10,-7,6)$

8.2 Compositions and Inverse Transformations

▶ In Exercises 1–2, determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1. ◀

1. (a) $T: R^2 \to R^2$, where $T(x, y) = (y, x)$

(b) $T: R^2 \to R^3$, where $T(x, y) = (x, y, x + y)$

(c) $T: R^3 \to R^2$, where $T(x, y, z) = (x + y + z, x - y - z)$

A linear transformation $\mathrm{T} : \mathrm{A} \to \mathrm{B}$ is said to be one-to-one if, $\mathrm{Ker}(\mathrm{T}) = \{\theta\}$
Where,
$\mathrm{Ker}(\mathrm{T}) = \{\mathrm{x} \in \mathrm{A} : \mathrm{T}(\mathrm{x}) = \theta\}$

1.a)
Here,
$\mathrm{T}(\mathrm{x}, \mathrm{y}) = (\mathrm{y}, \mathrm{x})$
Now,
$\mathrm{T}(\mathrm{x}, \mathrm{y}) = \theta$ gives
$\mathrm{T}(\mathrm{x}, \mathrm{y}) = (0, 0)$
$(\mathrm{y}, \mathrm{x}) = (0, 0)$
$\Rightarrow \mathrm{x} = \mathrm{y} = 0$
$\therefore \mathrm{Ker}(\mathrm{T}) = \{(0, 0)\}$
Hence, T is one-to-one.

1.b)
Here,
$\mathrm{T}(\mathrm{x}, \mathrm{y}) = (\mathrm{x}, \mathrm{y}, \mathrm{x} + \mathrm{y})$
Now,
$\mathrm{T}(\mathrm{x}, \mathrm{y}) = \theta$ gives
$\mathrm{T}(\mathrm{x}, \mathrm{y}) = (0, 0, 0)$
$(\mathrm{x}, \mathrm{y}, \mathrm{x} + \mathrm{y}) = (0, 0, 0)$
$\Rightarrow \mathrm{x} = \mathrm{y} = 0$
$\therefore \mathrm{Ker}(\mathrm{T}) = \{(0, 0)\}$
Hence, T is one-to-one.

1.c)
Here,
$\mathrm{T}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x} + \mathrm{y} + \mathrm{z}, \mathrm{x} - \mathrm{y} - \mathrm{z})$
Now,
$\mathrm{T}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = \theta$ gives
$\mathrm{T}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (0, 0)$
$(\mathrm{x} + \mathrm{y} + \mathrm{z}, \mathrm{x} - \mathrm{y} - \mathrm{z}) = (0, 0)$
$\Rightarrow \mathrm{x} + \mathrm{y} + \mathrm{z} = 0$
$\mathrm{x} - \mathrm{y} - \mathrm{z} = 0$
Take, $\mathrm{x} = 0, \mathrm{y} = 2, \mathrm{z} = -2$, then it also satisfying above
$\therefore \mathrm{Ker}(\mathrm{T}) \neq \{(0, 0, 0)\}$
Hence, T is not one-to-one.

5. Use the given information to determine whether the linear transformation is one-to-one.

(a) $T: V \to W$; $\text{nullity}(T) = 0$

(b) $T: V \to W$; $\text{rank}(T) = \dim(V)$

(c) $T: V \to W$; $\dim(W) < \dim(V)$

a.

Consider the given linear transformation,

$\mathrm{T} : \mathrm{V} \to \mathrm{W}; \text{nullity}(\mathrm{T}) = 0$

It is required to determine whether the linear transformation is one-to-one or not.

The null space consists of all vectors in V that map to the zero vector in W. If the null space is zero-dimensional, it implies that the only vector mapping to the zero vector in W is the zero vector in V.
Since, T maps distinct vectors in V to distinct vectors in W. Therefore, the linear transformation T is one-to-one.

b.

Consider the given linear transformation,

$\mathrm{T} : \mathrm{V} \to \mathrm{W}; \text{rank}(\mathrm{T}) = \dim(\mathrm{V})$

It is required to determine whether the linear transformation is one-to-one or not.

If the rank of T is equal to the dimension of V, it implies that the column space of T spans the entire space W.
So, for every vector $\mathbf{w}$ in W, there exists at least one vector $\mathbf{v}$ in V such that $\mathrm{T}(\mathbf{v}) = \mathbf{w}$.
Since each vector in W is uniquely mapped to by a vector in V, the linear transformation T is one-to-one.

c.

Consider the given linear transformation,

$\mathrm{T} : \mathrm{V} \to \mathrm{W}; \dim(\mathrm{W}) < \dim(\mathrm{V})$

It is required to determine whether the linear transformation is one-to-one or not.

In this case, it is not possible for the linear transformation T to be one-to-one. This is because if W has fewer dimensions, there must be multiple vectors in V that map to the same vector in W.
there exists a non-trivial null space, and T cannot be injective (one-to-one).

6. Use the given information to determine whether the linear operator is one-to-one, onto, both, or neither.

(a) $T: V \to V$; nullity$(T) = 0$

(b) $T: V \to V$; rank$(T) < \dim(V)$

(c) $T: V \to V$; $R(T) = V$

(a) Gives that, $\mathrm{T} : \mathrm{V} \to \mathrm{V} : \mathrm{nullity(T)} = 0$.

i.e. $\mathrm{nullity(T)} = \dim(\mathrm{kerT}) = 0$.

$\Rightarrow \ker(\mathrm{T}) = \{\theta_\mathrm{v}\}$, where θ_v is the null vector of the vector space V.

Now, let $\alpha, \beta \in \mathrm{V}$ such that $\mathrm{T}(\alpha) = \mathrm{T}(\beta)$.

$$\mathrm{T}(\alpha) = \mathrm{T}(\beta)$$
$$\mathrm{T}(\alpha) - \mathrm{T}(\beta) = \theta_\mathrm{v}$$
$$\mathrm{T}(\alpha - \beta) = \theta_\mathrm{v}$$

Therefore, $\alpha - \beta \in \mathrm{Ker(T)}$.

$\Rightarrow \quad \alpha - \beta = \theta_\mathrm{v}$, since $\mathrm{Ker(T)} = \theta_\mathrm{v}$.

$\Rightarrow \quad \alpha = \beta$

Hence, for any arbitrary $\alpha, \beta \in \mathrm{V}$, we have $\mathrm{T}(\alpha) = \mathrm{T}(\beta) \Rightarrow \alpha = \beta$.

$\therefore$ T is one-one.

Therefore, if $\mathrm{nullity(T)} = 0$, then T is one-to-one.

As T is one-to-one and $\mathrm{T} : \mathrm{V} \to \mathrm{V}$, so T is onto.

Hence, T is both one-to-one and onto.

(b) Given that, $\mathbf{T}: \mathbf{V} \rightarrow \mathbf{V}; \mathbf{rank}(\mathbf{T}) < \mathbf{dim}\,(\mathbf{V})$.

From rank-nullity theorem, we know that $\mathbf{nullily}(\mathbf{T}) + \mathbf{rank}(\mathbf{T}) = \mathbf{dim}\,(\mathbf{V})$.

$\Rightarrow \mathbf{nullily}(\mathbf{T}) = \mathbf{dim}\,(\mathbf{V}) - \mathbf{rank}(\mathbf{T})$

Since, $\mathbf{rank}(\mathbf{T}) < \mathbf{dim}\,(\mathbf{V})$,
$\Rightarrow 0 < \mathbf{dim}\,(\mathbf{V}) - \mathbf{rank}(\mathbf{T})$
$\Rightarrow 0 < \mathbf{nullily}(\mathbf{T})$
$\Rightarrow \mathbf{nullily}(\mathbf{T}) \neq 0$.

Therefore, $\mathbf{T}$ is not one-to one.

As, $\mathbf{T}$ is not one-to-one.

Hence, $\mathbf{T}$ is not onto.

Therefore, $\mathbf{T}$ is neither one-to-one nor onto.

(c) Given that, $\mathbf{T}: \mathbf{V} \rightarrow \mathbf{V}; \mathbf{R}(\mathbf{T}) = \mathbf{V}$.

Here, $\mathbf{R}(\mathbf{T}) = \mathbf{V}$, so $\mathbf{T}$ is onto.

From rank-nullity theorem, we know that $\mathbf{nullily}(\mathbf{T}) + \mathbf{rank}(\mathbf{T}) = \mathbf{dim}\,(\mathbf{V})$

$\Rightarrow \mathbf{nullily}(\mathbf{T}) + \mathbf{rank}(\mathbf{T}) = \mathbf{rank}(\mathbf{T})$

$\Rightarrow \mathbf{nullily}(\mathbf{T}) = 0$.

So by part (a) , $\mathbf{T}$ is one-to-one.

Therefore, in this case, $\mathbf{T}$ is both one-to-one and onto.

8. Show that the linear transformation $T: P_2 \to P_2$ defined by $T(p(x)) = p(x+1)$ is one-to-one. Do you think that this transformation is onto?

$T: P_2 \to P_2$ defined by $T(a_0 + a_1 x + a_2 x^2)$

$= a_0 + a_1(x+1) + a_2(x+1)^2$

$= a_0 + a_1 x + a_1 + a_2 x^2 + a_2 + 2a_2 x$

$= (a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2 x^2$

Let $T(a_0 + a_1 x + a_2 x^2) = T(b_0 + b_1 x + b_2 x^2)$

i.e. $T(p(x)) = T(g(x))$

$= (a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2 x^2 = (b_0 + b_1 + b_2) + (b_1 + 2b_2)x + b_2 x^2$

$\Rightarrow a_2 = b_2$

$(a_1 + 2a_2) = (b_1 + 2b_2) \Rightarrow a_1 = b_1$

$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 \Rightarrow a_0 = b_0$

Hence $p(x) = g(x)$

$\Rightarrow$ T is one-to-one.

Yes this onto

$\because \forall\ ax^0 + bx^1 + cx^2,\ \exists$

polynomial $(a-b+c) + (b-2c)x + cx^2$

s.t. $T\left((a-b+c) + (b-2c)x + cx^2\right)$

$= a + bx + cx^2$

Another proving of " onto " applying the following Theorem:

THEOREM 8.2.2 *If V and W are finite-dimensional vector spaces with the same dimension, and if $T: V \to W$ is a linear transformation, then the following statements are equivalent.*

(*a*) *T is one-to-one.*

(*b*) $\ker(T) = \{\mathbf{0}\}$.

(*c*) *T is onto* [*i.e.*, $R(T) = W$].

8.4 Matrices for General Linear Transformations

1. Let $T: P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$.

(a) Find the matrix for T relative to the standard bases

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \quad \text{and} \quad B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

where

$$\mathbf{u}_1 = 1, \quad \mathbf{u}_2 = x, \quad \mathbf{u}_3 = x^2$$
$$\mathbf{v}_1 = 1, \quad \mathbf{v}_2 = x, \quad \mathbf{v}_3 = x^2, \quad \mathbf{v}_4 = x^3$$

(b) Verify that the matrix $[T]_{B',B}$ obtained in part (a) satisfies Formula (5) for every vector $\mathbf{x} = c_0 + c_1 x + c_2 x^2$ in P_2.

$$[T]_{B',B}[\mathbf{x}]_B = [T(\mathbf{x})]_{B'} \qquad (5)$$

It is given that:

$\mathrm{T(p(x)) = xp(x)}$

We have:

$\mathrm{T(1) = x}$
$\mathrm{T(x) = x^2}$
$\mathrm{T(x^2) = x^3}$

The standard basis is:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii) Let $\mathrm{u = a + bx + cx^2}$

Then,

$\mathrm{T(u) = ax + bx^2 + cx^3}$

$$\Rightarrow [\mathrm{T(u)}]_{\mathrm{B'}} = \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}$$

And,

$$\mathrm{A[u]_B} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}$$

Clearly,

$[\mathrm{T(u)}]_{\mathrm{B'}} = \mathrm{A[u]_B}$

5. Let $T: R^2 \rightarrow R^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$$

(a) Find the matrix $[T]_{B',B}$ relative to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

(b) Verify that Formula (5) holds for every vector in R^2.

Given the linear transformation $\mathrm{T}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$

The basis are $\mathrm{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right\}$

and $\mathrm{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

To find the matrix representation of the linear transformation T with respect to the bases $\mathrm{B} = \{\mathrm{u}_1, \mathrm{u}_2\}$ and $\mathrm{B}\prime = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$, calculate how T maps the basis vectors u_1 and u_2 in terms of the basis B'.

Explanation:

First find images of basis B under T.

$$\begin{aligned} \mathrm{T}(\mathrm{u}_1) &= \mathrm{T}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2(3) \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} \\ \mathrm{T}(\mathrm{u}_2) &= \mathrm{T}\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -2 + 2(4) \\ -(-2) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

Now writing these vectors in linear combination of basis B'

$$\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \frac{8}{3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

The coefficients of the elements of basis forms required matrix.

$$[\mathrm{T}]_{B',B} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \\ \frac{8}{3} & \frac{4}{3} \end{bmatrix}$$

Formula 4a is not given so can't proceed with b.

b) rank of domain < rank of codomain

So, T is one-one

hence, image of every vector in $\mathbb{R}^2$ is unique

or we can say T gives image of every vector in $\mathbb{R}^2$.

6. Let $T : R^3 \to R^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_1, x_1 - x_3)$$

(a) Find the matrix for T with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (0, 1, 1), \quad \mathbf{v}_3 = (1, 1, 0)$$

(b) Verify that Formula (8) holds for every vector $\mathbf{x} = (x_1, x_2, x_3)$ in R^3.

(c) Is T one-to-one? If so, find the matrix of T^{-1} with respect to the basis B.

$$[T]_B[\mathbf{x}]_B = [T(\mathbf{x})]_B \qquad (8)$$

Given

$[T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_1, x_1 - x_3)]$

$v_1 = (1, 0, 1), v_2 = (0, 1, 1), v_3 = (1, 1, 0)$

(a) Matrix T w.r.t. basic $B = (v_1, v_2, v_3)$

$T(1, 0, 1) = \{1 - 0, 0 - 1, 1 - 1\} = (1, -1, 0)$

$T(0, 1, 1) = \{0 - 1, 1 - 0, 0 - 1\} = (-1, 1, -1)$

$T(1, 1, 0) = \{1 - 1, 1 - 1, 1 - 0\} = (0, 0, 1)$

(1,-1,0) can be written as linear combination of v_1, v_2, v_3

$(1, -1, 0) = a(1, 0, 1) + b(0, 1, 1) + c(1, 1, 0) \to (1)$

On comparing x,y and z coordinates

$1 = a + c, b + c = -1, a + b = 0$

by solving

$c = 0, a = 1, b = -1$

By Equation (1)

$(1, -1, 0) = 1(1, 0, 1) - 1(0, 1, 1) + 0((1, 1, 0)$

Similarly

$(-1, 1, -1) = -\frac{3}{2}(1, 0, 1) + \frac{1}{2}(0, 1, 1) + \frac{1}{2}(1, 1, 0)$

$(0, 0, 1) = \frac{1}{2}(1, 0, 1) + \frac{1}{2}(0, 1, 1) - \frac{1}{2}(1, 1, 0)$

Matrix for w.r.t. basis B is

$$T = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T$$

I.e.

$$T = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(b) property 8 is $[T]_B[X]_B = [T(X)]_B$

Let's $X = (X_1, X_2, X_3)$

$(X_1, X_2, X_3) = a(1,0,1) + b(0,1,1) + c(1,1,0)$

$\Rightarrow a = \frac{x_1 - x_2 + x_3}{2}$

$b = \frac{x_2 - x_1 + x_3}{2}$

$c = \frac{x_1 + x_2 - x_3}{2}$

$$[X]_B = \begin{bmatrix} \frac{x_1 - x_2 + x_3}{2} \\ \frac{x_2 - x_1 + x_3}{2} \\ \frac{x_1 + x_2 - x_3}{2} \end{bmatrix}$$

(c) we first find determinants of $[T]_B$

$$|T_B| = \begin{vmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$= \left(-\frac{1}{4} - \frac{1}{4}\right) + \frac{3}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)$

$= 0$

det $(T_B) = 0$

It is a singular Matrix.

And We know singular Matrix does not have inverse i.e. it is non inevitable .

It is not one-one.

9. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ be the matrix for $T: R^2 \to R^2$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) Find $[T(\mathbf{v}_1)]_B$ and $[T(\mathbf{v}_2)]_B$.

(b) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.

(c) Find a formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

(d) Use the formula obtained in (c) to compute $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Solution(11.1.3) Given, $\overline{\mathrm{v}}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \overline{\mathrm{v}}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

and $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ is the matrix of T : $\mathrm{R}^2 \to \mathrm{R}^2$ with respect to the basis $\mathrm{B} = \{\overline{\mathrm{v}}_1, \overline{\mathrm{v}}_2\}$

(a) So, $[\mathrm{T}(\overline{\mathrm{v}}_1)\}_{\mathrm{B}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **and** $[\mathrm{T}(\overline{\mathrm{v}}_2)]_{\mathrm{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

(b) The linear transformation T may be written as

$$\mathrm{T}(\overline{\mathrm{v}}) = \mathrm{A}\overline{\mathrm{v}} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\overline{\mathrm{v}}$$

Therefore, $\mathrm{T}(\overline{\mathrm{v}}_1) = \mathrm{T}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+9 \\ -2+15 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$ Answer

and $\mathrm{T}(\overline{\mathrm{v}}_2) = \mathrm{T}\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1+12 \\ 2+20 \end{bmatrix} = \begin{bmatrix} 11 \\ 22 \end{bmatrix}$ Answer

(c) $\mathrm{Now}, \mathrm{T}(\overline{\mathrm{v}}) = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\overline{\mathrm{v}}$

$$\Rightarrow \mathrm{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ -2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and $\mathrm{T}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+3 \\ 0+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Now, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{x}_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathrm{x}_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \mathrm{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \mathrm{x}_1\mathrm{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \mathrm{x}_2\mathrm{T}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ as T is a linear transformation.

$\Rightarrow \mathrm{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \mathrm{x}_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \mathrm{x}_2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ -2x_1 + 5x_2 \end{bmatrix}$ (1) Answer

(d) Here, $\mathrm{x}_1 = \mathrm{x}_2 = 1$

Therefore, from (1) above,

$$\mathrm{T}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1+3\times 1 \\ -2\times 1+5\times 1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$ Answer

11. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ be the matrix for $T: P_2 \to P_2$ with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = 3x + 3x^2$, $\mathbf{v}_2 = -1 + 3x + 2x^2$, $\mathbf{v}_3 = 3 + 7x + 2x^2$.

(a) Find $[T(\mathbf{v}_1)]_B$, $[T(\mathbf{v}_2)]_B$, and $[T(\mathbf{v}_3)]_B$.

(b) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$.

(c) Find a formula for $T(a_0 + a_1 x + a_2 x^2)$.

(d) Use the formula obtained in (c) to compute $T(1 + x^2)$.

Solution:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$$

A is a matrix of $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ w.r.t. basis $\mathcal{B} = \{v_1, v_2, v_3\}$, where

$v_1 = 3x + 3x^2$, $v_2 = -1 + 3x + 2x^2$, $v_3 = 3 + 7x + 2x^2$

(i) solution.

$$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 \Rightarrow [v_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore [T(v_1)]_{\mathcal{B}} = [T]_{\mathcal{B}} [v_1]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

$$v_2 = 0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 \Rightarrow [v_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore [T(v_2)]_{\mathcal{B}} = [T]_{\mathcal{B}} [v_2]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$v_3 = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 \Rightarrow [v_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore [T(v_3)]_{\mathcal{B}} = [T]_{\mathcal{B}} [v_3]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$$

$$\therefore [T(v_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \quad [T(v_2)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \quad [T(v_3)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$$

ii) $\to$ $T(v_1) = 1 \cdot v_1 + 2 \cdot v_2 + 6 v_3$

$= 3x + 3x^2 - 2 + 6x + 4x^2 + 18 + 42x + 12x^2$

$= 19x^2 + 51x + 16$

$T(v_2) = 3 \cdot v_1 - 2v_3$

$= 9x + 9x^2 - 6 - 14x - 4x^2$

$= 5x^2 - 5x - 6$

$T(v_3) = -v_1 + 5v_2 + 4v_3$

$= -3x - 3x^2 - 5 + 15x + 10x^2 + 12 + 28x + 8x^2$

$= 15x^2 + 40x + 7$

$\therefore T(v_1) = 19x^2 + 51x + 16,$

$T(v_2) = 5x^2 - 5x - 6,$

$T(v_3) = 15x^2 + 40x + 7$

iii) We write $1+x^2$ as linear combination of v_1, v_2, v_3.

$1 + x^2 = a \cdot v_1 + b v_2 + c v_3$

$= a(3x + 3x^2) + b(-1 + 3x + 2x^2) + c(3 + 7x + 2x^2)$

$= (-b + 3c) + (3a + 3b + 7c)x + (3a + 2b + 2c)x^2$

$\therefore -b + 3c = 1$

$3a + 3b + 7c = 0$

$3a + 2b + 2c = 1$

$\therefore -b = 1 - 3c \Rightarrow b = 3c - 1$

$\therefore 3a + 3(3c-1) + 7c = 0 \Rightarrow 3a + 16c = 3$

$3a + 2(3c-1) + 2c = 1 \Rightarrow 3a + 8c = 3$

$8c = 0 \Rightarrow c = 0$

$\therefore b = -1$

$\therefore a = 1 \Rightarrow a = 1, b = -1, c = 0$

$\therefore 1 + x^2 = v_1 - v_2$

$\therefore T(1+x^2) = T(v_1) - T(v_2) = 19x^2 + 51x + 16 - 5x^2 + 5x + 6$

$= 14x^2 + 56x + 22$

$\therefore T(1+x^2) = 14x^2 + 56x + 22$

14. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for a vector space V. Find the matrix with respect to B for the linear operator $T : V \to V$ defined by $T(\mathbf{v}_1) = \mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_3$, $T(\mathbf{v}_3) = \mathbf{v}_4$, $T(\mathbf{v}_4) = \mathbf{v}_1$.

Solⁿ $T : V \to V$

$T(v_1) = v_2 = 0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$

$T(v_2) = v_3 = 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 + 0 \cdot v_4$

$T(v_3) = v_4 = 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 1 \cdot v_4$

$T(v_4) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$

Matrix of T w.r.t basis B is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

18. (***Calculus required***) Let $D: P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.

(a) Find the matrix for D relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2 in which $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = 2 - 3x,\ \mathbf{p}_3 = 2 - 3x + 8x^2$.

(b) Use the matrix in part (a) to compute $D(6 - 6x + 24x^2)$.

(a)

$p_1 = 1; p_2 = x; p_3 = x^2$

So $D(p_1) = 0 = 0 \times p_1 + 0 \times p_2 + 0 \times p_3$

$D(p_2) = 1 = 1 \times p_1 + 0 \times p_2 + 0 \times p_3$

$D(p_3) = 2x = 0 \times p_1 + 2 \times p_2 + 0 \times p_3$

So the matrix of D with respect to the basis B is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$p_1 = 2; p_2 = 2 - 3x; p_3 = 2 - 3x + 8x^2$

So $D(p_1) = 0$

$D(p_2) = -3 = -\frac{3}{2} p_1$

$D(p_3) = -3 + 16x = \frac{16}{3} \times 3x - 3 = \frac{16}{3}(2 - p_2) - 3 = -\frac{16}{3} p_2 + \frac{23}{6} p_1$

So the matrix of D with respect to the basis B is

$$\begin{bmatrix} 0 & -\frac{3}{2} & \frac{23}{6} \\ 0 & 0 & -\frac{16}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

(c) In terms of the basis B given in part (a), we can rewrite

$6 - 6x + 24x^2 = 6p_1 - 6p_2 + 24p_3$

Therefore using the matrix in part (a), we get

$$D(6 - 6x + 24x^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} [6, -6, 24]^t = [-6, 48, 0]^t = -6 + 48x$$

(d) In terms of the basis B given in part (b), we can write

$$6 - 6x + 24x^2 = 3p_3 - p_2 + p_1$$

Therefore using the matrix in part (b), we get

$$D(6 - 6x + 24x^2) = \begin{bmatrix} 0 & -\frac{3}{2} & \frac{23}{6} \\ 0 & 0 & -\frac{16}{3} \\ 0 & 0 & 0 \end{bmatrix} [3, -1, 1]^t = \left[\frac{16}{3}, -\frac{16}{3}, 0\right]^t = \frac{16}{3}p_1 - \frac{16}{3}p_2 = 16x$$

8.5 Similarity

3. Let $T: R^2 \to R^2$ be a linear operator, and let B and B' be bases for R^2 for which

$$[T]_B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P_{B\to B'} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Find the matrix for T relative to the basis B'.

THEOREM 8.5.2 *Let $T: V \to V$ be a linear operator on a finite-dimensional vector space V, and let B and B' be bases for V. Then*

$$[T]_{B'} = P^{-1}[T]_B P \tag{11}$$

where $P = P_{B'\to B}$ and $P^{-1} = P_{B\to B'}$.

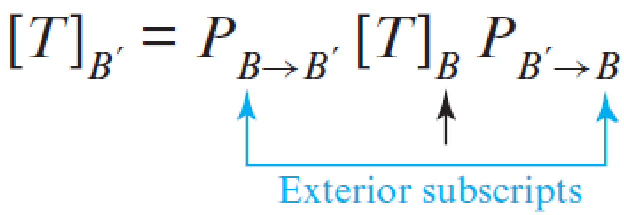


Figure 8.5.2

Solution:

Let $\mathrm{T}: \mathrm{R}^2 \to \mathrm{R}^2$ be a linear operator, and let B and B' be bases for R^2.

Given that, $[\mathrm{T}]_{\mathrm{B}'} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathrm{P}_{\mathrm{B}\to\mathrm{B}'} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

To find the matrix for T relative to the basis B:

Explanation:

The formula for finding $[\mathrm{T}]_{\mathrm{B}} = \mathrm{P}\,[\mathrm{T}]_{\mathrm{B}'}\,\mathrm{P}^{-1}$.

To find the P^{-1} :

The inverse of a 2×2 matrix can be found using the formula $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $ad - bc$ is the determinant.

Find the determinant.

$\det(\mathrm{P}) = 1$

Since the determinant is non-zero, the inverse exists.

Substitute the known values into the formula for the inverse.

$$\begin{aligned}
\mathrm{P}^{-1} &= \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\
&= 1\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 1 & 1\cdot -2 \\ 1\cdot -1 & 1\cdot 3 \end{bmatrix}
\end{aligned}$$

$$\therefore \mathrm{P}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Now, we calculate the matrix for T relative to the basis B,

$$
\begin{aligned}
[\mathrm{T}]_{\mathrm{B}} &= \mathrm{P}\,[\mathrm{T}]_{\mathrm{B'}}\,\mathrm{P}^{-1} \\
&= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 6+2 & 0+2 \\ 2+1 & 0+1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 8-2 & -16+6 \\ 3-1 & -6+3 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}
\end{aligned}
$$

$$
\therefore\ [\mathrm{T}]_{\mathrm{B}} = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}
$$

5.1 Eigenvalues and Eigenvectors

▶ In Exercises 1–4, confirm by multiplication that $\mathbf{x}$ is an eigenvector of A, and find the corresponding eigenvalue. ◀

2. $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Given matrix

$$A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$$

And eigen vector is

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given vector X is said to be an eigen vector if $AX = \lambda X$
Now to check

$$AX = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5-1 \\ 1+3 \end{bmatrix}$$

$$AX = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Comparing it with $AX = \lambda X$
We get

$$\lambda = 4$$

Eigen value is 4 ans eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4) Given matrix is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

And eigen vector is

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now to check eigen vector we find $\mathbf{AX}$

$$AX = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} 2-1-1 \\ -1+2-1 \\ -1-1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$AX = 0X$$

So eigen value is 0 and eigen vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

▶ In each part of Exercises 5–6, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix. ◀

6. (a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Question: 6(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0$

$4 + \lambda^2 - 4\lambda - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$ it is the characteristic equation.

Now solve the characteristic equation for the eigenvalues.

$\lambda^2 - 3\lambda - \lambda + 3 = 0$

$\lambda(\lambda - 3) - 1(\lambda - 3) = 0$

$(\lambda - 1)(\lambda - 3) = 0$

$\lambda = 1, 3$

Thus the eigenvalues are $\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 3$

And the eigenvectors can be obtained by $[A - \lambda I]X = 0$

For $\lambda = 1$; we get

$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + y = 0$

$\Rightarrow x = -y$

So $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thus the eigenspace for the corresponding value of $\lambda = 1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = 3$; we get

$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x + y = 0$

$\Rightarrow x = y$

So $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus the eigenspace for the corresponding value of $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The bases for the eigenspaces of the matrix are $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(b) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

Question: 6(b) $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

The characteristic equation is $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$\begin{vmatrix} 2-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 0 = 0$

$\lambda^2 - 4\lambda + 4 = 0$ it is the characteristic equation.

Now solve the characteristic equation for the eigenvalues.

$(\lambda - 2)^2 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 2, 2$

Thus the eigenvalues are $\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 2$

And the eigenvectors can be obtained by $[\mathbf{A} - \lambda \mathbf{I}]\mathbf{X} = 0$

For $\lambda = 2$; we get

$\begin{bmatrix} 2-2 & -3 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3\mathbf{y} = 0$

$\Rightarrow \mathbf{y} = 0$

And $\mathbf{x} = \mathbf{x}$

So $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathbf{x} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus the eigenspace for the corresponding value of $\lambda = 2$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The bases for the eigenspaces of the matrix are $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Question: $6(c)\ \mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

The characteristic equation is $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 0 = 0$

$\lambda^2 - 4\lambda + 4 = 0$ it is the characteristic equation.

Now solve the characteristic equation for the eigenvalues.

$(\lambda - 2)^2 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 2, 2$

Thus the eigenvalues are $\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 2$

And the eigenvectors can be obtained by $[\mathbf{A} - \lambda \mathbf{I}]\mathbf{X} = 0$

For $\lambda = 2$; we get

$\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

And $\mathbf{x} = \mathbf{x} \quad \text{and} \quad \mathbf{y} = 0$

Also $\mathbf{y} = \mathbf{y} \quad \text{and} \quad \mathbf{x} = 0$

So $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathbf{x} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

And $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = \mathbf{y} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus the eigenspace for the corresponding value of $\lambda = 2$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The bases for the eigenspaces of the matrix are $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

▶ In Exercises 7–12, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix. ◀

7. $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

We have given that the Matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

To find characteristics equation

Consider $|A - \lambda I| = 0$

Therefore, $\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$

Therefore, $(4-\lambda)\left[(1-\lambda)^2 - 0\right] + 0[-2(1-\lambda) - 0] + 1[0 - (-2)(1-\lambda)] = 0$

Therefore,

$(4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$

Therefore,

$(1-\lambda)(3-\lambda)(2-\lambda) = 0$ This is characteristics polynomials

Therefore , $\lambda = 1, 3, \quad \text{and} \quad 2$ are eigenvalues

Now to find eigenvector corresponding to eigenvalue

For $\lambda = 1$,

$$A - I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

To find null space of above Matrix

$R_2 = R_2 + \frac{2}{3}R_2 \quad \text{and} \quad R_3 = R_3 + \frac{2}{3}R_1$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$R_2 = \frac{3}{2}R_2$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$R_1 = R_1 - R_2 \quad \text{and} \quad R_3 = R_3 - \frac{2}{3}R_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore , $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Put, $y = t \quad \text{and} \quad x = 0, z = 0$

Therefore , $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t$

Therefore , $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is null space . This is eigenvector

Now for $\lambda = 3$

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

To find null space of above Matrix,

$$R_2 = R_2 + 2R_1 \quad \text{and} \quad R_3 = R_3 + 2R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Put $z = t \Rightarrow y = t \quad \text{and} \quad x = -t$

Therefore , $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t$

Therefore , $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is null space .

This is eigenvector

Similarly, for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

$$R_2 = R_2 + R_1 \quad \text{and} \quad R_3 = R_3 + R_1$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore ,

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $z = t \Rightarrow y = t \quad \text{and} \quad x = -\frac{1}{2}t$

Therefore , $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} t$

Therefore , $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \right\}$ is eigenvector

▶ In Exercises 15–16, find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula. [*Suggestion:* Work with the standard matrix for the operator.] ◀

16. $T(x, y, z) = (2x - y - z,\ x - z,\ -x + y + 2z)$

16. Given linear transformation is $\mathrm{T(x, y, z) = (2x - y - z, x - z, -x + y + 2z)}$.

To find standard matrix for T:

$\mathrm{T(1,0,0) = (2,1,-1)}$; $\mathrm{T(0,1,0) = (-1,0,1)}$; $\mathrm{T(0,0,1) = (-1,-1,2)}$

Standard matrix for T $= \mathrm{A} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$

Here eigenvalues and eigenvectors of A and T are same. Hence there basis for each eigenspace is same.(For this we only write column vector to row vector.)

To find eigenvalues of A:

$$\det(\mathrm{A} - \lambda \mathrm{I}) = \begin{bmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ -1 & 1 & 2-\lambda \end{bmatrix}$$

$= (2-\lambda)[-\lambda(2-\lambda)+1] + 1[2-\lambda-1] - [1-\lambda]$

$= (2-\lambda)[\lambda^2 - 2\lambda + 1] + (-\lambda+1) - (-\lambda+1)$

$= (2-\lambda)(\lambda^2 - \lambda - \lambda + 1)$

$= (2-\lambda)(\lambda(\lambda-1) - (\lambda-1))$

$= (2-\lambda)(\lambda-1)^2$

Eigenvalues of A are 1,2.

To find Basis for each eigenspace:

For $\lambda = 2$

$$\mathrm{A} - 2\mathrm{I} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \to \mathrm{R_2} \leftrightarrow \mathrm{R_1} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \to \mathrm{R_3} + \mathrm{R_1}, -\mathrm{R_2} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\to \mathrm{R_1} + 2\mathrm{R_2}, \mathrm{R_3} + \mathrm{R_2} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathrm{z} \text{ is a free variable. Put } \mathrm{z = t \Rightarrow x = -t, y = -t}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \mathrm{t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Basis for eigenspace corresponding to eigenvalue $\lambda = 2$ is $= \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 1$.

$$A + I = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow R_2 - R_1, R_3 + R_1 \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y, z$ are free variables. Put $y = s, z = t \Rightarrow x = s + t$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Basis for eigenspace corresponding to eigenvalue $\lambda = 1$ is $= \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$

33. Prove: If λ is an eigenvalue of an invertible matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector.

A value λ is an eigenvalue of a matrix A with corresponding eigenvector x iff they satisfy the equation

$\lambda x = Ax.$

We are given this to start with. Left-multiply both sides by A^{-1}, which we know to exist as A is specified as invertible:

$A^{-1}\lambda x = A^{-1}Ax.$

Scalar multiplication is commutative with matrix multiplication so the λ can move out in front. We also know that $A^{-1}A = I$, the identity matrix.

$\lambda A^{-1}x = Ix$

$\lambda A^{-1}x = x$

Multiply both sides by 1/λ:

$(1/\lambda)\lambda A^{-1}x = (1/\lambda)x$

$A^{-1}x = (1/\lambda)x.$

This exactly fits the equation needed to show that 1/λ is an eigenvalue for A^{-1} with corresponding eigenvector x.

38. (a) Prove that if A is a square matrix, then A and A^T have the same eigenvalues. [*Hint:* Look at the characteristic equation $\det(\lambda I - A) = 0$.]

Soln: Prove that if A is a square matrix, Then A and A^t have same eigenvalue.

Proof:-

We will use the fact that eigenvalues of a matrix are roots of its characteristic polynomial.

So if Ch(x) for A & A^T is same then it means that both have same ~~eigenvectors~~. eigenvalues.

[illegible]

Now Characteristic polynomial of A^T

$$Ch_{A^T}(x) = \det(\lambda I - A^T)$$

$$= \det(\lambda I^T - A^T) \quad [I = I^T]$$

$$= \det((\lambda I - A)^T)$$

$$= \det(\lambda I - A) \quad [\det(P) = \det(P^T) \text{ for any square matrix } P]$$

$$= Ch_A(x)$$

Since $Ch_{A^T}(x) = Ch_A(x)$

$\Rightarrow$ A^T & A have same eigenvalues.

(b) Show that A and A^T need not have the same eigenspaces. [*Hint:* Use the result in Exercise 30 to find a 2×2 matrix for which A and A^T have different eigenspaces.]

The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and its transpose A^T, have only one eigenvalue, namely 1. However, the eigenvectors of A are of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$, whereas the eigenvectors of A^T are of the form $\begin{bmatrix} 0 \\ a \end{bmatrix}$.

5.2 Diagonalization

In Exercises 1–4, show that A and B are not similar matrices.

DEFINITION 1 If A and B are square matrices, then we say that ***B is similar to A*** if there is an invertible matrix P such that $B = P^{-1}AP$.

Table 1 Similarity Invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant.

1. $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

Two matrices are similar if their determinants are the same, otherwise, they are said to be

Non-Similar matrices.

Solution: Determinant of A $= 2 \times 1 - 3 \times 1$

$$= 2 - 3 = -1$$

Determinant of B $= -2 \times 1 - 3 \times 0$

$$= -2 - 0 = -2$$

$$\text{Det. A} \neq \text{Det. B}$$

∴ A and B are not similar matrices.

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Expand the determinant by Column 1

$$\begin{aligned} \mathrm{Det.\,A} &= 1 \times \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 1(1-0) - 0 + 0 \\ &= 1 \end{aligned}$$

$$\mathrm{B} = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Expand the determinant by column 3

$$\begin{aligned} \mathrm{Det.\,B} &= 0 \times \begin{vmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{vmatrix} \\ &= 0 - 0 + 1\left(1 - \frac{1}{\cancel{2}} \times \cancel{2}\right) \\ &= (1-1) = 0 \end{aligned}$$

$$\mathrm{Det.\,A} \neq \mathrm{Det.\,B}$$

∴ A and B are not similar matrices.

▶ In Exercises 5–8, find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$. ◀

7. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

---

consider

$\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\left[(3-\lambda)(3-\lambda)\right] + 0 = 0$

$\Rightarrow (2-\lambda)(3-\lambda)(3-\lambda) = 0$

$\Rightarrow \lambda = 2, 3, 3$

These are eigenvalues

Now we find eigenvector corresponding to eigenvalues

$(a)\ \lambda_1 = 2$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$rref \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{2}R_1} \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow rref \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore to find eigenvector

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_3 = 0, v_2 = 0, v_1$ is arbritrary.

If $v_1 = t$

Therefore

$$V = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$$

$(b)\lambda_2 = 3$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rref \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow rref \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore to find eigenvector

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 + 2v_3 = 0 \Rightarrow v_1 = -2v_3, v_2, v_3$ is arbritrary.

If $v_2 = t, v_3 = s$

Therefore

$$V = \begin{bmatrix} -2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} s$$

$$\text{Eigenvalue}: 2, \text{Eigenvector}: \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Eigenvalue}: 3, \text{Eigenvector}: \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

We form the matrix P, whose i^{th} column is i^{th} eigenvector.

$$\Rightarrow \text{P} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We form the diagonal matrix D, whose element at row i is i^{th} eigenvalue.

$$\Rightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

These matrices has proprty that $D = P^{-1}AP$

9. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

(c) Is A diagonalizable? Justify your conclusion.

Remark Part (*a*) of Theorem 5.2.2 is a special case of a more general result: Specifically, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues, and if $S_1, S_2, \ldots, S_k$ are corresponding sets of linearly independent eigenvectors, then the *union* of these sets is linearly independent.

Given a matrix $\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

(a) Find eigenvalues of $\mathbf{A}$

For any matrix $\mathbf{M}$ eigenvalues are roots of $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ where $\mathbf{I}$ is the identity matrix.

Eigenvalues of $\mathbf{A}$ are $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \left| \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{vmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (4-\lambda)((3-\lambda)(4-\lambda) - 0 \times 2) - 0 + 1(2 \times 0 - 1(3-\lambda))$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (4-\lambda)^2(3-\lambda) - (3-\lambda) = (3-\lambda)\left((4-\lambda)^2 - 1\right) = (3-\lambda)(\lambda^2 - 8\lambda + 16 - 1)$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(\lambda^2 - 3\lambda - 5\lambda + 15) = (3-\lambda)(\lambda-3)(\lambda-5)$$

Roots of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$: roots of $(3-\lambda)^2(5-\lambda) = 0$ are $\lambda = 3, 3, 5$

Eigenvalues of $\mathbf{A}$ are $3, 5$

(b) For each eigenvalue λ find the rank of the matrix $\lambda\mathbf{I}-\mathbf{A}$

For $\lambda=3$, $\lambda\mathbf{I}-\mathbf{A}=3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}=\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}-\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}$

$$\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}3-4&0&-1\\-2&3-3&-2\\-1&0&3-4\end{bmatrix}=\begin{bmatrix}-1&0&-1\\-2&0&-2\\-1&0&-1\end{bmatrix}$$

We will perform row operations on $\begin{bmatrix}-1&0&-1\\-2&0&-2\\-1&0&-1\end{bmatrix}$ and find its rank.

Apply $\left\{\begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-R_1\end{matrix}\right\}:\begin{bmatrix}-1&0&-1\\-2+2&0&-2+2\\-1+1&0&-1+1\end{bmatrix}=\begin{bmatrix}-1&0&-1\\0&0&0\\0&0&0\end{bmatrix}$

So rank of $3\mathbf{I}-\mathbf{A}$ is 1. Nullity of the matrix $3\mathbf{I}-\mathbf{A}$ = dimension - rank = $3-1=2$

Nullity of $\lambda\mathbf{I}-\mathbf{A}$ will give the dimension of the eigenspace of eigenvalue λ

So dimension of the eigenspace of eigenvalue $\lambda=3$ is 2

For $\lambda=5$, $\lambda\mathbf{I}-\mathbf{A}=5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}-\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}$

$$\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}5-4&0&-1\\-2&5-3&-2\\-1&0&5-4\end{bmatrix}=\begin{bmatrix}1&0&-1\\-2&2&-2\\-1&0&1\end{bmatrix}$$

We will perform row operations on $\begin{bmatrix}1&0&-1\\-2&2&-2\\-1&0&1\end{bmatrix}$ and find its rank.

Apply $\left\{\begin{matrix}R_2\to R_2+2R_1\\R_3\to R_3+R_1\end{matrix}\right\}:\begin{bmatrix}1&0&-1\\-2+2&2&-2-2\\-1+1&0&1-1\end{bmatrix}=\begin{bmatrix}-1&0&-1\\0&2&-4\\0&0&0\end{bmatrix}$

So rank of $5\mathbf{I}-\mathbf{A}$ is 2. Nullity of the matrix $5\mathbf{I}-\mathbf{A}$ = dimension - rank = $3-2=1$

So dimension of the eigenspace of eigenvalue $\lambda=5$ is 1

(c) Is A diagonalizable?

A matrix is diagonalizable if the sum of dimensions of the eigenspaces of each eigenvalue of the matrix is equal to the order of the matrix.

Sum of dimensions of the eigenspaces of each eigenvalue of $\mathbf{A}$ = $\mathbf{2+1}$ = $\mathbf{3}$

Order of the matrix $\mathbf{A}$ = $\mathbf{3}$

So the sum of dimensions of the eigenspaces of each eigenvalue of $\mathbf{A}$ is equal to the order of $\mathbf{A}$.

So matrix $\mathbf{A}$ is **diagonalizable** consider

$\det(A-\lambda I)=0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[(3-\lambda)(4-\lambda)-1\right]+0+1\left[-(3-\lambda)\right]=0$$

$$\Rightarrow (4-\lambda)\left[(3-\lambda)(4-\lambda)-1\right]-(3-\lambda)=0$$

$$\Rightarrow -\lambda^3+11\lambda^2-39\lambda+45=0$$

$$\Rightarrow \lambda^3-11\lambda^2+39\lambda-45=0$$

$$\Rightarrow (\lambda-5)(\lambda-3)(\lambda-3)=0$$

$$\Rightarrow \lambda=5,3,3$$

These are eigenvalues

Part b)

Now we find eigenvector corresponding to eigenvalues

$(a)\lambda_1=5$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow rref \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore for $\lambda_1=5$

$Rank(\lambda I-A)=2$

Therefore to find eigenvector

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 = v_3, v_2 = 2v_3, v_3$ is arbitrary.

If $v_3 = t$

Therefore

$$V = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} t$$

$(b)\lambda_2 = 3$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow rref \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore for $\lambda_2 = 3$

$Rank(\lambda I - A) = 1$

Therefore to find eigenvector

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 + v_3 = 0 \Rightarrow v_1 = -v_3, v_2, v_3$ is arbitrary.

If $v_2 = t, v_3 = s$

Therefore

$$V = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s$$

Part C-

$$\text{Eigenvalue} : 5, \text{Eigenvector} : \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Eigenvalue} : 3, \text{Eigenvector} : \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Since for each eigenvalue of the matrix A we get

(algebraic multiplicity) $=$ (Geometric multiplicity)

Hence matrix A is diagonalisable.

We form the matrix P, whose i^{th} column is i^{th} eigenvector.

$$\Rightarrow \text{P} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We form the diagonal matrix D, whose element at row i is i^{th} eigenvalue.

$$\Rightarrow D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

These matrices has proprty that $D = P^{-1}AP$

▶ In Exercises 11–14, find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A, and find $P^{-1}AP$. ◀

11. $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

There is some terminology that is related to these ideas. If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the ***geometric multiplicity*** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the ***algebraic multiplicity*** of λ_0. The following theorem, which we state without proof, summarizes the preceding discussion.

THEOREM 5.2.4 Geometric and Algebraic Multiplicity

If A is a square matrix, then:

(*a*) *For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(*b*) *A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*

To find the eigenvalues of A :

Set up the formula to find the characteristic equation ,

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I_3) \\ &= \det \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \end{aligned}$$

Set the characteristic polynomial equal to 0 to find the eigenvalues λ .

$$\begin{aligned} -\lambda^3 + 6\lambda^2 - 11\lambda + 6 &= 0 \\ (\lambda-1)(\lambda-2)(\lambda-3) &= 0 \\ \lambda &= 1, 2, 3 \end{aligned}$$

Therefore, the eigenvalues of A are : $1, 2$ and 3 and all have algebraic multiplicity 1 and since all are distinct , So the geometric multiplicity is also 1 for each eigenvalue.

To find eigenvector corresponding to each eigenvalue:

Explanation:

The eigenvector is equal to the null space of the matrix minus the eigenvalue times the identity matrix .

Find the eigenvector using the eigenvalue $\lambda = 1$. Let $\mathbf{v}_1$ be the eigenvector corresponding to $\lambda = 1$, then

$$\begin{bmatrix} -1-1 & 4 & -2 \\ -3 & 4-1 & 0 \\ -3 & 1 & 3-1 \end{bmatrix} \mathbf{v} = 0$$

$$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \mathbf{v} = 0$$

Multiply each element of R_1 by $-\frac{1}{2}$, $\mathrm{R}_2 = \mathrm{R}_2 - \frac{3}{2}\mathrm{R}_1$, $\mathrm{R}_3 = \mathrm{R}_3 - \frac{3}{2}\mathrm{R}_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & -5 & 5 \end{bmatrix}$$

Multiply each element of R_2 by $-\frac{1}{3}$, $\mathrm{R}_3 = \mathrm{R}_3 - \frac{5}{3}\mathrm{R}_2$, $\mathrm{R}_1 = \mathrm{R}_1 - \frac{2}{3}\mathrm{R}_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Use the result matrix to declare the final solution to the system of equations.

$\mathbf{x} - \mathbf{z} = 0$

$\mathbf{y} - \mathbf{z} = 0$

Write the solution as a linear combination of vectors.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the eigenvector corresponding to $\lambda = 1$ is : $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Similarly, find the eigenvector using the eigenvalue $\lambda = 2$.

$$\mathbf{v}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

Find the eigenvector using the eigenvalue $\lambda = 3$.

$$\mathbf{v}_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{bmatrix}$$

And the matrix $\mathbf{P} = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} \\ 1 & 1 & \frac{3}{4} \\ 1 & 1 & 1 \end{bmatrix}$

To find the inverse of $\mathbf{P}$ set up a 3×6 matrix where the left half is the original matrix and the right half is its identity matrix.

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} & 1 & 0 & 0 \\ 1 & 1 & \frac{3}{4} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the reduced row echelon form.

Perform the row operation $\mathrm{R}_2 = \mathrm{R}_2 - \mathrm{R}_1$, $\mathrm{R}_3 = \mathrm{R}_3 - \mathrm{R}_1$.

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & -3 & 3 & 0 \\ 0 & \frac{1}{3} & \frac{3}{4} & -1 & 0 & 1 \end{bmatrix}$$

Perform the row operation $R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & -3 & 3 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & -1 & 1 \end{bmatrix}$$

Multiply each element of R_3 by 4 , $R_2 = R_2 - \frac{3}{8}R_3$, $R_1 = R_1 - \frac{1}{16}R_3$

$$\begin{bmatrix} 1 & \frac{2}{3} & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 9 & -6 \\ 0 & 0 & 1 & 0 & -4 & 4 \end{bmatrix}$$

Perform the row operation $R_1 = R_1 - \frac{2}{3}R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -5 & 3 \\ 0 & 1 & 0 & -3 & 9 & -6 \\ 0 & 0 & 1 & 0 & -4 & 4 \end{bmatrix}$$

The right half of the reduced row echelon form is the inverse.

$$P^{-1} = \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix}$$

The matrix D is formed by the eigenvalues of A :

The matrix D is : $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

21. Find A^n if n is a positive integer and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow (3-\lambda)\left((2-\lambda)(3-\lambda)-1\right) + 1(\lambda-3) = 0$$

$$\Rightarrow (3-\lambda)(6-5\lambda+\lambda^2-1) + (\lambda-3) = 0$$

$$\Rightarrow -(\lambda-3)(\lambda^2-5\lambda+5) + (\lambda-3) = 0$$

$$\Rightarrow (\lambda-3)(-\lambda^2+5\lambda-5+1) = 0$$

$$\Rightarrow (\lambda-3)(-\lambda^2+5\lambda-4) = 0$$

$\Rightarrow \lambda = 3$ or $-\lambda^2 + 5\lambda - 4 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda - 1) = 0$

$\Rightarrow \lambda = 4$ or $\lambda = 1$

So, $\lambda = 1, 3, 4$

for $\lambda = 1$

$(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 2x - y = 0 \Rightarrow y = 2x.$

$-x + y - z = 0$

$-y + 2z = 0 \Rightarrow y = 2z.$

$\Rightarrow 2x = 2z$

$\Rightarrow z = x.$

So, the eigen vector corresponding to the eigen value $\lambda = 1$ is (x, y, z)

$= (x, 2x, x) = (1, 2, 1)x, x \neq 0$

for $\lambda = 3$

$(A - 3I)X = 0$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -y = 0$

$\Rightarrow -x - y - z = 0$

$\Rightarrow -x - z = 0$

$\Rightarrow -x = z$

So, the eigen vector corresponding to eigen value $\lambda = 3$ is $(x, y, z) = (x, 0, -x)$

$= (1, 0, -1)x, \; x \neq 0$

for $\lambda = 4$

$(A - 4I)X = 0$

$$\Rightarrow \begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -x - y = 0 \Rightarrow y = -x$

$-x - 2y - z = 0$

$-y - z = 0 \Rightarrow z = -y = -(-x) = x$

So, the eigen vector corresponding to $\lambda = 4$ is $(x, y, z) = (x, -x, x) = (1, -1, 1)x, \; x \neq 0$

$$\text{So, } P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$|P| = 1(-1) - 1(3) + 1(-2) = -1 - 3 - 2 = -6$

$$P^{-1} = \frac{\text{adj } P}{|P|} = \frac{(\text{Cofactor of } P)^T}{|P|}$$

$$\text{Cofactor of } P = \begin{bmatrix} -1 & -3 & -2 \\ -2 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}$$

$$\text{adj } P = \begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix},$$

$$P^{-1} = \frac{1}{-6}\begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

We know, $A = PDP^{-1}$

$$A^2 = (PDP^{-1})(PDP^{-1})$$

$$= PD^2P^{-1}$$

Similarly $A^n = PD^nP^{-1}$.

So, $D^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$, $n \in N$.

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix} \frac{\begin{pmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{pmatrix}}{-6}$$

$$= -\frac{1}{6} \begin{pmatrix} 1 & 3^n & 4^n \\ 2 & 0 & -4^n \\ 1 & -3^n & 4^n \end{pmatrix} \begin{pmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{pmatrix}$$

$$A^n = -\frac{1}{6} \begin{pmatrix} -1-3^{n+1}-2\cdot 4^n & -2+2\cdot 4^n & -1+3^{n+1}-2\cdot 4^n \\ -2+2\cdot 4^n & -4-2\cdot 4^n & -2+2\cdot 4^n \\ -1+3^{n+1}-2\cdot 4^n & -2+2\cdot 4^n & -1-3^{n+1}-2\cdot 4^n \end{pmatrix}$$

37. Prove that if A is diagonalizable, then so is A^k for every positive integer k.

Since $\mathbf{A}$ is diagonalizable, there exists an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.
First we show that, $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$, where k is a positive integer.
Let $\mathbf{P}(k)$ be the statement $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$.
Now $\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})$
$= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1}$
$= \mathbf{PD}^2\mathbf{P}^{-1}$, since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$
So $\mathbf{A}^2 = \mathbf{PD}^2\mathbf{P}^{-1}$
So $\mathbf{P}(2)$ is true.
Let us assume that $\mathbf{P}(m)$ is true .
So $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$
Now $\mathbf{A}^{m+1} = \mathbf{A}^m \cdot \mathbf{A}$
$= (\mathbf{PD}^m\mathbf{P}^{-1}) \cdot (\mathbf{PDP}^{-1})$
$= \mathbf{PD}^m(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1}$
$= \mathbf{PD}^m\mathbf{DP}^{-1}$
$= \mathbf{PD}^{m+1}\mathbf{P}^{-1}$
So $\mathbf{P}(m+1)$ is true if $\mathbf{P}(m)$ is true.
So by mathematical induction $\mathbf{P}(k)$ is true for all positive integers. k.
Hence $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$

Since $\mathbf{D}$ is a diagonal matrix, $\mathbf{D}^k$ is also a diagonal matrix.
So there exists a invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}^k$ such that $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$.
This shows that the matrix $\mathbf{A}^k$ is diagonalizable.

CHAPTER 5: Eigenvalues and Eigenvectors

Useful links:

https://atozmath.com/Menu/MatrixAlgebra.aspx