[Solution Key]

King Saud University
College of Sciences
Department of Mathematics
Semester 461 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed.***

Question 1 [Marks 1× 10]**:** Choose the correct answer:

(i) If the rows of a 3×4 matrix are linearly dependent, then the maximum dimension for *col*(A) is:

(a) 1 (b) ✓2 (c) 3 (d) 4

(ii) If A is a nonzero 4×7 matrix, then the possible values for *nullity*(A) are:

(a) 2,3,4,5,6 (b) 3,4,5,6,7 (c) ✓3,4,5,6 (d) 1,2,3,4

(iii) If u and v are nonzero vectors in an inner product space with $d(u, v) = d(u, -v)$, then u is orthogonal to:

(a) u (b) $u + v$ (c) $u - v$ (d) ✓v.

(iv) If $\{u, v\}$ is linearly independent and $\{u, v, w\}$ is linearly dependent, then:

(a) $\{u, w\}$ is linearly independent. (b) $\{v, w\}$ is linearly independent. (c) ✓$w \in span\{u, v\}$ (d) $u \in span\{v, w\}$.

(v) If $\begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}$ is the transition matrix from the basis $\{u, (1,1)\}$ to the basis $\{(5,4), v\}$ for $\mathbb{R}^2$, then u is equal to:

(a) (1,3) (b) $(4, -1)$. (c) ✓(14,11) (d) (3,0).

(vi) If $S = \{v_1 = (1,1), v_2 = (1,0)\}$ is a basis for $\mathbb{R}^2$ and the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is such that $T(v_1) = (1, -2)$ and $T(v_2) = (-4,1)$, then $T(5,-3)$ equals:

(a) ✓ (-35, 14) (b) (2, -10) (c) (2, 5) (d) (2, 10).

(vii) The set $\{(-3, 4, 0), (4, x, 0), (0, 0, x)\}$ of vectors in the Euclidean space $\mathbb{R}^3$ is orthogonal *iff* :

(a) $x = 1$ (b) ✓ $x = 3$ (c) $x =$ -3 (d) $x = 5$

(viii) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T(1,0) = (1,2)$ and $T(1,1) = (5, -3)$, then its standard matrix is:

(a) ✓$\begin{bmatrix} 1 & 4 \\ 2 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 5 \\ 2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 5 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 6 \\ 2 & -1 \end{bmatrix}$

(ix) The eigenvalues of a square matrix A are the same as the eigenvalues of:

(a) A^2. (b) ✓A^T. (c) *RREF*(A). (d) *adj*(A)

(x) If A is diagonalizable matrix, then *det*(A) equals:

(a) *The sum of the eigen values of* A (b) ✓*The product of the eigen values of* A (c) *Zero* (d) *Number of columns in* A.

Question 2 [Marks 2 + 2 + 2]:

(a) Let a matrix $\boldsymbol{A}$ satisfy $\boldsymbol{A}^2 + \boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Then show that A is invertible.

Solution: $|A|\,|A+I| = |A(A+I)| = |A^2 + A| = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9 \neq 0$ [Mark 1]

$\Rightarrow |A| \neq 0$; which means that the matrix A is invertible. [Mark 1]

(b) Consider $\boldsymbol{B}, \boldsymbol{C} \in \mathbf{M}_3(\mathbb{R})$ with $|\mathbf{B}| = 2|\mathbf{C}| = 1$. Then evaluate $|3\,\boldsymbol{C}\,\mathbf{B}\,adj(B^{-3})|$.

Solution: $adj(B) = |B|B^{-1} \Rightarrow adj(B^{-3}) = |B|^{-3}B^3$. [Mark 1]

Hence, $|3\,\boldsymbol{C}\,\mathbf{B}\,adj(B^{-3})| = |3\,\boldsymbol{C}\,\mathbf{B}\,(|B|^{-3}B^3)| = |3\,\boldsymbol{C}| = 27/2$. [Mark 1]

(c) Find the values of a, b such that the following system of linear equations

$$x - 2y + 3z = 4$$
$$3x - 4y + 5z = b$$
$$2x - 3y + az = 5$$

has: (i) no solution (ii) unique solution.

Solution: $[A|\mathrm{I}] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 3 & -4 & 5 & b \\ 2 & -3 & a & 5 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & (b-12)/2 \\ 0 & 0 & a-4 & (6-b)/2 \end{array}\right]$. [Mark 1]

Hence, (i) $a = 4$ and $b \neq 6$ (ii) $a \neq 4$. [Mark 1]

Question 3 [Marks 3 + 3 + 2]:

(a) Find a subset $\boldsymbol{B}$ of $\boldsymbol{G} = \{(1,1,-4,-3), (2,0,2,-2), (1,2,-9,-5)\}$ that forms a basis for $span(\boldsymbol{G})$. Then express each vector in $\boldsymbol{G} - \boldsymbol{B}$ as a linear combination of vectors in $\boldsymbol{B}$.

Solution: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ -4 & 2 & -9 \\ -3 & -2 & -5 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (REF)$. [Mark 1]

Hence, $\boldsymbol{B} = \{(1,1,-4,-3), (2,0,2,-2)\} \subseteq \boldsymbol{G}$ forms a basis for $span(\boldsymbol{G})$. [Mark 1]

Moreover, $(1,2,-9,-5) = 2(1,1,-4,-3) - \frac{1}{2}(2,0,2,-2)$. [Mark 1]

(b) Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$. Then find a basis for the column space $col(\boldsymbol{A})$ and dimension of the null space $\boldsymbol{N}(\boldsymbol{A})$.

Solution: $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (REF)$. [Mark 1]

Hence, $\{(1,-1,0,1), (0,1,2,1), (3,-3,1,4)\}$ is a basis for the column space $col(\boldsymbol{A})$ [Mark 1]

and $dim(\boldsymbol{N}(\boldsymbol{A})) = 5 - 3 = 2$. [Mark 1]

(c) Let B and $\boldsymbol{B'}$ be two ordered bases for $\mathbb{R}^2$ with a transition matrix $\boldsymbol{P}_{\mathrm{B} \to \mathrm{B'}} = \begin{bmatrix} 5 & 3 \\ -1 & -1 \end{bmatrix}$ from B to $\boldsymbol{B'}$. If $[\boldsymbol{v}]_{\mathbf{B'}} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$ is the coordinate vector of a vector $\boldsymbol{v} \in \mathbb{R}^2$ relative to the basis $\boldsymbol{B'}$. Then find $[\boldsymbol{v}]_{\boldsymbol{B}}$.

Solution: $\boldsymbol{P}_{B'\to B} = (\boldsymbol{P}_{B\to B'})^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -1 \end{bmatrix}^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -3 \\ 1 & 5 \end{bmatrix}$. [Mark 1]

Hence, $[\boldsymbol{v}]_{\boldsymbol{B}} = \boldsymbol{P}_{B'\to B}[\boldsymbol{v}]_{\mathbf{B}'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. [Mark 1]

Question 4 [Marks 2 + 3 + 5]**:**

(a) Let $\{v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,0), v_4\}$ be the orthonormal basis obtained by applying the Gram-Schmidt algorithm on the basis $\{u_1 = (3,0,0,0), u_2 = (3,3,0,0), u_3 = (3,3,3,0), u_4 = (3,3,3,3)\}$ of Euclidean inner product space $\mathbb{R}^4$. Then find the vector v_4.

Solution: According to the Gram-Schmidt algorithm,

$$w_4 = u_4 - \frac{<u_4,v_1>}{||v_1||^2}v_1 - \frac{<u_4,v_2>}{||v_2||^2}v_2 - \frac{<u_4,v_3>}{||v_3||^2}v_3$$ [Mark 1]

$= (3,3,3,3) - 3(1,0,0,0) - 3(0,1,0,0) - 3(0,0,1,0) = (0,0,0,3).$ [Mark 0.5]

Hence, $v_4 = \frac{1}{||w_4||}w_4 = \frac{1}{3}(0,0,0,3) = (0,0,0,1).$ [Mark 0.5]

(b) Let v_0 be any fixed vector in an inner product space V of dimension n and $T: V \to \mathbb{R}$ be the linear transformation defined by $T(\mathrm{v}) = < v, v_0 >$ for all $v \in V$. If $v_0 \,\epsilon\, Ker(T)$, then show that $nullity(T) = n$.

Solution: If $v_0 \,\epsilon\, Ker(T)$ then $0 = T(v_0) = < v_0, v_0 >$; which means $v_0 = 0$. [Mark 1]

So, $T(v) = < v, v_0 > = < v, 0 > = 0$ for all $v \in V$; meaning that $Ker(T) = V$. [Mark 1]

Thus, $nullity(T) = \dim(Ker(\boldsymbol{T})) = \dim(V) = n$. [Mark 1]

(c) Let $\boldsymbol{B} = \{u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (0,0,1)\}$ and $\boldsymbol{C} = \{v_1 = (1,1,1)), v_2 = (1,1,0), v_3 = (1,0,0)\}$ be two ordered bases for $\mathbb{R}^3$. Let the linear transformation $\boldsymbol{T}: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$\boldsymbol{T}(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 - x_1, x_1 - x_3)\}.$$

Find the matrix $[\boldsymbol{T}]_{\boldsymbol{B}}$ of the transformation $\boldsymbol{T}$ relative to the basis $\boldsymbol{B}$ and then use it to find the matrix $[\boldsymbol{T}]_{\boldsymbol{C}}$.

Solution: $[T(u_1)]_B = [(1,-1,1)]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Similarly, $[T(u_2)]_B == \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $[T(u_3)]_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. [Mark 1]

So, $[T]_B = [\,[T(u_1)]_B\ \ [T(u_2)]_B\ \ [T(u_3)]_B] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. [Mark 1]

Next, $P_{C\to B} = [\,[v_1]_B\ \ [v_2]_B\ \ [v_3]_B] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ [Mark 1]

and $P_{B\to C} = (P_{C\to B})^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. [Mark 1]

Hence, $[T]_C = P_{B\to C}[T]_B P_{C\to B} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$. [Mark 1]

Question 5 [Marks 2 + 1 + 3]**:** Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. Then:

(a) Find the eigenvalues of $\boldsymbol{A}$.

Solution: $-(\mathbf{1} + \lambda)(1-\lambda)(2-\lambda) = \begin{vmatrix} \mathbf{1}-\lambda & 0 & 3 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = |\boldsymbol{A} - \lambda \mathrm{I}| = 0$ [Mark 1]

$\Rightarrow$ the eigenvalues of A are $-1, 1, 2$. [Mark 1]

(b) Is the matrix $\boldsymbol{A}$ diagonalizable? Justify your answer.

Solution: Since the matrix A is of size 3×3 having 3 different eigenvalues, it is diagonalizable. [Mark 1]

(c) Find a diagonal matrix $\boldsymbol{D}$ and an invertible matrix such that $\boldsymbol{P^{-1}AP = D}$.

Solution: It is easily seen that $E_{-1} = <(-9,1,6)>$, $E_1 = <(-1,1,0)>$ and $E_2 = <(-1,1,0)>$. [Mark 1]

Hence, $P = \begin{bmatrix} -9 & -1 & -1 \\ 1 & 1 & 1 \\ 6 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ [Mark 1]

with $\boldsymbol{P^{-1}} = \begin{bmatrix} 0 & 0 & \frac{1}{6} \\ -1 & 0 & -\frac{3}{2} \\ 1 & 1 & \frac{4}{3} \end{bmatrix}$ satisfying $\boldsymbol{P^{-1}AP = D}$. [Mark 1]

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