[Solution Key]

King Saud University
College of Sciences
Department of Mathematics
Semester 471 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Question 1 [Marks 10]**:** Which of the given choices are correct?

(i) If a matrix A is symmetric and $A^T = -A$, then A must be:
a) identity b) non-singular c) inverse of itself d) ✔ zero.

(ii) If A is a 4×4 matrix such that $adj(A) = A^{-1}$, then the determinant $|adj(A)|$ is equal to:
a) ✔ $|A|^2$ b) $3|A|$ c) 4 d) $2|A|$.

(iii) If the matrix of coefficients in the linear system $AX = B$ is invertible, then the system must have:
a) infinitely many solutions b) ✔ a unique solution c) no solution d) zero solution.

(iv) Let $B = \{(2,1),(2,3)\}$ and $C = \{(0,1),(2,0)\}$ be ordered bases of a vector space V, then the transition matrix $\boldsymbol{P}_{\boldsymbol{B\to C}}$ from B to C is:
a) $\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}$ b) $\begin{bmatrix}1 & -3\\ -1 & 1\end{bmatrix}$ c) ✔ $\begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}$ d) $\begin{bmatrix}-2 & 6\\ 2 & -2\end{bmatrix}$.

(v) If $A = \begin{bmatrix}1 & 4 & 5 & 2\\ 2 & 1 & 3 & 0\\ -1 & 3 & 2 & 2\end{bmatrix}$, then $rank(A)$ is:
a) 0 b) 1 c) ✔ 2 d) 3.

(vi) If $A = \begin{bmatrix}-1 & 3\\ 4 & 1\end{bmatrix}$, $B = \begin{bmatrix}5 & 2\\ 0 & 1\end{bmatrix}$ are vectors in the vector space $M_2(\mathbb{R})$ with the standard inner product, then the angle θ between A and B is equal to:
a) $\theta = cos^{-1}(\frac{2}{3\sqrt{3}})$ b) $\theta = cos^{-1}\frac{\sqrt{2}}{30}$ c) ✔ $\theta = cos^{-1}\frac{2}{9\sqrt{10}}$ d) $\theta = cos^{-1}\frac{2}{\sqrt{30}}$.

(vii) If $T: M_2(\mathbb{R}) \to \mathbb{R}^2$ is the linear transformation defined by $T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = (a,b),\ \forall\, a,b,c,d \in \mathbb{R}$, then $ker(T)$ is equal to:
a) $\left\{\begin{bmatrix}s & t\\ 0 & 0\end{bmatrix} : s,t \in \mathbb{R}\right\}$ b) $\left\{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right\}$ c) ✔ $\left\{\begin{bmatrix}0 & 0\\ s & t\end{bmatrix} : s,t \in \mathbb{R}\right\}$ d) $\{(0,0)\}$.

(viii) If $\{(-3r+4s, r-s, r, s) \mid r,s \in \mathbb{R}\}$ is the solution set of the homogeneous system $A\boldsymbol{X} = O$ and T_A is the linear transformation given by $T_A(X) = AX$, then:
a) ✔ $nullity(T_A) = 2$ b) $nullity(A) = 0$ c) $rank(A^T) = 3$ (d) $ker(T_A) = \{0\}$.

(ix) If the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(x_1,x_2) = (x_1 - x_2, x_2 - x_1, 3x_2)$, then the induced matrix $[\boldsymbol{T}]_{\boldsymbol{C,B}}$ with respect to the ordered basis $B = \{(2,0),(1,1)\}$ of $\mathbb{R}^2$ and the ordered standard basis C of $\mathbb{R}^3$ is:
a) $\begin{bmatrix}1 & -1\\ -1 & 1\\ 0 & 3\end{bmatrix}$ b) ✔ $\begin{bmatrix}2 & 0\\ -2 & 0\\ 0 & 3\end{bmatrix}$ c) $\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 3\end{bmatrix}$ d) $\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$.

(x) If $A = \begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$, then the eigenvalues of A^4 are:
a) ✔ 16, 81 b) 2,3 c) 4, 9 d) −1, 2, 3.

Question 2 [Marks 2 + 2 + 2]:

(a) Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$. Find the invertible matrix C that satisfies $AC^{-1} = B$.

Solution: $AC^{-1} = B \Rightarrow C = B^{-1}A = \begin{bmatrix} 0 & \frac{-1}{3} & \frac{2}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{-2}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} & \frac{-1}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & \frac{1}{3} & \frac{-5}{3} \end{bmatrix}$.

(b) Find all the values of α for which the matrix $\begin{bmatrix} \alpha & 1 & 0 \\ \alpha+2 & 2 & 1 \\ \alpha^2 & 2 & 3 \end{bmatrix}$ is non-invertible.

Solution: $\begin{bmatrix} \alpha & 1 & 0 \\ \alpha+2 & 2 & 1 \\ \alpha^2 & 2 & 3 \end{bmatrix}$ is non-invertible $\Leftrightarrow \begin{vmatrix} \alpha & 1 & 0 \\ \alpha+2 & 2 & 1 \\ \alpha^2 & 2 & 3 \end{vmatrix} = 0 \Leftrightarrow \alpha = -3, 2$.

(c) Consider the matrix $A = \begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & -3 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$. Find *rank*(*A*) and $nullity(A^T)$.

Solution: $A = \begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & -3 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$ gives *rank*(*A*) = 3, and so

$nullity(A^T) = \text{number of columns} - rank(A^T) = 3 - rank(A) = 3 - 3 = 0.$

Question 3 [Marks 2 + (3 + 2 + 2)]:

(a) Consider the vector subspace $E = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0,\ 2x + 3y = 0\}$ of the vector space $\mathbb{R}^3$. Find a basis B of E and a basis C of $\mathbb{R}^3$ containing B.

Solution: $(x,y,z) \in E \Leftrightarrow (x,y,z) = t(-3,2,1)$ for all $t \in \mathbb{R}$. Hence, $B = \{(-3,2,1)\}$ is a basis of E. Next,

$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ means $C = \{(-3,2,1), (1,0,0), (0,1,0)\}$ is basis of $\mathbb{R}^3$ containing B.

(b) Consider the vectors $u_1 = (1,1,0)$, $u_2 = (0,1,1)$ and $u_3 = (1,1,-1)$ in the Euclidean space $\mathbb{R}^3$.

(i) Use the Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

Solution: By the Gram-Schmidt process: $w_1 = (1,1,0)$, so that $||w_1|| = \sqrt{2}$ and $v_1 = \frac{1}{\sqrt{2}}(1,1,0)$. Next,

$w_2 = u_2 - \frac{<u_2,\ w_1>}{||w_1||^2}w_1 = \left(-\frac{1}{2},\frac{1}{2},1\right)$, so that $||w_2|| = \sqrt{\frac{3}{2}}$ and $v_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{1}{2},\frac{1}{2},1\right)$. Finally,

$w_3 = u_3 - \frac{<u_3,\ w_1>}{||w_1||^2}w_1 - \frac{<u_3,\ w_2>}{||w_2||^2}w_2 = \left(-\frac{1}{3},\frac{1}{3},-\frac{1}{3}\right)$, so that $||w_3|| = \sqrt{\frac{1}{3}}$ and $v_3 = \sqrt{3}\left(-\frac{1}{3},\frac{1}{3},-\frac{1}{3}\right)$.

Thus, $\left\{v_1 = \frac{1}{\sqrt{2}}(1,1,0), v_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{1}{2},\frac{1}{2},1\right), v_3 = \sqrt{3}\left(-\frac{1}{3},\frac{1}{3},-\frac{1}{3}\right)\right\}$ is the required orthonormal basis of $\mathbb{R}^3$.

(ii) Express the vector $u = (1, 2, -2)$ as the linear combination of v_1, v_2, v_3.

Solution: Since $\left\{v_1 = \frac{1}{\sqrt{2}}(1,1,0), v_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{1}{2},\frac{1}{2},1\right), v_3 = \sqrt{3}\left(-\frac{1}{3},\frac{1}{3},-\frac{1}{3}\right)\right\}$ is an orthonormal basis of $\mathbb{R}^3$, we have

$(1,2,-2) = u = \sum_{i=1}^{3} < u,\ v_i > \ v_i = \frac{3}{\sqrt{2}}v_1 - \frac{\sqrt{3}}{\sqrt{2}}v_2 + \sqrt{3}\, v_3.$

(iii) Find the angle θ between the vectors u and v_1.

Solution: $\theta = cos^{-1}\frac{<u,v_1>}{||u||\,||v_1||\sqrt{2}} = cos^{-1}\frac{\frac{3}{\sqrt{2}}}{3} = cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$

Question 4 [Marks 2 + 2 + 3]:

Let $[T]_B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ be the matrix of the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ relative to the ordered basis $B = \{v_1 = (1,3), v_2 = (1,4)\}$. Find:

(i) $[T(v_1)]_B$ and $[T(v_2)]_B$.

Solution: $[[T(v_1)]_B\ [T(v_2)]_B] = [T]_B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \Rightarrow [T(v_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $[T(v_2)]_B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(ii) $T(v_1)$ and $T(v_2)$.

Solution: As seen above, $[T(v_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $[T(v_2)]_B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, where $B = \{v_1 = (1,3), v_2 = (1,4)\}$. Hence,

$T(v_1) = 1v_1 - 2v_2 = 1(1,3) - 2(1,4) = (-1,-5)$. Similarly, $T(v_2) = (8, 29)$.

(iii) a formula for $T(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

Solution: For any fixed, $(x,y) \in \mathbb{R}^2$, there exists unique scalars α_1 and α_2 such that:

$(x,y) = \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(1,3) + \alpha_2(1,4) = (\alpha_1 + \alpha_2, 3\alpha_1 + 4\alpha_2) \Rightarrow \alpha_1 + \alpha_2 = x, 3\alpha_1 + 4\alpha_2 = y.$

Hence, $\alpha_1 = 4x - y$ and $\alpha_2 = -3x + y$. Thus, by Part (ii), for all $(x, y) \in \mathbb{R}^2$:

$$T(x,y) = T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2) = (4x - y)(-1,-5) + (-3x + y)(8\,,29)$$
$$= (-28x + 9y\,, -107x + 34y)\,.$$

Question 5 [Marks 4 + 2 + 2]**:**

Consider the matrix $A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

(i) Find the eigenvalues of A and bases for the corresponding eigenspaces.

Solution: Eigenvalues: Real number λ is an eigenvalue of the given matrix $A \Leftrightarrow |A - \lambda I| = 0 \Leftrightarrow \lambda = -1, 0, 1$.

Eigen spaces: $(x, y, z) \epsilon\, E_{-1} \Leftrightarrow A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = - \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow (A + I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow (x, y, z) = z(-3,1,1)$. So, $E_{-1} = < (-3, 1, 1) >$. Similarly, $E_0 = < (-1, 1, 0) >$ and $E_1 = < (0, 1, 0) >$.

(ii) Is the matrix A diagonalizable? Justify your answer.

Solution: From the solution of Part (i), we know that the 3×3 matrix A has three different eigenvalues. Hence, the given matrix A is diagonalizable.

(iii) Compute A^9.

Solution: $\begin{bmatrix} -3 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}^{-1} \boldsymbol{A} \begin{bmatrix} -3 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^9 = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^9 \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -3 \\ 1 & 0 & 2 \end{bmatrix} = A.$

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