First Semester	**midterm Exam**	**King Saud University**
(without calculators)	**Time: 8 to 10 am**	**College of Science**
Wednesday 2-3-1444	**240 Math**	**Math. Department**

Q1: (a) Find the inverse of $F=\begin{bmatrix}3&2&0\\2&1&-1\\1&1&0\end{bmatrix}$ in two different ways. Then find the inverse of $(2\sqrt{2}F)$. (8 marks)

Answer: The first way:

$$[F\,|\,I]=\left[\begin{array}{ccc|ccc}3&2&0&1&0&0\\2&1&-1&0&1&0\\1&1&0&0&0&1\end{array}\right]\xrightarrow{R_{13}}\left[\begin{array}{ccc|ccc}1&1&0&0&0&1\\2&1&-1&0&1&0\\3&2&0&1&0&0\end{array}\right]$$

$$\xrightarrow[-3R_{13}]{-2R_{12}}\left[\begin{array}{ccc|ccc}1&1&0&0&0&1\\0&-1&-1&0&1&-2\\0&-1&0&1&0&-3\end{array}\right]\xrightarrow[-1R_{23}]{1R_{21}}\left[\begin{array}{ccc|ccc}1&0&-1&0&1&-1\\0&-1&-1&0&1&-2\\0&0&1&1&-1&-1\end{array}\right]$$

$$\xrightarrow[1R_{32}]{1R_{31}}\left[\begin{array}{ccc|ccc}1&0&0&1&0&-2\\0&-1&0&1&0&-3\\0&0&1&1&-1&-1\end{array}\right]\xrightarrow{-1R_{2}}\left[\begin{array}{ccc|ccc}1&0&0&1&0&-2\\0&1&0&-1&0&3\\0&0&1&1&-1&-1\end{array}\right]=[I\,|\,F^{-1}]$$

The second way:

$$\det(F)=\det\begin{bmatrix}3&2&0\\2&1&-1\\1&1&0\end{bmatrix}=-(-1)\begin{vmatrix}3&2\\1&1\end{vmatrix}=3-2=1$$

$$adj\,(F)=\begin{bmatrix}1&-1&1\\0&0&-1\\-2&3&-1\end{bmatrix}^T=\begin{bmatrix}1&0&-2\\-1&0&3\\1&-1&-1\end{bmatrix}$$

$$F^{-1}=\frac{1}{|F|}adj\,(F)=adj\,(F)=\begin{bmatrix}1&0&-2\\-1&0&3\\1&-1&-1\end{bmatrix}$$

and

$$(2\sqrt{2}F)^{-1}=\frac{1}{2\sqrt{2}}F^{-1}=\frac{1}{2\sqrt{2}}\begin{bmatrix}1&0&-2\\-1&0&3\\1&-1&-1\end{bmatrix}$$

Q2: Solve the following linear system By Crammer's Rule: (5 marks)

$$2x_1+4x_2+2x_3=4$$
$$x_1+3x_2+3x_3=2$$
$$x_1+3x_2+5x_3=4$$

Answer:

$$\det(A)=\begin{vmatrix}2&4&2\\1&3&3\\1&3&5\end{vmatrix}\overset{-2R_{21}}{\underset{-1R_{23}}{=}}\begin{vmatrix}0&-2&-4\\1&3&3\\0&0&2\end{vmatrix}=2\begin{vmatrix}0&-2\\1&3\end{vmatrix}=2(2)=4$$

$$\det(A_1)=\begin{vmatrix}4&4&2\\2&3&3\\4&3&5\end{vmatrix}\overset{-2R_{21}}{\underset{-2R_{23}}{=}}\begin{vmatrix}0&-2&-4\\2&3&3\\0&-3&-1\end{vmatrix}=(-2)\begin{vmatrix}-2&-4\\-3&-1\end{vmatrix}=-2(-10)=20$$

$$\det(A_2)=\begin{vmatrix}2&4&2\\1&2&3\\1&4&5\end{vmatrix}\overset{-2R_{21}}{\underset{-1R_{23}}{=}}\begin{vmatrix}0&0&-4\\1&2&3\\0&2&2\end{vmatrix}=(-4)\begin{vmatrix}1&2\\0&2\end{vmatrix}=-4(2)=-8$$

$$\det(A_3)=\begin{vmatrix}2&4&4\\1&3&2\\1&3&4\end{vmatrix}\overset{-2R_{21}}{\underset{-1R_{23}}{=}}\begin{vmatrix}0&-2&0\\1&3&3\\0&0&2\end{vmatrix}=(2)\begin{vmatrix}0&-2\\1&3\end{vmatrix}=2(2)=4$$

So

$$x_1=\frac{|A_1|}{|A|}=\frac{20}{4}=5$$

$$x_2=\frac{|A_2|}{|A|}=\frac{-8}{4}=-2$$

$$x_3=\frac{|A_3|}{|A|}=\frac{4}{4}=1$$

Q3: Let V={0} and define addition and scalar multiplication as follows:

0+0=0 and k0=0 for all scalars k. Show that V is a vector space. (5 marks)

Answer: Since 0+0=0∈V and k0=0∈V for all scalars k, V is closed under addition and scalar multiplication. Now, for all u,v,w∈V and k,m∈$\mathbb{R}$, we have:

1- u+v=0+0=v+u
2- (u+v)+w=(0+0)+0=0+0=0+(0+0)=u+(v+w)
3- u+0=0+0=0+u=0=u
4- -u=0 as u+(-u)=0+0=0
5- k(u+v)=k(0+0)=k0=0=0+0=k0+k0=ku+kv
6- (k+m)u=(k+m)0=0=0+0=k0+m0=ku+mu
7- K(mu)=k(m0)=k0=0=(km)0=(km)u
8- 1u=1(0)=0.

Q4: In the vector space $\mathbb{R}^3$, show that the vector (2,3,4) is a linear combination of the vectors (1,2,3) and (1,0,-1). (3 marks)

Answer: Assume (2,3,4)=a(1,2,3)+b(1,0,-1), where a and b are constants. So

$$(2,3,4)=(a,2a,3a)+(b,0,-b)=(a+b,2a,3a-b)$$
$$\Rightarrow a+b=2$$
$$2a=3$$
$$3a-b=4$$
$$\Rightarrow a=\frac{3}{2}, b=\frac{1}{2}$$

Therefore, (2,3,4) is a linear combination of the vectors (1,2,3) and (1,0,-1).

Q5: Use the Wronskian to show that the vectors: 1, x and sin(x) are linearly independent in the vector space $C^{\infty}(-\infty,\infty)$. (2 marks)

Answer: As

$$W(x)=\begin{vmatrix} 1 & x & \sin(x) \\ 0 & 1 & \cos(x) \\ 0 & 0 & -\sin(x) \end{vmatrix} = -\sin(x)$$
$$W(\tfrac{\pi}{2}) = -\sin(\tfrac{\pi}{2}) = -1 \neq 0$$

So the vectors 1, x and sin(x) are linearly independent.

Q6: (a) Prove that if a square matrix A has a row of zeros, then |A|=0. (1 mark)

Answer: Suppose A is of order n and the row of zeros is the row number i. Computing the determinant using the cofactor expansion, we get that:

$$\det(A)=\sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} 0C_{ij} = 0$$

(b) Show that if A is a symmetric matrix, then A^2 is symmetric. (1 mark)

Answer: $(A^2)^T=(AA)^T=A^TA^T=AA= A^2$,

or

Since A is symmetric and commutes with itself, then $AA= A^2$ is symmetric.

(c) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(1 mark)

Answer: $(A^{-1})^T=(A^T)^{-1}=A^{-1}$.

(d) If A is an invertible matrix of size 3×3 and |A|=2, then find $|2((A^T)^2)^{-1}|$.

(2 marks)

Answer:

$$\left|2\left(\left(A^T\right)^2\right)^{-1}\right| = 2^3\left|\left(\left(A^T\right)^2\right)^{-1}\right| = 8\times\frac{1}{\left|\left(A^T\right)^2\right|}$$

$$= \frac{8}{\left|A^T\right|^2} = \frac{8}{\left|A\right|^2} = \frac{8}{2^2} = \frac{8}{4} = 2$$

(e) Prove that the solution set W of a homogeneous linear system A**x** = **0** of m equations in n unknowns is a subspace of $\mathbb{R}^n$. (2 marks)

Answer: 1- Since A0=0, so 0∈W≠ϕ.

2- If x,y∈W, then they are solutions of the system. So:

A(x+y)=Ax+Ay=0+0=0 and x+y∈W.

3- If x∈W, k∈$\mathbb{R}$, then A(kx)=k(Ax)=k0=0 and kx∈W.

1, 2 and 3 implies that W a subspace of $\mathbb{R}^n$.