[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 462 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Solution of Question 1: **Correct choices:**

(i) Let P_2 be the vector space of all real polynomials in one variable of degree ≤ 2 and $S = \{u, v\} \subseteq P_2$. If E denotes the set of all linear combinations of the vectors in S, then the set E is equal to:

(a) $\{\alpha u + v \mid \alpha \in \mathbb{R}\}$ (b) $\{u + \beta v \mid \beta \in \mathbb{R}\}$ (c) P_2 (d) ✓ a vector space. [Mark 1]

(ii) If $W = \{w_1, w_2, w_3, w_4\}$ spans the vector space V, then:

(a) $dim(V) = 3$ (b) $dim(V) = 4$ (c) $dim(V) > 4$ (d) ✓ $dim(V) \leq 4$. [Mark 1]

(iii) Consider the vector space $\mathbb{R}^2$ with ordered basis $B = \{(1,0), (1,2)\}$. If $v \in \mathbb{R}^2$ with the coordinate vector $[v]_B = \begin{bmatrix}1\\2\end{bmatrix}$, then:

(a) $v = (1,0)$ (b) ✓ $v = (3,4)$ (c) $v = (1,2)$ (d) $v = (2,2)$. [Mark 1]

(iv) Which of the following matrices cannot be a transition matrix?

(a) $\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$ (b) $\begin{bmatrix}0&0&3\\0&2&0\\1&0&0\end{bmatrix}$ (c) ✓ $\begin{bmatrix}3&0&0\\0&2&0\\1&0&0\end{bmatrix}$ (d) $\begin{bmatrix}1&0&3\\0&2&2\\1&0&1\end{bmatrix}$ [Mark 1]

(v) If A is an invertible matrix of order 3, then $rank(A)$ is equal to:

(a) 0 (b) 1 (c) 2 (d) ✓ 3. [Mark 1]

Question 2:

Consider the matrix $A = \begin{bmatrix}1&2&3\\2&4&6\\1&1&2\end{bmatrix}$.

(a) Find a basis B_1 for the null space $N(A)$.

Solution: $B_1 = \{(-1,-1,1)\}$ is a basis for $N(A)$. [Marks 3]

(b) Find a basis B_2 for the column space $col(A)$.

Solution: $B_2 = \{(1,2,1), (2,4,1)\}$ is a basis for $col(A)$. [Marks 2]

(c) Find nullity and rank of the matrix A.

Solution: From Part (a), $nullity(A) = 1$. From Part (b), $rank(A) = 2$. [Marks 1 + 1]

(d) Show that $B_1 \cup B_2$ is a basis for the vector space $\mathbb{R}^3$.

Solution: Since $B_1 \cup B_2 = \{(-1,-1,1), (1,2,1), (2,4,1)\}$ is linearly independent and $dim(\mathbb{R}^3) = 3$, $B_1 \cup B_2$ is a basis for $\mathbb{R}^3$. [Marks 1 + 1 + 1]

Question 3:

Consider a vector space E of dimension 3. Let $B = \{u_1, u_2, u_3\}$ and $C = \{v_1, v_2, v_3\}$ be two ordered bases for E such that the transition matrix ${}_CP_B = \begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix}$ from B to C. Then, compute:

(a) Transition matrix ${}_BP_C$ from C to B.

Solution: $\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&0&1&0&1&0\\1&1&0&0&0&1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&0&0&-1&1&1\\0&1&0&1&-1&0\\0&0&1&1&0&-1\end{array}\right] \Rightarrow {}_BP_C = {}_CP^{-1}{}_B = \begin{bmatrix}-1&1&1\\1&-1&0\\1&0&-1\end{bmatrix}$. [Marks 1.5 + 1.5]

(b) Coordinate vectors $[v_1 - v_2]_C$, $[v_1 - v_2]_B$ and $[v]_C$, where $v = v_1 - 2u_2 + v_3$.

Solution: $v_1 - v_2 = v_1 + (-1)v_2 + 0v_3 \Rightarrow [v_1 - v_2]_C = \begin{bmatrix}1\\-1\\0\end{bmatrix}$. [Mark 1]

$[v_1 - v_2]_B = {}_BP_C\,[v_1 - v_2]_C = \begin{bmatrix}-1&1&1\\1&-1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}-2\\2\\1\end{bmatrix}$. [Marks 2 + 1]

${}_CP_B = \begin{bmatrix}1&1&1\\1&0&1\\1&1&0\end{bmatrix}$ gives $u_2 = v_1 + v_3$. Then, $v = v_1 - 2u_2 + v_3 = -v_1 - v_3$. Hence, $[v]_C = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$. [Marks 1.5+.5+ 1]

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