First Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Sunday 5-5-1442	**240 Math**	**Math. Department**

Q1: Suppose (1,2,3) is a solution of the following linear system:

$$x_1 + 2x_2 - x_3 = b_1$$
$$2x_1 + 3x_2 - 3x_3 = b_2$$

Find the **values** of b_1, b_2. (2 marks)

Answer: b_1=1+4-3=2 and b_2=2+6-9=-1

Q2: Show that the matrix A is invertible, where $A^2 + 3A = B$ and det(B)=2. (2 marks)

Answer: A(A+3I)=B implies |A||A+3I|=|B|=2 which implies |A| is nonzero and hence A is invertible.

Q3: Let V be the subspace of $\mathbb{R}^3$ **spanned** by the set S={ v_1=(1, 2,3), v_2=(2, 4,6), v_3=(4, 6, 6)}. Find a **subset** of S that forms a basis of V. (4 marks)

Answer:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 3 & 6 & 6 \end{bmatrix} \xrightarrow[-3R_{13}]{-2R_{12}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -6 \end{bmatrix}$$
$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{6R_{23}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since columns 1 and 3 have leading ones, then v_1 and v_3 forms a basis of V.

Q4: Show that $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ is diagonalizable and find a matrix P that diagonalizes A. (6 marks)

Answer:

$$0=\begin{vmatrix}\lambda-1 & -2 & 2\\ 0 & \lambda-1 & 0\\ 0 & -2 & \lambda+1\end{vmatrix}=(\lambda-1)(\lambda-1)(\lambda+1)$$

$$\lambda=\pm1$$

At λ=1

$$\begin{bmatrix}0 & -2 & 2\\ 0 & 0 & 0\\ 0 & -2 & 2\end{bmatrix}\xrightarrow{-1R_{13}}\begin{bmatrix}0 & -2 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\xrightarrow{-\frac{1}{2}R_1}\begin{bmatrix}0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1=s\in\mathbb{R},x_2=x_3=t\in\mathbb{R}$$

$$E_1=\left\{\begin{bmatrix}s\\ t\\ t\end{bmatrix}:s,t\in\mathbb{R}\right\}=\left\{s\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}+t\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}:s,t\in\mathbb{R}\right\}$$

At λ=-1

$$\begin{bmatrix}-2 & -2 & 2\\ 0 & -2 & 0\\ 0 & -2 & 0\end{bmatrix}\xrightarrow[-1R_{23}]{-1R_{21}}\begin{bmatrix}-2 & 0 & 2\\ 0 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\xrightarrow[-\frac{1}{2}R_2]{-\frac{1}{2}R_1}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1=x_3=t\in\mathbb{R},x_2=0$$

$$E_{-1}=\left\{t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}:t\in\mathbb{R}\right\}$$

Since A has three independent Eigen vectors, it is diagonalizable and

$$P=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

Q5: Assume that the vector space $\mathbb{R}^3$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the following basis vectors (1,-2,0), (2,1,-1), (0,1,1) into an **orthonormal basis**. (8 marks)

Answer: Let v_1=(1,-2,0) , v_2=(2,1,-1) , v_3=(0,1,1) .

Now define u_1, u_2 and u_3 as follows:

$$u_1 = v_1 = (1,-2,0)$$

$$u_2 = v_2 - \frac{<v_2,u_1>}{\|u_1\|^2}u_1 = (2,1,-1) - 0 = (2,1,-1)$$

$$u_3 = v_3 - \frac{<v_3,u_2>}{\|u_2\|^2}u_2 - \frac{<v_3,u_1>}{\|u_1\|^2}u_1$$

$$= (0,1,1) - 0 - \tfrac{-2}{5}(1,-2,0) = (\tfrac{2}{5},\tfrac{1}{5},1)$$

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{5}}(1,-2,0)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}}(2,1,-1)$$

$$w_3 = \frac{u_3}{\|u_3\|} = \frac{\sqrt{5}}{\sqrt{6}}(\tfrac{2}{5},\tfrac{1}{5},1)$$

So $\{w_1, w_2, w_3\}$ is the wanted orthonormal basis.

Q6: Let V be an inner product space, let v_o be any fixed vector in V, and let $T: V \to \mathbb{R}$ be the map defined by $T(v) = \langle v, v_o \rangle$ for all v in V. Show that:

(a) T is a linear transformation. (4 marks)

(b) If $v_o \in \ker(T)$, then $v_o = 0$ and $\ker(T) = V$. (2 marks)

Answer: (a) For any u and v in V and any real number k we have:

(1) $T(u+v)=<u+v,v_o>=<u,v_o>+<v,v_o>=T(u)+T(v)$

(2) $T(ku)= <ku,v_o>=k<u,v_o>=kT(u)$

(b) v_o belongs to ker(T) implies that $0=T(v_o)=< v_o, v_o>$ which implies that $v_o=0$. So for all v in V: $T(v)= <v,v_o>=<v,0>=0$ and hence ker(T)=V.

Q7: Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by:

$T(x_1,x_2) = (3x_1 - x_2, -2x_1, x_1 + x_2)$.

(a) Find $[T]_{S,B}$ where S is the standard basis of $\mathbb{R}^3$ and B={v_1=(1,1),v_2=(1,0)}. (4 marks)

(b) Show that T is one-to-one. (2 marks)

Answer: (a) T(1,1)=(2,-2,2) and T(1,0)=(3,-2,1). Hence

$$[T]_{S,B} = \left[[T(1,1)]_S \mid [T(1,0)]_S \right] = \begin{bmatrix} 2 & 3 \\ -2 & -2 \\ 2 & 1 \end{bmatrix}$$

(b) (0,0,0)=T(x,y)=(3x-y,-2x,x+y) implies 3x-y=0, -2x=0, x+y=0. So x=0 and hence y=0. Thus ker(T)={0} and T is 1-1.

Q8: Show that:

(a) If $T:V \to W$ is a linear transformation, then the kernel of T is a subspace of V. (2 marks)

Answer: T(0)=0 implies ker(T) is not empty.

For all u and v in ker(T) and a real number k we have:

T(u+v)=T(u)+T(v)=0+0=0, so u+v is in ker(T).

T(ku)=kT(u)=k0=0, so ku is in ker(T).

Thus, ker(T) is a subspace of V.

(b) If 1 and -1 are the eigenvalues of a square matrix A of order 2, then we have that A^{100} = I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (2 marks)

Answer: As 1 and -1 are distinct eigenvalues, So A is diagonalizable and A is similar to $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ such that $A=PDP^{-1}$. So $A^{100} = PD^{100}P^{-1} = PIP^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) If *u* and *v* are orthogonal vectors in an inner product space, then:
$\|u+v\|^2 = \|u\|^2 + \|v\|^2$. (2 marks)

Answer: As *u* and *v* are orthogonal, so <u,v>=0 and hence:

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v \rangle \\ &= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle \\ &= \langle u,u \rangle + \langle v,v \rangle = \|u\|^2 + \|v\|^2 \end{aligned}$$