[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 452 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Question 1: [Marks: 3 + 4 + 3]

(a). Let $A=\begin{bmatrix}1 & 2 & 2\\ 2 & -2 & 1\\ 2 & 1 & -2\end{bmatrix}$. Compute A^2 and then use A^2 to find A^{-1}.

Solution: $A^2=\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}=9\mathbf{I}$ and so $A^{-1}=\frac{1}{9}A=\frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & -2 & 1\\ 2 & 1 & -2\end{bmatrix}$. (Marks 1 + 2)

(b). Let $A=\begin{bmatrix}1 & 1 & -1 & -2\\ 1 & 1 & 0 & -2\\ -1 & 0 & 1 & 1\\ -2 & 0 & 0 & 1\end{bmatrix}$. Find A^{-1} and then use A^{-1} to find $adj(A)$.

Solution: $[A|I]=\left[\begin{array}{cccc|cccc}1 & 1 & -1 & -2 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & -2 & 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]\sim\ldots\sim\left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 1 & -1 & 1 & -1\\ 0 & 1 & 0 & 0 & 3 & -2 & 3 & -1\\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1\end{array}\right]$ and so $A^{-1}=\begin{bmatrix}1 & -1 & 1 & -1\\ 3 & -2 & 3 & -1\\ -1 & 1 & 0 & 0\\ 2 & -2 & 2 & -1\end{bmatrix}$. ((Marks 1.5)

Next, $det(A)=1$. Hence, $adj(A)=det(A)A^{-1}=\begin{bmatrix}1 & -1 & 1 & -1\\ 3 & -2 & 3 & -1\\ -1 & 1 & 0 & 0\\ 2 & -2 & 2 & -1\end{bmatrix}$. (Marks 1 + 1.5)

(c). Let A be a 3×3 matrix with $det(A)=2$. Evaluate $det(adj(A))$.

Solution: $det(adj(A))=det(det(A)A^{-1})=(det(A))^3(\det(A))^{-1}=4$. (Marks 1 + 1.5 + 0.5)

Question 2: [Marks: 4 + 4]

(a). Let $A=\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}$. Find all matrices $M=\begin{bmatrix}x & y\\ z & t\end{bmatrix}$ such that

Solution: $AM=MA$ implies $\begin{cases} y-z=0\\ x-3z-t=0\\ x-3y-t=0.\end{cases}$ This system of linear equations has solution set (Mark 1.5)

$\{(3z+t,z,z,t)|\ t,z\in\mathbb{R}\}$. Hence, $M=\begin{bmatrix}3z+t & z\\ z & t\end{bmatrix}$ for all $t,z\in\mathbb{R}$. (Marks 2 + 0.5)

(b). Find the values of a, b and c so that (1,−2,3) is the solution of following system of linear equations:

$$\begin{aligned}ax+2by+cz&=6\\ ax+6by+cz&=-2\\ 3ax+4by+cz&=-8.\end{aligned}$$

Solution: Since (1,−2,3) is the solution of the above given system, we get the following system of linear equations:

$$\begin{aligned}a-4b+3c&=6\\ a-12b+3c&=-2\\ 3a-8b+3c&=-8.\end{aligned}$$ (Marks 2)

Which has the unique solution $a=-5$, $b=1$ and $c=5$. (Marks 2)

Question 3: [Marks: 3 + 4]

(a). Use the Gauss-Jordan elimination method to solve the linear system $AX=B$, where:

$$A=\begin{bmatrix}2 & -1 & -4 & 3\\ 3 & -2 & -5 & 4\\ 3 & -3 & -2 & 0\end{bmatrix},\ X=\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix}\ \text{and}\ B=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}.$$

Solution: $[A|I]=\left[\begin{array}{cccc|c}2 & -1 & -4 & 3 & 1\\ 3 & -2 & -5 & 4 & 1\\ 3 & -3 & -2 & 0 & 1\end{array}\right]\sim\ldots\sim\left[\begin{array}{cccc|c}1 & 0 & 0 & -7 & 4\\ 0 & 1 & 0 & -5 & 3\\ 0 & 0 & 1 & -3 & 1\end{array}\right]$ (RREF). (Marks 1.5)

Hence, solution set of the given linear system is {(4+7t, 3+5t, 1+3t, t) | t ∈ ℝ}. (Marks 1.5)

(b). Find all the non-trivial solutions of the following homogeneous system:

$$\begin{aligned}2x+2y+4z&=0\\ w-y-3z&=0\\ 2w+3x+y+z&=0\\ -2w+x+3y-2z&=0.\end{aligned}$$

Solution: $[A|I]=\left[\begin{array}{cccc|c}0 & 2 & 2 & 4 & 0\\ 1 & 0 & -1 & -3 & 0\\ 2 & 3 & 1 & 1 & 0\\ -2 & 1 & 3 & -2 & 0\end{array}\right]\sim\ldots\sim\left[\begin{array}{cccc|c}1 & 0 & -1 & -3 & 0\\ 0 & 1 & 1 & -8 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$. (Marks 1.5)

Hence, the set of non-trivial solutions of the given linear system is {(t, - t, t, 0) | 0 ≠ t ∈ ℝ. (Marks 2 + 0.5)

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