Example 2.12

Find the MGF of the r.v. X, then use it to find the first four moments. Where

$$f(x) = \frac{x}{2}; \quad 0 < x < 2$$

Solution

$M_X(t) = E(e^{tX}) = \int_0^2 \frac{xe^{tx}}{2}\, dx.$

Use integration by parts:

$u = \frac{x}{2} \qquad dv = e^{tx}dx$

$du = \frac{1}{2}dx \qquad v = \frac{e^{tx}}{t}$

Hence,

$M_X(t) = \frac{xe^{tx}}{2t}|_0^2 - \int_0^2 \frac{e^{tx}}{2t}dx = \frac{e^{2t}}{t} - \frac{e^{tx}}{2t^2}|_0^2 = \frac{e^{2t}}{t} - \frac{e^{2t}}{2t^2} + \frac{1}{2t^2}.$

Since the derivative of $M_X(t)$ does not exist at $t = 0$, we will use the Taylor series form. Thus, we have to put the MGF on the form

If we put t=0 ,then $M_X(t) = \frac{e^{2t}}{t} - \frac{e^{2t}}{2t^2} + \frac{1}{2t^2}(at\ t = 0) = \frac{e^{2t}}{0} + \frac{e^{2t}}{2(0)} + \frac{1}{2x0^2} = not\ exist$

Then,

<u>We will use Taylor series to find the value of MGF $M_X(t)$:</u>

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} E(X^k)\, \frac{t^k}{k!} = 1 + E(X)t + E(X^2)\frac{t^2}{2!} + E(X^3)\frac{t^3}{3!} + \cdots.$$

<u>نستخدم المتسلسلة لحل MGF:</u> قيمة $M_X(t)$ اللي تم حسابها بالمثال كانت كالتالي :

$M_X(t) = \frac{e^{2t}}{t} + \frac{e^{2t}}{2t^2} + \frac{1}{2t^2} = \frac{1}{2t^2} + \frac{e^{2t}}{t} + \frac{e^{2t}}{2t^2} = \frac{1}{2t^2} + e^{2t}\left(\frac{1}{t} + \frac{1}{2t^2}\right) = \frac{1}{2t^2} + e^{2t}\left(\frac{2t}{2t^2} + \frac{1}{2t^2}\right) =$

$= \frac{1}{2t^2} + e^{2t}\left(\frac{2t-1}{2t^2}\right)$

$$M_X(t) = \frac{1}{2t^2} + e^{2t}\left(\frac{2t-1}{2t^2}\right)$$

Use the **Taylor series :**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Let x =2t , then the **Taylor series :**

$$e^{2t} = 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \frac{(2t)^6}{6!} \dots\dots\dots\dots$$

Now, substitute series e^{2t} in MGF $M_X(t)$:

$$M_X(t) = \frac{1}{2t^2} + e^{2t}\left(\frac{2t-1}{2t^2}\right)$$

$$M_X(t) = \frac{1}{2t^2} + e^{2t}\left(\frac{2t-1}{2t^2}\right) = \frac{1}{2t^2} + \left(\frac{2t-1}{2t^2}\right)\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \frac{(2t)^6}{6!} \dots\dots\dots\right)$$

$$= \frac{1}{2t^2} + \left(\frac{2t-1}{2t^2}\right)\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \frac{(2t)^6}{6!} \dots\dots\dots\right)$$

$$= \frac{1}{2t^2} + \left(\frac{2t-1}{2t^2}\right)\left(1 + 2t + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!} + \frac{2^5t^5}{5!} + \frac{2^6t^6}{6!} \dots\dots\dots\right)$$

$$= \frac{1}{2t^2} + \frac{2t-1}{2t^2} + \frac{2t-1}{t} + 2t - 1 + (2t-1)\frac{2^2t}{3!} + (2t-1)\frac{2^3t^2}{4!} + (2t-1)\frac{2^4t^3}{5!} + (2t-1)\frac{2^5t^4}{6!} \dots\dots\dots$$

$$= \frac{1}{2t^2} + \frac{1}{t} - \frac{1}{2t^2} + 2 - \frac{1}{t} + 2t - 1 + \frac{2^3t^2}{3!} - \frac{2^2t}{3!} + \frac{2^4t^3}{4!} - \frac{2^3t^2}{4!} + \frac{2^5t^4}{5!} - \frac{2^4t^3}{5!} + \frac{2^6t^5}{6!} - \frac{2^5t^4}{6!} \dots\dots\dots$$

$$= 1 + 2t + \frac{2^3t^2}{3!} - \frac{2^2t}{3!} + \frac{2^4t^3}{4!} - \frac{2^3t^2}{4!} + \frac{2^5t^4}{5!} - \frac{2^4t^3}{5!} + \frac{2^6t^5}{6!} - \frac{2^5t^4}{6!} + \cdots \dots\dots$$

$$= 1 + 2t - \frac{2^2t}{3!} + \frac{2^3t^2}{3!} - \frac{2^3t^2}{4!} + \frac{2^4t^3}{4!} - \frac{2^4t^3}{5!} + + \frac{2^5t^4}{5!} - \frac{2^5t^4}{6!} + \frac{2^6t^5}{6!} \dots\dots\dots\dots\dots\dots\dots$$

$$= 1 + \left(2 - \frac{2^2}{3!}\right)t + \left(\frac{2^3}{3!} - \frac{2^3}{4!}\right)t^2 + \left(\frac{2^4}{4!} - \frac{2^4}{5!}\right)t^3 + \left(\frac{2^5}{5!} - \frac{2^5}{6!}\right)t^4 + \left(\frac{2^6}{6!} - \frac{2^6}{7!}\right)t^5 + \cdots \dots\dots\dots\dots\dots\dots$$

$$= 1 + \frac{4}{3}t + 1t^2 + \frac{8}{15}t^3 + \frac{2}{9}t^4 + \frac{8}{15}t^5 \dots\dots\dots = 1 + \frac{4}{3}t + 2\frac{t^2}{2!} + \frac{16}{5}\frac{t^3}{3!} + \frac{16}{3}\frac{t^4}{4!} + \frac{8}{15}t^3 + \frac{8}{105}t^4 + 64\frac{t^5}{5!} \dots\dots\dots$$

By compare with the Taylor series for the first 4 terms with

$$M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2!} + E(X^3)\frac{t^3}{3!} + E(X^4)\frac{t^4}{4!} + E(X^5)\frac{t^5}{5!} + E(X^6)\frac{t^6}{6!} + \dots\dots$$

$$M_X(t) = 1 + \frac{4}{3}t + 2\frac{t^2}{2!} + \frac{16}{5}\frac{t^3}{3!} + \frac{16}{3}\frac{t^4}{4!} + \frac{8}{15}t^3 + \frac{8}{105}t^4 + 64\frac{t^5}{5!} \dots\dots\dots$$

Then , the first four moments are

$$E(X) = \frac{4}{3}$$

$$E(X^2) = 2$$

$$E(X^3) = \frac{16}{5}$$

$$E(X^4) = \frac{16}{3}$$

Mean and variance :

The mean = μ = $E(X) = \frac{4}{3}$

Variance = σ^2= $E(X^2) - E(X)^2$

$$= 2 - \frac{4}{3}^2 = \frac{2}{9}$$