First Semester	**Second Exam**	**King Saud University**
(without calculators)	**Time allowed: 1 h and 30 m**	**College of Science**
Monday 15-4-1445	**240 Math**	**Math. Department**

Q1: **(a)** Show that the vector w=(1,2,3)$\in$span{(1,2,2), (2,4,8)}. (3 marks)
(b) Let V=M_{nn} and W is the set of all symmetric matrices of degree n. Prove that W is a subspace of V. (3 marks)

Q2: **(a)** Use the Wronskian to show that 1, x, x^3 are linearly independent in the vector space $C^2(-\infty,\infty)$. (2 marks)

(b) show that the set S={(1,1,2), (2,1,1), (1,1,0)} forms a basis for $\mathbb{R}^3$ and then find the vector w whereas $(w)_S$=(1,2,3). (4 marks)

Q3: **(a)** Let B={(1,2),(2,5)} and B'={(1,1),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (3 marks).
(b) Find a basis for the column space of the matrix:

$$A=\begin{bmatrix} 1 & 2 & 6 & -1 \\ 2 & 4 & 4 & 6 \\ 3 & 6 & 10 & 5 \end{bmatrix}$$

and **deduce** nullity(A^T) without solving any linear system. (4 marks)

Q4: **(a)** Let S={$\mathbf{v_1}$, $\mathbf{v_2}$,…,$\mathbf{v_n}$} be a basis for a vector space **V**. Suppose **u** is a vector in **V** such that

$$\mathbf{u}=|A_1|\mathbf{v_1}+2|A_2|\mathbf{v_2}+3|A_3|\mathbf{v_3}+\ldots+n|A_n|\mathbf{v_n}$$

where, A_i is a matrix of order 2 for all i$\in${1,2,…,n}. Find $(\mathbf{u})_S$ (1 mark)

(b) If S={$\mathbf{v_1}$, $\mathbf{v_2}$,…,$\mathbf{v_n}$} is a basis for a vector space **V**, then prove that every vector **v** in **V** can be expressed in the form $\mathbf{v}=c_1\mathbf{v_1}+c_2\mathbf{v_2}+\ldots+c_n\mathbf{v_n}$ in exactly one way, where c_1, c_2, …, c_n are real numbers. (2 marks)

(c) Show that rank(A)=rank(A^T) for any matrix A. (1 mark)

(d) If u and v are linearly independent, then show that u+v and u-v are linearly independent. (2 marks)

Solutions:

A1(a):

$$\begin{bmatrix}1&2&1\\2&4&2\\2&8&3\end{bmatrix}\xrightarrow[(-2)R_{13}]{(-2)R_{12}}\begin{bmatrix}1&2&1\\0&0&0\\0&4&1\end{bmatrix}\xrightarrow{R_{23}}\begin{bmatrix}1&2&1\\0&4&1\\0&0&0\end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_2}\begin{bmatrix}1&2&1\\0&1&\frac{1}{4}\\0&0&0\end{bmatrix}\xrightarrow{(-2)R_{21}}\begin{bmatrix}1&0&\frac{1}{2}\\0&1&\frac{1}{4}\\0&0&0\end{bmatrix}$$

$$\Rightarrow (1,2,3)=\tfrac{1}{2}(1,2,2)+\tfrac{1}{4}(2,4,8)$$

A1(b): For all A,B∈W and k∈$\mathbb{R}$:

1- W is not empty since $0^T=0$. Hence 0∈W
2- $(A+B)^T=A^T+B^T=A+B$. So A+B∈W.
3- $(kA)^T=kA^T=kA$. So kA∈W
   1, 2 and 3 implies that W is a subspace of $V=M_{nn}$.

A2(a):

$$W(x)=\begin{vmatrix}1&x&x^3\\0&1&3x^2\\0&0&6x\end{vmatrix}=6x$$

$$W(1)=6\neq 0$$

So 1, x, x^3 are linearly independent.

A2(b):

$$\begin{vmatrix}1&2&1\\1&1&1\\2&1&0\end{vmatrix}\underset{(-2)R13}{\overset{(-1)R_{12}}{=}}\begin{vmatrix}1&2&1\\0&-1&0\\0&-3&-2\end{vmatrix}=1(-1)(-2)=2\neq 0$$

So the vectors (1,1,2), (2,1,1), (1,1,0) form a basis for $\mathbb{R}^3$. Now,

$$w=(1,1,2)+2(2,1,1)+3(1,1,0)=(8,6,4)$$

A3(a):

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 0 \end{array}\right] \xrightarrow{(-2)R_{12}} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\xrightarrow{(-2)R_{21}} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 10 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$= [I \mid P_{B'\to B}]$$

$$P_{B'\to B} = \begin{bmatrix} 3 & 10 \\ -1 & -4 \end{bmatrix}$$

A3(b):

$$A = \begin{bmatrix} 1 & 2 & 6 & -1 \\ 2 & 4 & 4 & 6 \\ 3 & 6 & 10 & 5 \end{bmatrix} \xrightarrow[(-3)R_{13}]{(-2)R_{12}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & -8 & 8 \end{bmatrix}$$

$$\xrightarrow{(-1)R_{23}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-1/8)R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the leading ones, $\{[1\ \ 2\ \ 3]^T, [6\ \ 4\ \ 10]^T\}$ is a basis of col(A).

Now, rank(A)+nullity(A^T)=m

So nullity(A^T)=m- rank(A)=3-2=1

A4(a): $(u)_S=(|A_1|,2|A_2|,3|A_3|,\ldots,n|A_n|)$

A4(b): Suppose $v \in V$ has two expressions:

$v = c_1v_1 + c_2v_2 + \cdot\cdot\cdot + c_nv_n$ and $v = k_1v_1 + k_2v_2 + \cdot\cdot\cdot + k_nv_n$, so

$0= (c_1-k_1)v_1 + (c_2-k_2)v_2 + \cdot\cdot\cdot + (c_n-k_n)v_n$

But $S = \{v_1, v_2, \ldots, v_n\}$ is a basis, so it is linearly independent. Thus,

c_1-k_1= c_2-k_2=…= c_n-k_n=0 and hence c_i=k_i for all i∈{1,2,…,n} and hence v has exactly one expression.

A4(c): rank(A)=dim(row(A))=dim(col(A^T))=rank(A^T).

A4(d): Observe that:

$$
\begin{aligned}
& a(u+v)+b(u-v)=0 \\
& \Rightarrow (a+b)u+(a-b)v=0 \\
& L.I. \Rightarrow a+b=0 \;\&\; a-b=0 \\
& \Rightarrow 2a=0 \Rightarrow a=0 \Rightarrow b=0
\end{aligned}
$$

So, u+v and u-v are linearly independent.