[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 461 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time:** $1\frac{1}{2}$ **hrs.**

Note: Scientific calculators are not allowed.

Question 1: [Marks: 1+1+1+1+1]

Which of the given choices are correct?

(i) Let $A = \{u_1, u_2, u_3, u_4\}$ is a subset of $\mathbb{R}^3$. Then the set A must be:

a) a subspace of $\mathbb{R}^3$ b)✓ linearly dependent c) linearly independent d) a basis of $\mathbb{R}^3$.[Mark 1]

(ii) For the matrix $M = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, which of the following statements is true?

a) $nullity(M) = 2$ b) $rank(M) = 3$ c)✓ $nullity(M) = 3$ d) $rank(M) = 0$. [Mark 1]

(iii) Let $B = \{(1,0,0,1), (-1,1,0,1), (0,0,1,1)\}$ and $C = \{v_1, v_2, v_3\}$ be two ordered bases for a vector subspace of the Euclidean space $\mathbb{R}^4$. If ${}_BP_C = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ is the transition matrix from C to B, then the vector v_3 is equal to:

a) (1,1,0) b) (2,0,0) c) (2,0,0,2) d)✓ (0,1,0,2). [Mark 1]

(iv) Let u and v be any two vectors in a real inner product space $(V, <,>)$ such that $\|u\| = 3 = \|v\|$. Which of the following statements is true?

a) $|<u,v>| \le 6$ b) $|<u,v>| > 6$ c) $|<u,v>| > 9$ d)✓ $|<u,v>| \le 9$. [Mark 1]

(v) Let $W = \{w_1, w_2, w_3, w_{4,}, w_5\}$ be an orthogonal set of nonzero vectors in an inner product space E of dimension 5. Then W must be:

a)✓ a basis for E b) a subspace of E c) equal to E d) linearly dependent. [Mark 1]

Question 2: [Marks: 3 + 3 + 4]

(a) Show that $F = \{(x, y, z) \in \mathbb{R}^3; x - y + z = 0, \ 2x + y - z = 0, x + y + z = 0\}$ is a vector subspace of $\mathbb{R}^3$.

Solution: Clearly, $(0,0,0) \in F$. [Mark 0.5]

Next, for all $\alpha, \beta \in \mathbb{R}$, $(x_1, y_1, z_1), (x_2, y_2, z_2) \in F$, we have:

$$(\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2) = \alpha(x_1 - y_1 + z_1) + \beta(x_2 - y_2 + z_2) = 0,$$
$$2(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) - (\alpha z_1 + \beta z_2) = \alpha(2x_1 + y_1 - z_1) + \beta(2x_2 + y_2 - z_2) = 0,$$
$$(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2) = \alpha(x_1 + y_1 + z_1) + \beta(x_2 + y_2 + z_2) = 0.$$ [Mark 1.5]

Hence, F is a vector subspace of $\mathbb{R}^3$. [Mark 1]

(b) Show that $V = \{\begin{bmatrix} 0 & 2x - y \\ x & y \end{bmatrix} : x, y \in \mathbb{R}\}$ is a real vector space under usual addition and scalar multiplication of matrices. Also find $dim(V)$.

Solution: Clearly, $V \subseteq M_2(\mathbb{R})$. [Mark 0.5]

Next,, $\alpha \begin{bmatrix} 0 & 2x_1 - y_1 \\ x_1 & y_1 \end{bmatrix} + \beta \begin{bmatrix} 0 & 2x_2 - y_2 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 0 & 2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) \\ \alpha x_1 + \beta y_1 & \alpha x_2 + \beta y_2 \end{bmatrix}$ for all $\alpha, \beta, x_1, y_1 \in \mathbb{R}$.

Thus, V is a vector subspace of $M_2(\mathbb{R})$; so, it is a vector space. [Marks 0.5 + 0.5]

Next, $\begin{bmatrix} 0 & 2x - y \\ x & y \end{bmatrix} = x \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ for all $x, y \in \mathbb{R}$. [Mark 0.5]

Moreover, $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ are linearly independent matrices. Hence, $\{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\}$ is a basis for V. So, $dim(V) = 2$. [Marks 0.5 + 0.5]

(c) Let $G = \{(0,0,3,3), (1,0,0,1), (-1,1,0,1), (0,0,1,1)\} \subseteq \mathbb{R}^4$. Find a basis B for $span(G)$ with $B \subseteq G$ and then find a basis C for $\mathbb{R}^4$ such that $B \subseteq C$.

Solution: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{B} := \{(1,0,0,1), (-1,1,0,1), (0,0,1,1)\}$ is a basis for $span(\boldsymbol{G})$ with $\boldsymbol{B} \subseteq \boldsymbol{G}$.

[Marks 1+1]

Next, $\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \Rightarrow C := \{(1,0,0,1), (-1,1,0,1), (0,0,1,1), (1,00,0)\}$

is a basis for $\mathbb{R}^4$ with $\boldsymbol{B} \subseteq C$. [Marks 1+1]

Question 3: [Marks: 4 + 3 + 3]

(a) Let $\boldsymbol{B} = \{(0,1,1), (1,1,0), (1,0,1)\}$ and $\boldsymbol{C} = \{(1,1,0), (1,0,2), (1,1,1)\}$ be two ordered bases for the Euclidean space $\mathbb{R}^3$ and $[v]_C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Then construct the transition matrix ${}_{\boldsymbol{B}}\boldsymbol{P}_{\boldsymbol{C}}$ from basis $\boldsymbol{C}$ to $\boldsymbol{B}$, and then find the coordinate vector $[\boldsymbol{v}]_{\boldsymbol{B}}$.

Solution: ${}_{\boldsymbol{B}}\boldsymbol{P}_{\boldsymbol{C}} = [[(1,1,0)]_{\boldsymbol{B}} \;\; [(1,0,2)]_{\boldsymbol{B}} \;\; [(1,1,1)]_{\boldsymbol{B}}] = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$ so that $[\boldsymbol{v}]_{\boldsymbol{B}} = {}_{\boldsymbol{B}}\boldsymbol{P}_{\boldsymbol{C}}[v]_C = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

[Marks (1+0.5+0.5+0.5) + (1+0.5)]

(b) Consider the vector space $\boldsymbol{M}_2(\mathbb{R})$ of 2×2 real matrices with the inner product:

$$< \mathbf{A}, \mathbf{B} > := trace(\mathbf{A}\boldsymbol{B}^T), \; \forall \; \mathbf{A}, \mathbf{B} \in \boldsymbol{M}_2(\mathbb{R}).$$

If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then compute the angle θ between the matrices **A** and **B**.

Solution: Since $< \boldsymbol{A}, \boldsymbol{B} > = trace\;(\boldsymbol{AB}^T) = trace\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = trace\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0,$

[Mark 1]

we get the asked angle: $\theta = cos^{-1}\frac{< \boldsymbol{A}, \boldsymbol{B} >}{||\boldsymbol{A}||\,||\boldsymbol{B}||} = cos^{-1}\frac{0}{||\boldsymbol{A}||\,||\boldsymbol{B}||} = cos^{-1}0 = \frac{\pi}{2}$. [Marks 1 + 0.5 + 0.5]

(c) Let x and y be nonzero orthogonal vectors in an inner product space. Then show that $\{x, y\}$ is linearly independent and $||x+y||^2 = ||x||^2 + ||y||^2$.

Solution: The given orthogonality of the vectors x and y means $< x, y > = 0$. If $\alpha x + \beta y = 0$ with $\alpha, \beta \in \mathbb{R}$, then $\alpha||x||^2 = \alpha < x, x > + \beta < x, y > = < x, \alpha x + \beta y > = < x, 0 > = 0$, and so $\alpha = 0$ since $x \neq 0$; similarly, $\beta||x||^2 = \alpha < y, x > + \beta < y, y > = < y, \alpha x + \beta y > = < y, 0 > = 0$ gives $\beta = 0$. Thus, the set $\{x, y\}$ is linearly independent. [Marks 0.5 + 1]

Moreover, $||x+y||^2 = <x+y, x+y> = <x,x> + 2<x,y> + <y,y> = ||x||^2 + ||y||^2$. [Marks 1.5]

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