First Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Wednesday 15-4-1444	**240 Math**	**Math. Department**

Q1: Find all the values of m such that the following system has a unique solution:

$$x_1 + x_2 - x_3 = 1$$
$$x_2 - 3x_3 = 1 \qquad \text{(2 marks)}$$
$$(m^2+1)x_3 = m^2 - 4$$

Answer: The determinant of the coefficient matrix shouldn't be zero.

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & m^2+1 \end{vmatrix} = m^2 + 1 \neq 0 \forall m \in \mathbb{R}$$

Or

The augmented matrix of the system

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & m^2+1 & m^2-4 \end{array}\right] \xrightarrow{\frac{1}{m^2+1}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & \frac{m^2-4}{m^2+1} \end{array}\right]$$

So the system has a unique solution for every $m \in \mathbb{R}$.

Q2: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set $S=\{v_1=(1,0,1,1), v_2=(-2,2,0,0), v_3=(-1,4,3,3), v_4=(-5,4,-1,-1)\}$. Find a **subset** of S that forms a basis of V. Then **express** each vector that is not in the basis as a linear combination of the basis vectors. (4 marks)

Answer: Putting the vectors as columns in the following matrix:

$$\begin{bmatrix} 1 & -2 & -1 & -5 \\ 0 & 2 & 4 & 4 \\ 1 & 0 & 3 & -1 \\ 1 & 0 & 3 & -1 \end{bmatrix} \xrightarrow[(-1)R_{14}]{(-1)R_{13}} \begin{bmatrix} 1 & -2 & -1 & -5 \\ 0 & 2 & 4 & 4 \\ 0 & 2 & 4 & 4 \\ 0 & 2 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow[(-1)R_{24}]{\substack{1R_{21} \\ (-1)R_{23}}} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\{v_1, v_2\}$ is a basis of V. So $v_3=3v_1+2v_2$ and $v_4=-v_1+2v_2$.

Q3: If $W=\{(a,b,c,d)\in \mathbb{R}^4 : b=a-c\}$, then show that W is a **subspace** of $\mathbb{R}^4$. Also, find a **basis** of W and **dim(W)**. (5 marks)

Answer: 1- $(0,0,0,0)\in W$. So $W\neq\emptyset$.

2- Suppose $u=(a_1,b_1,c_1,d_1), v=(a_2,b_2,c_2,d_2)\in W$. So $b_1=a_1-c_1$ and So $b_2=a_2-c_2$. Now, $u+v=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)$ and $b_1+b_2= a_1-c_1+a_2-c_2=(a_1+a_2)-(c_1+c_2)$. So $u+v\in W$.

3- Suppose $u=(a_1,b_1,c_1,d_1), k\in\mathbb{R}$. Now, $ku=(ka_1,kb_1,kc_1,kd_1)$ and $kb_1=k(a_1-c_1)= ka_1-kc_1$. So $ku\in W$. 1,2 and 3 imply that W is a subspace of $\mathbb{R}^4$. Now, the general element of W is written as (a,a-c,c,d) and $a,c,d\in \mathbb{R}$. So (a,a-c,c,d)=a(1,1,0,0)+c(0,-1,1,0)+d(0,0,0,1) and the vectors (1,1,0,0), (0,-1,1,0) and (0,0,0,1) generate W. As a(1,1,0,0)+c(0,-1,1,0)+d(0,0,0,1)=(0,0,0,0) implies (a,a-c,c,d)=(0,0,0,0) and hence a=c=d=0, the vectors (1,1,0,0), (0,-1,1,0) and (0,0,0,1) are linearly independent. So the set {(1,1,0,0),(0,-1,1,0),(0,0,0,1)} is a basis of W with dim(W)=3.

Q4: Let B={(1,0),(1,1)} and B'={(1,3),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (3 marks).

Answer:

$$[B\,|\,B']=\left[\begin{array}{cc|cc}1&1&1&2\\0&1&3&0\end{array}\right]\xrightarrow{(-1)R_{21}}\left[\begin{array}{cc|cc}1&0&-2&2\\0&1&3&0\end{array}\right]=[I\,|\,P_{B'\to B}]\Rightarrow P_{B'\to B}=\begin{bmatrix}-2&2\\3&0\end{bmatrix}$$

Q5: Show that $A=\begin{bmatrix}2&0&-1\\1&1&0\\-3&0&0\end{bmatrix}$ is diagonalizable and find the matrix P that diagonalizes A. (6 marks)

Answer: The characteristic equation:

$$0=\det(\lambda I-A)=\det\left(\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}2&0&-1\\1&1&0\\-3&0&0\end{bmatrix}\right)=\begin{vmatrix}\lambda-2&0&1\\-1&\lambda-1&0\\3&0&\lambda\end{vmatrix}$$

$$=(\lambda-2)\lambda(\lambda-1)-3(\lambda-1)=(\lambda-1)(\lambda(\lambda-2)-3)$$

$$=(\lambda-1)(\lambda^2-2\lambda-3)=(\lambda-1)(\lambda+1)(\lambda-3)$$

The Eigenvalues are λ=1,−1,3. Since they are distinct, A is diagonalizable. To find P, take the equation (λI-A)x=0 and substitute λ=1,−1,3, respectively as follows:

$$\lambda I-A=\begin{bmatrix}\lambda-2&0&1\\-1&\lambda-1&0\\3&0&\lambda\end{bmatrix}$$

$$\lambda=1\Rightarrow(1)I-A=\begin{bmatrix}-1&0&1\\-1&0&0\\3&0&1\end{bmatrix}\xrightarrow[3R_{13}]{(-1)R_{12}}\begin{bmatrix}-1&0&1\\0&0&-1\\0&0&4\end{bmatrix}\xrightarrow[4R_{23}]{(-1)R_{1}}\begin{bmatrix}1&0&-1\\0&0&-1\\0&0&0\end{bmatrix}$$

$$\Rightarrow y=t,x=z=0\;\&\;t=1\Rightarrow C_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\lambda=-1\Rightarrow(-1)I-A=\begin{bmatrix}-3&0&1\\-1&-2&0\\3&0&-1\end{bmatrix}\xrightarrow[-1R_{2}]{1R_{13}}\begin{bmatrix}-3&0&1\\1&2&0\\0&0&0\end{bmatrix}$$

$$\xrightarrow{3R_{21}}\begin{bmatrix}0&6&1\\1&2&0\\0&0&0\end{bmatrix}\xrightarrow{-\frac{1}{3}R_{12}}\begin{bmatrix}0&6&1\\1&0&-\frac{1}{3}\\0&0&0\end{bmatrix}$$

$$\Rightarrow x=\tfrac{1}{3}t,y=-\tfrac{1}{6}t,z=t\;\&\;t=6\Rightarrow C_2=\begin{bmatrix}2\\-1\\6\end{bmatrix}$$

$$\lambda=3\Rightarrow(3)I-A=\begin{bmatrix}1&0&1\\-1&2&0\\3&0&3\end{bmatrix}\xrightarrow[-3R_{13}]{1R_{12}}\begin{bmatrix}1&0&1\\0&2&1\\0&0&0\end{bmatrix}$$

$$\Rightarrow x=-t,y=-\tfrac{1}{2}t\;\&\;t=2\Rightarrow C_3=\begin{bmatrix}-2\\-1\\2\end{bmatrix}$$

So $P=\begin{bmatrix}0&2&-2\\1&-1&-1\\0&6&2\end{bmatrix}$.

Q6: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis $\{u_1=(1,0,0), u_2=(3,1,-1), u_3=(0,3,1)\}$ into an **orthonormal basis**. (5 marks)

Answer:

$$u_1=(1,0,0), u_2=(3,1,-1), u_3=(0,3,1)$$

$$v_1=u_1=(1,0,0)$$

$$v_2=u_2-\frac{<u_2,v_1>}{\|v_1\|^2}v_1$$

$$(3,1,-1)-\frac{<(3,1,-1),(1,0,0)>}{\|(1,0,0)\|^2}(1,0,0)=(3,1,-1)-\frac{3}{1}(1,0,0)$$

$$=(3,1,-1)-(3,0,0)=(0,1,-1)$$

$$v_3=u_3-\frac{<u_3,v_2>}{\|v_2\|^2}v_2-\frac{<u_3,v_1>}{\|v_1\|^2}v_1$$

$$=(0,3,1)-\frac{<(0,3,1),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1)-\frac{<(0,3,1),(1,0,0)>}{\|(1,0,0)\|^2}(1,0,0)$$

$$=(0,3,1)-\tfrac{2}{2}(0,1,-1)-0=(0,3,1)-(0,1,-1)=(0,2,2)$$

$$w_1=\frac{v_1}{\|v_1\|}=(1,0,0)$$

$$w_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt{2}}(0,1,-1)$$

$$w_3=\frac{v_3}{\|v_3\|}=\frac{1}{\sqrt{8}}(0,2,2)=\frac{1}{2\sqrt{2}}(0,2,2)=\frac{1}{\sqrt{2}}(0,1,1)$$

Q7: Let M_{22} be the vector space of square matrices of order 2, and let $T: M_{22}\to M_{22}$ be the mapping defined by $T(A)=A-A^T$ for all matrices A in M_{22}.

(a) Show that T is a linear operator. (2 marks)

Answer: For all $A,B\in M_{22}$, $k\in \mathbb{R}$:

1- $T(A+B)=A+B-(A+B)^T= A+B-(A^T+B^T)= A+B-A^T-B^T= A-A^T +B-B^T=T(A)+T(B)$

2- $T(kA)= kA-(kA)^T= kA-kA^T= k(A-A^T)=kT(A)$

So T is linear.

(b) Find ker(T). (1 mark)

Answer: $\ker(T)=\{A\in M_{22}\,|\,T(A)=0\}=\{A\in M_{22}\,|\,A-A^T=0\}=\{A\in M_{22}\,|\,A=A^T\}$.

(c) Find a basis for ker(T). (2 marks)

$\ker(T)=\{A\in M_{22}\,|\,A=A^T\}$. So ker(T) is the set of all symmetric matrices. Then:

$$\ker(T)=\left\{\begin{bmatrix} a & b \\ b & c\end{bmatrix} \,|\, a,b,c\in\mathbb{R}\right\}=\left\{\begin{bmatrix} a & 0 \\ 0 & 0\end{bmatrix}+\begin{bmatrix} 0 & b \\ b & 0\end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & c\end{bmatrix} \,|\, a,b,c\in\mathbb{R}\right\}$$

$$=\left\{a\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} \,|\, a,b,c\in\mathbb{R}\right\}$$

So the vectors $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ generate ker(T). They are also linearly independent as $a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ implies that $\begin{bmatrix}a & b\\b & c\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ and hence a=b=c=0. So $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is a basis of ker(T).

Q8: Let $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by: $T(x_1,x_2)=(x_1-2x_2,x_1-2x_2,-x_2)$.

(a) Find $[T]_{S,B}$ where $S=\{u_1=(1,1,1),u_2=(1,1,0),u_3=(1,0,0)\}$ is a basis of $\mathbb{R}^3$ and $B=\{v_1=(1,1),v_2=(1,0)\}$ is a basis of $\mathbb{R}^2$. (2 marks)

Answer: T(1,1)=(−1,−1,−1) and T(1,0)=(1,1,0). So T(1,1)= $-u_1$ and T(1,0)=u_2 and hence $[T(1,1)]_S=(-1,0,0)$ and $[T(1,0)]_S=(0,1,0)$. Therefore, $[T]_{S,B}=\begin{bmatrix}-1 & 0\\0 & 1\\0 & 0\end{bmatrix}$.

(b) Find rank(T). (2 marks)

Answer: As rank(T)=dim(R(T)), we will find a basis for R(T). Since T(1,1) and T(1,0) generate R(T) and neither is a scalar multiple of the other, so {T(1,1),T(1,0)} is a basis of R(T) and hence rank(T)=2.

Q9: (a) Let $T:\mathbb{R} \longrightarrow \mathbb{R}^2$ be a linear transformation such that T(3)=(1,2). **Find** T(x) for all $x\in\mathbb{R}$. (1 mark)

Answer: Let $x\in\mathbb{R}$. Then x=3a, where $a\in\mathbb{R}$ (as {3} is a basis of $\mathbb{R}$). So a=(x/3) and then T(x)=T(3a)=aT(3)=a(1,2)=(a,2a)=(x/3,(2x)/3).

(b) If *u* and *v* are orthogonal vectors in an inner product space, then prove that:

$\|u+v\|^2 = \|u\|^2 + \|v\|^2$. (2 marks)

Answer: As *u* and *v* are orthogonal, so <u,v>=0 and hence:

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \langle u,u\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2\end{aligned}$$

(c) If B is a 5×9 matrix with rank(B)=3, then find nullity(B^T). (1 mark)

Answer: nullity(B^T)=5−rank(B^T)= 5−rank(B)= 5−3=2

(d) If $A = \begin{bmatrix}2 & 0\\3 & -1\end{bmatrix}$, then find the **<u>eigenvalues</u>** of A^{444}. (1 mark)

Answer: Since A Is lower triangular, so its eigenvalues are 2 and −1. Therefore, the eigenvalues of A^{444} are 2^{444} and $(-1)^{444}=1$.

(e) Let F: $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined as follows: $F((x,y),(z,w))=x^2+y^2$. Show that F is **<u>not</u>** an inner product function on $\mathbb{R}^2$. (1 mark)

Answer: Since $F((1,0),(1,1))=1^2+0^2=1\neq 2=1^2+1^2=F((1,1),(1,0))$, so F is not an inner product function on $\mathbb{R}^2$.