[**Solution Key**]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS
Semester 461 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 **Max. Time:** $1\frac{1}{2}$ **hrs.**

Question 1: [Marks: 5]

Determine whether the following statements are true or false and justify your answer:

(i) If A and B are symmetric matrices compatible for the product AB, then AB is also symmetric.

False: for example, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ are symmetric but $AB = \begin{bmatrix} 5 & 5 \\ -1 & 1 \end{bmatrix}$ is not symmetric.

(ii) If the matrix A^2 is invertible, then A itself is invertible.

True: A is not invertible $\Rightarrow |A| = 0 \Rightarrow |A^2| = |A|^2 = 0 \Rightarrow A^2$ is not invertible.

(iii) If the matrix $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & x \end{bmatrix}$ is its own adjoint, then $x = 3$.

True: $adj\left(\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & x \end{bmatrix}\right) = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & x \end{bmatrix} \Rightarrow x = c_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = 3.$

(iv) If square matrices A and B are compatible for the product AB, then $|AB| = |BA|$.

True: $|AB| = |A||B| = |B||A| = |BA|.$

(v) If $A = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$, then $(-\frac{1}{11}, \frac{2}{11})$ is a solution of the equation $A^{-1} = xA + yI_2$.

True: $A^{-1} = \frac{1}{|A|}adj((A) = \frac{1}{11}\begin{bmatrix} 1 & -2 \\ 5 & 1 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix} + \frac{2}{11}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = xA + yI_2.$

Question 2: [Marks: 4 + 3 + 3]

(a) Find the matrix A if $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

Solution: $\boldsymbol{A} = (A^{-1})^{-1} = 2\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 7 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & -2 \end{bmatrix}. = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$

(b) Show that $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ -(b+c-a) & -(c+a-b) & -(a+b-c) \end{vmatrix} = 0.$

Solution: $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ -(b+c-a) & -(c+a-b) & -(a+b-c) \end{vmatrix} \overset{A'_{23}}{=} \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ a & b & c \end{vmatrix}$

$\overset{A'_{32}}{=} \begin{vmatrix} 1 & 1 & 1 \\ a+b+c & b+c+a & c+a+b \\ a & b & c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = 0.$

(c) Let A be a square matrix of size n with $|A| = 3$ and $|adj(A)| = 27$. Find n.

Solution: $adj(A) = |A|A^{-1} \Rightarrow |adj(A)| = |A|^{n-1}$. Hence, $3^3 = 27 = 3^{n-1}$ gives $n = 3 + 1 = 4$.

Question 3: [Marks: 3 + 3 + 4]

(a) Solve the matrix equation: $XA = B$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ -4 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 1 \\ 8 & 1 & -5 \\ 4 & 3 & -3 \end{bmatrix}$.

Solution: Since $|A| = -3$, A^{-1} exists. $XA = B \Rightarrow X = BA^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 8 & 1 & -5 \\ 4 & 3 & -3 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 3 & 6 & 3 \\ 2 & 1 & 1 \\ 4 & 8 & 5 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$

(b) Solve the following system of linear equations by using Cramer's Rule:

$$\begin{aligned} x + 3y + 3z &= 1 \\ x + 3y + 5z &= -1 \\ x + 2y - z &= 2. \end{aligned}$$

Solution: $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & -1 \end{vmatrix} = 2$, $|A_x| = \begin{vmatrix} 1 & 3 & 3 \\ -1 & 3 & 5 \\ 2 & 2 & -1 \end{vmatrix} = -10$, $|A_y| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 2 & -1 \end{vmatrix} = 6$ and $|A_z| = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & -1 \\ 1 & 2 & 2 \end{vmatrix} = -2.$

Hence. $x = \frac{|A_x|}{|A|} = -\frac{10}{2} = -5$, $y = \frac{|A_y|}{|A|} = \frac{6}{2} = 3$ and $z = \frac{|A_z|}{|A|} = -\frac{2}{2} = -1.$

(c) Find the value of m for which the following system of linear equations admits a unique solution and then find this uniquely existing solution.

$$\begin{aligned} x + y + z &= 1 \\ x + y + 2z &= 0 \\ 2x - y - z &= -1 \\ x - 2y + z &= m. \end{aligned}$$

Solution: $[A{:}B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 1 & 2 & : & 0 \\ 2 & -1 & -1 & : & -1 \\ 1 & -2 & 1 & : & m \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 1 & : & 1 \\ 0 & 0 & 1 & : & -1 \\ 0 & 0 & 0 & : & m+5 \end{bmatrix}$. So, $m = -5$ gives a unique solution $x = 0, y = 2, z = -1$.

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$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[A_{13}^{-1}]{A_{12}^{1}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 2 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{M_2^{\frac{1}{2}}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & -3 & -1 & 0 & 1 \end{array}\right] \xrightarrow[A_{21}^{1}]{A_{23}^{-2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -5 & -2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{M^{-\frac{1}{5}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \end{array}\right] \xrightarrow[A_{32}^{-1}]{A_{31}^{-2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{10} & \frac{1}{10} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \end{array}\right]$$

$$\Rightarrow A^{-1} = 2\begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{2}{5} \\ \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & \frac{-2}{5} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ -4 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[A_{13}^{4}]{A_{12}^{-2}} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -2 & 1 & 0 \\ 0 & 8 & -1 & 4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{M_2^{-\frac{1}{5}}} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 8 & -1 & 4 & 0 & 1 \end{array}\right] \xrightarrow{A_{23}^{-8}} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} & \frac{8}{5} & 1 \end{array}\right] \xrightarrow{M_3^{\frac{5}{3}}} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{8}{3} & \frac{5}{3} \end{array}\right]$$

$$\xrightarrow[A_{31}^{1}]{A_{32}^{\frac{1}{5}}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{7}{3} & \frac{8}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{10}{15} & \frac{5}{15} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{4}{3} & \frac{8}{3} & \frac{5}{3} \end{array}\right] \xrightarrow{A_{21}^{-2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{3} & \frac{6}{3} & \frac{3}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{4}{3} & \frac{8}{3} & \frac{5}{3} \end{array}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 6 & 3 \\ 2 & 1 & 1 \\ 4 & 8 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & -1 & -1 \\ 1 & -2 & 1 & m \end{array}\right] \xrightarrow[A_{14}^{-1}]{A_{12}^{-1},\ A_{13}^{-2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -3 & -3 & -3 \\ 0 & -3 & 0 & m-1 \end{array}\right] \xrightarrow[I_{21}]{M_3^{-\frac{1}{3}}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -3 & 0 & m-1 \end{array}\right]$$

$$\xrightarrow{A_{24}^{3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & m+2 \end{array}\right] \xrightarrow{A_{34}^{-3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & m+5 \end{array}\right]$$