[Solution Key]

KING SAUD UNIVERSITY

COLLEGE OF SCIENCES

DEPARTMENT OF MATHEMATICS

Semester 471 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Question 2: [Marks: 3 + 2]

I. Select the correct choice:

(i) If W = {(1,1),(1,2)}, then W is:

(a) a vector space (b) a subspace of $\mathbb{R}^2$ (c) ✓ a basis of $\mathbb{R}^2$ (d) linearly dependent. [Mark 1]

(ii) If B = {u, v, w} is an orthogonal set of non-zero vectors in an inner product space E and αu + βv + γw = 0 for some scalars α, β and γ, then:

(a) ✓ α = β = γ = 0 (b) u = v = w (c) α = γ, v = o, u = w (d) B is linearly dependent. [Mark 1]

(iii) If A is an invertible matrix of order 3, then $rank(A)$ is equal to:

(a) 0 (b) 1 (c) 2 (d) ✓ 3. [Mark 1]

II. Give an example for each of the following. You don't have to prove your answers.

(i) A linearly dependent subset of $\mathbb{R}^3$ consisting of three vectors in which every two vectors are linearly independent.

Example: $\{(1,0,0),(0,1,0),(1,1,0)\}$. [Mark 1]

(ii) A matrix having *rank* 3 and *nullity* 4.

Example: $\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\end{bmatrix}$. [Mark 1]

Question 2: [Marks: 4 + 3]

Let $A=\begin{bmatrix}1&1&2&-3&5\\2&3&0&-1&-6\\3&4&2&-4&-1\end{bmatrix}$. Then:

(i) Find bases for $row(A)$ and $col(A)$.

(ii) Express the last three columns of the matrix A as linear combinations of the first two.

Solution: (i) From $\mathrm{RREF}(A)=\begin{bmatrix}1&0&6&-8&21\\0&1&-4&5&-16\\0&0&0&0&0\end{bmatrix}$: [Marks 2]

Basis for $\mathrm{row}(A)=\{(1,0,6,-8,21),(0,1,-4,5,-16)\}$, Basis for $\mathrm{col}(A)=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\3\\4\end{bmatrix}\right\}$. [Marks 1+1]

(ii) $\begin{bmatrix}2\\0\\2\end{bmatrix}=6\begin{bmatrix}1\\2\\3\end{bmatrix}-4\begin{bmatrix}1\\3\\4\end{bmatrix}$, $\begin{bmatrix}-3\\-1\\-4\end{bmatrix}=-8\begin{bmatrix}1\\2\\3\end{bmatrix}+5\begin{bmatrix}1\\3\\4\end{bmatrix}$ and $\begin{bmatrix}5\\-6\\-1\end{bmatrix}=21\begin{bmatrix}1\\2\\3\end{bmatrix}-16\begin{bmatrix}1\\3\\4\end{bmatrix}$. [Marks 1+1+1]

Question 3: [Marks: 3 + 4]

(i) Let $F=\{p=2-x,\ q=5+3x-x^2+2x^3,\ r=1+7x+x^2+4x^3\}$. Find the real numbers α, β, γ so that $-3+5x^2+2x^3=\alpha p+\beta q+\gamma r$. Also, show that the set F is linearly independent in the vector space P_3.

(ii) Let $G=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ be a basis of the vector space $\mathbb{R}^6$. Let A be a 4×6 matrix such that $A(v_1)$ and $A(v_2)$ are linearly independent vectors in $\mathbb{R}^4$ and $A(v_i)=0$, for all $i=3,4,5,6$. Show that the null space $N(A)=span(\{v_3,v_4,v_5,v_6\})$. Also, find $nullity(A^T)$.

Solution: (i) Substituting the values of polynomials p, q, r in $-3+5x^2+2x^3=\alpha p+\beta q+\gamma r$, and then equating the coefficients of the same degree terms, we get that: $-3=2\alpha+5\beta+\gamma$, $0=-\alpha+3\beta+7\gamma$, $5=0\alpha-\beta+\gamma$, $2=0\alpha+2\beta+4\gamma$.

Solving this linear system, we obtain: $\alpha=5$, $\beta=-3$ and $\gamma=2$. [Marks 2]

Also, $ap+bq+cr=0\Rightarrow a=b=c=0$. Hence, the set F is linearly independent in the vector space P_3. [Mark 1]

(ii) $u\in N(A)\Rightarrow 0=A(u)=A\left(\sum_{i=1}^{6}\alpha_i v_i\right)=\sum_{i=1}^{6}\alpha_i A(v_i)=\alpha_1A(v_1)+\alpha_2A(v_2)\Rightarrow\alpha_1=\alpha_2=0$. So, $u=\sum_{i=3}^{6}\alpha_i v_i$.

Hence, $N(A)\subseteq span(\{v_3,v_4,v_5,v_6\})\subseteq N(A)$.Thus, $N(A)=span(\{v_3,v_4,v_5,v_6\})$, and so: [Marks 2]

$nullity(A^T)=4-rank(A^T)=4-rank(A)=4-(6-nullity(A))=4-(6-4)=4-2=2$. [Marks 2]

Question 4: [Marks: 3 + 3]

(i) Consider a vector space $\boldsymbol{E}$ of dimension 3. Let $B=\{u_1,u_2,u_3\}$ and $C=\{v_1,v_2,v_3\}$ be two ordered bases for $\boldsymbol{E}$ such that the transition matrix ${}_C\boldsymbol{P}_B=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$ from B to C. Compute the coordinate vector $[v_1-2v_2+3v_3]_B$.

(ii) Consider the matrices $A=\begin{bmatrix}2&0\\0&-1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\0&2\end{bmatrix}$ in the inner product space $M_2(\mathbb{R})$, consisting of all real matrices of size 2, with inner product $<A,B>=trace\,(AB^T)$. Find the angle between the matrices A and B. Also, verify the Pythagorean theorem for the same matrices.

Solution: (i) ${}_B\boldsymbol{P}_C=({}_C\mathbf{P}_B)^{-1}=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}^{-1}=\begin{bmatrix}-1&1&1\\1&0&-1\\1&-1&0\end{bmatrix}$ and $[v_1-2v_2+3v_3]_C=\begin{bmatrix}1\\-2\\3\end{bmatrix}$. Hence: [Marks 2]

$[v_1-2v_2+3v_3]_B={}_B\boldsymbol{P}_C[v_1-2v_2+3v_3]_C=\begin{bmatrix}-1&1&1\\1&0&-1\\1&-1&0\end{bmatrix}\begin{bmatrix}1\\-2\\3\end{bmatrix}=\begin{bmatrix}0\\-2\\3\end{bmatrix}$. [Mark 1]

(ii) $<A,B>=trace\,(AB^T)=trace\left(\begin{bmatrix}2&0\\0&-2\end{bmatrix}\right)=0$, so that the angle θ between the matrices A and B:

$\theta=\cos^{-1}\frac{<A,B>}{||A||\,||B||}=\cos^{-1}0=\frac{\pi}{2}$. Hence, A and B are orthogonal matrices. [Mark 1]

Next, $||A||=\sqrt{<A,A>}=\sqrt{trace(AA^T)}=\sqrt{trace(\begin{bmatrix}4&0\\0&1\end{bmatrix})}=\sqrt{5}$. Similarly, $||B||=\sqrt{trace(\begin{bmatrix}1&0\\0&4\end{bmatrix})}=\sqrt{5}$ and

$||A+B||=||\begin{bmatrix}3&0\\0&1\end{bmatrix}||=\sqrt{trace(\begin{bmatrix}9&0\\0&1\end{bmatrix})}=\sqrt{10}$. So, $||A+B||^2=10=5+5=||A||^2+||B||^2$; as required. [Marks 2]

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