[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 461 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hrs.**

Question 1: [Marks: 5]

Determine whether the following statements are true or false:

(i) If A is a square matrix and $A^2 = 0$, then $(I + A)^{-1} = I - A$. [True] [Mark 1]

(ii) If A and B are row equivalent square matrices, then $|A| = |B|$. [False] [Mark 1]

(iii) If $Aadj(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then $|A| = 9$. [False] [Mark 1]

(iv) There is a homogeneous linear equation for which (1, 0, 2) is a solution but not (2, 0, −4). [True] [Mark 1]

(v) If $RREF(A)$ has a zero row, then $AX = B$ must have infinitely many solutions. [False] [Mark 1]

Question 2: [Marks: 4 + 3 + 3]

(a) Find inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and find the matrix B satisfying the equation $BA = A^2 + 5A$.

Solution: $[A|I] = \begin{bmatrix} 2 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 \\ 0 & 0 & 3 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & \frac{1}{3} \end{bmatrix}$, and so $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$. [Marks 2]

Next, $B = (BA)A^{-1} = (A^2 + 5A)A^{-1} = A + 5I = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. [Marks 2]

(b) Let A be an invertible matrix. Show that $(adj(A))^{-1} = adj(A^{-1})$.

Solution: $adj(A) = |A|\, A^{-1} \Longrightarrow (adj(A^{-1}))^{-1} = (|A|^{-1} A)^{-1} = |A|\, A^{-1} = adj(A) \Longrightarrow (adj(A))^{-1} = adj(A^{-1})$.

[Marks 1+1+1]

(c) Show that the matrix $\begin{bmatrix} 1 & 0 & 1+x \\ 1 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix}$ is invertible for all $x\varepsilon\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

Solution: Since $\begin{vmatrix} 1 & 0 & 1+x \\ 1 & 1 & 0 \\ -x & 0 & 1 \end{vmatrix} = \mathbf{1 + x(x+1) \neq 0}$ **for all** $x\varepsilon\mathbb{R}$, $\begin{bmatrix} 1 & 0 & 1+x \\ 1 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix}$ is invertible for all $x\varepsilon\mathbb{R}$.

[Marks 2+1]

Question 3: [Marks: 2 + 4 + 4]

(a) If E is an elementary matrix, then show that the linear system $EX = O$ has only the trivial solution.

Solution: Since E being an elementary matrix is invertible, $X = E^{-1}(EX) = E^{-1}O = O$; only trivial solution.

[Marks 1+1]

(b) Solve the following system of linear equations:

$$x + y = 1$$
$$x + 2y + z = -1$$
$$x + 3y - z = 2.$$

Solution: Observe that $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = \mathbf{-3} \neq$, $|A_x| = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{-4}$, $|A_y| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \mathbf{1}$,

and $|A_z| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix} = \mathbf{5}$. **So, by Cramer's Rule,** $x = \frac{|A_x|}{|A|} = \frac{4}{3}$, $y = \frac{|A_y|}{|A|} = \frac{-1}{3}$, $z = \frac{|A_z|}{|A|} = \frac{-5}{3}$.

[Marks 1+1.5+1.5]

(c) What conditions must a, b, c, and d satisfy for the following system to be consistent?

$$x_1 + x_2 - x_4 = a$$
$$x_2 - x_3 - 2x_4 = b$$
$$2x_1 + 2x_3 + 2x_4 = c$$
$$2x_1 + x_2 + x_3 = d.$$

Solution: Since $[A|I] = \begin{bmatrix} 1 & 1 & 0 & -1 & : & a \\ 0 & 1 & -1 & -2 & : & b \\ 2 & 0 & 2 & 2 & : & c \\ 2 & 1 & 1 & 0 & : & d \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 & : & a \\ 0 & 1 & -1 & -2 & : & b \\ 0 & 0 & 0 & 0 & : & c-2a+2b \\ 0 & 0 & 0 & 0 & : & d-2a+b \end{bmatrix}$, the given system would be consistent for real numbers a, b, c, d satisfying the conditions $c = 2(a-b)$; $d = c + b$. [Marks 2+2]

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