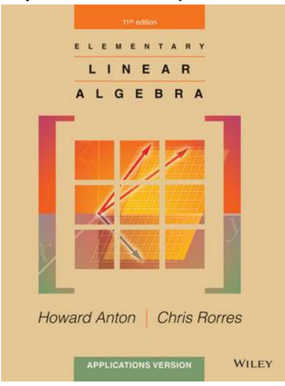
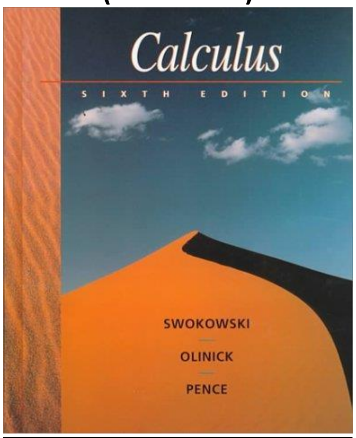
King Saud University

جامعة الملك سعود

King Saud University
College of sciences
Department of Mathematics

Math-107
Exercises (solved)

Elementary Linear Algebra by Howard Anton, Chris Rorres, 11th Edition	Calculus by Swokowski, Olinick, and Pence (6th Edition)
	

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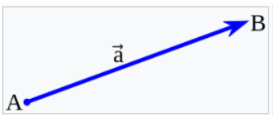
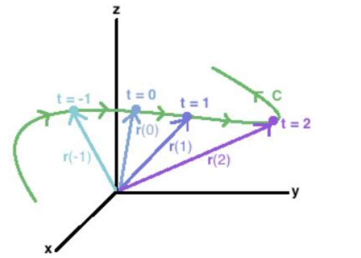
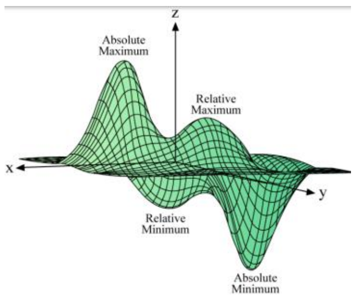
YouTube**:The file contains video codes explaining the problems in the file.**



A vector pointing from *A* to *B*



A curve C in three-dimensions represents by a vector-valued function **r(t)**, where sample values t=-1, t=0, t=1, and t=2 are arbitrarily plotted.



Systems of Linear Equations and Matrices

⑧ Solve each of the following system using Gauss-Gordan elimination method

(a) $2x_1 - 3x_2 = -2$

$2x_1 + x_2 = 1$

$3x_1 + 2x_2 = 1$

Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

$$\left(\begin{array}{ccc|c} 2 & -3 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 0 & 1 \end{array}\right)$$

$R_1 / 2 \rightarrow R_1$ (divide the 1 row by 2)

$$\left(\begin{array}{ccc|c} 1 & -1.5 & 0 & -1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 0 & 1 \end{array}\right)$$

$R_2 - 2R_1 \rightarrow R_2$ (multiply 1 row by 2 and subtract it from 2 row); $R_3 - 3R_1 \rightarrow R_3$ (multiply 1 row by 3 and subtract it from 3 row)

$$\left(\begin{array}{ccc|c} 1 & -1.5 & 0 & -1 \\ 0 & 4 & 0 & 3 \\ 0 & 6.5 & 0 & 4 \end{array}\right)$$

$R_2 / 4 \rightarrow R_2$ (divide the 2 row by 4)

$$\left(\begin{array}{ccc|c} 1 & -1.5 & 0 & -1 \\ 0 & 1 & 0 & 0.75 \\ 0 & 6.5 & 0 & 4 \end{array}\right)$$

$R_1 + 1.5R_2 \rightarrow R_1$ (multiply 2 row by 1.5 and add it to 1 row); $R_3 - 6.5R_2 \rightarrow R_3$ (multiply 2 row by 6.5 and subtract it from 3 row)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0.125 \\ 0 & 1 & 0 & 0.75 \\ 0 & 0 & 0 & -0.875 \end{array}\right)$$

Answer:
The system of equations has no solution because: $0 \neq -0.875$

(b) $4x_1 - 8x_2 = 12$

$3x_1 - 6x_2 = 9$

$-2x_1 + 4x_2 = -6$

Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

$$\left(\begin{array}{ccc|c} 4 & -8 & 0 & 12 \\ 3 & -6 & 0 & 9 \\ -2 & 4 & 0 & -6 \end{array}\right)$$

$R_1 / 4 \rightarrow R_1$ (divide the 1 row by 4)

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 3 & -6 & 0 & 9 \\ -2 & 4 & 0 & -6 \end{array}\right)$$

$R_2 - 3R_1 \rightarrow R_2$ (multiply 1 row by 3 and subtract it from 2 row); $R_3 + 2R_1 \rightarrow R_3$ (multiply 1 row by 2 and add it to 3 row)

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Answer:

The system of equations has a solution set:

$\{\ x_1 - 2x_2 = 3$

System has infinitely many solutions

Put $x_2 = t$, t *any real number then* $x_1 = 3 + 2t$

So the solution of the system is: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 + 2t \\ t \end{bmatrix}$ *where* $t \epsilon \mathbb{R}$

③ Solve the system by any method

(a)
$$2x - y - 3z = 0$$
$$-x + 2y - 3z = 0$$
$$x + y + 4z = 0$$

Gauss Elimination Back Substitution method

Converting given equations into matrix form

$$\left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 + 0.5 \times R_1$

$$= \left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ 0 & 1.5 & -4.5 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 0.5 \times R_1$

$$= \left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ 0 & 1.5 & -4.5 & 0 \\ 0 & 1.5 & 5.5 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ 0 & 1.5 & -4.5 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right]$$

i.e.

$2x - y - 3z = 0 \rightarrow (1)$

$1.5y - 4.5z = 0 \rightarrow (2)$

$10z = 0 \rightarrow (3)$

Now use back substitution method

From (3)

$10z = 0$

$\Rightarrow z = \frac{0}{10} = 0$

From (2)

$1.5y - 4.5z = 0$

$\Rightarrow 1.5y - 4.5(0) = 0$

$\Rightarrow 1.5y = 0$

$\Rightarrow y = \frac{0}{1.5} = 0$

From (1)

$2x - y - 3z = 0$

$\Rightarrow 2x - (0) - 3(0) = 0$

$\Rightarrow 2x = 0$

$\Rightarrow x = \frac{0}{2} = 0$

Solution using back substitution method.

$x = 0, y = 0$ and $z = 0$

Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination)

$$\left(\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right)$$

$R_1 / 2 \rightarrow R_1$ (divide the 1 row by 2)

$$\left(\begin{array}{ccc|c} 1 & -0.5 & -1.5 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right)$$

$R_2 + 1\ R_1 \rightarrow R_2$ (multiply 1 row by 1 and add it to 2 row); $R_3 - 1\ R_1 \rightarrow R_3$ (multiply 1 row by 1 and subtract it from 3 row)

$$\left(\begin{array}{ccc|c} 1 & -0.5 & -1.5 & 0 \\ 0 & 1.5 & -4.5 & 0 \\ 0 & 1.5 & 5.5 & 0 \end{array}\right)$$

$R_2 / 1.5 \rightarrow R_2$ (divide the 2 row by 1.5)

$$\left(\begin{array}{ccc|c} 1 & -0.5 & -1.5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1.5 & 5.5 & 0 \end{array}\right)$$

$R_1 + 0.5\ R_2 \rightarrow R_1$ (multiply 2 row by 0.5 and add it to 1 row); $R_3 - 1.5\ R_2 \rightarrow R_3$ (multiply 2 row by 1.5 and subtract it from 3 row)

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right)$$

$R_3 / 10 \rightarrow R_3$ (divide the 3 row by 10)

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$R_1 + 3\ R_3 \rightarrow R_1$ (multiply 3 row by 3 and add it to 1 row); $R_2 + 3\ R_3 \rightarrow R_2$ (multiply 3 row by 3 and add it to 2 row)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

Make a check:

$2\cdot0 - 0 - 3\cdot0 = 0 + 0 + 0 = 0$
$-0 + 2\cdot0 - 3\cdot0 = 0 + 0 + 0 = 0$
$0 + 0 + 4\cdot0 = 0 + 0 + 0 = 0$

Check completed successfully.

Answer:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

Matrices and Matrix Operations

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$2E^T - 3D^T$$

$$2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 9 \\ 15 & 0 & 6 \\ 6 & 3 & 12 \end{bmatrix} =$$

$$\begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$$

⑦ Let A be invertible matrix such that $A^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, find A.

$$A = (A^{-1})^{-1} = \frac{1}{2*5-(-1)*3}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{1}{13} \\ \frac{-3}{13} & \frac{2}{13} \end{pmatrix}$$

⑤ (b) Find A^{-1} where $A = \begin{pmatrix} -3 & 6 \\ 4 & 5 \end{pmatrix}$ by elementary row operations.

Solution:

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{cc|cc} -3 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_1 / -3 \to R_1$ (divide the 1 row by -3)

$$\left(\begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 4 & 5 & 0 & 1 \end{array}\right)$$

$R_2 - 4\,R_1 \to R_2$ (multiply 1 row by 4 and subtract it from 2 row)

$$\left(\begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 0 & 13 & \frac{4}{3} & 1 \end{array}\right)$$

$R_2 / 13 \to R_2$ (divide the 2 row by 13)

$$\left(\begin{array}{cc|cc} 1 & -2 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array}\right)$$

$R_1 + 2\,R_2 \to R_1$ (multiply 2 row by 2 and add it to 1 row)

$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{5}{39} & \frac{2}{13} \\ 0 & 1 & \frac{4}{39} & \frac{1}{13} \end{array}\right)$$

Answer:

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{pmatrix}$$

(a) Find A^{-1}, where $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$ by elementar

row operations.

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_1 / \frac{1}{5} \rightarrow R_1$ (divide the 1 row by $\frac{1}{5}$)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right)$$

$R_2 - \frac{1}{5} R_1 \rightarrow R_2$ (multiply 1 row by $\frac{1}{5}$ and subtract it from 2 row); $R_3 - \frac{1}{5} R_1 \rightarrow R_3$ (multiply 1 row by $\frac{1}{5}$ and subtract it from 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right)$$

$R_2 \leftrightarrow R_3$ (interchange the 2 and 3 rows)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right)$$

$R_2 / -1 \to R_2$ (divide the 2 row by -1)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right)$$

$R_1 - 1\,R_2 \to R_1$ (multiply 2 row by 1 and subtract it from 1 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right)$$

$R_3 / \frac{1}{2} \to R_3$ (divide the 3 row by $\frac{1}{2}$)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right)$$

$R_1 + \frac{3}{2} R_3 \to R_1$ (multiply 3 row by $\frac{3}{2}$ and add it to 1 row); $R_2 + \frac{1}{2} R_3 \to R_2$ (multiply 3 row by $\frac{1}{2}$ and add it to 2 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right)$$

Answer:

$$A^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

③ By $A^{-1}b$, Solve the system:

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 3$$

Finding A^{-1}:

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_2 - 2R_1 \rightarrow R_2$ (multiply 1 row by 2 and subtract it from 2 row); $R_3 - 2R_1 \rightarrow R_3$ (multiply 1 row by 2 and subtract it from 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right)$$

$R_2 / -4 \rightarrow R_2$ (divide the 2 row by -4)

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0.25 & 0.5 & -0.25 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right)$$

$R_1 - 3R_2 \rightarrow R_1$ (multiply 2 row by 3 and subtract it from 1 row); $R_3 + 3R_2 \rightarrow R_3$ (multiply 2 row by 3 and add it to 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0.25 & -0.5 & 0.75 & 0 \\ 0 & 1 & 0.25 & 0.5 & -0.25 & 0 \\ 0 & 0 & -0.25 & -0.5 & -0.75 & 1 \end{array}\right)$$

$R_3 / -0.25 \rightarrow R_3$ (divide the 3 row by -0.25)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0.25 & -0.5 & 0.75 & 0 \\ 0 & 1 & 0.25 & 0.5 & -0.25 & 0 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right)$$

$R_1 - 0.25R_3 \rightarrow R_1$ (multiply 3 row by 0.25 and subtract it from 1 row); $R_2 - 0.25R_3 \rightarrow R_2$ (multiply 3 row by 0.25 and subtract it from 2 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right)$$

Answer:

$$A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{B}$$

so

$$\mathbf{X} = \mathbf{A}^{-1} \cdot \mathbf{B}$$

Find a solution:

$$\mathbf{X} = \mathbf{A}^{-1} \cdot \mathbf{B} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} (-1)\cdot 4 + 0\cdot(-1) + 1\cdot 3 \\ 0\cdot 4 + (-1)\cdot(-1) + 1\cdot 3 \\ 2\cdot 4 + 3\cdot(-1) + (-4)\cdot 3 \end{pmatrix} = \begin{pmatrix} -4 + 0 + 3 \\ 0 + 1 + 3 \\ 8 - 3 - 12 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -7 \end{pmatrix}$$

Answer:

$$\begin{cases} x_1 = -1 \\ x_2 = 4 \\ x_3 = -7 \end{cases}$$

(18) Find conditions on b's must satisfy for the system to be consistent

$$x_1 - 2x_2 - x_3 = b_1$$
$$-4x_1 + 5x_2 + 2x_3 = b_2$$
$$-4x_1 + 7x_2 + 4x_3 = b_3$$

Adjoin the identity matrix onto the right of the original matrix, so that you have **A** on the left side and the identity matrix on the right side. It will look like this:

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -4 & 5 & 2 & 0 & 1 & 0 \\ -4 & 7 & 4 & 0 & 0 & 1 \end{array}\right)$$

Now find the inverse matrix. Using elementary row operations to transform the left side of the resulting matrix to the identity matrix.

$R_2 + 4R_1 \to R_2$ (multiply 1 row by 4 and add it to 2 row); $R_3 + 4R_1 \to R_3$ (multiply 1 row by 4 and add it to 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -3 & -2 & 4 & 1 & 0 \\ 0 & -1 & 0 & 4 & 0 & 1 \end{array}\right)$$

$R_2 / -3 \to R_2$ (divide the 2 row by -3)

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 0 & 4 & 0 & 1 \end{array}\right)$$

$R_1 + 2R_2 \to R_1$ (multiply 2 row by 2 and add it to 1 row); $R_3 + 1R_2 \to R_3$ (multiply 2 row by 1 and add it to 3 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & -\frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{8}{3} & -\frac{1}{3} & 1 \end{array}\right)$$

$R_3 / \frac{2}{3} \to R_3$ (divide the 3 row by $\frac{2}{3}$)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & -\frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 4 & -0.5 & 1.5 \end{array}\right)$$

$R_1 - \frac{1}{3} R_3 \to R_1$ (multiply 3 row by $\frac{1}{3}$ and subtract it from 1 row); $R_2 - \frac{2}{3} R_3 \to R_2$ (multiply 3 row by $\frac{2}{3}$ and subtract it from 2 row)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -0.5 & -0.5 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 4 & -0.5 & 1.5 \end{array}\right)$$

Answer:

$$\mathbf{A}^{-1} = \begin{pmatrix} -3 & -0.5 & -0.5 \\ -4 & 0 & -1 \\ 4 & -0.5 & 1.5 \end{pmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

In general, we say that a linear system is ***consistent*** if it has at least one solution.

Since $\mathbf{A}^{-1}$ exists, the system AX=b has unique solution X=$A^{-1}b$ regradless of values of b.So no conditions on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Determinants

Evaluate $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}$

$$+(-2)\begin{vmatrix} 1 & -2 \\ 8 & 4 \end{vmatrix} - (7)\begin{vmatrix} 5 & -2 \\ 3 & 4 \end{vmatrix} + (6)\begin{vmatrix} 5 & 1 \\ 3 & 8 \end{vmatrix} =$$

$$(-2)(4-(-16)) - 7(20-(-6)) + 6(40-3) =$$

$$-2(20) - 7(26) + 6(37) =$$

$$-40 - 182 + 222 = 0$$

Solve for x : $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$

L.H.S= $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = x(1-x) - (-3) = x - x^2 + 3$

R.H.S= $+(1)\begin{vmatrix} x & -6 \\ 3 & x-5 \end{vmatrix} - (0)\begin{vmatrix} 2 & -6 \\ 1 & x-5 \end{vmatrix} + (-3)\begin{vmatrix} 2 & x \\ 1 & 3 \end{vmatrix} =$

$$x(x-5) - (-18) - 0 - 3(6-x) =$$

$$x^2 - 5x + 18 - 18 + 3x =$$

$$x^2 - 2x$$

$$\Rightarrow \quad x - x^2 + 3 = x^2 - 2x$$

$$\Rightarrow \quad 0 = 2x^2 - 3x - 3$$

$$\Rightarrow \quad x = \frac{3+\sqrt{33}}{4} \text{ or } x = \frac{3-\sqrt{33}}{4}$$

y Find $\det \begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

Solution:

Transform matrix to upper triangular form, using elementary row operations and properties of a matrix determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} =$$

$R_2 - 5\,R_1 \rightarrow R_2$ (multiply 1 row by 5 and subtract it from 2 row); $R_3 + 1\,R_1 \rightarrow R_3$ (multiply 1 row by 1 and add it to 3 row); $R_4 - 2\,R_1 \rightarrow R_4$ (multiply 1 row by 2 and subtract it from 4 row)

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix} =$$

$R_4 - 12\,R_2 \rightarrow R_4$ (multiply 2 row by 12 and subtract it from 4 row)

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix} =$$

$R_4 + 36\,R_3 \rightarrow R_4$ (multiply 3 row by 36 and add it to 4 row)

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix} = 1\cdot 1\cdot(-3)\cdot(-13) = 39$$

4) Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

(i) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$, (ii) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

Solution:

scan me

5/ Without directly evaluating, show that:

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution:

scan me

⑥ Find A^{-1}, by using adj$A$, $A=\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

DEFINITION 1 If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by adj(A).

adj(A)=C^t

THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

Finding det(A):

$$\textbf{det(A)} = +(2)\begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} - (5)\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} + (5)\begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} =$$

$$(2)(-3-0) - 5(-3-0) + 5(-4-(-2)) =$$

$$2(-3) - 5(-3) + 5(-2) =$$

$$-6 + 15 - 10 = -1$$

The determinant of **A** is not zero, therefore the inverse matrix $\mathbf{A}^{-1}$ exist. To calculate the inverse matrix find additional minors and cofactors of matrix **A**

- Find the minor M_{11} and the cofactor C_{11}. In matrix **A** cross out row 1 and column 1.

  $$M_{11} = \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} = -3$$

  Show detailed calculation of the determinant

  $C_{11} = (-1)^{1+1}M_{11} = -3$
- Find the minor M_{12} and the cofactor C_{12}. In matrix **A** cross out row 1 and column 2.

  $$M_{12} = \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = -3$$

  Show detailed calculation of the determinant

  $C_{12} = (-1)^{1+2}M_{12} = 3$
- Find the minor M_{13} and the cofactor C_{13}. In matrix **A** cross out row 1 and column 3.

  $$M_{13} = \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} = -2$$

  Show detailed calculation of the determinant

  $C_{13} = (-1)^{1+3}M_{13} = -2$
- Find the minor M_{21} and the cofactor C_{21}. In matrix **A** cross out row 2 and column 1.

  $$M_{21} = \begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} = -5$$

  Show detailed calculation of the determinant

  $C_{21} = (-1)^{2+1}M_{21} = 5$

- Find the minor M_{22} and the cofactor C_{22}. In matrix **A** cross out row 2 and column 2.

  $$M_{22} = \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} = -4$$

  Show detailed calculation of the determinant

  $C_{22} = (-1)^{2+2} M_{22} = -4$
- Find the minor M_{23} and the cofactor C_{23}. In matrix **A** cross out row 2 and column 3.

  $$M_{23} = \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} = -2$$

  Show detailed calculation of the determinant

  $C_{23} = (-1)^{2+3} M_{23} = 2$
- Find the minor M_{31} and the cofactor C_{31}. In matrix **A** cross out row 3 and column 1.

  $$M_{31} = \begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} = 5$$

  Show detailed calculation of the determinant

  $C_{31} = (-1)^{3+1} M_{31} = 5$
- Find the minor M_{32} and the cofactor C_{32}. In matrix **A** cross out row 3 and column 2.

  $$M_{32} = \begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} = 5$$

  Show detailed calculation of the determinant

  $C_{32} = (-1)^{3+2} M_{32} = -5$
- Find the minor M_{33} and the cofactor C_{33}. In matrix **A** cross out row 3 and column 3.

  $$M_{33} = \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} = 3$$

  Show detailed calculation of the determinant

Show detailed calculation of the determinant

$$C_{33} = (-1)^{3+3}M_{33} = 3$$

Write matrix of cofactors:

$$\mathbf{C} = \begin{pmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{pmatrix}$$

Transposed matrix of cofactors:

$$\mathbf{C}^T = \begin{pmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{pmatrix}$$

Find inverse matrix:

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det \mathbf{A}} = \begin{pmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{pmatrix}$$

7/ Solve by Crammer's rule; where it applies

$$x - 4y + z = 4$$
$$4x - y + 2z = -1$$
$$2x + 2y - 3z = -20$$

$$\Delta = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = -55$$

Show detailed calculation of the determinant

$$\Delta_1 = \begin{vmatrix} 4 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 146$$

Show detailed calculation of the determinant

$$\Delta_2 = \begin{vmatrix} 1 & 4 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 29$$

Show detailed calculation of the determinant

$$\Delta_3 = \begin{vmatrix} 1 & -4 & 4 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = -250$$

Show detailed calculation of the determinant

$$x = \frac{\Delta_1}{\Delta} = \frac{146}{-55} = -\frac{146}{55}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{29}{-55} = -\frac{29}{55}$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-250}{-55} = \frac{50}{11}$$

10.3 THE DOT PRODUCT

Exer. 1–10: Given $\mathbf{a} = \langle -2, 3, 1\rangle$, $\mathbf{b} = \langle 7, 4, 5\rangle$, and $\mathbf{c} = \langle 1, -5, 2\rangle$, find the number.

(9) $\text{comp}_{\mathbf{b}}\,(\mathbf{a} + \mathbf{c})$

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle, \qquad \mathbf{b} = \langle b_1, b_2, b_3\rangle.$$

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$

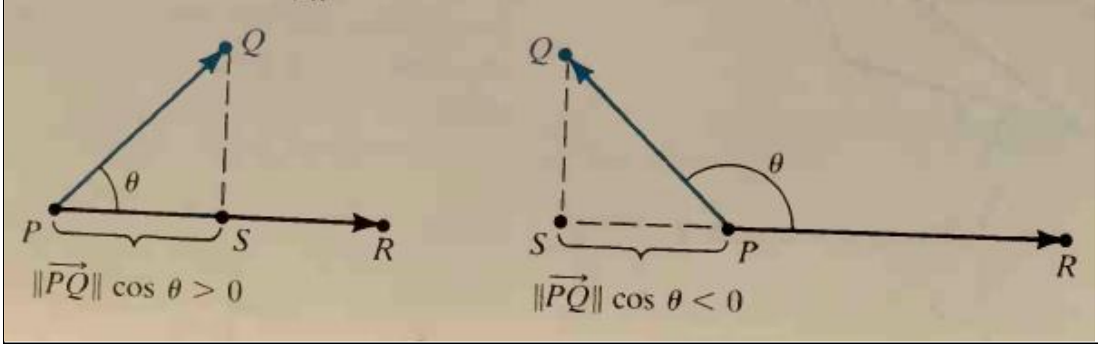
The **dot product** $\mathbf{a} \cdot \mathbf{b}$ of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in V_3 with $\mathbf{b} \neq \mathbf{0}$. The **component of a along b**, denoted by $\text{comp}_{\mathbf{b}}\,\mathbf{a}$, is

$$\text{comp}_{\mathbf{b}}\,\mathbf{a} = \mathbf{a} \cdot \frac{1}{\|\mathbf{b}\|}\mathbf{b}.$$

Figure 10.36 $\text{comp}_{\overrightarrow{PR}}\ \overrightarrow{PQ}$



$$comp_b(a + c) = \langle -1, -2, 3\rangle \cdot \frac{\langle 7, 4, 5\rangle}{\sqrt{49 + 16 + 25}} = \frac{(-7 - 8 + 15)}{\sqrt{90}} = 0$$

Exer. 17 – 18: Find all values of c such that a and b are orthogonal.

17 $\mathbf{a} = \langle c, -2, 3 \rangle$, $\mathbf{b} = \langle c, c, -5 \rangle$

Theorem 10.21 Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Solution:

$$a \text{ and } b \text{ are orthogonal } \Leftrightarrow$$

$$a \cdot b = 0 \Leftrightarrow c^2 - 2c - 15 = 0$$

$$\Leftrightarrow (c-5)(c+3) = 0$$

$$\Leftrightarrow c = 5 \text{ or } c = -3$$

Exer. 19–24: Given points $P(3, -2, -1)$, $Q(1, 5, 4)$, $R(2, 0, -6)$, and $S(-4, 1, 5)$, find the indicated quantity.

(22) The angle between $\overrightarrow{QS}$ and $\overrightarrow{RP}$

Theorem 10.7

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any points, the vector **a** in V_2 that corresponds to $\overrightarrow{P_1P_2}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1 \rangle.$$

Theorem 10.19

If θ is the angle between nonzero vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta.$$

$$P(3, -2, -1), Q(1, 5, 4), R(2, 0, -6), S(-4, 1, 5)$$

$$\overrightarrow{QS} = S - Q = \langle -4 - 1, 1 - 5, 5 - 4 \rangle = \langle -5, -4, 1 \rangle$$

$$\Rightarrow \ \|\overrightarrow{QS}\| = \sqrt{25 + 16 + 1} = \sqrt{42}$$

$$\overrightarrow{RP} = P - R = \langle 3 - 2, (-2) - 0, (-1) - (-6) \rangle = \langle 1, -2, 5 \rangle$$

$$\Rightarrow \ \|\overrightarrow{RP}\| = \sqrt{1 + 4 + 25} = \sqrt{30}$$

$$\overrightarrow{QS} \cdot \overrightarrow{RP} = (-5)(1) + (-4)(-2) + 1(5) = 8$$

$$\cos(\theta) = \frac{\overrightarrow{QS} \cdot \overrightarrow{RP}}{\|\overrightarrow{QS}\|\|\overrightarrow{RP}\|} = \frac{8}{\sqrt{42}\sqrt{30}} \approx 0.2254$$

$$\theta = \cos^{-1}(0.2254) \approx 76.97^{\circ} \ or \ \approx (180^{\circ} - 76.97^{\circ}) = 103.03^{\circ}$$

Exer. 25–26: If the vector a represents a constant force, find the work done when its point of application moves along the line segment from P to Q.

25 $\mathbf{a} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$; $P(4, 0, -7)$, $Q(2, 4, 0)$

Definition 10.26

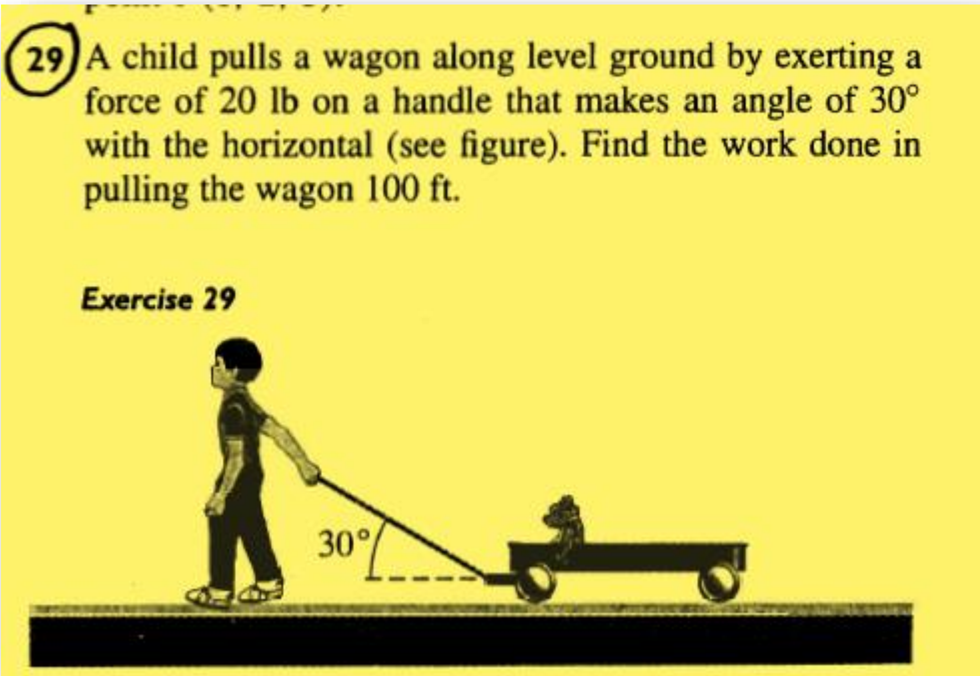
The **work done by a constant force** $\overrightarrow{PQ}$ as its point of application moves along the vector $\overrightarrow{PR}$ is $\overrightarrow{PQ} \cdot \overrightarrow{PR}$.

$$P(4, 0, -7), Q(2, 4, 0)$$

$$\overrightarrow{PQ} = Q - P = \langle 2-4, 4-0, 0-(-7) \rangle = \langle -2, 4, 7 \rangle = -2i + 4j + 7k$$

$$W = a \cdot \overrightarrow{PQ} = (-i + 5j - 3k) \cdot (-2i + 4j + 7k)$$

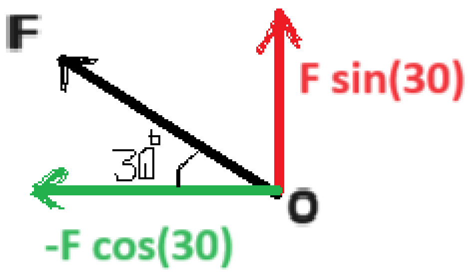
$$(-1)(-2) + 5(4) + (-3)(7) = 2 + 20 - 21 = 1 \quad Joul$$



(29) A child pulls a wagon along level ground by exerting a force of 20 lb on a handle that makes an angle of 30° with the horizontal (see figure). Find the work done in pulling the wagon 100 ft.

Exercise 29

Solution:



$$\vec{F} = -F\cos(30^\circ)\,\boldsymbol{i} + F\sin(30^\circ)\,\boldsymbol{j}$$

$$= -20\,\frac{\sqrt{3}}{2}\,\boldsymbol{i} + 20\,\frac{1}{2}\,\boldsymbol{j}$$

$$= -10\sqrt{3}\,\boldsymbol{i} + 10\,\boldsymbol{j}$$

$$\boldsymbol{d} = -100\boldsymbol{i} + 0.\boldsymbol{j}$$

$$W = \vec{F}\cdot \boldsymbol{d} = -10\sqrt{3}(-100) + 10(0) = 1000\sqrt{3} \approx 1732 \quad ft - lb$$

36 Refer to Exercise 35.

(a) Find the direction cosines of $\mathbf{a} = \langle -2,\ 1,\ 5 \rangle$.

(b) Find the direction angles and the direction cosines of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

(c) Find two unit vectors that satisfy the condition

$$\cos\alpha = \cos\beta = \cos\gamma.$$

35 The *direction angles* of a nonzero vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ are defined as the angles α, β, and γ between the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively, and the vector $\mathbf{a}$. The *direction cosines* of $\mathbf{a}$ are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Prove the following:

(a) $\cos\alpha = \dfrac{a_1}{\|\mathbf{a}\|}, \quad \cos\beta = \dfrac{a_2}{\|\mathbf{a}\|}, \quad \cos\gamma = \dfrac{a_3}{\|\mathbf{a}\|}$

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Solution:

***(a)* :**

$$\|a\| = \sqrt{4+1+25} = \sqrt{30}$$

$$cos(\alpha) = \frac{-2}{\sqrt{30}} = -0.999979692241$$

$$\Rightarrow\ \alpha = cos^{-1}(0.999979692241) = 179.634851630589^{\circ}$$

$$cos(\beta) = \frac{1}{\sqrt{30}} = 0.999994923047$$

$$\Rightarrow\ \beta = cos^{-1}(0.999994923047) = 0.18257419122^{\circ}$$

$$cos(\gamma) = \frac{5}{\sqrt{30}} = 0.99987307876$$

$$\Rightarrow\ \gamma = cos^{-1}(0.99987307876) = 0.912870929223^{\circ}$$

(c)

$$u_1 = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$$

$$u_2 = \langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \rangle$$

10.4 THE VECTOR PRODUCT

Exer. 11 – 12: Use the vector product to show that **a** and **b** are parallel.

12 $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \qquad \mathbf{b} = -6\mathbf{i} + 3\mathbf{j} - 12\mathbf{k}$

Corollary 10.31 Two vectors **a** and **b** are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ -6 & 3 & -12 \end{vmatrix}$$

$$= i\begin{vmatrix} -1 & 4 \\ 3 & -12 \end{vmatrix} - j\begin{vmatrix} 2 & 4 \\ -6 & -12 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix}$$

$$= i(12 - 12) - j(-24 + 24) + k(6 - 6) = 0i + 0j + 0k$$

$$\therefore a \text{ and } b \text{ are parallel.}$$

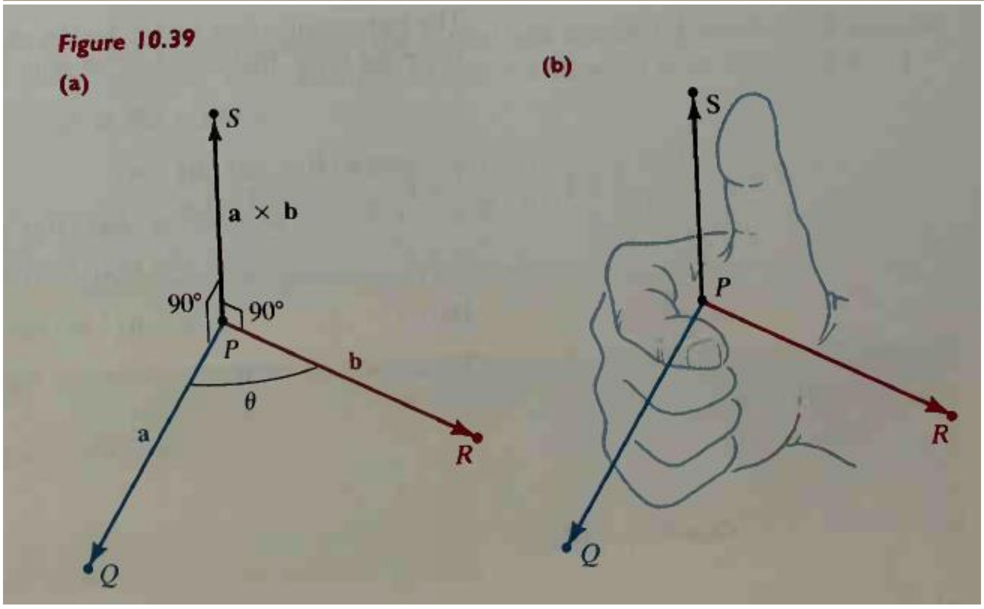
Exer. 15 – 18: **(a)** Find a vector perpendicular to the plane determined by P, Q, and R. **(b)** Find the area of the triangle PQR.

16 $P(-3, 0, 5)$, $Q(2, -1, -3)$, $R(4, 1, -1)$

$$P(-3, 0, 5), Q(2, -1, -3), R(4, 1, -1)$$

Theorem 10.29 The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(a)



A vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular (orthogonal) to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = Q - P = \langle 2 - (-3), -1 - 0, -3 - 5 \rangle = \langle 5, -1, -8 \rangle$$

$$\overrightarrow{PR} = R - P = \langle 4 - (-3), 1 - 0, -1 - 5 \rangle = \langle 7, 1, -6 \rangle$$

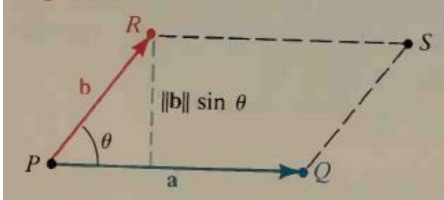
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 5 & -1 & -8 \\ 7 & 1 & -6 \end{vmatrix}$$

$$= i \begin{vmatrix} -1 & -8 \\ 1 & -6 \end{vmatrix} - j \begin{vmatrix} 5 & -8 \\ 7 & -6 \end{vmatrix} + k \begin{vmatrix} 5 & -1 \\ 7 & 1 \end{vmatrix}$$

$$= i(6 + 8) - j(-30 + 56) + k(5 + 7) = 14i - 26j + 12k$$

$(\boldsymbol{b})$:

Figure 10.40



To interpret $\|\mathbf{a} \times \mathbf{b}\|$ geometrically, let us represent $\mathbf{a}$ and $\mathbf{b}$ by vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ having the same initial point P. As in Figure 10.40, let S be the point such that segments PQ and PR are adjacent sides of a parallelogram with vertices P, Q, R, and S. An altitude of the parallelogram is $\|\mathbf{b}\| \sin\theta$, and hence its area is $\|\mathbf{a}\|\ \|\mathbf{b}\| \sin\theta$. Comparing this with Theorem (10.30), we see that *the magnitude of the vector product* $\mathbf{a} \times \mathbf{b}$ *equals the area of the parallelogram determined by* $\mathbf{a}$ *and* $\mathbf{b}$.

Area of triangle=$\frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{14^2 + (-26)^2 + 12^2} = \frac{1}{2}\sqrt{1016} \approx 15.94$ **unite** ^{2}

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Exer. 19 – 20: Refer to Example 3. Find the distance from P to the line through Q and R.

(19) $P(3, 1, -2)$, $Q(2, 5, 1)$, $R(-1, 4, 2)$

EXAMPLE 3 Find a formula for the distance d from a point R to a line l.

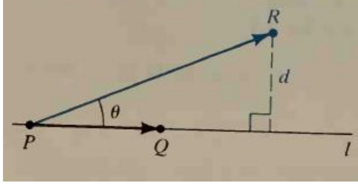
SOLUTION Let P and Q be points on l, as shown in Figure 10.42, and let θ be the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Since $d = \|\overrightarrow{PR}\| \sin\theta$, we obtain

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\| \sin\theta = \|\overrightarrow{PQ}\|\, d.$$

Hence,

$$d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PQ}\|}.$$

Figure 10.42



$$P(3, 1, -2), Q(2, 5, 1), R(-1, 4, 2)$$

$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|}$$

$$\overrightarrow{QP} = P - Q = \langle 3 - 2, 1 - 5, -2 - 1 \rangle = \langle 1, -4, -3 \rangle$$

$$\overrightarrow{QR} = R - Q = \langle -1 - 2, 4 - 5, 2 - 1 \rangle = \langle -3, -1, 1 \rangle$$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & -4 & -3 \\ -3 & -1 & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} -4 & -3 \\ -1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & -3 \\ -3 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & -4 \\ -3 & -1 \end{vmatrix}$$

$$= i(-4 - 3) - j(1 - 9) + k(-1 - 12) = -7i + 8j - 13k$$

$$\Rightarrow \|\overrightarrow{QP} \times \overrightarrow{QR}\| = \sqrt{(-7)^2 + (8)^2 + (-13)^2} = \sqrt{49 + 64 + 169} = \sqrt{282}$$

$$\Rightarrow \|\overrightarrow{QR}\| = \sqrt{(-3)^2 + (-1)^2 + (1)^2} = \sqrt{9 + 1 + 1} = \sqrt{11}$$

$$d = \frac{\sqrt{282}}{\sqrt{11}} = \sqrt{\frac{282}{11}} \text{ unite}$$

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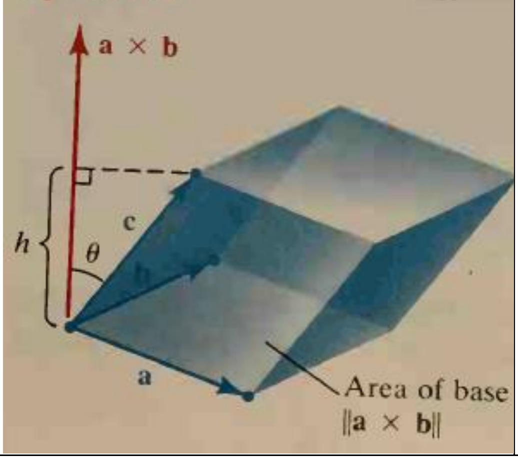
Exer. 22–23: Use Example 4 and Exercise 21 to find the volume of the box having adjacent sides *AB*, *AC*, and *AD*.

(22) $A(0, 0, 0)$, $B(1, -1, 2)$, $C(0, 3, -1)$, $D(3, -4, 1)$

Figure 10.43



$$V = (\text{area of base})(\text{altitude})$$
$$= \|\mathbf{a} \times \mathbf{b}\| \, \|\mathbf{c}\| \, |\cos\theta|$$
$$= |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|,$$

$$V = \left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD}\right|$$

$$\overrightarrow{AB} = B - A = \langle 1-0, -1-0, 2-0 \rangle = \langle 1, -1, 2 \rangle$$

$$,\overrightarrow{AC} = C - A = \langle 0-0, 3-0, -1-0 \rangle = \langle 0, 3, -1 \rangle,$$

$$\overrightarrow{AD} = D - A = \langle 3-0, -4-0, 1-0 \rangle = \langle 3, -4, 1 \rangle$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

$$V = \left|\begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 3 & -4 & 1 \end{vmatrix}\right|$$

$$= \left| +(1)\begin{vmatrix} 3 & -1 \\ -4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix} + (2)\begin{vmatrix} 0 & 3 \\ 3 & -4 \end{vmatrix} \right|$$

$$= |(3-4) + (0+3) + 2(0-9)|$$

$$= |-1 + 3 - 18| = |-16| = 16 \ \text{unite}^3$$

10.5 LINES AND PLANES

Exer. 1–4: Find parametric equations for the line through P parallel to a.

(4) $P(1, 2, 3)$; $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Theorem 10.34

Parametric equations for the line through $P_1(x_1, y_1, z_1)$ parallel to $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ are

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t; \quad t \text{ in } \mathbb{R}.$$

$$\boldsymbol{P}(1, 2, 3), \quad \boldsymbol{a} = 1\,\boldsymbol{i} + 2\,\boldsymbol{j} + 3\,\boldsymbol{k}$$

Parametric equations of the line:

$$x = 1 + 1.t, \quad y = 2 + 2t, \quad z = 3 + 3t \quad . \ t \in \mathbb{R}$$

(Every value of " t " gives a point on the line).

Exer. 11 – 14: Determine whether the two lines intersect, and if so, find the point of intersection.

(11) $x = 1 + 2t, \quad y = 1 - 4t, \quad z = 5 - t$

$x = 4 - v, \quad y = -1 + 6v, \quad z = 4 + v$

Let $x = x$ and $y = y \Rightarrow 1 + 2t = 4 - v$ and $1 - 4t = -1 + 6v \Rightarrow$

$$\begin{matrix} 2t + v = 3 \\ -4t - 6v = -2 \end{matrix} \Rightarrow v = -1\,, t = 2$$

Now plug $v = -1\,, t = 2$ in z (of line 1) and z (of line 2)

$$\begin{matrix} (\text{of line } 1): z = 5 - 2 = 3 \\ (\text{of line } 2): z = 4 + (-1) = 3 \end{matrix} \Rightarrow z = z \Rightarrow \text{line 1 and line 2 intersect.}$$

To find the point of intersection, go to any of the two lines:

Line 2 (say): $x = 4 - (-1) = 5, y = -1 + 6(-1) = -7, z = 4 + (-1) = 3$

$\therefore$ The point of intersection is $(5, -7, 3)$

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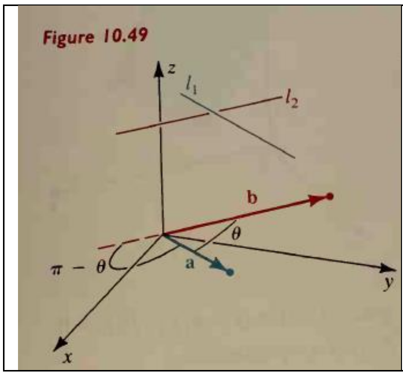
Exer. 15 – 18: Equations for two lines l_1 and l_2 are given. Find the angles between l_1 and l_2.



Let θ be the angle between nonzero vectors **a** and **b** and let l_1 and l_2 be lines that are parallel to the position vectors of **a** and **b** (see Figure 10.49).

(i) The angles between l_1 and l_2 are θ and $\pi - \theta$.

16) $x = 5 + 3t, \quad y = 4 - t, \quad z = 3 + 2t$

$x = -t, \quad y = 1 - 2t, \quad z = 3 + t$

$$\boldsymbol{a} = \langle 3, -1, 2 \rangle \,, \boldsymbol{b} = \langle -1, -2, 1 \rangle$$

$$\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \, \|\boldsymbol{b}\|} = \frac{3(-1) + (-1)(-2) + 2(1)}{\sqrt{9+1+4}\,\sqrt{1+4+1}} = \frac{1}{\sqrt{14}\,\sqrt{6}} \approx 0.1091$$

$$\theta = \cos^{-1}(0.1091) \approx 83.74^{\circ} \text{ and } \pi - 83.74^{\circ} = 96.26^{\circ}$$

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17 $\dfrac{x-1}{-3} = \dfrac{y+2}{8} = \dfrac{z}{-3}; \quad \dfrac{x+2}{10} = \dfrac{y}{10} = \dfrac{z-4}{-7}$

$$\boldsymbol{a} = \langle -3, 8, -3 \rangle, \boldsymbol{b} = \langle 10, 10, -7 \rangle$$

$$\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \, \|\boldsymbol{b}\|} = \frac{(-3)(10) + 8(10) + (-3)(-7)}{\sqrt{9+64+9}\sqrt{100+100+49}} = \frac{71}{\sqrt{82}\sqrt{249}} \approx 0.49688$$

$$\theta = \cos^{-1}(0.49688) \approx 60.21^{\circ} \text{ and } \pi - 60.21^{\circ} = 119.79^{\circ}$$

Exer. 19–26: Find an equation of the plane that satisfies the stated conditions.

19 Through $P(6,\ -7,\ 4)$ and parallel to

(a) the xy-plane **(b)** the yz-plane **(c)** the xz-plane

Theorem 10.36

An equation of the plane through $P_1(x_1, y_1, z_1)$ with normal vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0.$$

Parallel planes have parallel normal vectors.

(a) $\langle 0,0,1\rangle \perp xy-plane$ ***and the sought plane*** $\parallel xy - plane$

$\Rightarrow\ \langle 0,0,1\rangle \perp \boldsymbol{the\ sought\ plane} \Rightarrow \boldsymbol{a} = \langle 0,0,1\rangle = \langle a_1, a_2, a_3\rangle$

$P(6,-7,4) = (x_1, y_1, z_1)$

Equation of ***the sought*** **plane is :** $0(x-6) + 0\big(y-(-7)\big) + 1(z-4) = 0$

$z = 4$. **(This plane contains all points of the form (a, b, 4) and is parallel to** $xy - plane$**).**

(b) $\langle 1,0,0\rangle \perp yz - plane$ ***and the sought plane*** $\parallel yz - plane$

$\Rightarrow\ \langle 1,0,0\rangle \perp \boldsymbol{the\ sought\ plane} \Rightarrow \boldsymbol{a} = \langle 1,0,0\rangle = \langle a_1, a_2, a_3\rangle$

$P(6,-7,4) = (x_1, y_1, z_1)$

Equation of ***the sought*** **plane is :** $1(x-6) + 0\big(y-(-7)\big) + 0(z-4) = 0$

$x = 6$. **(This plane contains all points of the form (6, b, c) and is parallel to** $yz - plane$ **).**

(c) $\langle 0,1,0\rangle \perp xz - plane$ ***and the sought plane*** $\parallel xz - plane$

$\Rightarrow\ \langle 0,1,0\rangle \perp \boldsymbol{the\ sought\ plane} \Rightarrow \boldsymbol{a} = \langle 0,1,0\rangle = \langle a_1, a_2, a_3\rangle$

$P(6,-7,4) = (x_1, y_1, z_1)$

Equation of ***the sought*** **plane is :** $0(x-6) + 1\big(y-(-7)\big) + 0(z-4) = 0$

$y = -7$. **(This plane contains all points of the form (a, -7, c) and is parallel to** $xz - plane$ **).**

23 Through $P(2, 5, -6)$ and parallel to the plane $3x - y + 2z = 10$

Parallel planes have parallel normal vectors.

$\langle 3, -1, 2\rangle \perp$ ***the plane*** $3x - y + 2z = 10$ ***and***

the sought plane $\parallel$ ***the plane*** $3x - y + 2z = 10$

$\Rightarrow \ \langle 3, -1, 2\rangle \perp$ ***the sought plane*** $\Rightarrow a = \langle 3, -1, 2\rangle = \langle a_1, a_2, a_3\rangle$

$P(2, 5, -6) = (x_1, y_1, z_1)$

Equation of our plane is :

$$3(x - 2) + (-1)(y - 5) + 2(z - (-6)) = 0 \quad \Rightarrow$$

$$3x - 6 - y + 5 + 2z + 12 = 0. \quad \Rightarrow$$

$$3x - y + 2z + 11 = 0$$

Exer. 27 – 28: Find an equation of the plane through P, Q, and R.

(27) $P(1,\ 1,\ 3)$, $Q(-1,\ 3,\ 2)$, $R(1,\ -1,\ 2)$

$$P(1,1,3), Q(-1,3,2), R(1,-1,2)$$

A vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular (orthogonal) to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ $\Rightarrow$

A vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular (normal) *to the plane determined by the points* P, Q and R

$$\overrightarrow{PQ} = Q - P = \langle -1-1, 3-1, 2-3 \rangle = \langle -2, 2, -1 \rangle$$

$$\overrightarrow{PR} = R - P = \langle 1-1, -1-1, 2-3 \rangle = \langle 0, -2, -1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -2 & 2 & -1 \\ 0 & -2 & -1 \end{vmatrix} = -4i - 2j + 4k = \langle -4, -2, 4 \rangle$$

$P(1,1,3) = (x_1, y_1, z_1)$

Equation of our plane is :

$$(-4)(x-1) + (-2)(y-1) + 4(z-3) = 0 \quad \Rightarrow$$

$$-4x + 4 - 2y + 2 + 4z - 12 = 0. \quad \Rightarrow$$

$$-4x - 2y + 4z - 6 = 0 \quad \Rightarrow$$

$$2x + y - 2z = -3$$

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Exer. 29–36: Sketch the graph of the equation in an *xyz*-coordinate system.

29 (a) $x = 3$ (b) $y = -2$ (c) $z = 5$

30 (a) $x = -4$ (b) $y = 0$ (c) $z = -\frac{2}{3}$

31 $2x + y - 6 = 0$

32 $3x - 2z - 24 = 0$

33 $2y - 3z - 9 = 0$

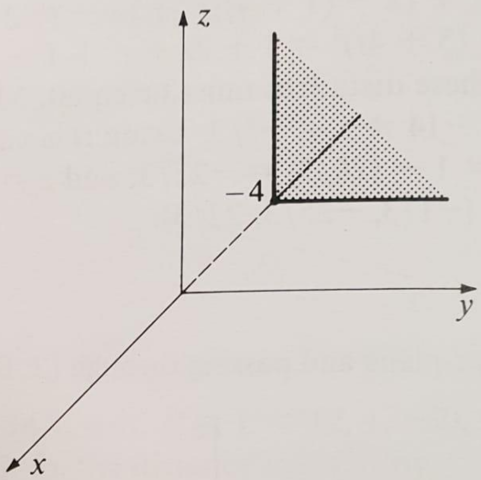
34 $5x + y - 4z + 20 = 0$

35 $2x - y + 5z + 10 = 0$

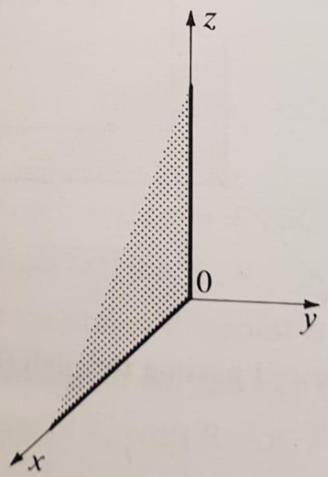
36 $x + y + z = 0$

30:

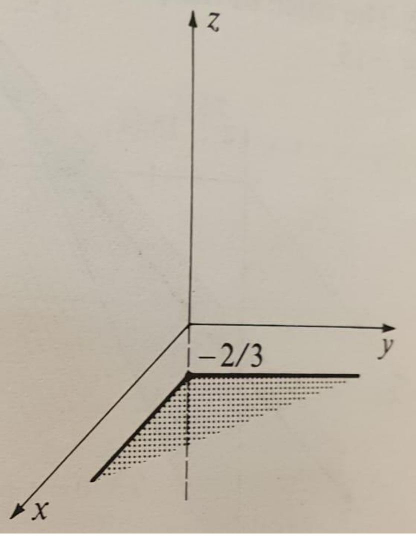
(a) A plane parallel to the yz-plane and passing through $(-4, 0, 0)$.



(b) This is the xz-plane.



(c) A plane parallel to the xy-plane passing through $(0, 0, -2/3)$.



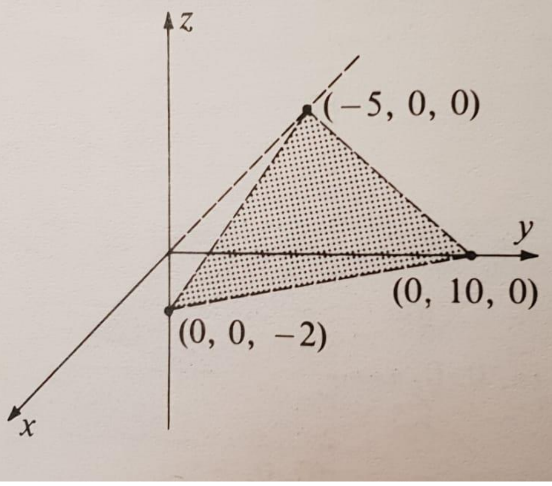
35:

Intersection with x-axis: $\boldsymbol{put\ y = z = 0 \ \Rightarrow 2x + 10 = 0 \ \Rightarrow x = -5 \ \Rightarrow}$ $\boldsymbol{point\ of}$ **intersection** $(\mathbf{-5, 0, 0})$

Intersection with y-axis: $\boldsymbol{put\ x = z = 0 \ \Rightarrow -y + 10 = 0 \ \Rightarrow y = 10 \Rightarrow}$ $\boldsymbol{point\ of}$ **intersection**$(\mathbf{0, 10, 0})$

Intersection with z-axis: $\boldsymbol{put\ x = y = 0 \ \Rightarrow 5z + 10 = 0 \ \Rightarrow z = -2 \Rightarrow}$ $\boldsymbol{point\ of}$ **intersection**$(\mathbf{0, 0, -2})$

See Example 3 of this section. The points of intersection of the plane with the coordinate axes are $(-5, 0, 0)$, $(0, 10, 0)$, and $(0, 0, -2)$.



Exer. 43 – 46: Find a symmetric form for the line through P_1 and P_2.

43 $P_1(5, -2, 4)$, $P_2(2, 6, 1)$

44 $P_1(-3, 1, -1)$, $P_2(7, 11, -8)$

45 $P_1(4, 2, -3)$, $P_2(-3, 2, 5)$

46 $P_1(5, -7, 4)$, $P_2(-2, -1, 4)$

Symmetric Form for a Line **10.39**

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$$

$\langle a_1, a_2, a_3 \rangle = \overrightarrow{P_1P_2} = P_2 - P_1 = \langle -7, 6, 0 \rangle$ is a direction vector of the line.

Choose $P_1(5, -7, 4)$ (say)

Symmetric form of the line is: $\frac{x-5}{-7} = \frac{y-(-7)}{6} = \frac{z-0}{4} \quad \Rightarrow \quad \frac{x-5}{-7} = \frac{y+7}{6} = \frac{z}{4}$

Exer. 47 – 50: Find parametric equations for the line of intersection of the two planes.

47 $x + 2y - 9z = 7, \quad 2x - 3y + 17z = 0$

(48) $2x + 5y + 16z = 13, \quad -x - 2y - 6z = -5$

By linear algebra: **Gauss Elimination Back Substitution method for the system:**

$$\begin{cases} 2x + 5y + 16z = 13 \\ -x - 2y - 6z = -5 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & 16 & 13 \\ -1 & -2 & -6 & -5 \end{array}\right)$$

$R_1 / 2 \to R_1$ (divide the 1 row by 2)

$$\left(\begin{array}{ccc|c} 1 & 2.5 & 8 & 6.5 \\ -1 & -2 & -6 & -5 \end{array}\right)$$

$R_2 + 1\,R_1 \to R_2$ (multiply 1 row by 1 and add it to 2 row)

$$\left(\begin{array}{ccc|c} 1 & 2.5 & 8 & 6.5 \\ 0 & 0.5 & 2 & 1.5 \end{array}\right)$$

$R_2 / 0.5 \to R_2$ (divide the 2 row by 0.5)

$$\left(\begin{array}{ccc|c} 1 & 2.5 & 8 & 6.5 \\ 0 & 1 & 4 & 3 \end{array}\right)$$

System has infinitely many solutions

Put $z = t\,, t \text{ any real number then } y = 3 - 4t \Rightarrow x = 2t - 1$

$x = -1 + 2t$

$y = 3 - 4t$ **(parametric equations of the line "intersection of two planes")**

$z = t$

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Exer. 51–52: Refer to Example 13. Find the distance from P to the plane.

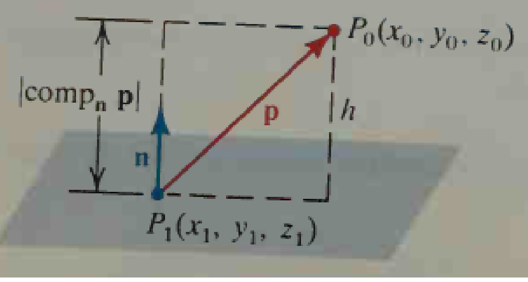
51 $P(1, -1, 2)$; $3x - 7y + z - 5 = 0$

52 $P(3, 1, -2)$; $2x + 4y - 5z + 1 = 0$

52:

EXAMPLE 13 Find a formula for the distance h from a point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$.

Figure 10.55



$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

$$P_0(3, 1, -2) = (x_0, y_0, z_0),$$

Distance from $P_0(3, 1, -2)$ to the plane :

$$h = \frac{|2(3) + 4(1) + (-5)(-2) + 1|}{\sqrt{4 + 16 + 25}} = \frac{21}{\sqrt{45}} \approx 3.13 \text{ unite}$$

Exer. 53 – 54: Show that the two planes are parallel and find the distance between the planes.

53 $4x - 2y + 6z = 3, \qquad -6x + 3y - 9z = 4$

Definition 10.38

Two planes with normal vectors **a** and **b** are

(i) **parallel** if **a** and **b** are parallel

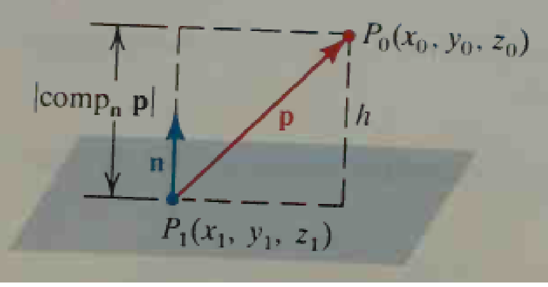
(ii) **orthogonal** if **a** and **b** are orthogonal

$$\boldsymbol{a} = \langle 4, -2, 6 \rangle \ \boldsymbol{normal\ on\ plane}\ 1, \boldsymbol{b} = \langle -6, 3, -9 \rangle \ \boldsymbol{normal\ on\ plane}\ 2$$

$$\frac{4}{-6} = \frac{-2}{3} = \frac{6}{-9} \ \Rightarrow \boldsymbol{a} = \frac{-2}{3}\boldsymbol{b} \ \Rightarrow \boldsymbol{a} \parallel \boldsymbol{b} \ \Rightarrow \boldsymbol{plane}1\ \boldsymbol{is\ parallel\ to\ plane}\ 2$$

EXAMPLE 13 Find a formula for the distance h from a point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$.

Figure 10.55



$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

To find the distance take a point on plane 1: Put $x = y = 0 \ \Rightarrow z = \frac{1}{2} \ \Rightarrow$

$\boldsymbol{P_0}\left(0, 0, \frac{1}{2}\right) = (x_0, y_0, z_0), \quad \langle -6, 3, -9 \rangle = \langle a, b, c \rangle \ \boldsymbol{vector\ normal\ on\ plane}\ 2$

Distance from $P_0\left(0, 0, \frac{1}{2}\right)$ to plane 2 : $h = \frac{|(-6)(0) + 3(0) + (-9)\left(\frac{1}{2}\right) + (-4)|}{\sqrt{36 + 9 + 81}} = \frac{\frac{17}{2}}{\sqrt{126}} \approx$

0.7572 unite

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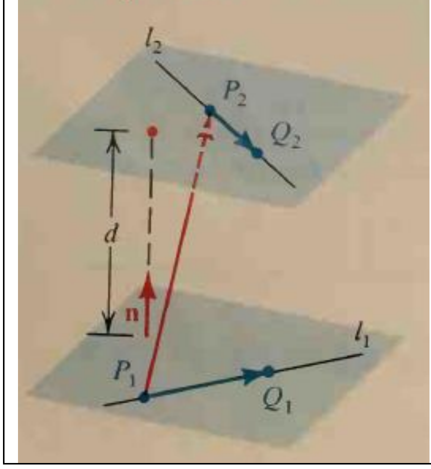
Exer. 55–56: Refer to Example 14. Let l_1 be the line through A and B, and let l_2 be the line through C and D. Find the shortest distance between l_1 and l_2.

55 $A(1, -2, 3)$, $B(2, 0, 5)$; $C(4, 1, -1)$, $D(-2, 3, 4)$

Figure 10.56



$$d = |\text{comp}_{\mathbf{n}} \overrightarrow{P_1P_2}| = |\mathbf{n} \cdot \overrightarrow{P_1P_2}|$$
$$= \frac{1}{\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\|} |(\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}) \cdot \overrightarrow{P_1P_2}|.$$

$$\overrightarrow{AB} = \langle 1, 2, 2\rangle, \overrightarrow{CD} = \langle -6, 2, 5\rangle, \overrightarrow{AC} = \langle 3, 3, -4\rangle$$

$$\overrightarrow{AB} \times \overrightarrow{CD} = \langle 6, -17, 14\rangle \quad \Rightarrow \|\overrightarrow{AB} \times \overrightarrow{CD}\| = \sqrt{521}$$

$$(\overrightarrow{AB} \times \overrightarrow{CD}) \cdot \overrightarrow{AC} = -89$$

$$d = \frac{1}{\|\overrightarrow{AB} \times \overrightarrow{CD}\|} |(\overrightarrow{AB} \times \overrightarrow{CD}) \cdot \overrightarrow{AC}| = \frac{89}{\sqrt{521}} \approx 3.8992 \text{ unite}$$

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Exer. 57 – 58: Find an equation of the plane that contains the point P and the line.

57 $P(5, 0, 2)$; $x = 3t + 1$, $y = -2t + 4$, $z = t - 3$

58 $P(4, -3, 0)$; $x = t + 5$, $y = 2t - 1$, $z = -t + 7$

58:

A vector $\boldsymbol{a} = \langle 1, 2, -1 \rangle$ (directed vector of the line) lies on the plane.

Take a point Q on the line (So it is on the plane): Set $t = 0$

$$\Rightarrow \quad x = 5, y = -1, z = 7$$

$$Q(5, -1, 7) \Rightarrow \overrightarrow{PQ} = Q - P = \langle 1, 2, 7 \rangle$$

Now $\overrightarrow{PQ} \times \boldsymbol{a}$ is a normal vector on the plane.

$$\overrightarrow{PQ} \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 7 \\ 1 & 2 & -1 \end{vmatrix} = -16\boldsymbol{i} + 8\boldsymbol{j} + 0.\boldsymbol{k} = \langle -16, 8, 0 \rangle$$

$$P(4, -3, 0)$$

Equation of the plane: $-16(x - 4) + 8(y - 3) + 0.(z - 0) = 0 \quad \Rightarrow$

$$-16x + 64 + 8y - 24 = 0 \quad \Rightarrow$$

$$2x - y - 5 = 0$$

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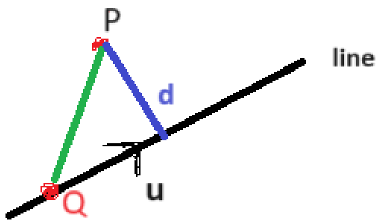
Exer. 61–62: Find the distance from the point P to the line.

61 $P(2, 1, -2)$; $x = 3 - 2t$, $y = -4 + 3t$, $z = 1 + 2t$

62 $P(3, 1, -1)$; $x = 1 + 4t$, $y = 3 - t$, $z = 3t$



The direction vector of the line: $\vec{u} = \langle 4, -1, 3 \rangle$

To find a point Q on the line, let $t = 0$ and obtain the point $Q(1, 3, 0)$

$$d = \frac{\|\overrightarrow{PQ} \times \vec{u}\|}{\|\vec{u}\|}$$

$$\overrightarrow{PQ} = Q - P = \langle 1 - 3, 3 - 1, 0 - (-1) \rangle = \langle -2, 2, 1 \rangle$$

$$\overrightarrow{PQ} \times \vec{u} = \begin{vmatrix} i & j & k \\ -2 & 2 & 1 \\ 4 & -1 & 3 \end{vmatrix} = 7i + 10j - 6k = \langle 7, 10, -6 \rangle$$

$$\|\overrightarrow{PQ} \times \vec{u}\| = \sqrt{49 + 100 + 36} = \sqrt{185}$$

$$\|\vec{u}\| = \sqrt{16 + 1 + 9} = \sqrt{26}$$

$$d = \frac{\sqrt{185}}{\sqrt{26}} = \sqrt{\frac{185}{26}} \approx 2.67 \text{ unite}$$

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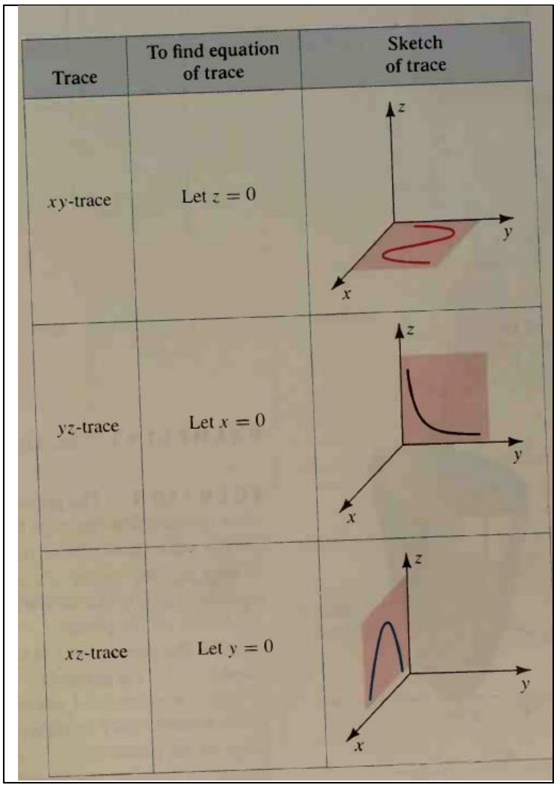
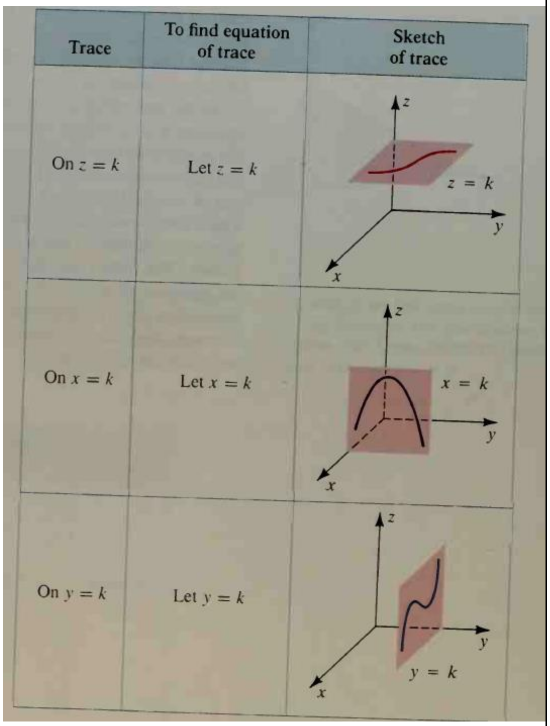
10.6 SURFACES



Trace	To find equation of trace	Sketch of trace
xy-trace	Let $z = 0$	
yz-trace	Let $x = 0$	
xz-trace	Let $y = 0$	



Trace	To find equation of trace	Sketch of trace
On $z = k$	Let $z = k$	
On $x = k$	Let $x = k$	
On $y = k$	Let $y = k$	

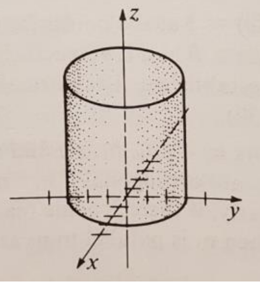
Exer. 1–8: Sketch the graph of the cylinder in an *xyz*-coordinate system.

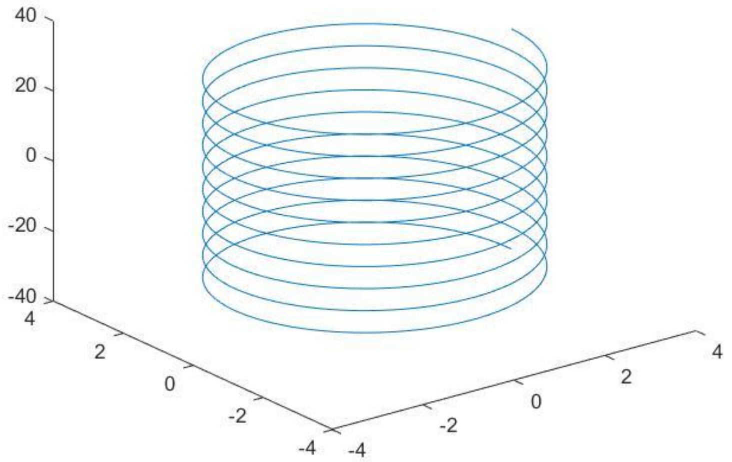
(1) $x^2 + y^2 = 9$	(6) $x^2 - 4y = 0$

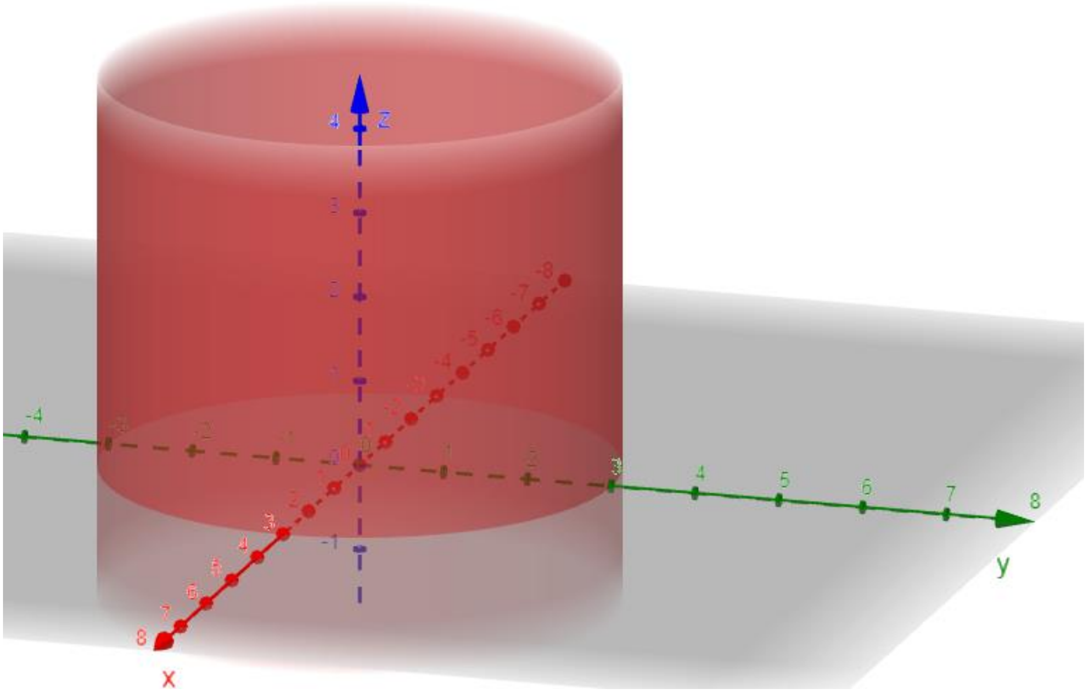
1:

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$x^2 + y^2 = 9$	circle	
$yz - plane\ (x = 0)$	$y = \pm 3$	Two lines	
$xz - plane\ (y = 0)$	$x = \pm 3$	Two lines	
On $z = k$ *(plane* $\parallel$ $xy - plane$*)*	$x^2 + y^2 = 9$	circle	

The directrix of the cylinder is the circle $x^2 + y^2 = 9$ in the xy-plane. The rulings are parallel to the z-axis. See Example 2 of this section.



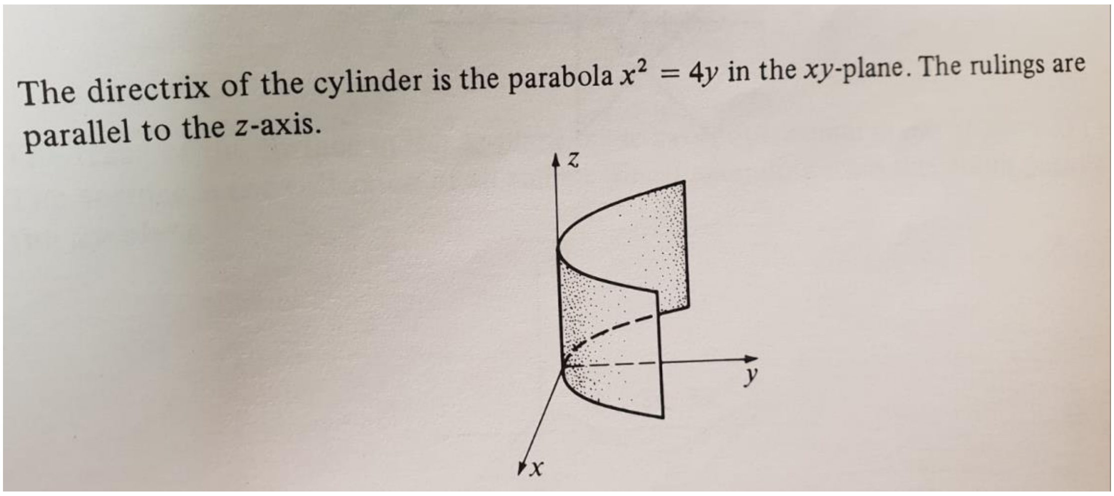


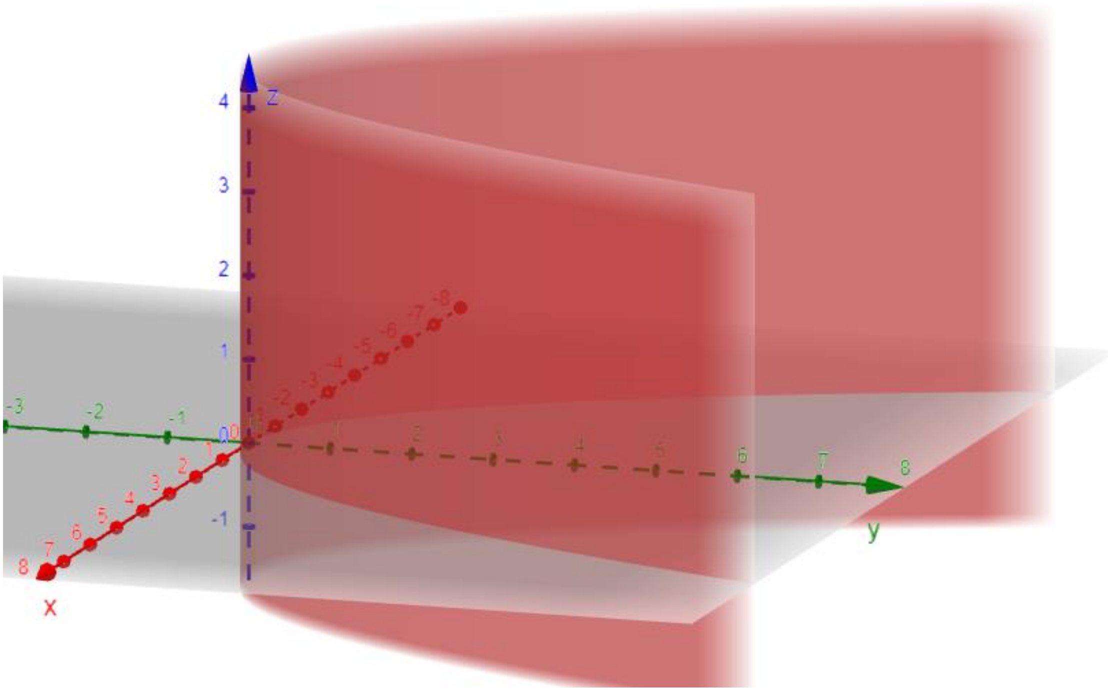


Right cylinder, its axis is $z - axis$

6:

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$x^2 - 4y = 0$	parabola	
$yz - plane\ (x = 0)$	$y = 0$	line	
$xz - plane\ (y = 0)$	$x = 0$	line	
On $z = k$ (plane $\parallel$ $xy - plane$)	$x^2 - 4y = 0$	parabola	


The directrix of the cylinder is the parabola $x^2 = 4y$ in the xy-plane. The rulings are parallel to the z-axis.

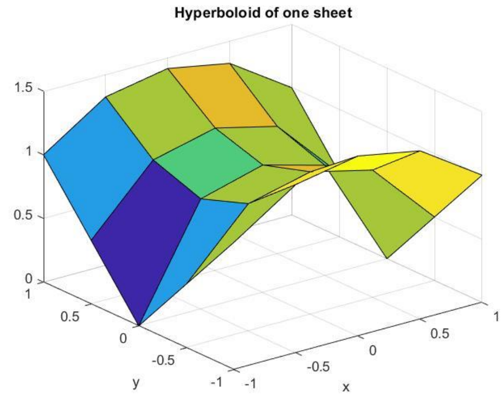


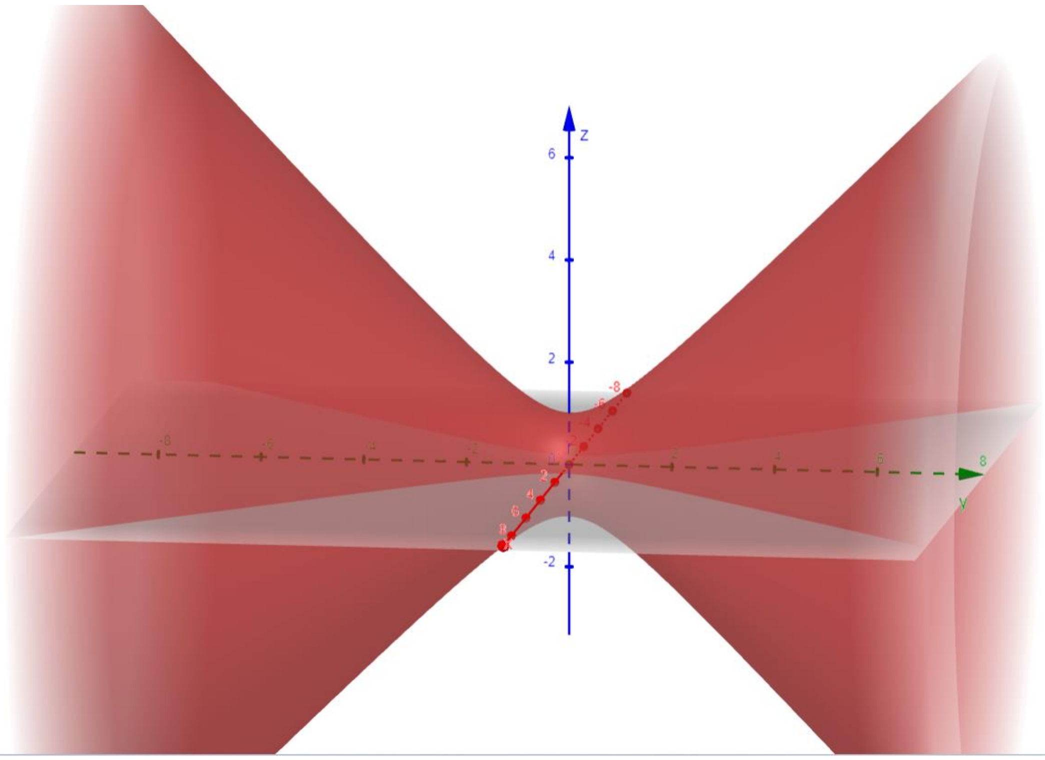
Cylinder

(24) (a) $z^2 + x^2 - y^2 = 1$ (b) $y^2 + \frac{z^2}{4} - x^2 = 1$

24 (a)

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$x^2 - y^2 = 1$	hyperbola	
$yz - plane\ (x = 0)$	$z^2 - y^2 = 1$	hyperbola	
$xz - plane\ (y = 0)$	$x^2 + z^2 = 1$	circle	
On $y = k$ (plane ∥ $xz - plane$)	$x^2 + z^2 = 1 + k^2$	circle	

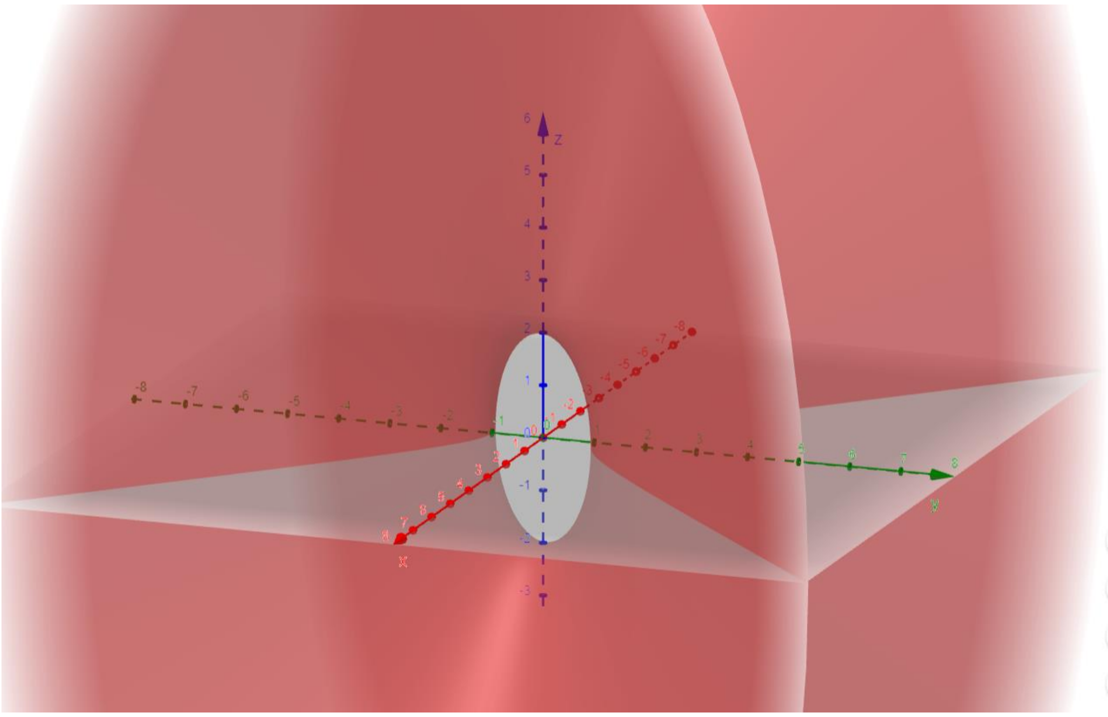


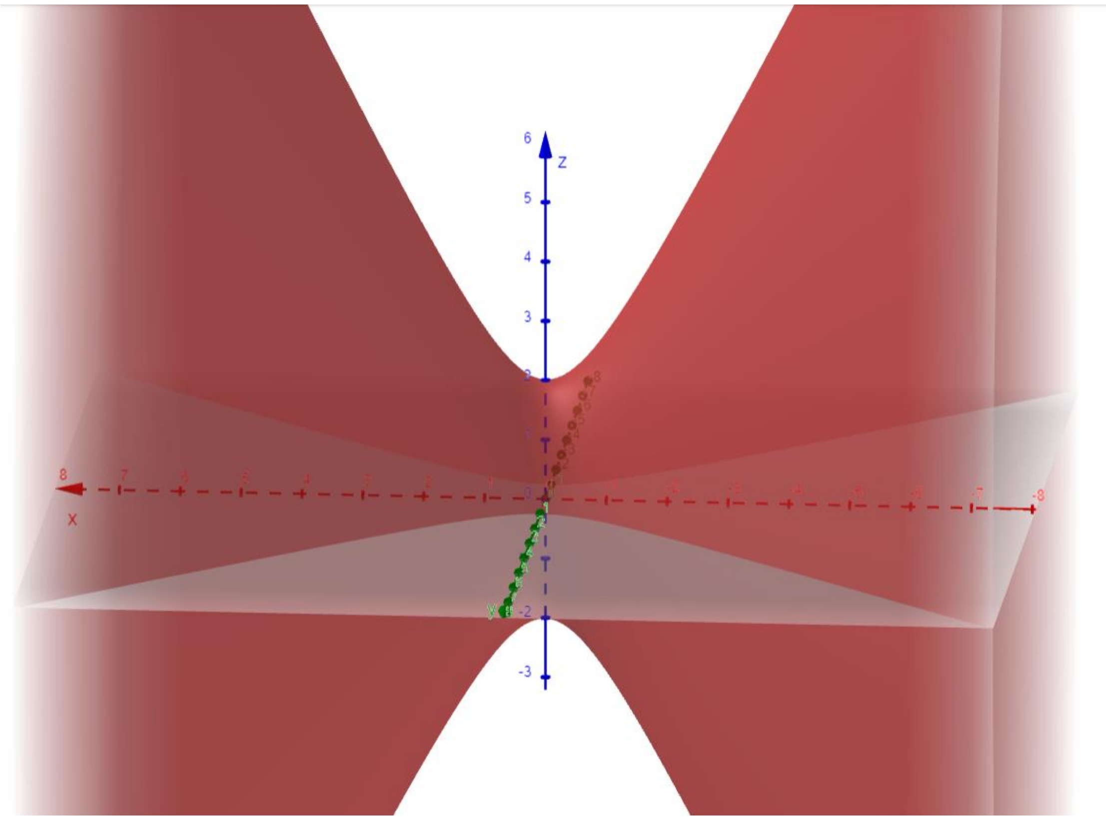


Hyperboloid of one sheet, its axis is $y - axis$.

24 (b)

Trace	Equation of trace	Description	Sketch of trace
$xy-plane\ (z=0)$	$y^2-x^2=1$	hyperbola	
$yz-plane\ (x=0)$	$y^2+\frac{z^2}{4}=1$	ellipse	
$xz-plane\ (y=0)$	$\frac{z^2}{4}-x^2=1$	hyperbola	
On $x=k$ (plane ∥ $yz-plane$)	$y^2+\frac{z^2}{4}=1+k^2$	ellipse	

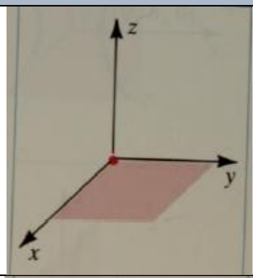
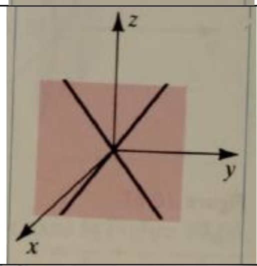
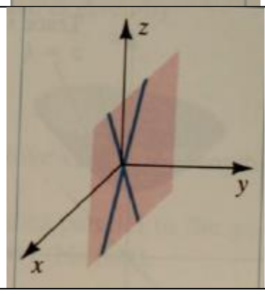


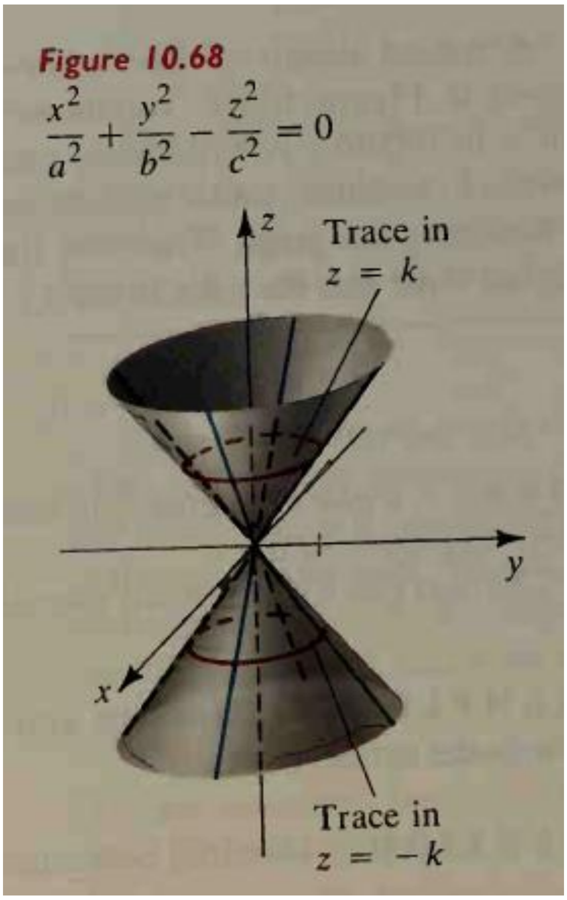


Hyperboloid of one sheet, its axis is $x-axis$.

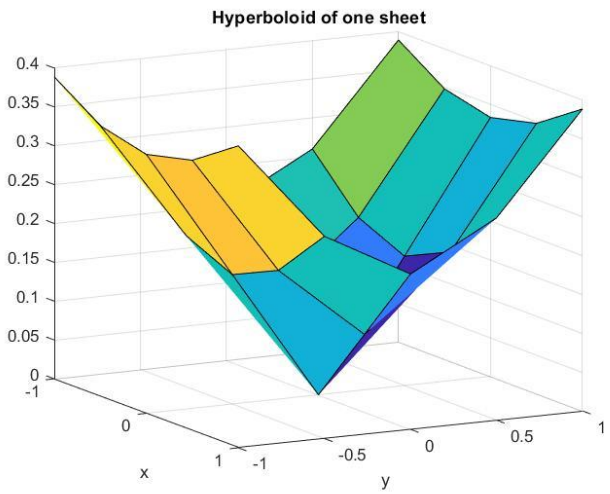
(28) (a) $\frac{x^2}{25} + \frac{y^2}{9} - z^2 = 0$ (b) $x^2 = 4y^2 + z^2$

28 (a)

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$\frac{x^2}{25} + \frac{y^2}{9} = 0$	(0,0)	
$yz - plane\ (x = 0)$	$y = \pm 3z$	Two lines	
$xz - plane\ (y = 0)$	$x = \pm 5z$	Two lines	
On $z = k$ *(plane* $\parallel$ $xy - plane$ *)*	$\frac{x^2}{25} + \frac{y^2}{9} = k^2$	ellipse	



Cone, its axis is $z-axis$



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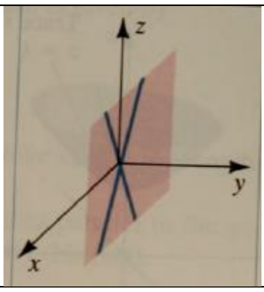
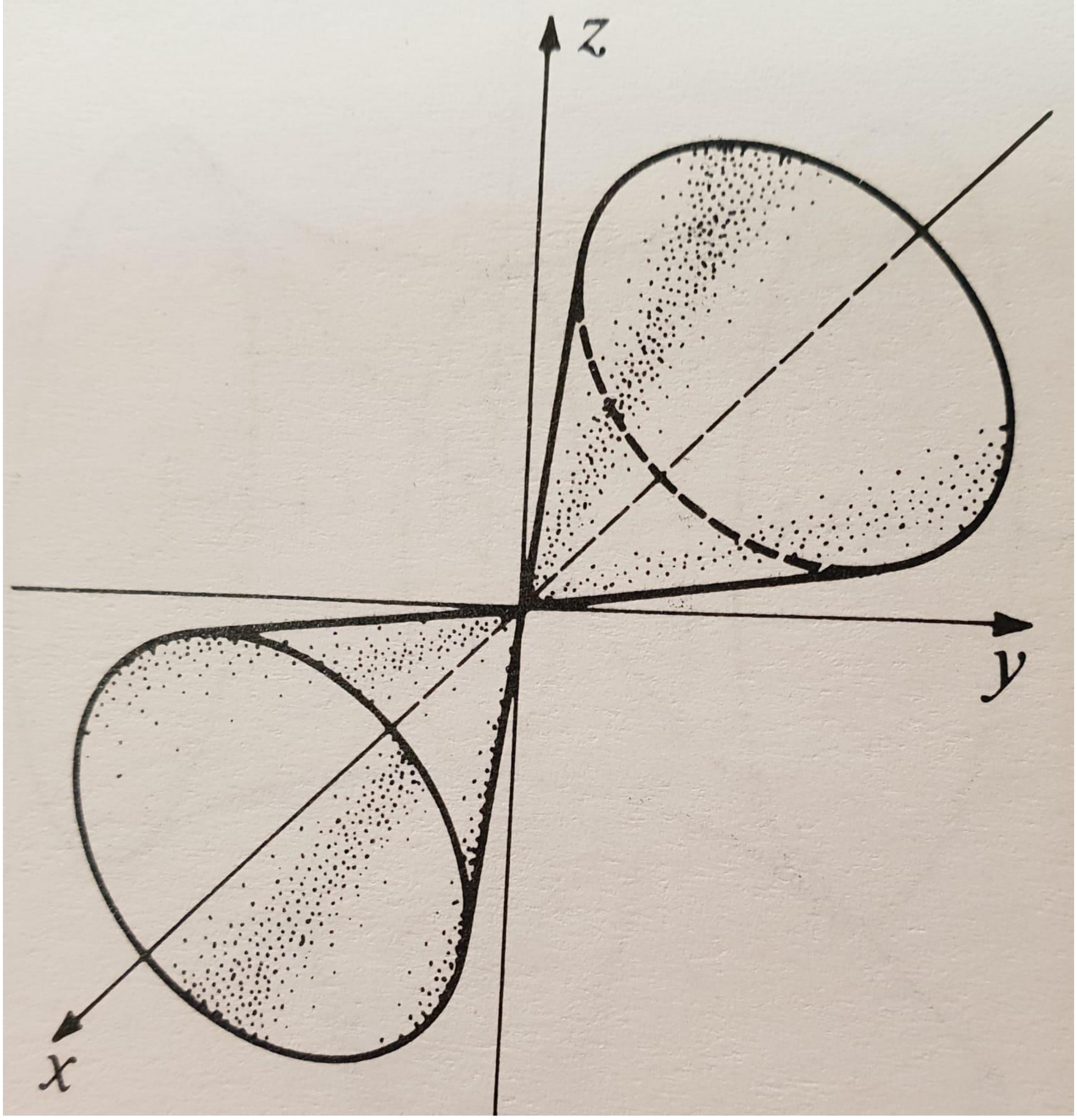
28 (b)

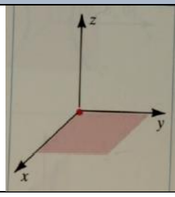
Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$x = \pm 2y$	Two lines	
$yz - plane\ (x = 0)$	$4y^2 + z^2 = 0$	(0,0)	
$xz - plane\ (y = 0)$	$x = \pm z$	Two lines	
On $x = k$ (plane ∥ $yz - plane$)	$4y^2 + z^2 = k^2$	ellipse	

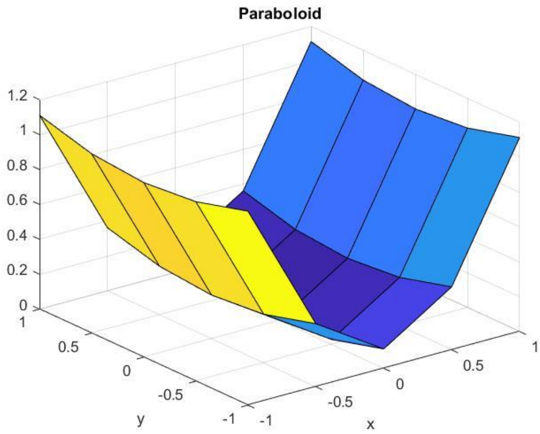


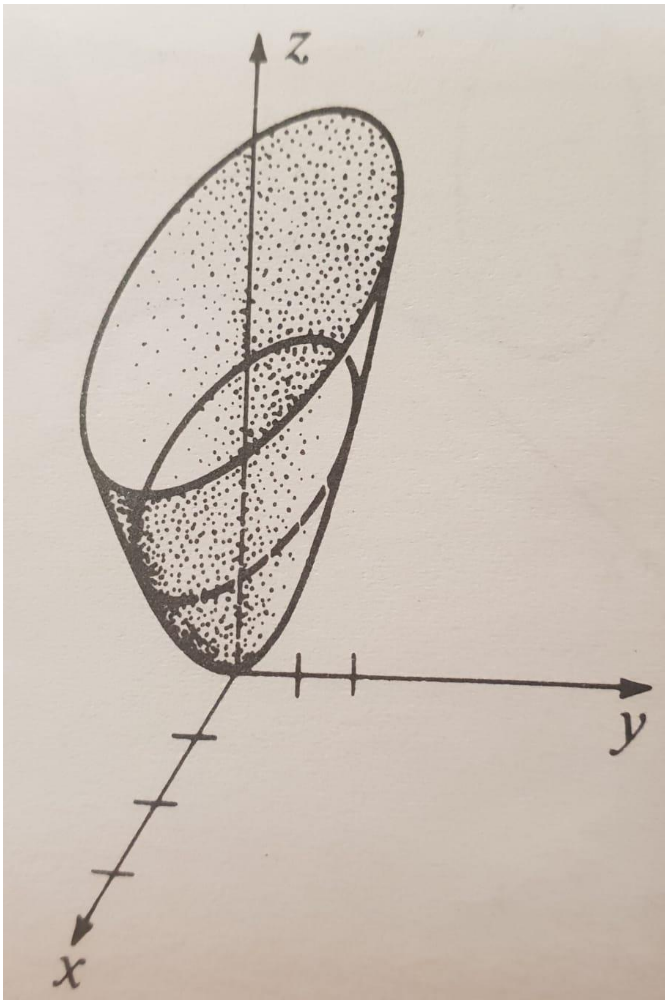
Cone, its axis is $x - axis$

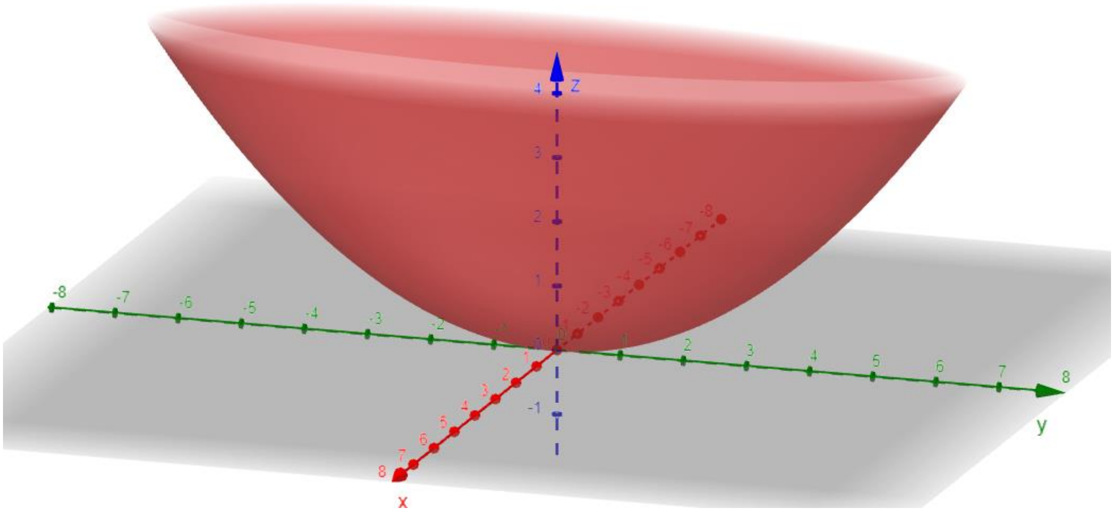
30 **(a)** $z = x^2 + \frac{y^2}{9}$ **(b)** $\frac{z^2}{25} + \frac{y^2}{9} - x = 0$

30 (a)

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$x^2 + \frac{y^2}{9} = 0$	$(0, 0)$	
$yz - plane\ (x = 0)$	$z = \frac{y^2}{9}$	parabola	
$xz - plane\ (y = 0)$	$z = x^2$	parabola	
On $z = k$ *(plane* $\parallel$ $xy - plane$ *)*	$x^2 + \frac{y^2}{9} = k$	ellipse	



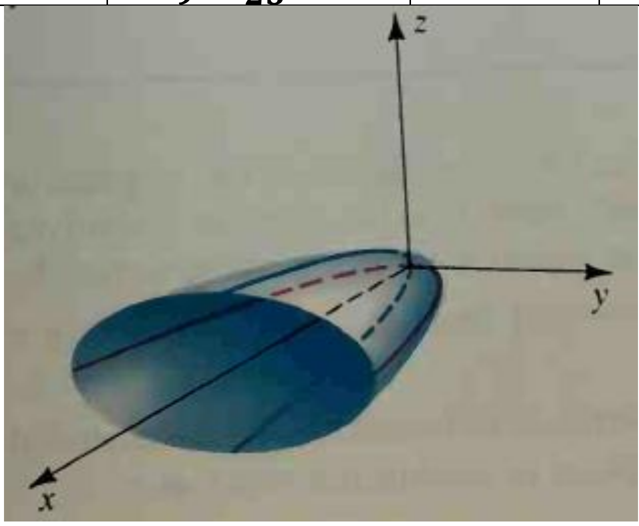


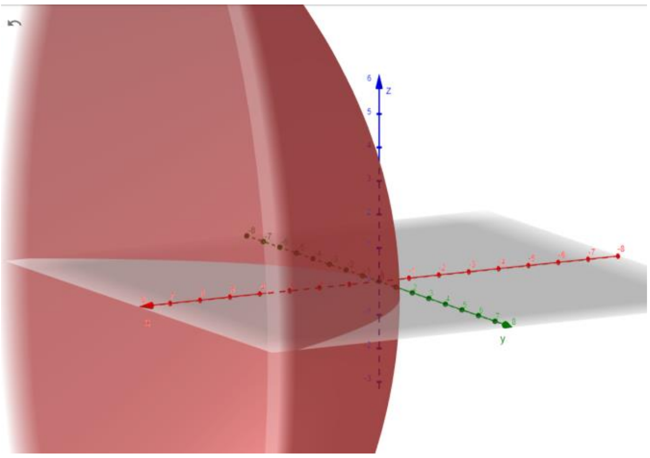


Paraboloid, its axis is $z-axis$

30 (b)

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$\frac{y^2}{9} = x$	parabola	
$yz - plane\ (x = 0)$	$0 = \frac{y^2}{9} + \frac{z^2}{25}$	$(0, 0)$	
$xz - plane\ (y = 0)$	$\frac{z^2}{25} = x$	parabola	
On $x = k$ $_{(plane \parallel yz - plane)}$	$\frac{y^2}{9} + \frac{z^2}{25} = k$	ellipse	





Paraboloid, its axis is $x - axis$

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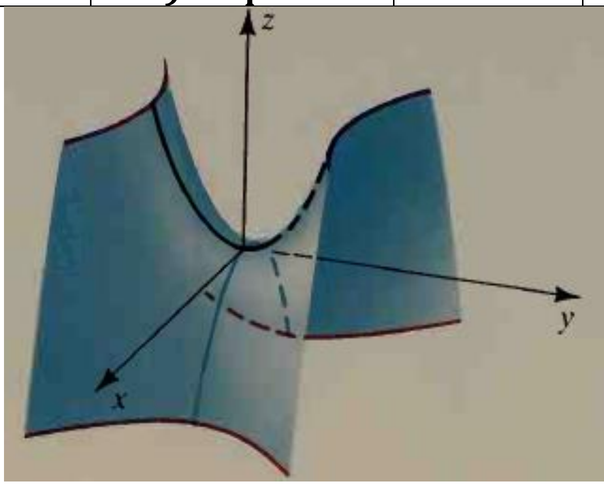
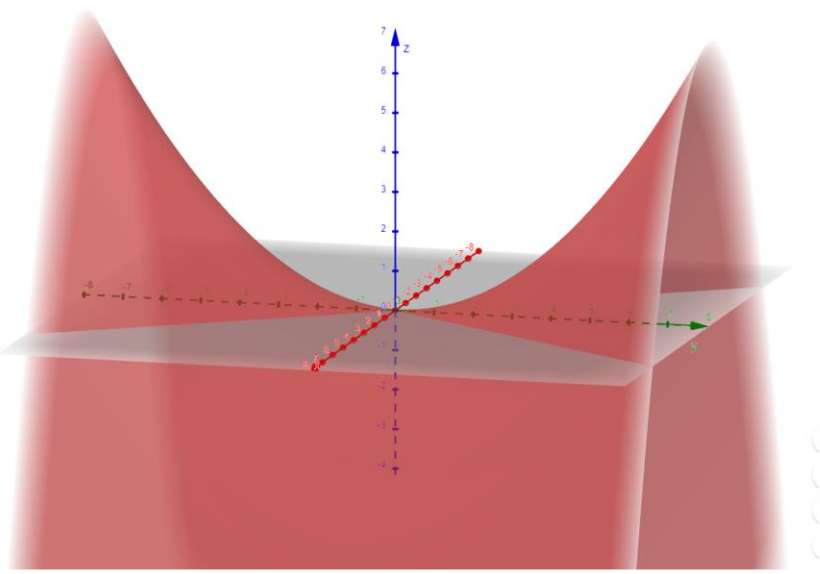
(32) (a) $z = \frac{y^2}{9} - \frac{x^2}{4}$　　　(b) $z = \frac{x^2}{4} - \frac{y^2}{9}$

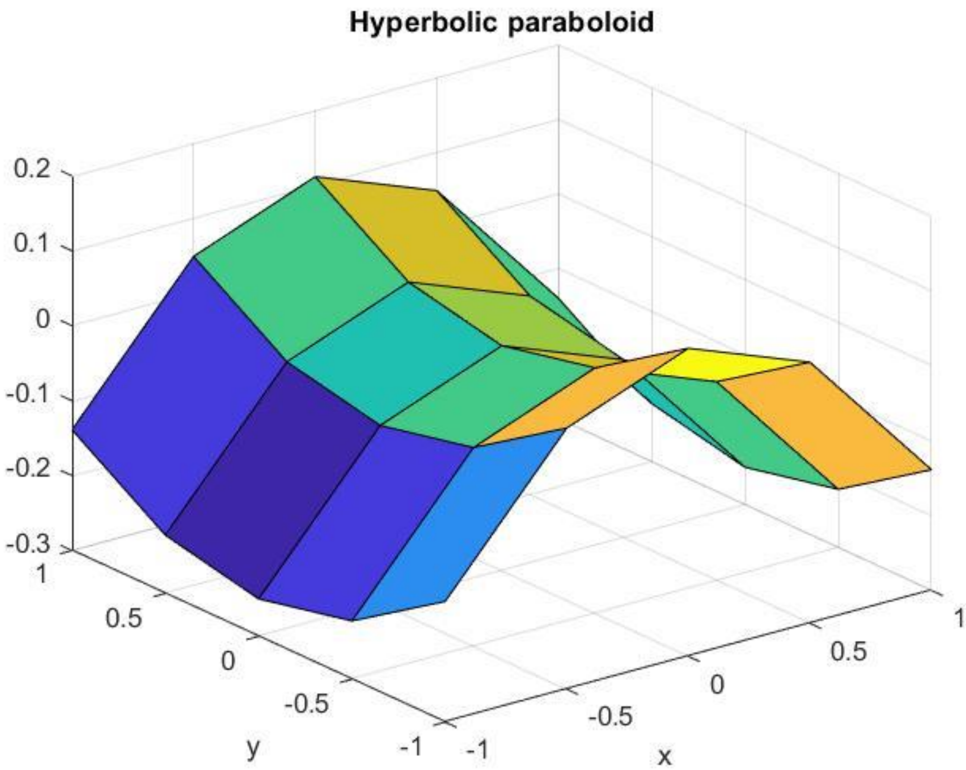
32 (a)

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$y = \pm\frac{3}{2}x$	Two lines	
$yz - plane\ (x = 0)$	$z = \frac{y^2}{9}$	parabola	
$xz - plane\ (y = 0)$	$z = -\frac{x^2}{4}$	parabola	
On $z = k$ (plane ∥ $xz - plane$)	$\frac{y^2}{9} - \frac{x^2}{4} = k$	hyperbola	



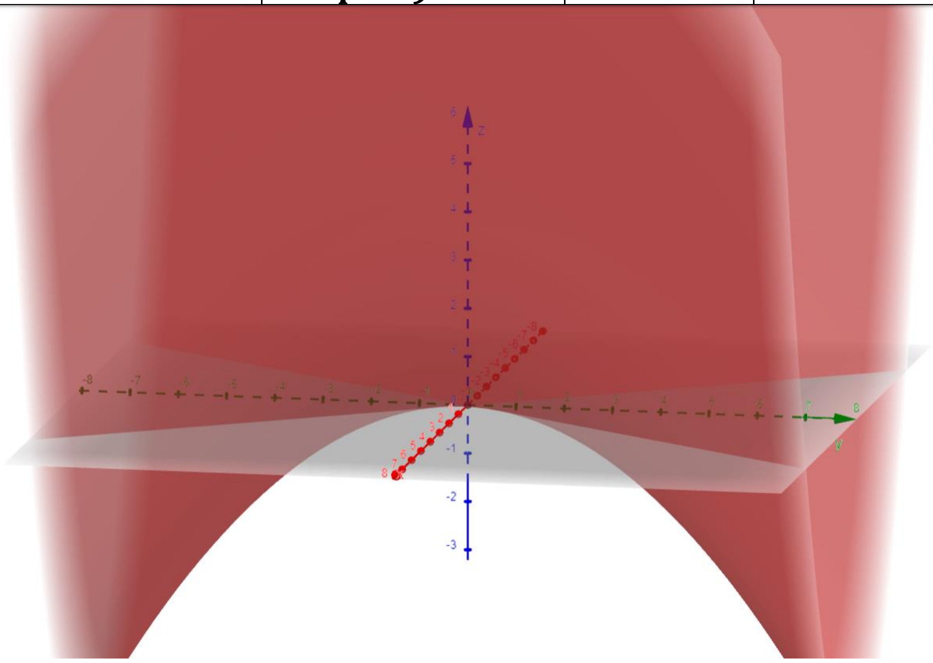


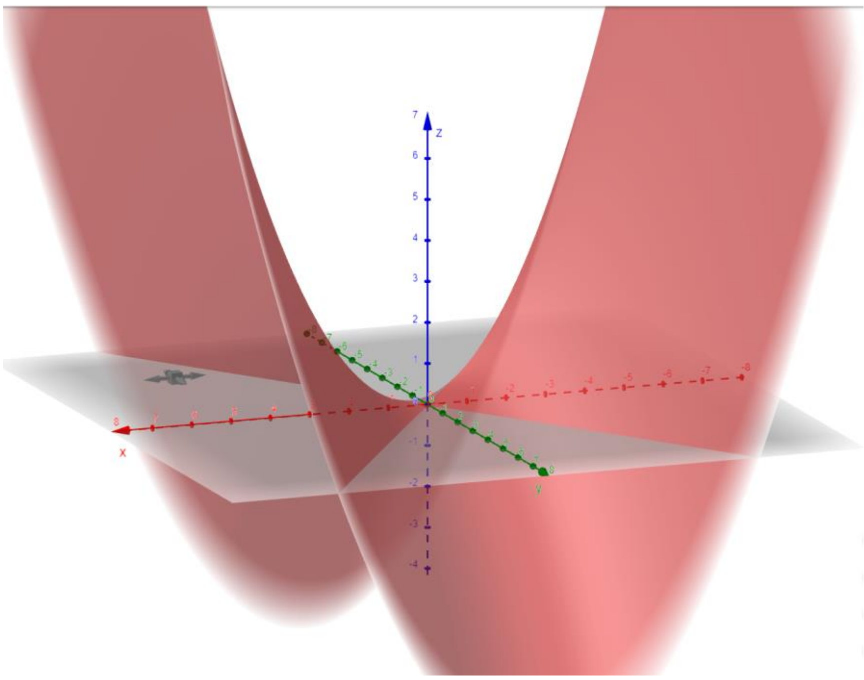
hyperbolic paraboloid (saddle-shaped surface)


Hyperbolic paraboloid
0.2
0.1
0
-0.1
-0.2
-0.3
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x

32 (b)

Trace	Equation of trace	Description	Sketch of trace
$xy-plane\ (z=0)$	$y=\pm\frac{3}{2}x$	**Two lines**	
$yz-plane\ (x=0)$	$z=-\frac{y^2}{9}$	**parabola**	
$xz-plane\ (y=0)$	$z=\frac{x^2}{4}$	**parabola**	
On $z=k$ *(plane ∥ $xz-plane$)*	$\frac{x^2}{4}-\frac{y^2}{9}=k$	**hyperbola**	

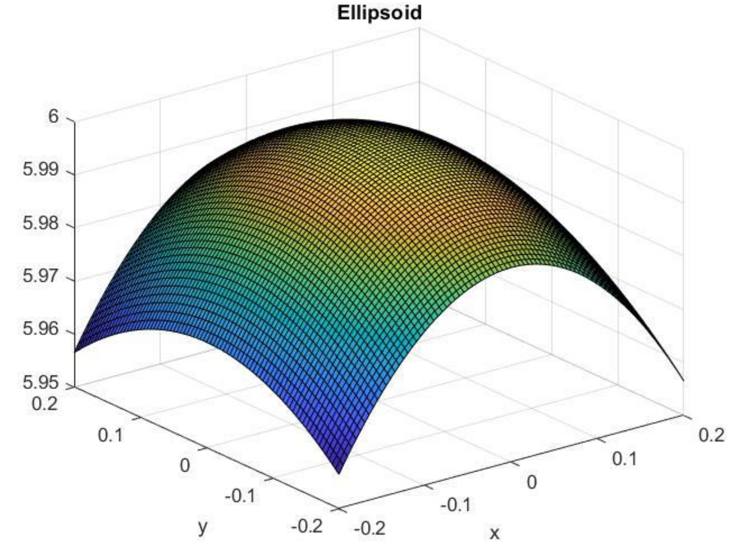




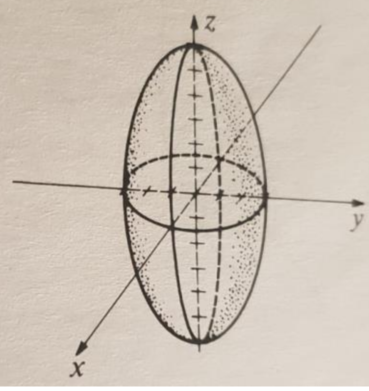
hyperbolic paraboloid **(saddle-shaped surface)**

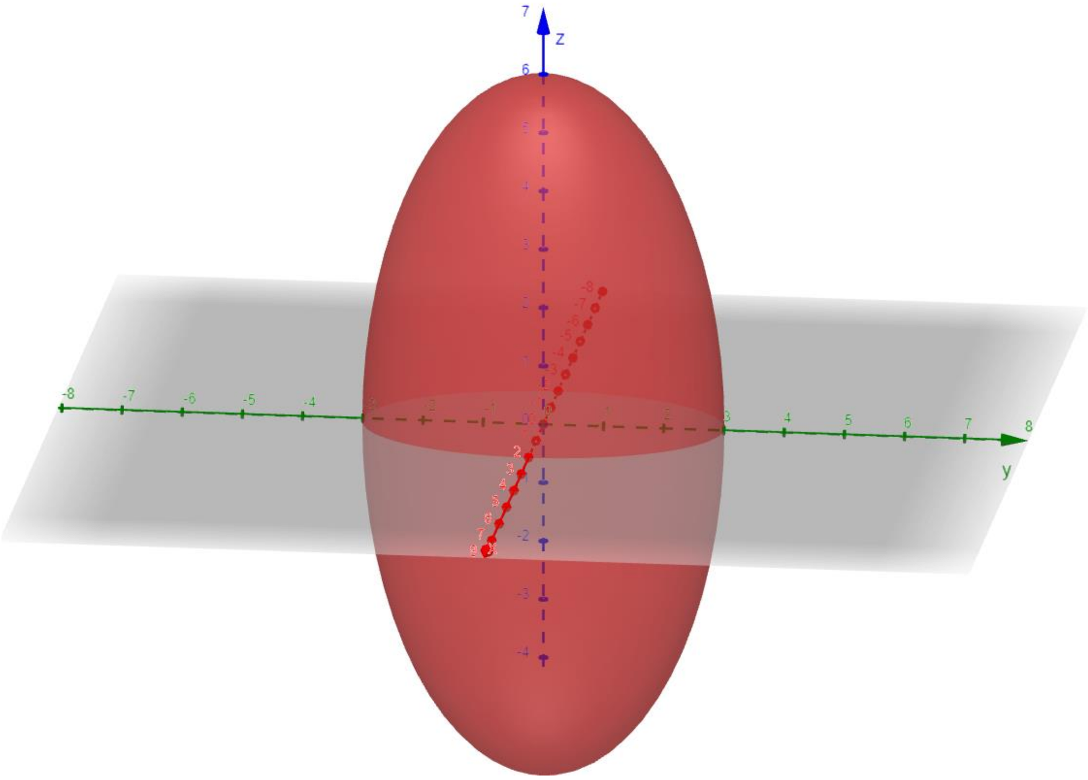
(39) $9x^2 + 4y^2 + z^2 = 36$

Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	ellipse	
$yz - plane\ (x = 0)$	$\frac{z^2}{36} + \frac{y^2}{9} = 1$	ellipse	
$xz - plane\ (y = 0)$	$\frac{z^2}{36} + \frac{x^2}{4} = 1$	ellipse	



Upon division by 36, the equation can be written $(x^2/4) + (y^2/9) + (z^2/36) = 1$. From (14.43), this is an ellipsoid.





Ellipsoid

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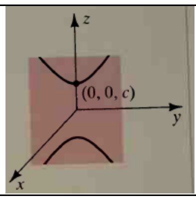
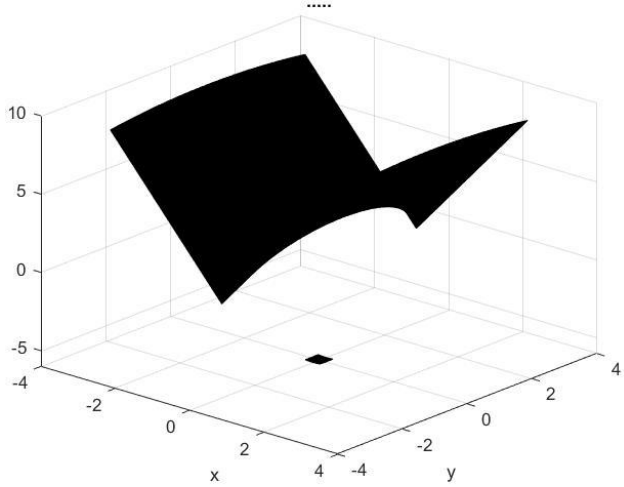
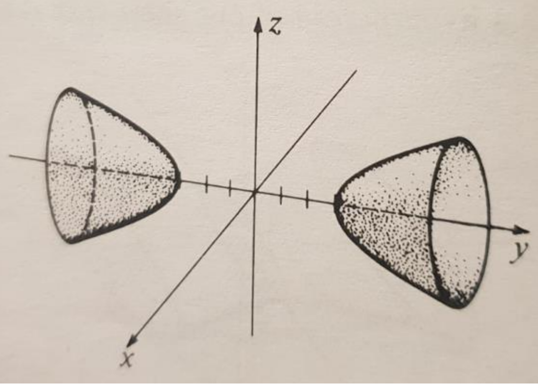
$$45)\quad y^2 - 9x^2 - z^2 - 9 = 0$$

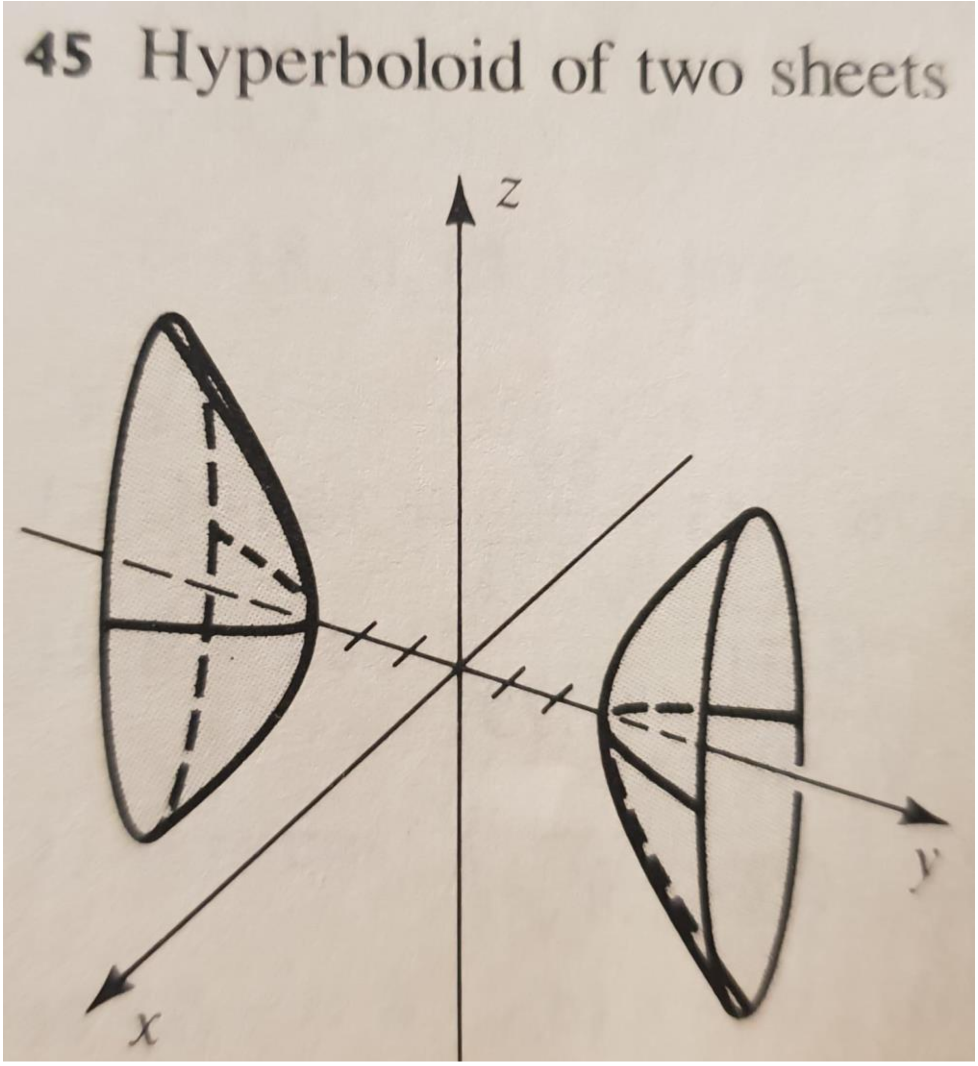
45:

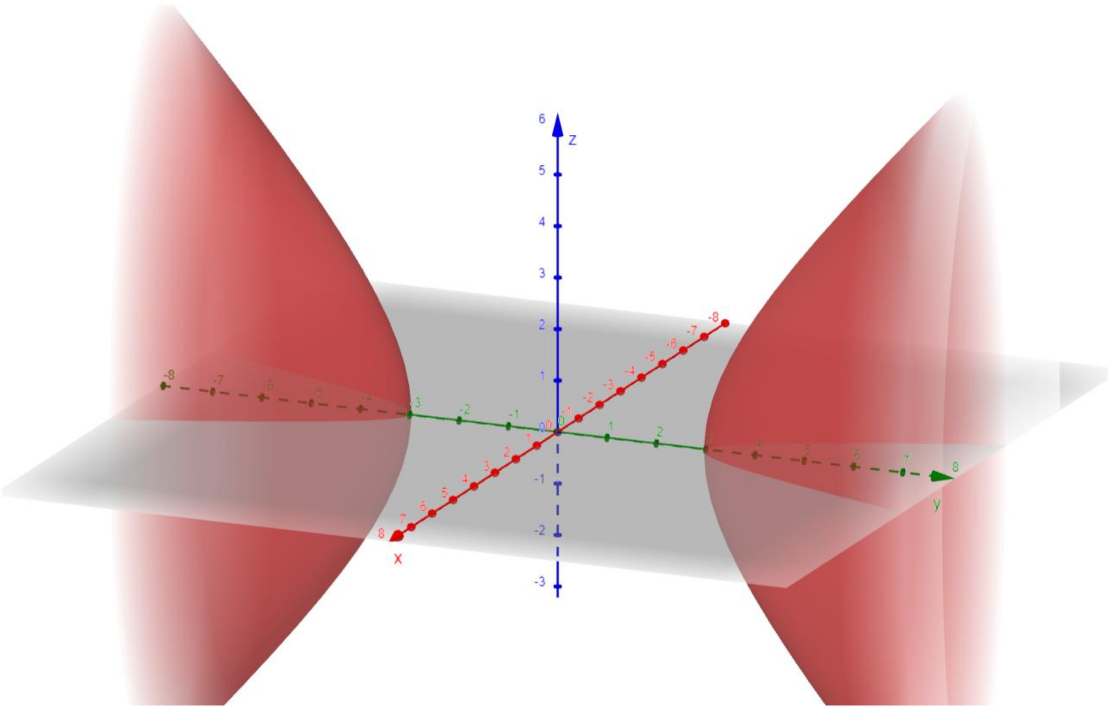
Trace	Equation of trace	Description	Sketch of trace
$xy - plane\ (z = 0)$	$\frac{y^2}{9} - \frac{x^2}{1} = 1$	hyperbola	
$yz - plane\ (x = 0)$	$\frac{y^2}{9} - \frac{z^2}{9} = 1$	hyperbola	
$xz - plane\ (y = 0)$	$\frac{z^2}{9} + \frac{x^2}{1} = -1$	No locus	
On $y = k$ *(plane ∥ xz − plane)*	$\frac{z^2}{9} + \frac{x^2}{1} = k^2 - 9$	Ellipse $\lvert k\rvert > 3$	



Upon dividing by 9, the equation can be written $(y^2/9) - (x^2/1) - (z^2/9) = 1$. Comparing (14.45), this is a hyperboloid of two sheets with axis on the y-axis.







Hyperboloid of two sheets, its axis is $y - axis$

11.1 VECTOR-VALUED FUNCTIONS AND SPACE CURVES

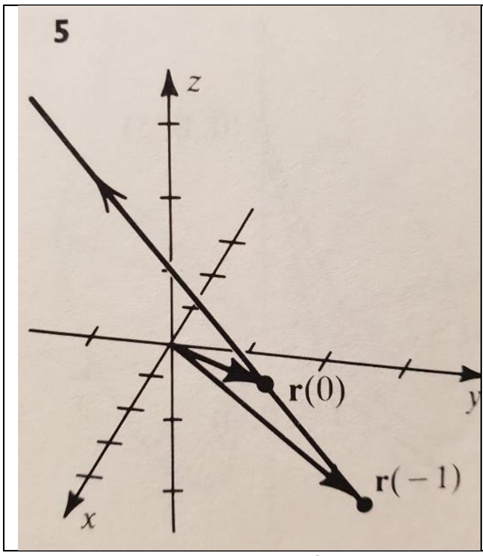
Exer. 1–8: (a) Sketch the two vectors listed after the formula for $\mathbf{r}(t)$. (b) Sketch, on the same coordinate system, the curve C determined by $\mathbf{r}(t)$, and indicate the orientation for the given values of t.

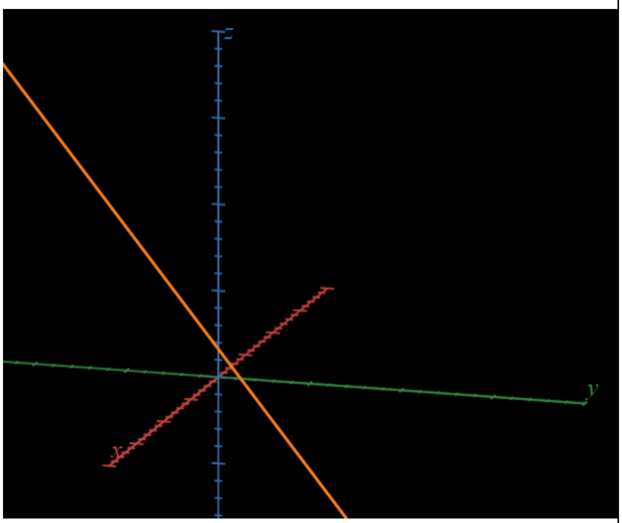
(5) $\mathbf{r}(t) = (3 + t)\mathbf{i} + (2 - t)\mathbf{j} + (1 + 2t)\mathbf{k}$,
$\mathbf{r}(-1)$, $\mathbf{r}(0)$; $t \geq -1$

$$r(-1) = \langle 3 + (-1), 2 - (-1), 1 + 2(-1)\rangle = \langle 2, 3, -1\rangle$$

$$r(0) = \langle 3 - 0, 2 - 0, 1 + 2(0)\rangle = \langle 3, 2, 1\rangle$$

The orientation of C is the direction determined by increasing values of t.





$$C: \quad \begin{aligned} x &= 3 + t \\ y &= 2 - t \\ z &= 1 + 2t \end{aligned} \qquad (\textit{parametric equations of the line})$$

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Exer. 21–26: Find the arc length of the parametrized curve. Estimate with numerical integration if needed, and express answers to four decimal places of accuracy.

21 $x = 5t, \quad y = 4t^2, \quad z = 3t^2; \quad 0 \le t \le 2$

22 $x = t^2, \quad y = t \sin t, \quad z = t \cos t; \quad 0 \le t \le 1$

22:

Theorem 11.3

If a curve C has a smooth parametrization

$$x = f(t), \quad y = g(t), \quad z = h(t); \quad a \le t \le b$$

and if C does not intersect itself, except possibly for $t = a$ and $t = b$, then the length L of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

$\approx 23.234.$

$(x')^2 + (y')^2 + (z')^2 = (2t)^2 + (t \cos t + \sin t)^2 + (-t \sin t + \cos t)^2 = 4t^2 + t^2 \cos^2 t + 2t \sin t \cos t + \sin^2 t + t^2 \sin^2 t - 2t \sin t \cos t + \cos^2 t = 4t^2 + t^2(\cos^2 t + \sin^2 t) + (\sin^2 t + \cos^2 t) = 5t^2 + 1$. Then, using (15.2), $L = \int_0^1 \sqrt{5t^2 + 1}\, dt$. Let $u = \sqrt{5}t$ so that $dt = du/\sqrt{5}$. $t = 0, 1 \Rightarrow u = 0, \sqrt{5}$. Making these changes and using Formula 21 in the Table of Integrals, $L = (1/\sqrt{5}) \int_0^{\sqrt{5}} \sqrt{1 + u^2}\, du = (1/\sqrt{5})[(u/2)\sqrt{1 + u^2} + (1/2) \ln|u + \sqrt{1 + u^2}|]_0^{\sqrt{5}}$ $= (1/\sqrt{5})[(\sqrt{5}/2)\sqrt{1 + 5} + (1/2) \ln|\sqrt{5} + \sqrt{1 + 5}|] - (1/\sqrt{5})[0 + (1/2) \ln|0 + \sqrt{1 + 0}|]$ $= (1/2)\sqrt{6} + [1/(2\sqrt{5})] \ln(\sqrt{5} + \sqrt{6})$ since $\ln 1 = 0$. Note that the answer given in the Even Answer Supplement reduces to this because $\ln(1 + \sqrt{6/5}) = \ln[(\sqrt{5} + \sqrt{6})/\sqrt{5}]$ $= \ln(\sqrt{5} + \sqrt{6}) - \ln\sqrt{5}$ and $\ln(\sqrt{1/5}) = -\ln\sqrt{5}$.

11.2 LIMITS, DERIVATIVES, AND INTEGRALS

Exer. 21 – 22: A curve C is given parametrically. Find two unit tangent vectors to C at P.

21 $x = e^{2t}, \quad y = e^{-t}, \quad z = t^2 + 4; \quad P(1, 1, 4)$

22 $x = \sin t + 2, \quad y = \cos t, \quad z = t; \quad P(2, 1, 0)$

22:

We may think of the curve as determined by the vector function $\mathbf{r}(t) = \langle 2 + \sin t, \cos t, t\rangle$. The point $P(2,1,0)$ occurs when $t = 0$. Now, $\mathbf{r}'(t) = \langle \cos t, -\sin t, 1\rangle$ and $\mathbf{r}'(0) = \langle 1,0,1\rangle$, a tangent vector to the curve at P. Since $|\mathbf{r}'(0)| = \sqrt{2}$, the two unit tangent vectors are $\pm(1/\sqrt{2})\langle 1,0,1\rangle$.

Exer. 31 – 34: Find $\mathbf{r}(t)$ subject to the given conditions.

33 $\mathbf{r}''(t) = 6t\mathbf{i} - 12t^2\mathbf{j} + \mathbf{k}, \quad \mathbf{r}'(0) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{r}(0) = 7\mathbf{i} + \mathbf{k}$

In the following solution: the symbol "u " is the same of "r"

$\mathbf{u}''(t) = 6t\mathbf{i} - 12t^2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u}'(t) = 3t^2\mathbf{i} - 4t^3\mathbf{j} + t\mathbf{k} + \mathbf{c}$ where $\mathbf{c}$ is a constant vector. To find $\mathbf{c}$, note that $\mathbf{u}'(0) = \mathbf{c}$ and that the problem requires $\mathbf{u}'(0) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Hence $\mathbf{c} = \langle 1, 2, -3\rangle$ and $\mathbf{u}'(t) = (3t^2 + 1)\mathbf{i} - (4t^3 - 2)\mathbf{j} + (t - 3)\mathbf{k}$. This in turn by a second integration implies $\mathbf{u}(t) = (t^3 + t)\mathbf{i} - (t^4 - 2t)\mathbf{j} + [(t^2/2) - 3t]\mathbf{k} + \mathbf{b}$ where $\mathbf{b}$ is a constant vector. To find $\mathbf{b}$, note that $\mathbf{u}(0) = \mathbf{b}$ and that the problem requires $\mathbf{u}(0) = 7\mathbf{i} + 0\mathbf{j} + \mathbf{k}$. Hence $\mathbf{b} = \langle 7, 0, 1\rangle$ and $\mathbf{u}(t) = (t^3 + t + 7)\mathbf{i} - (t^4 - 2t)\mathbf{j} + [(t^2/2) - 3t + 1]\mathbf{k}$.

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Exer. 35 – 36: If a curve C has a tangent vector **a** at a point P, then the *normal plane* to C at P is the plane through P with normal vector **a**. Find an equation of the normal plane to the given curve at P.

(35) $x = e^t, \quad y = te^t, \quad z = t^2 + 4; \quad P(1, 0, 4)$

$$r(t) = xi + yj + zk = e^t i + te^t j + (t^2 + 4)k$$

Tangent vector: $r'(t) = e^t i + (1.e^t + te^t)j + 2tk$

$$P(1,0,4) = (x, y, z) \Rightarrow x = 1 = e^t \quad \Rightarrow t = 0$$

Normal vector of the normal plane: plug $t = 0$ in $r'(t) \Rightarrow \langle 1, 1, 0 \rangle$

Equation of the normal plane: $1(x - 1) + 1(y - 0) + 0(z - 4) = 0$

$$\Rightarrow \quad x + y - 1 = 0$$

11.3 CURVILINEAR MOTION

Exer. 9 – 16: If $\mathbf{r}(t)$ is the position vector of a moving point P, find its velocity, acceleration, and speed at the given time t.

(15) $\mathbf{r}(t) = (1 + t)\mathbf{i} + 2t\mathbf{j} + (2 + 3t)\mathbf{k}; \quad t = 2$

Velocity: $v(t) = r'(t) = \langle 1, 2, 3 \rangle$

Acceleration: $a(t) = r''(t) = \langle 0, 0, 0 \rangle$

At time $t = 1$: $v(1) = \langle 1, 2, 3 \rangle$

$a(1) = \langle 0, 0, 0 \rangle$

Speed: $\|r'(1)\| = \|\langle 1, 2, 3 \rangle\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

v

11.4 CURVATURE

Exer. 1–6: **(a)** Find the unit tangent and normal vectors T(t) and N(t) for the curve C determined by r(t). **(b)** Sketch the graph of C, and show T(t) and N(t) for the given value of t.

(3) $\mathbf{r}(t) = t^3\mathbf{i} + 3t\mathbf{j}$; $t = 1$

Unit Tangent Vector 11.14

$$\mathbf{T}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)$$

Principal Unit Normal Vector 11.15

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{T}'(t)\|}\mathbf{T}'(t)$$

a- $r'(t) = 3t^2 i + 3j \quad \Rightarrow \|r'(t)\| = \sqrt{9t^4+9} = 3\sqrt{t^4+1}$

$$\mathrm{T}(t) = \frac{3t^2 i + 3j}{3\sqrt{t^4+1}} = \frac{t^2 i + j}{\sqrt{t^4+1}} = \frac{t^2}{\sqrt{t^4+1}} i + \frac{1}{\sqrt{t^4+1}} j \quad \Rightarrow \|\mathrm{T}(t)\| = \sqrt{\frac{t^4}{t^4+1} + \frac{1}{t^4+1}} = 1$$

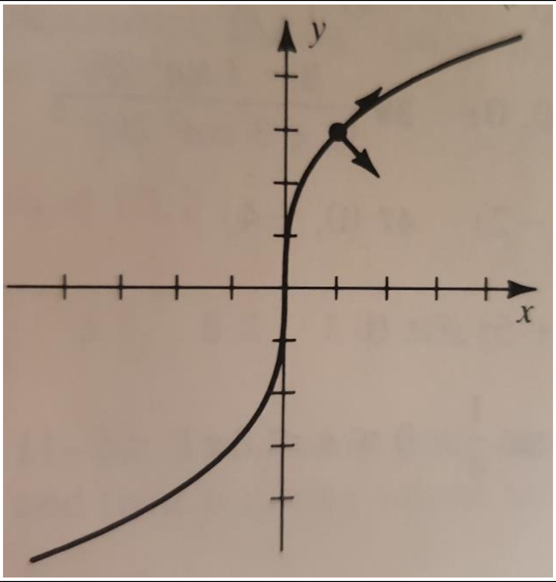
$$\mathrm{T}'(t) = \frac{\sqrt{t^4+1}\,.\,2t - t^2.\dfrac{4t^3}{2\sqrt{t^4+1}}}{t^4+1} i - \frac{\dfrac{4t^3}{2\sqrt{t^4+1}}}{t^4+1} j = \frac{2t}{(t^4+1)^{3/2}} i - \frac{2t^3}{(t^4+1)^{3/2}} j$$

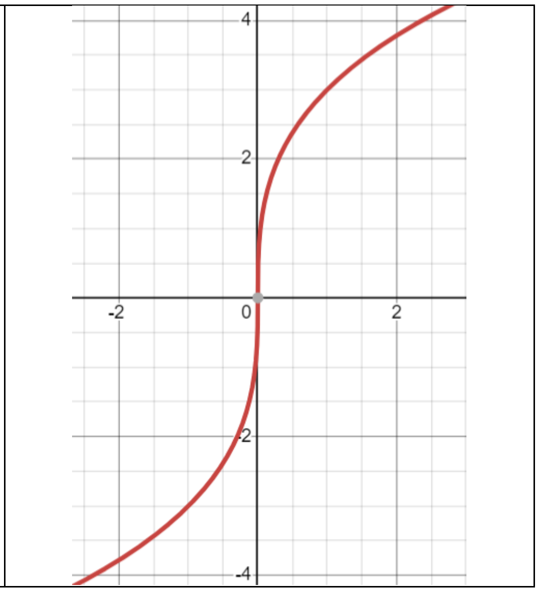
$$\|\mathrm{T}'(t)\| = \sqrt{\frac{4t^2}{(t^4+1)^3} + \frac{4t^6}{(t^4+1)^3}} = \sqrt{\frac{4t^2(1+t^4)}{(t^4+1)^3}} = \sqrt{\frac{4t^2}{(t^4+1)^2}} = \frac{2t}{t^4+1}$$

$$\mathrm{N}(t) = \frac{\dfrac{2t}{(t^4+1)^{3/2}} i - \dfrac{2t^3}{(t^4+1)^{3/2}} j}{\dfrac{2t}{t^4+1}} = \frac{1}{\sqrt{t^4+1}} i - \frac{t^2}{\sqrt{t^4+1}} j$$

$$\mathrm{T}(1) = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j\,, \qquad \mathrm{N}(1) = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

b- For sketch: $x = t^3, y = 3t \Rightarrow t = \frac{y}{3} \Rightarrow x = \frac{y^3}{27}$





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Exer. 7 – 18: Find the curvature of the curve at P.

(7) $y = 2 - x^3$; $P(1,\ 1)$

7:

Theorem **11.18**

If a smooth curve C is the graph of $y = f(x)$, then the curvature K at $P(x, y)$ is

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

$$y' = -3x^2 \ \Rightarrow\ y'' = -6x$$

$$K = \frac{|-6x|}{[1 + (-3x^2)^2]^{3/2}} = \frac{6|x|}{[1 + 9x^4]^{3/2}} \ \Rightarrow\ \text{At } P(1, 1) = (x, y) \Rightarrow x = 1:$$

$$K = \frac{6}{10^{3/2}}$$

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(14) $x = t + 1, \qquad y = t^2 + 4t + 3;\ P(1,\ 3)$

14:

Theorem 11.19

If a plane curve C has a parametrization $x = f(t)$, $y = g(t)$ and if f'' and g'' exist, then the curvature K at $P(x, y)$ is

$$K = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{[(f'(t))^2 + (g'(t))^2]^{3/2}}.$$

$$f(t) = t + 1 \ \Rightarrow\ f'(t) = 1 \ \Rightarrow\ f''(t) = 0$$

$$g(t) = t^2 + 4t + 3 \ \Rightarrow\ g'(t) = 2t + 4 \ \Rightarrow\ g''(t) = 2$$

$$P(1,3) = (x,y) \Rightarrow 1 = x \ \Rightarrow 1 = 1 + t \ \Rightarrow t = 0$$

$$K = \frac{|1(2) - (2t+1)(0)|}{(1^2 + (2t+4)^2)^{3/2}} = \frac{2}{(1 + (2t+4)^2)^{3/2}} \quad \Rightarrow\ At\ t = 0:\ K = \frac{2}{17^{3/2}}$$

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Exer. 19 – 22: For the given curve and point P, (a) find the radius of curvature, (b) find the center of curvature, and (c) sketch the graph and the circle of curvature for P.

(21) $y = e^x$; $P(0, 1)$

a-

If the curvature K at a point P on a curve C is not 0, then the circle of radius $\rho = 1/K$ whose center lies on the concave side of C and that has the same tangent line at P as C is the **circle of curvature** of the curve C at the point P. Its radius ρ and center are the **radius of curvature** and **center of curvature**, respectively, for P. According to Examples 5 and 7, the

$$y' = e^x \Rightarrow y'' = e^x$$

$$K = \frac{|e^x|}{[1+(e^x)^2]^{3/2}} = \frac{e^x}{[1+e^{2x}]^{3/2}} \Rightarrow \text{At } P(0,1) = (x,y) \Rightarrow x = 0:$$

$$K = \frac{1}{2^{3/2}} = \frac{1}{\sqrt{8}} \Rightarrow \rho = \frac{1}{K} = \frac{1}{\frac{1}{\sqrt{8}}} = \sqrt{8}$$

b-

Let $P(x, y)$ be a point on the graph of $y = f(x)$ at which $K \neq 0$. If (h, k) is the center of curvature for P, show that

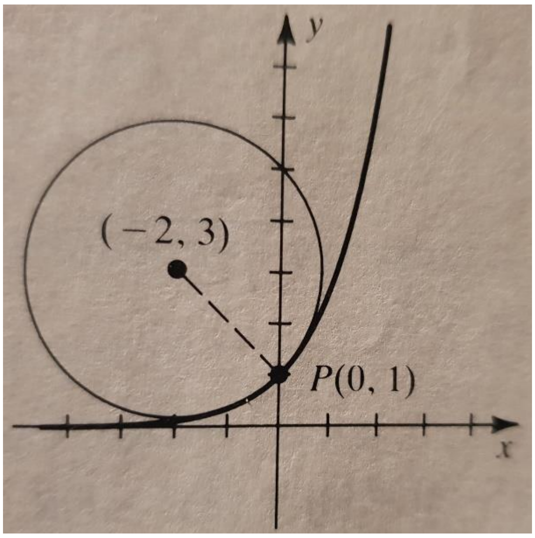
$$h = x - \frac{y'[1+(y')^2]}{y''}, \qquad k = y + \frac{[1+(y')^2]}{y''}.$$

$$\text{At } x = 0, y = 1 : h = 0 - \frac{1[1+1]}{1} = -2 , \quad k = 1 + \frac{[1+1]}{1} = 3 \Rightarrow$$

center of cuvarture is $(-2, 3)$

c- **Equation of circle of curvature is:** $(x-h)^2+(y-k)^2=\rho^2$

$$(x+2)^2+(y-3)^2=8$$



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Exer. 27 – 32: Find the points on the given curve at which the curvature is a maximum.

27 $y = e^{-x}$ **28** $y = \cosh x$

27:

$$y' = -e^{-x} \Rightarrow y'' = e^{-x}$$

$$K(x) = \frac{|e^{-x}|}{[1 + (-e^{-x})^2]^{3/2}} = \frac{e^{-x}}{[1 + e^{-2x}]^{3/2}} \Rightarrow$$

$$K(x) = \frac{e^{-x}}{\left[1 + \frac{1}{e^{2x}}\right]^{3/2}} = \frac{e^{-x}(e^{2x})^{3/2}}{[1 + e^{2x}]^{3/2}} = \frac{e^{-x}e^{3x}}{[1 + e^{2x}]^{3/2}} = \frac{e^{2x}}{[1 + e^{2x}]^{3/2}}$$

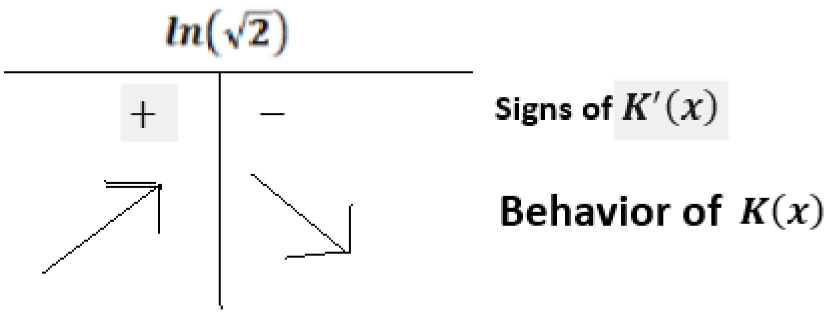
$$K'(x) = \frac{[1 + e^{2x}]^{3/2}.2e^{2x} - e^{2x}.\frac{3}{2}(1 + e^{2x})^{\frac{1}{2}}e^{2x}.2}{(1 + e^{2x})^3}$$

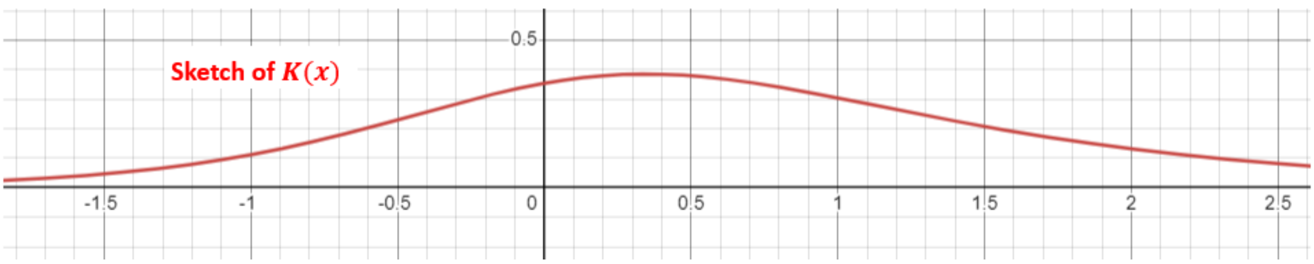
$$= \frac{(1+e^{2x})2.e^{2x} - 3e^{4x}}{(1+e^{2x})^{5/2}} = \frac{2e^{2x} - e^{4x}}{(1+e^{2x})^{5/2}} = 0 \Rightarrow$$

$$2e^{2x} - e^{4x} = 0 \Rightarrow 2 - e^{2x} = 0 \Rightarrow 2 = e^{2x} \Rightarrow ln(2) = 2x \Rightarrow$$

$$x = \frac{1}{2}ln(2) = ln(\sqrt{2}) \text{ is critical number}$$

" Test of the first derivative"





So at $x = ln(\sqrt{2})$ there is local max of $K(x)$ "Curvarture".

$$x = ln(\sqrt{2}) \Rightarrow y = e^{-ln(\sqrt{2})} = e^{ln(\sqrt{2})^{-1}} = \frac{1}{\sqrt{2}}$$

The point at which the curvature is maximum is: $(x, y) = (ln(\sqrt{2}), \frac{1}{\sqrt{2}})$

Exer. 33 – 36: Find the points on the graph of the equation at which the curvature is 0.

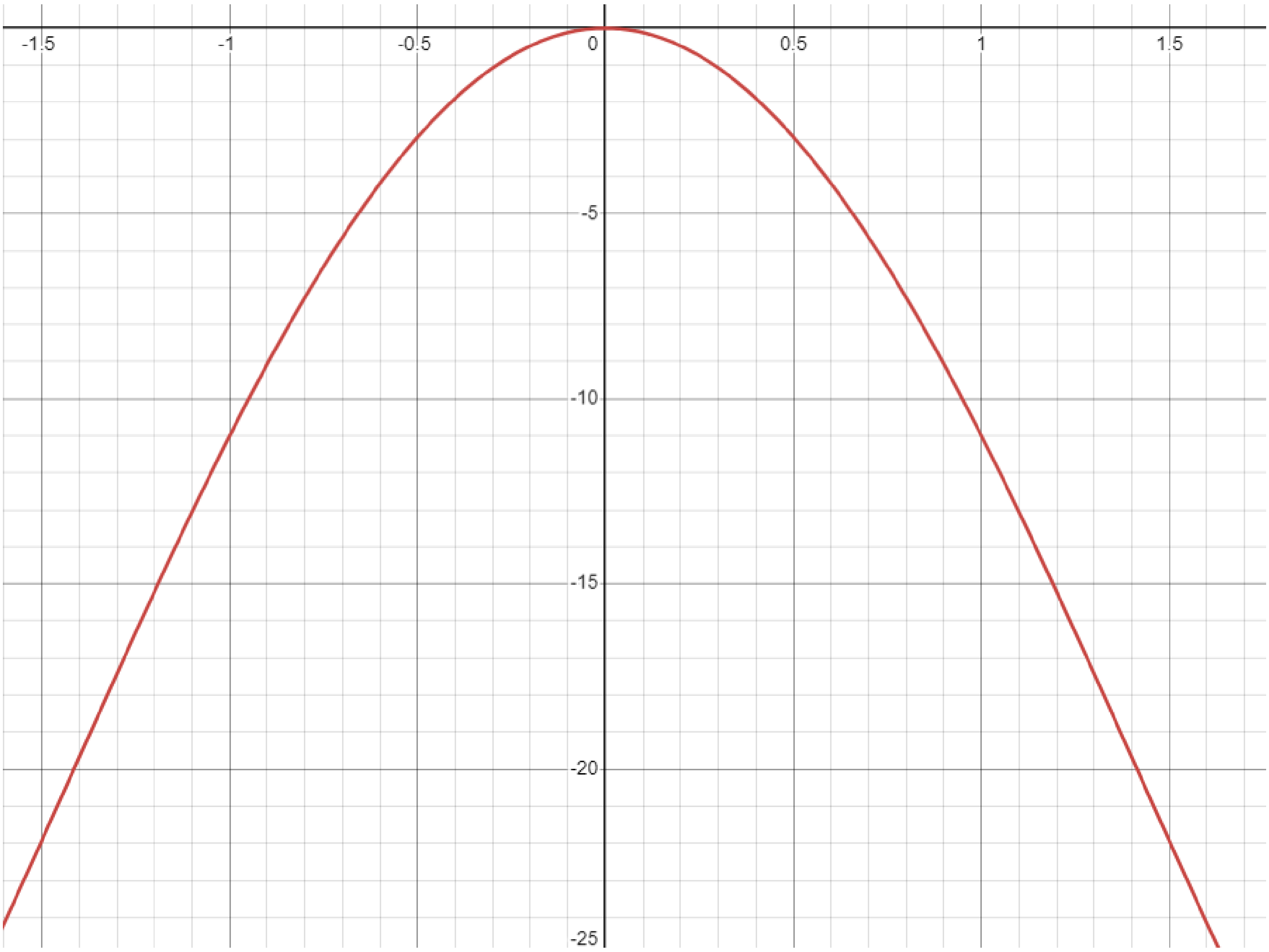
33 $y = x^4 - 12x^2$ **34** $y = \tan x$

$$y' = 4x^3 - 24x \Rightarrow y'' = 12x^2 - 24$$

$$K = \frac{|12x^2 - 24|}{[1 + (4x^3 - 24x)^2]^{3/2}} = 0 \Rightarrow 12x^2 - 24 = 0 \Rightarrow x = \pm\sqrt{2} \Rightarrow$$

$$y = (\pm\sqrt{2})^4 - 12(\pm\sqrt{2})^2 = 4 - 24 = -20 \quad \Rightarrow$$

The points on the graph at which the curvature is 0 are: $(\sqrt{2}, -20), (-\sqrt{2}, -20)$



Sketch of: $y = x^4 - 12x^2$

Exer. 42–46: Use the formulas in Exercise 41 to find the center of curvature for the point P on the graph of the equation. (Refer to Exercises 7–11.)

45 $y = \ln(x - 1); \quad P(2, 0)$

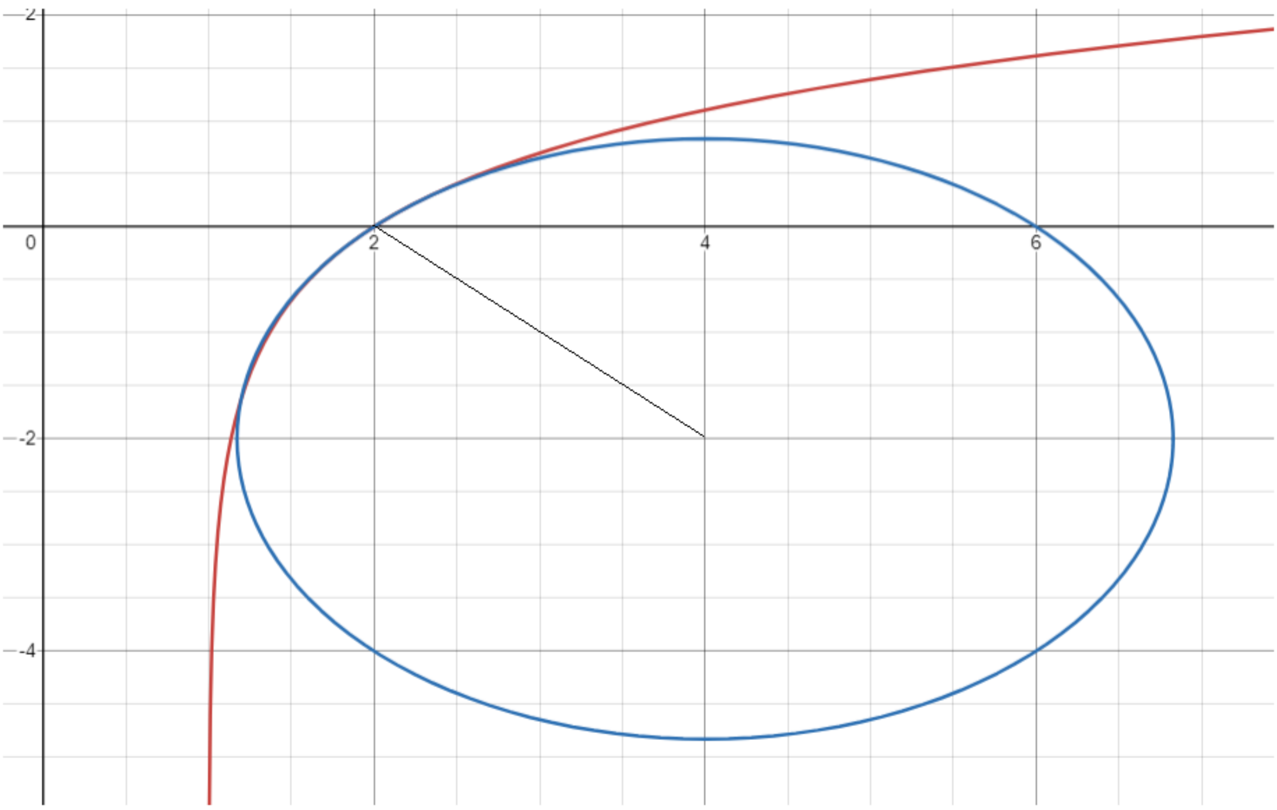
$$y' = \frac{1}{x-1} \Rightarrow y'' = \frac{-1}{(x-1)^2}$$

Let $P(x, y)$ be a point on the graph of $y = f(x)$ at which $K \neq 0$. If (h, k) is the center of curvature for P, show that

$$h = x - \frac{y'[1 + (y')^2]}{y''}, \qquad k = y + \frac{[1 + (y')^2]}{y''}.$$

At $x = 2$, $y = 0$: $h = 2 - \frac{1[1+1]}{-1} = 4$, $k = 0 + \frac{[1+1]}{-1} = -2 \Rightarrow$

center of cuvarture is $(4, -2)$



11.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

Exer. 1–8: Find general formulas for the tangential and normal components of acceleration and for the curvature of the curve C determined by $\mathbf{r}(t)$.

(7) $\mathbf{r}(t) = 4\cos t\mathbf{i} + 9\sin t\mathbf{j} + t\mathbf{k}$

Tangential Component of Acceleration 11.22

$$a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$$

Normal Component of Acceleration 11.23

$$a_N = K\left(\frac{ds}{dt}\right)^2 = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

Theorem 11.25

Let a space curve C have the parametrization $x = f(t)$, $y = g(t)$, $z = h(t)$, where f'', g'', and h'' exist. The curvature K at the point $P(x, y, z)$ on C is

$$K = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = a_N\frac{1}{\|\mathbf{r}'(t)\|^2}.$$

$$r'(t) = -4\sin t\, i + 9\cos t\, j + k \Rightarrow r''(t) = -4\cos t\, i - 9\sin t\, j + 0.k$$

$$r'(t)\cdot r''(t) = 16\sin t\cos t - 81\cos t\sin t + 0 = -65\sin t\cos t$$

$$r'(t)\times r''(t) = \begin{vmatrix} i & j & k \\ -4\sin t & 9\cos t & 1 \\ -4\cos t & -9\sin t & 0 \end{vmatrix} = 9\sin t\, i - 4\cos t\, j + 36k$$

$$\|r'(t)\| = \sqrt{16\sin^2(t) + 81\cos^2(t) + 1}\,,$$

$$\|r'(t)\times r''(t)\| = \sqrt{81\sin^2(t) + 16\cos^2(t) + 1296}$$

$$a_T = \frac{-65\sin t\cos t}{\sqrt{16\sin^2(t) + 81\cos^2(t) + 1}}$$

$$a_N = \frac{\sqrt{81 sin^2(t) + 16\, cos^2(t) + 1296}}{\sqrt{16\, sin^2(t) + 81\, cos^2(t) + 1}}$$

$$K = \frac{\sqrt{81 sin^2(t) + 16\, cos^2(t) + 1296}}{\left(\sqrt{16\, sin^2(t) + 81\, cos^2(t) + 1}\right)^3}$$

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9 A point moves along the parabola $y = x^2$ such that the horizontal component of velocity is always 3. Find the tangential and normal components of acceleration at $P(1, 1)$.

10 Work Exercise 9 if the point moves along the graph of $y = 2x^3 - x$.

10:

In the following solution: "TC" is the same of a_T, and "NC" is the same of a_N

Let the path of motion be expressed by a vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ where $g(t) = 2[f(t)]^3 - f(t)$. $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ gives the velocity vector. Hence, the horizontal component of velocity is $f'(t) = 3 \Rightarrow f(t) = 3t + c$. We choose $c = -2$ so that $f(1) = 1$. Thus $f(t) = 3t - 2$ and $g(t) = 2(3t-2)^3 - (3t-2)$. Then $\mathbf{r}'(t) = 3\mathbf{i} + [18(3t-2)^2 - 3]\mathbf{j}$ and $\mathbf{r}''(t) = 108(3t-2)\mathbf{j}$. $|\mathbf{r}'(t)| = \sqrt{9 + [18(3t-2)^2 - 3]^2}$. At $t = 1$, $\mathbf{r}'(1) = 3\mathbf{i} + 15\mathbf{j}$, $\mathbf{r}''(1) = 108\mathbf{j}$, and $|\mathbf{r}'(1)| = \sqrt{234}$. By (15.16), TC $= 1620/\sqrt{234}$. $\mathbf{r}'(1) \times \mathbf{r}''(1)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 15 & 0 \\ 0 & 108 & 0 \end{vmatrix} = 324\mathbf{k}$$

and so, by (15.17), NC $= 324/\sqrt{234}$.

12.2 LIMITS AND CONTINUITY

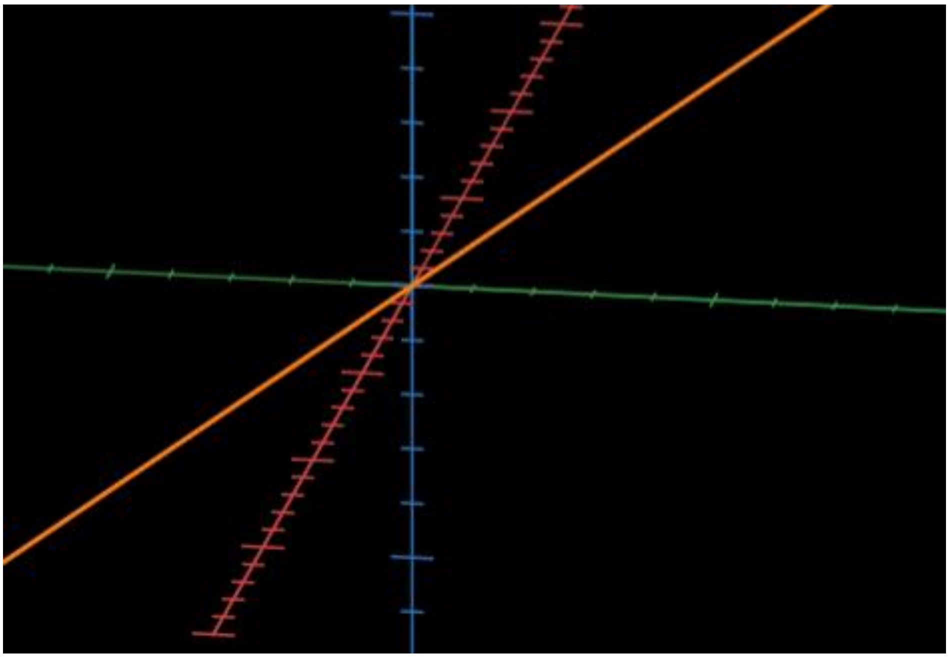
Exer. 11 – 20: Show that the limit does not exist.

(17) $\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}$

On the $x - axis, y = z = 0$: $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{0}{x^2} = \lim_{(x,y,z)\to(0,0,0)} 0 = 0$

On the line: , $x = y = z = t$: $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{3t^2}{3t^2} = \lim_{(x,y,z)\to(0,0,0)} 1 = 1$

Since different paths to $(0, 0, 0)$ produce different limiting values, the limit itself does not exist.



Sketch of the line: , $x = y = z = t$

Exer. 21 – 24: Use polar coordinates to find the limit, if it exists.

(21) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

$$x = r\cos(\theta), y = r\sin(\theta)$$

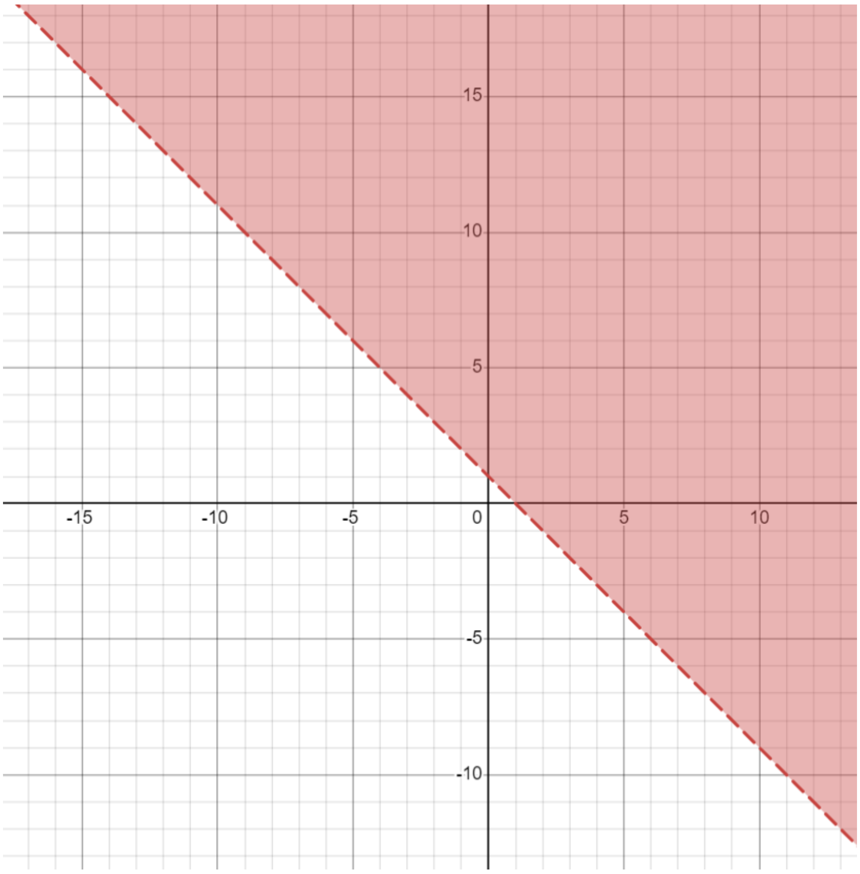
As $(x, y) \to (0,0), (r,\theta) \to (0,\theta) \Rightarrow$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^3\cos(\theta)\sin^2(\theta)}{r^2} = \lim_{(r,\theta)\to(0,\theta)} r\cos(\theta)\sin^2(\theta) = 0$$

Exer. 25 – 28: Describe the set of all points in the *xy*-plane at which *f* is continuous.

25. $f(x, y) = \ln(x + y - 1)$

For $f(x,y) = \ln(x + y - 1)$ to be defined, the argument must be positive, i.e., $x + y - 1 > 0$ or $y > 1 - x$. The ln function is continuous everywhere it is defined. Therefore, f is continuous on $\{(x,y) \mid y > 1 - x\}$.



Exer. 29–32: Describe the set of all points in an *xyz*-coordinate system at which *f* is continuous.

29 $f(x, y, z) = \dfrac{1}{x^2 + y^2 - z^2}$

30 $f(x, y, z) = \sqrt{xy}\tan z$

31 $f(x, y, z) = \sqrt{x-2}\ln(yz)$

30:

f is continuous on $\{(x,y,z) \mid xy > 0, z \neq (\pi/2) + n\pi\}$ which excludes points where the radicand is negative and the tangent is undefined.

31:

f **is continous on** $\{(x, y, z) \mid x - 2 \geq 0, yz > 0\} = \{(x, y, z) \mid x \geq 2, yz > 0\}$

which excludes points where radical is negative and ln is undefined.

12.3 PARTIAL DERIVATIVES

Notations for Partial Derivatives 12.9

If $w = f(x, y)$, then

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}$$

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \frac{\partial w}{\partial x} = w_x$$

$$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \frac{\partial w}{\partial y} = w_y.$$

Second Partial Derivatives 12.11

$$\frac{\partial}{\partial x} f_x = (f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y} f_x = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$\frac{\partial}{\partial x} f_y = (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}$$

$$\frac{\partial}{\partial y} f_y = (f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Third and higher partial derivatives are defined in similar fashion. For example,

$$\frac{\partial}{\partial x} f_{xx} = f_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3},$$

$$\frac{\partial}{\partial x} f_{xy} = f_{xyx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial x\, \partial y\, \partial x},$$

(25) If $w = 3x^2y^3z + 2xy^4z^2 - yz$, find w_{xyz}.

$w_x = 6xy^3z + 2y^4z^2 - 0 \Rightarrow w_{xy} = (\partial/\partial y)w_x = 18xy^2z + 8y^3z^2 \Rightarrow w_{xyz} = (\partial/\partial z)w_{xy}$
$= 18xy^2 + 16y^3z.$

throughout the domain of f. Prove that the given function is harmonic.

(33) $f(x, y) = \ln\sqrt{x^2 + y^2}$

A function f of x and y is *harmonic* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$f(x,y) = \ln(x^2 + y^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2 + y^2)$$

$f_x(x,y) = [(1/2)(x^2 + y^2)^{-1/2}\, 2x]/(\sqrt{x^2 + y^2}) = x/(x^2 + y^2)$

$f_{xx}(x,y) = (y^2 - x^2)/(x^2 + y^2)^2$

$f_y(x,y) = y/(x^2 + y^2),\ f_{yy}(x,y) = (x^2 - y^2)/(x^2 + y^2)^2$

$f_{xx} + f_{yy} = 0$

(39) If $w = e^{-c^2 t}\sin cx$, show that $w_{xx} = w_t$ for every real number c.

$w_x = ce^{-c^2 t}\cos(cx)$

$w_{xx} = -c^2 e^{-c^2 t}\sin(cx)$

$w_t = -c^2 e^{-c^2 t}\sin(cx) = w_{xx}.$

12.4 INCREMENTS AND DIFFERENTIALS

Exer. 7–18: Find *dw*.

(12) $w = \ln(x^2 + y^2) + x \tan^{-1} y$

Definition 12.15

Let $w = f(x, y)$, and let Δx and Δy be increments of x and y, respectively.

(i) The **differentials** dx **and** dy of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y.$$

(ii) The **differential** dw of the dependent variable w is

$$dw = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy.$$

$$dw = (\partial/\partial x)\,[\ln(x^2 + y^2) + x \tan^{-1} y]\,dx + (\partial/\partial y)[\ln(x^2 + y^2) + x \tan^{-1} y]\,dy$$

$$= \left(\frac{2x}{x^2 + y^2} + \tan^{-1} y\right) dx + \left(\frac{2y}{x^2 + y^2} + \frac{x}{1 + y^2}\right) dy.$$

Exer. 19 – 22: Use differentials to approximate the change in f if the independent variables change as indicated.

(19) $f(x, y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$; $(-2, 3)$ to $(-2.02, 3.01)$

$\boldsymbol{dx = \Delta x = -2.02 - (-2) = -0,02}$, $\boldsymbol{dy = \Delta y = 3.01 - 3 = 0.01}$

$f_x(x,y) = 2x - 9x^2y^2 + 4 \Rightarrow f_x(-2,3) = -324.$

$f_y(x,y) = -6x^3y - 6y^2 \Rightarrow f_y(-2,3) = 90.$

$df = f_x dx + f_y dy \Rightarrow df = (-324)(-.02) + (90)(.01) = 6.48 + .9 = 7.38.$

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Exer. 39–40: Prove that f is differentiable throughout its domain.

39 $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

Theorem 12.17

If $w = f(x, y)$ and if f_x and f_y are continuous on a rectangular region R, then f is differentiable on R.

$f(x, y) = (x^2 - y^2)/(x^2 + y^2) \Rightarrow f_x = \partial f/\partial x$
$= [(x^2 + y^2)(2x) - (x^2 - y^2)(2x)]/(x^2 + y^2)^2 = 4xy^2/(x^2 + y^2)^2$.
Also $f_y = \partial f/\partial y = [(x^2 + y^2)(-2y) - (x^2 - y^2)(2y)]/(x^2 + y^2)^2 = -2yx^2/(x^2 + y^2)^2$.
Now the domain of $f(x, y)$ consists of all pairs of real numbers except (0, 0). Both f_x and f_y are continuous except at (0, 0). [See the comment about vanishing denominators of rational functions preceding (16.5).] Hence, by (16.13), $f(x, y)$ is differentiable on its domain. [Any portion of the domain can be included in a rectangle that excludes (0, 0).]

42 Let

$$f(x, y, z) = \begin{cases} \dfrac{xyz}{x^3 + y^3 + z^3} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

(a) Prove that f_x, f_y, and f_z exist at (0, 0, 0).

(b) Prove that f is not differentiable at (0, 0, 0).

Definition 12.8

Let f be a function of two variables. The **first partial derivatives of** f **with respect to** x **and** y are the functions f_x and f_y such that

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Theorem 12.18

If a function f of two variables is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

The proof imitates the steps in the proof of problem **39**. Using (16.7) for each of f_x, f_y, f_z proves that each exists. For example, at (0, 0, 0) $f_x = \partial f/\partial x$

$$= \lim_{h \to 0} \left[\frac{f(0 + h, 0, 0) - f(0, 0, 0)}{h}\right] = \lim_{h \to 0} \left[\frac{0}{h}\right] = \lim_{h \to 0} [0] = 0.$$

Now consider $\lim\limits_{(x, y, z) \to (0,0,0)} f(x, y, z)$ along the path $y = 0$ in the xy-plane (where $z = 0$ also) gives $\lim\limits_{(x, y, z) \to (0,0,0)} f(x, y, z) = \lim\limits_{(x, y, z) \to (0,0,0)} [0/x^3]$ $= \lim\limits_{(x, y, z) \to (0,0,0)} [0] = 0$. But, along the line in three-space $x = y = z$, $\lim\limits_{(x, y, z) \to (0,0,0)} f(x, y, z) = \lim\limits_{(x, y, z) \to (0,0,0)} [x^3/3x^3] = \lim\limits_{(x, y, z) \to (0,0,0)} [1/3] = 1/3$. Hence the limit does not exist, and $f(x, y, z)$ is not continuous at (0, 0, 0). By the contrapositive of (16.14), since $f(x, y, z)$ is not continuous at (0, 0, 0), it also is not differentiable there.

12.5 CHAIN RULES

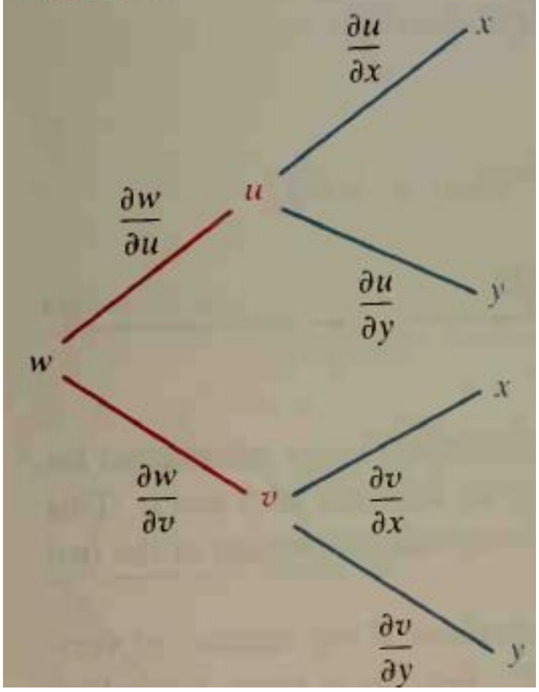
Chain Rules 12.21

If $w = f(u, v)$, with $u = g(x, y)$, $v = h(x, y)$, and if f, g, and h are differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}.$$

Figure 12.39



Use a chain rule in Exercises 1–14.

Exer. 1 – 2: Find $\partial w/\partial x$ and $\partial w/\partial y$.

1 $w = u \sin v$; $u = x^2 + y^2$, $v = xy$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = (\sin v)(2x) + (u \cos v)(y) = 2x \sin v + uy \cos v$$

$$= 2x \sin(xy) + y(x^2 + y^2)\cos(xy).$$

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Exer. 15 – 18: Use partial derivatives to find dy/dx if $y = f(x)$ is determined implicitly by the given equation.

(17) $6x + \sqrt{xy} = 3y - 4$

Theorem 12.22

If an equation $F(x, y) = 0$ determines, implicitly, a differentiable function f of one variable x such that $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

$f(x,y) = 6x + \sqrt{x}\sqrt{y} - 3y + 4$; $f_x(x,y) = 6 + \sqrt{y}/(2\sqrt{x})$; $f_y(x,y) = -3 + \sqrt{x}/(2\sqrt{y})$; $y' = -f_x(x,y)/f_y(x,y)$.

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Exer. 19–22: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(x, y)$ is determined implicitly by the given equation.

(20) $xz^2 + 2x^2y - 4y^2z + 3y - 2 = 0$

Theorem 12.23

If an equation $F(x, y, z) = 0$ determines an implicit differentiable function f of two variables x and y such that $z = f(x, y)$ for every (x, y) in the domain of f, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}.$$

$f_x(x,y,z) = z^2 + 4xy;\ f_y(x,y,z) = 2x^2 - 8yz + 3;\ f_z(x,y,z) = 2xz - 4y^2;$

$z_x = -f_x/f_z = -(z^2 + 4xy)/(2xz - 4y^2);\ z_y = -f_y/f_z = -(2x^2 - 8yz + 3)/(2xz - 4y^2).$

$z_x = -f_x/f_z = -(e^{yz} - 2yze^{xz} + 3yze^{xy})/(xye^{yz} - 2xye^{xz} + 3e^{xy});$

$z_y = -(xze^{yz} - 2e^{xz} + 3xze^{xy})/(xye^{yz} - 2xye^{xz} + 3e^{xy}).$

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37 If $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 .$$

$w_r = w_x\, x_r + w_y\, y_r = w_x \cos\theta + w_y \sin\theta$; $w_\theta = w_x\, x_\theta + w_y\, y_\theta = -w_x\, r\sin\theta + w_y\, r\cos\theta$.

$(w_r)^2 + (r^{-1} w_\theta)^2 = w_x^{\,2}\cos^2\theta + 2w_x\, w_y \sin\theta\cos\theta + w_y^{\,2}\sin^2\theta + w_x^{\,2}\sin^2\theta$

$- 2w_x\, w_y \sin\theta\cos\theta + w_y^{\,2}\cos^2\theta = w_x^{\,2}(\cos^2\theta + \sin^2\theta) + w_y^{\,2}(\sin^2\theta + \cos^2\theta)$.

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12.6 DIRECTIONAL DERIVATIVES

Exer. 11–24: Find the directional derivative of f at the point P in the indicated direction.

(21) $f(x, y, z) = z^2 e^{xy}$; $P(-1, 2, 3)$, $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

Definition 12.26

Let f be a function of two variables. The **gradient** of f (or of $f(x, y)$) is the vector function given by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

Directional Derivative (Gradient Form) 12.27

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{3\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}}{\sqrt{9 + 1 + 25}} = \frac{3\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}}{\sqrt{35}}$$

$f_x = z^2 y e^{xy}$, $f_y = z^2 x e^{xy}$, $f_z = 2z e^{xy}$; $\nabla f(-1,2,3) = 18e^{-2}\,\mathrm{i} - 9e^{-2}\,\mathrm{j} + 6e^{-2}\,\mathbf{k}$.
$D_{\mathbf{a}} f(-1,2,3) = e^{-2}\,(54 - 9 - 30)/\sqrt{35} = 15e^{-2}/\sqrt{35}$.

Exer. 25 – 28: (a) Find the directional derivative of f at P in the direction from P to Q. **(b)** Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f in that direction. **(c)** Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f in that direction.

28 $f(x, y, z) = \dfrac{x}{y} - \dfrac{y}{z}$; $P(0, -1, 2)$, $Q(3, 1, -4)$

Gradient Theorem 12.28

Let f be a function of two variables that is differentiable at the point $P(x, y)$.

(i) The maximum value of $D_{\mathbf{u}} f(x, y)$ at $P(x, y)$ is $\|\nabla f(x, y)\|$.

(ii) The maximum rate of increase of $f(x, y)$ at $P(x, y)$ occurs in the direction of $\nabla f(x, y)$.

Corollary 12.29

Let f be a function of two variables that is differentiable at the point $P(x, y)$.

(i) The minimum value of $D_{\mathbf{u}} f(x, y)$ at the point $P(x, y)$ is $-\|\nabla f(x, y)\|$.

(ii) The minimum rate of increase (or maximum rate of decrease) of $f(x, y)$ at the point $P(x, y)$ occurs in the direction of $-\nabla f(x, y)$.

$f_x = 1/y$, $f_y = -x/y^2 - 1/z$, $f_z = y/z^2$; $\nabla f(0,-1,2) = -\mathbf{i} - (1/2)\mathbf{j} - (1/4)\mathbf{k}$; $\overrightarrow{PQ} = \langle 3,2,-6 \rangle$, $\mathbf{u} = \overrightarrow{PQ}/7$. $D_{\overrightarrow{PQ}}\ f(0,-1,2) = (-6-2+3)/14 = -5/14$. Maximal direction is $\nabla f(0,-1,2)$; maximum rate is $|\nabla f(0,-1,2)| = \sqrt{21}/4$. As the comment after Theorem (16.26) indicates, the minimum increase has direction $\langle 1, 1/2, 1/4 \rangle$ and the minimum increase is $-\sqrt{21}/4$. Expressed as unit vectors, the directions of maximal and minimal increase are respectively $(\sqrt{21}/21)\langle -4,-2,-1 \rangle$ and $(\sqrt{21}/21)\langle 4,2,1 \rangle$.

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12.7 TANGENT PLANES AND NORMAL LINES

Exer. 1 – 10: Find equations for the tangent plane and the normal line to the graph of the equation at the point P.

(1) $4x^2 - y^2 + 3z^2 = 10$; $P(2, -3, 1)$

Gradient of $f(x, y, z)$ **12.31**

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

Corollary 12.34

An equation for the tangent plane to the graph of $F(x, y, z) = 0$ at the point $P_0(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The line perpendicular to the tangent plane at a point $P_0(x_0, y_0, z_0)$ on a surface S is a **normal line** to S at P_0. If S is the graph of $F(x, y, z) = 0$, then the normal line is parallel to the vector $\nabla F(x_0, y_0, z_0)$.

Let $F(x,y,z)$ be the left side of the equation rewritten as $4x^2 - y^2 + 3z^2 - 10 = 0$. Then $\nabla F(x,y,z) = \langle 8x, -2y, 6z\rangle$ and at $P(2, -3, 1)$, $\nabla F(2, -3, 1) = \langle 16, 6, 6\rangle$. This is a normal vector for the tangent plane and a direction vector for the normal line. Thus, using P, we get: $16(x - 2) + 6(y + 3) + 6(z - 1) = 0$ for the tangent plane and $(x - 2)/16 = (y + 3)/6 = (z - 1)/6$ for the normal line.

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12.8 EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

Definition 12.38

Let f be a function of two variables. A pair (a, b) is a **critical point** of f if either

(i) $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or

(ii) $f_x(a, b)$ or $f_y(a, b)$ does not exist.

Definition 12.39

Let f be a function of two variables that has continuous second partial derivatives. The **discriminant** D of f is given by

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2.$$

Test for Local Extrema 12.40

Let f be a function of two variables that has continuous second partial derivatives throughout an open disk R containing (a, b). If $f_x(a, b) = f_y(a, b) = 0$ and $D(a, b) > 0$, then $f(a, b)$ is

(i) a local maximum of f if $f_{xx}(a, b) < 0$

(ii) a local minimum of f if $f_{xx}(a, b) > 0$

Theorem 12.41

Let f have continuous second partial derivatives throughout an open disk R containing (a, b). If $f_x(a, b) = f_y(a, b) = 0$ and $D(a, b)$ is negative, then $P(a, b, f(a, b))$ is a saddle point on the graph of f.

Exer. 1 – 20: Find the extrema and saddle points of f.

13 $f(x, y) = \frac{1}{2}x^4 - 2x^3 + 4xy + y^2$

The solution: scan me

13 SP: $(0, 0, f(0, 0))$; min: $f(4, -8) = -64$, $f(-1, 2) = -\frac{3}{2}$

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Exer. 23 – 28: Find the maximum and minimum values of f on R. (Refer to Exercises 3–8 for local extrema.)

(27) $f(x, y) = x^3 + 3xy - y^3$;
the triangular region R with vertices $(1, 2)$, $(1, -2)$, and $(-1, -2)$

The two equations are $f_x = 3x^2 + 3y = 0$ and $f_y = 3x - 3y^2 = 0$. The second reduces to $x = y^2$. Substituting into the first gives $y^4 + y = 0 = y(y^3 + 1) \Rightarrow y = 0, y = -1 \Rightarrow x = 0, x = 1$. $f_{xx} = 6x, f_{xy} = 3, f_{yy} = -6y$. For $x = 0, y = 0, g = (0)(3) - 9 < 0 \Rightarrow f(0,0)$ is not an extremum. For $(1,-1)$, $g = (6)(6) - 9 > 0 \Rightarrow f(1,-1)$ is a local minimum.

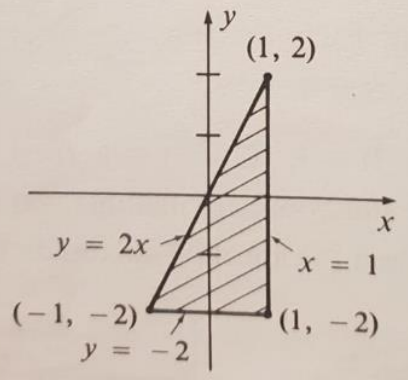
, $(1, -1)$ was determined to be a local minimum of $f(x, y)$ on the plane. But $(1, -1)$ is not interior to triangular region R given as the domain of $f(x, y)$ in this problem. So $f(x, y)$ has no local extrema interior to R. The boundaries of the triangular region are $x = 1$, $y = -2$, and $y = 2x$. The extrema on the boundaries are found as in Chapter 4 since, on each boundary, f can be written as a function of one variable there. Thus:

(i) $\partial f(1, y)/\partial y = D_y(-y^3 + 3y + 1) = -3y^2 + 3 = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = 1$ or $y = -1$ for both of which $x = 1$. By the Second Derivative Test, $y = 1$ is a maximum, and $y = -1$ is a minimum. $f(1, 1) = 3$ and $f(1, -1) = -1$.

(ii) $\partial f(x, -2)/\partial x = D_x(x^3 - 6x + 8) = 3x^2 - 6 = 0 \Rightarrow x^2 - 2 = 0 \Rightarrow x = \sqrt{2}$ or $x = -\sqrt{2}$ both of which are outside of R.

(iii) $\partial f(x, 2x)/\partial x = D_x(-7x^3 + 6x^2) = -21x^2 + 12x = 0 \Rightarrow x = 0$ or $x = 4/7$.
When $x = 0$, $y = 2(0) = 0$. When $x = 4/7$, $y = 2(4/7) = 8/7$.
By the Second Derivative Test, $x = 0$ is a minimum, and $x = 4/7$ is a maximum.
$f(0, 0) = 0$ and $f(4/7, 8/7) = 32/49$.

At the corners of R, $f(1, 2) = -1$, $f(1, -2) = 3$, and $f(-1, -2) = 13$.
Hence, comparing the local and boundary maxima and minima, the absolute minimum is $f(1, 2) = f(1, -1) = -1$, and the absolute maximum is $f(-1, -2) = 13$.



12.9 LAGRANGE MULTIPLIERS

Exer. 1 – 10: Use Lagrange multipliers to find the extrema of f subject to the stated constraints.

(1) $f(x, y) = y^2 - 4xy + 4x^2$; $x^2 + y^2 = 1$

Lagrange's Theorem 12.42

Suppose that f and g are functions of two variables having continuous first partial derivatives and that $\nabla g \neq \mathbf{0}$ throughout a region of the xy-plane. If f has an extremum $f(x_0, y_0)$ subject to the constraint $g(x, y) = 0$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Corollary 12.43

The points at which a function f of two variables has relative extrema subject to the constraint $g(x, y) = 0$ are included among the points (x, y) determined by the first two coordinates of the solutions (x, y, λ) of the system of equations

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = 0 \end{cases}$$

Letting $g(x, y) = x^2 + y^2 - 1$ and $\nabla f = \lambda \nabla g$, equations (16.36) take on the form $-4y + 8x = 2x\lambda$, $2y - 4x = 2y\lambda$, $x^2 + y^2 - 1 = 0$. The first equation plus twice the second yields $0 = 2x\lambda + 4y\lambda = 2\lambda(x + 2y) \Rightarrow \lambda = 0$ or $x = -2y$. If $\lambda = 0$, then from either the first or second equation $y = 2x$ which, when substituted into the third, produces $5x^2 = 1$ and the two solutions $P_1(1/\sqrt{5}, 2/\sqrt{5})$, $P_2(-1/\sqrt{5}, -2/\sqrt{5})$. If $x = -2y$, substitution into the third equation produces $5y^2 = 1$, and we get two more solutions $P_3(2/\sqrt{5}, -1/\sqrt{5})$, $P_4(-2/\sqrt{5}, 1/\sqrt{5})$. From the form of $f(x, y) = (y - 2x)^2$, we see that f has the minimum value of 0 at P_1 and P_2 (where $y = 2x$) and the maximum value of 5 at P_3 and P_4.

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(7) $f(x, y, z) = x^2 + y^2 + z^2$;

$x - y = 1, \qquad y^2 - z^2 = 1$

Let $g(x, y, z) = x - y - 1$, $h(x, y, z) = y^2 - z^2 - 1 = 0$, and set $\nabla f = \lambda \nabla g + \mu \nabla h$. This gives us the five equations $2x = \lambda$, $2y = -\lambda + 2y\mu$, $2z = -2z\mu$, $x - y - 1 = 0$, $y^2 - z^2 - 1 = 0$. From the third equation, $2z(1 + \mu) = 0$ and hence $z = 0$ or $\mu = -1$. If $z = 0$, then $y^2 - z^2 - 1 = y^2 - 1 = 0 \Rightarrow y = \pm 1$; and from $x - y - 1 = 0$, we get $x = 2$ or $x = 0$. This gives us the two solutions $P_1(2, 1, 0)$ and $P_2(0, -1, 0)$. If $\mu = -1$, then from the second equation $\lambda = 2y\mu - 2y = -4y$; and since $\lambda = 2x$, we have $2x = -4y$ or $x = -2y$. Using the fourth equation, $x - y - 1 = -2y - y - 1 = 0$ $\Rightarrow y = -1/3$, but when this value is used in the fifth equation we get $1/9 - z^2 - 1 = 0$ or $z^2 = -8/9$, with no solutions. Thus, P_1 and P_2 are the only solutions and f attains a local minimum at each. This is clear from the fact that $f = d^2(O, P)$ and that the plane, $x - y = 1$, and the cylinder, $y^2 - z^2 = 1$, intersect in a curve consisting of a right branch (for $y > 1$) and a left branch for ($y \leq -1$). P_1 is the point on the right branch closest to the origin [$f(P_1) = 5$], and P_2 is the point on the left branch closest to the origin [$f(P_2) = 1$].

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(12) Find the point on the line of intersection of the planes $x + 3y - 2z = 11$ and $2x - y + z = 3$ that is closest to the origin.

We will apply method of Lagrange's multipliers.

Consider $f(x, y, z) = d^2 = x^2 + y^2 + z^2$,

$$g(x, y, z) = x + 3y - 2z - 11, h(x, y, z) = 2x - y + z - 3$$

and let $\nabla f = \lambda \nabla g + \mu \nabla h$.

This leads to the system of five equations:

$$\begin{gathered} 2x = \lambda + 2\mu \\ 2y = 3\lambda - \mu \\ 2z = -2\lambda + \mu \\ x + 3y - 2z - 11 = 0 \\ 2x - y + z - 3 = 0 \end{gathered} \quad \Rightarrow$$

From Eq 4: $\frac{\lambda+2\mu}{2} + 3\frac{3\lambda-\mu}{2} - 2\frac{-2\lambda+\mu}{2} - \frac{22}{2} = 0 \quad \Rightarrow \quad 14\lambda + 9\mu = 22$

From Eq 5: $2\frac{\lambda+2\mu}{2} - \frac{3\lambda-\mu}{2} + \frac{-2\lambda+\mu}{2} - \frac{6}{2} = 0 \quad \Rightarrow \quad -\lambda + 2\mu = 2$

Solving the two equations $\begin{matrix} 14\lambda + 9\mu = 22 \\ -\lambda + 2\mu = 2 \end{matrix}$ simultaneously yields

$$\lambda = 0.7027, \mu = 1.3514 \quad \Rightarrow$$

$$x = \frac{0.7027 + 2(1.3514)}{2} = 1.70275$$

$$y = \frac{3(0.7027) - 1.3514}{2} = 0.37835$$

$$z = \frac{-2(0.7027) + 1.3514}{2} = -0.027$$

The point is: $(x, y, z) = (\,1.70275, 0.37835, -0.027)$