

[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS
Semester 461 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25**Max. Time: $1\frac{1}{2}$ hrs.****Question 1: [Marks: 5]**

Determine whether the following statements are true or false and justify your answer:

- (i) If
- A
- and
- B
- are symmetric matrices compatible for the product
- AB
- , then
- AB
- is also symmetric.

False: for example, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ are symmetric but $AB = \begin{bmatrix} 5 & 5 \\ -1 & 1 \end{bmatrix}$ is not symmetric.

- (ii) If the matrix
- A^2
- is invertible, then
- A
- itself is invertible.

True: A is not invertible $\Rightarrow |A| = 0 \Rightarrow |A^2| = |A|^2 = 0 \Rightarrow A^2$ is not invertible.

- (iii) If the matrix
- $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & x \end{bmatrix}$
- is its own adjoint, then
- $x = 3$
- .

True: $\text{adj} \left(\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & x \end{bmatrix} \right) = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & x \end{bmatrix} \Rightarrow x = c_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = 3$.

- (iv) If square matrices
- A
- and
- B
- are compatible for the product
- AB
- , then
- $|AB| = |BA|$
- .

True: $|AB| = |A||B| = |B||A| = |BA|$.

- (v) If
- $A = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$
- , then
- $(-\frac{1}{11}, \frac{2}{11})$
- is a solution of the equation
- $A^{-1} = xA + yI_2$
- .

True: $A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{11} \begin{bmatrix} 1 & -2 \\ 5 & 1 \end{bmatrix} = -\frac{1}{11} \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix} + \frac{2}{11} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = xA + yI_2$.**Question 2: [Marks: 4 + 3 + 3]**

- (a) Find the matrix
- A
- if
- $A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{bmatrix}$
- .

Solution: $A = (A^{-1})^{-1} = 2 \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 7 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & -2 \end{bmatrix}$.

- (b) Show that
- $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ -(b+c-a) & -(c+a-b) & -(a+b-c) \end{vmatrix} = 0$
- .

Solution: $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ -(b+c-a) & -(c+a-b) & -(a+b-c) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ a & b & c \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a+b+c & b+c+a & c+a+b \\ a & b & c \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = 0$.

- (c) Let
- A
- be a square matrix of size
- n
- with
- $|A| = 3$
- and
- $|\text{adj}(A)| = 27$
- . Find
- n
- .

Solution: $\text{adj}(A) = |A|A^{-1} \Rightarrow |\text{adj}(A)| = |A|^{n-1}$. Hence, $3^3 = 27 = 3^{n-1}$ gives $n = 3 + 1 = 4$.**Question 3: [Marks: 3 + 3 + 4]**

- (a) Solve the matrix equation:
- $XA = B$
- , where
- $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ -4 & 0 & 3 \end{bmatrix}$
- and
- $B = \begin{bmatrix} -2 & 1 & 1 \\ 8 & 1 & -5 \\ 4 & 3 & -3 \end{bmatrix}$
- .

Solution: Since $|A| = -3$, A^{-1} exists. $XA = B \Rightarrow X = BA^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 8 & 1 & -5 \\ 4 & 3 & -3 \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 3 & 6 & 3 \\ 2 & 1 & 1 \\ 4 & 8 & 5 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

- (b) Solve the following system of linear equations by using Cramer's Rule:

$$\begin{aligned} x + 3y + 3z &= 1 \\ x + 3y + 5z &= -1 \\ x + 2y - z &= 2. \end{aligned}$$

Solution: $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & -1 \end{vmatrix} = 2$, $|A_x| = \begin{vmatrix} 1 & 3 & 3 \\ -1 & 3 & 5 \\ 2 & 2 & -1 \end{vmatrix} = -10$, $|A_y| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 2 & -1 \end{vmatrix} = 6$ and $|A_z| = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & -1 \\ 1 & 2 & 2 \end{vmatrix} = -2$.Hence, $x = \frac{|A_x|}{|A|} = -\frac{10}{2} = -5$, $y = \frac{|A_y|}{|A|} = \frac{6}{2} = 3$ and $z = \frac{|A_z|}{|A|} = -\frac{2}{2} = -1$.

- (c) Find the value of
- m
- for which the following system of linear equations admits a unique solution and then find this uniquely existing solution.

$$\begin{aligned} x + y + z &= 1 \\ x + y + 2z &= 0 \\ 2x - y - z &= -1 \\ x - 2y + z &= m. \end{aligned}$$

Solution: $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 1 & 2 & : & 0 \\ 2 & -1 & -1 & : & -1 \\ 1 & -2 & 1 & : & m \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 1 & : & 1 \\ 0 & 0 & 1 & : & -1 \\ 0 & 0 & 0 & : & m+5 \end{bmatrix}$. So, $m = -5$ gives a unique solution $x = 0, y = 2, z = -1$.***** /**