[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 471 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 Max.Time: 1.5 hrs

Question 1: [Marks: 5]

Determine whether the following statements are true or false:

(i) If A and B are diagonal matrices of same size, then $AB = BA$.

True:: $\begin{bmatrix} a_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & a_{n,n} \end{bmatrix}\begin{bmatrix} b_{1,1} & 0 & 0 & \dots & 0 \\ 0 & b_{2,2} & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2}b_{2,2} & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & a_{n,n}b_{n,n} \end{bmatrix} = \begin{bmatrix} b_{1,1} & 0 & 0 & \dots & 0 \\ 0 & b_{2,2} & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & b_{2,2} \end{bmatrix}\begin{bmatrix} a_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & a_{n,n} \end{bmatrix}$.

(ii) If A and B are $n{\times}n$ matrices and A is singular, then AB is also singular.

True: $|AB| = |A||B| = 0|A| = 0$ (since A being singular satisfies $|A| = 0$). and so AB is singular.

(iii) If a matrix A satisfies $A^2 = I_n$, then $A = I_n$ or $A = -I_n$.

False: For example, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(iv) Any square matrix A satisfies $RREF(A) = I$ *iff* A is invertible.

True: $|RREF(A)| = 1$ *iff* $|A| \neq 0$. Hence, $RREF(A) = I$ *iff* A is invertible.

(v) If the linear system $AX = B$ has solutions $u = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, then $w = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ is another solution of the same system.

True: $A\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = A\begin{bmatrix} \frac{2+4}{2} \\ \frac{2+4}{2} \\ \frac{2+4}{2} \end{bmatrix} = \frac{1}{2}\left(A\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + A\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}\right) = \frac{1}{2}(B + B) = B.$

Question 2: [Marks: 5 + 5]

(a) Let the matrix $A = \begin{bmatrix} a & a & a & a \\ b & a & a & a \\ b & b & a & a \\ b & b & b & a \end{bmatrix}$, where $a \neq 0$ and $b \neq a$. Show that $|A| = a(a-b)^3$ and deduce that the matrix A is invertible.

Solution: $|A| = \begin{vmatrix} a & a & a & a \\ b & a & a & a \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = a\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & a & a \\ b & b & a & a \\ b & b & b & a \end{vmatrix} = a\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-b & a-b & a-b \\ 0 & 0 & a-b & a \\ 0 & 0 & 0 & a-b \end{vmatrix} = a(a-b)^3.$

Now, $|A| = a(a-b)^3 \neq 0$ since $a \neq 0$ and $b \neq a$. Thus, A is invertible.

(b) Let A be the matrix with $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Find: $(A^T)^{-1}$, $|2A|$, $adj(A)$, $RREF(A)$. Also find the solution of the homogeneous linear system $AX = O$.

Solution: $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 3 & 3 & 8 \end{bmatrix}$. $|2A| = 2^3|A| = 8\left|(A^{-1})^{-1}\right| = 8\,|A^{-1}|^{-1} = \frac{8}{|A^{-1}|} = \frac{8}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix}} = \frac{8}{-1} = -8.$

$adj(A) = |A|A^{-1} = \frac{1}{|A^{-1}|}A^{-1} = -A^{-1} = -\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Since $|A| = -1 \neq 0$, A is invertible; hence, $RREF(A) = I$.

Finally, the invertibility of the matrix A gives $X = A^{-1}O = O$.

Question 3: [Marks: 5 + 5]

(a) Solve the following system of equations by the Gauss-Jordan elimination method:

$$\begin{aligned} w + x + y + z &= 6 \\ w \quad\;\; + y + z &= 4 \\ w \quad\;\; + y \quad\;\; &= 2. \end{aligned}$$

Solution: $\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{6} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{4} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \end{array}\right] \sim \left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{2} \end{array}\right]$ **(RREF)**

$\Rightarrow$ the variable $\boldsymbol{y}$ is a free and so $\{(\mathbf{2-t,2,t,2}) | \boldsymbol{t} \in \mathbb{R}\}$ **is the solution set of the given system.**

(b) Under what condition on α and β, the following system of equations has no non- trivial solution?

$$\begin{aligned} x \;+ \quad\;\; 2z &= 0 \\ \alpha x + 8y + 3z &= 0 \\ \beta y + 5z &= 0. \end{aligned}$$

Solution: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ \alpha & 8 & 3 & 0 \\ 0 & \beta & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{3-2\alpha}{8} & 0 \\ 0 & 0 & 40-(3-2\alpha)\beta & 0 \end{array}\right]$. Hence, there is no non-trivial solution if $(3-2\alpha)\beta \neq 40$.